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Multinomial Probability Distribution and Quantum Deformed Algebras

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Abstract. An examination is conducted on the multinomial coefficients derived from generalized quantum deformed algebras, and on their recurrence relations. The $\mathcal{R}(p,q)$-deformed multinomial probability distribution and the negative $\mathcal{R}(p,q)$-deformed multinomial probability distribution are constructed, and the recurrence relations are determined. From our general result, we deduce particular cases that correspond to quantum algebras considered in the literature.

1. Introduction

The q-deformations of the Vandermonde formula, the Cauchy formula and the univariate discrete probability distributions were investigated in [2]. Their limiting distributions were derived, the q-deformed multinomial coefficient was defined, and recurrence relations for these coefficients were deduced. Then, in [3], the q-deformed multinomial and negative q-deformed multinomial probability distributions of the first and second kind were presented [3].

Now, let p and q be two positive real numbers such that $0 < q < p < 1$. We

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consider a meromorphic function $\mathcal{R}$ defined on $\mathbb{C}\times\mathbb{C}$ by [7]:

$$\mathcal{R}(u,v)=\sum_{s,t=-l}^{\infty} r_{st}u^s v^t, \tag{1.1}$$

with an eventual isolated singularity at the zero, where r_{st} are complex numbers, $l\in\mathbb{N}\cup\{0\}$, $\mathcal{R}(p^n,q^n)>0,\forall n\in\mathbb{N}$, and $\mathcal{R}(1,1)=0$ by definition. We denote by $\mathbb{D}_R$ the bidisk

$$\begin{aligned}\mathbb{D}_R &:= \prod_{j=1}^{2}\mathbb{D}_{R_j}\\ &= \left\{e=(e_1,e_2)\in\mathbb{C}^2 : |e_j|<R_j\right\},\end{aligned}$$

where R is the convergence radius of the series (1.1) defined by Hadamard formula[12]:

$$\limsup_{s+t\longrightarrow\infty}\sqrt[s+t]{|r_{st}|R_1^s\,R_2^t}=1.$$

We denote by $\mathcal{O}(\mathbb{D}_R)$ the set of holomorphic functions defined on $\mathbb{D}_R$.

The $\mathcal{R}(p,q)$-deformed numbers are given by [7]

$$[n]_{\mathcal{R}(p,q)}:=\mathcal{R}(p^n,q^n),\qquad n\in\mathbb{N}, \tag{1.2}$$

by which the $\mathcal{R}(p,q)$-deformed factorials are defined as

$$[n]!_{\mathcal{R}(p,q)}:=\begin{cases}1 \quad \text{for} \quad n=0\\ \mathcal{R}(p,q)\cdots\mathcal{R}(p^n,q^n) \quad \text{for} \quad n\geq 1,\end{cases}$$

and the $\mathcal{R}(p,q)$-deformed binomial coefficients as

$$\left[\begin{array}{c} m\\ n\end{array}\right]_{\mathcal{R}(p,q)}:=\frac{[m]!_{\mathcal{R}(p,q)}}{[n]!_{\mathcal{R}(p,q)}[m-n]!_{\mathcal{R}(p,q)}},\quad (m,n)\in\mathbb{N}\cup\{0\};\quad m\geq n.$$

The linear operators on $\mathcal{O}(\mathbb{D}_R)$ are defined by

$$\begin{aligned}Q &: \varphi\longmapsto Q\varphi(z):=\varphi(qz)\\ P &: \varphi\longmapsto P\varphi(z):=\varphi(pz),\end{aligned}$$

and the $\mathcal{R}(p,q)$-deformed derivative given as

$$\partial_{\mathcal{R},p,q}:=\partial_{p,q}\frac{p-q}{P-Q}\mathcal{R}(P,Q)=\frac{p-q}{pP-qQ}\mathcal{R}(pP,qQ)\partial_{p,q},$$

where $\partial_{p,q}$ is the (p,q)-derivative

$$\partial_{p,q}:\varphi\longmapsto\partial_{p,q}\varphi(z):=\frac{\varphi(pz)-\varphi(qz)}{z(p-q)}.$$

We spoke of the quantum algebra associated with the $\mathcal{R}(p,q)$-deformation. It is a quantum algebra, $\mathcal{A}_{\mathcal{R}(p,q)}$, generated by the set of operators $\{1, A, A^\dagger, N\}$ satisfying the following commutation relations [8]:

$$\begin{aligned} AA^\dagger &= [N+1]_{\mathcal{R}(p,q)}, & A^\dagger A &= [N]_{\mathcal{R}(p,q)} \\ [N,\ A] &= -A, & [N,\ A^\dagger] &= A^\dagger. \end{aligned} \tag{1.3}$$

Its realization on $\mathcal{O}(\mathbb{D}_R)$ is given by

$$A^\dagger := z, \qquad A := \partial_{\mathcal{R}(p,q)}, \qquad N := z\partial_z$$

where $\partial_z := \frac{\partial}{\partial z}$ is the usual derivative on $\mathbb{C}$. Let us recall some notions useful in this paper.

The model deformation structure functions $\tau_i, i \in \{1,2\}$, depending on the deformation parameters p and q were introduced in [5].

For $a, b \in \mathbb{N}$, the $\mathcal{R}(p,q)$-deformed shifted factorial is defined by [5]:

$$\big(a \oplus b\big)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n}\big(a\,\tau_1^{i-1} + b\,\tau_2^{i-1}\big), \quad \text{with} \quad \big(a \oplus b\big)^0_{\mathcal{R}(p,q)} := 1.$$

Analogously,

$$\big(a \ominus b\big)^n_{\mathcal{R}(p,q)} := \prod_{i=1}^{n}\big(a\,\tau_1^{i-1} - b\,\tau_2^{i-1}\big), \quad \text{with} \quad \big(a \ominus b\big)^0_{\mathcal{R}(p,q)} := 1.$$

Furthermore, the $\mathcal{R}(p,q)$-deformed factorial of a of order r is defined by[6]:

$$[a]_{r,\mathcal{R}(p,q)} = \prod_{i=1}^{r}[a-i+1]_{\mathcal{R}(p,q)}, \quad r \in \mathbb{N}, \tag{1.4}$$

and the following relations hold :

$$[a]_{\mathcal{R}(p^{-1},q^{-1})} \;=\; (\tau_1\,\tau_2)^{1-a}\,[a]_{\mathcal{R}(p,q)}, \tag{1.5}$$

$$[a]_{\mathcal{R}(p^{-1},q^{-1})}! = (\tau_1\tau_2)^{-\binom{r}{2}}\,[a]_{\mathcal{R}(p,q)}!, \tag{1.6}$$

and

$$[a]_{r,\mathcal{R}(p^{-1},q^{-1})} = (\tau_1\,\tau_2)^{-ar+\binom{r+1}{2}}\,[a]_{r,\mathcal{R}(p,q)}. \tag{1.7}$$

The $\mathcal{R}(p,q)$-deformed of orthogonal polynomials and basic univariate discrete distributions of probability theory were defined and discussed by Hounkonnou and Melong [5]. Relevant $\mathcal{R}(p,q)$-deformed factorial moments of a random variable and associated expressions of mean and variance established, and recurrence relations for the probability distributions were derived, recovering known results as particular cases. Furthermore, the multivariate probability distributions (Pólya, inverse Pólya, hypergeometric and negative hypergeometric) of the generalized quantum deformed algebras were constructed. Their corresponding bivariate probability distributions and properties were derived and determined [11].

Our aims are to construct the multinomial coefficients, the multinomial probability distribution and properties corresponding to the $\mathcal{R}(p,q)$-deformed quantum algebras [8].

This paper is organized as follows: Section 2 is focussed on multinomial coefficients associated to $\mathcal{R}(p,q)$-deformed quantum algebras. Alternate presentations and their recurrence relations are derived. In Section 3, we contruct the $\mathcal{R}(p,q)$-deformed multinomial probability distributions of the first and second kinds. Section 4 is dedicated to particular cases of our results corresponding to known quantum algbras. We make some concluding remarks in Section 5.

2. $\mathcal{R}(p,q)$-deformed Multinomial Formulae

In this section we investigate the multinomial coefficients, multinomial formula and negative multinomial formula in the framework of the $\mathcal{R}(p,q)$-deformed quantum algebras. The recurrence relations are also determined.

Theorem 2.1. *The $\mathcal{R}(p,q)$-deformed multinomial coefficient*

$$\begin{bmatrix} x \\ r_1,r_2,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} = \frac{[x]_{r_1+r_2+\cdots+r_k,\mathcal{R}(p,q)}}{[r_1]_{\mathcal{R}(p,q)}![r_2]_{\mathcal{R}(p,q)}!\cdots[r_k]_{\mathcal{R}(p,q)}!} \tag{2.1}$$

satisfies the recurrence relation

$$\begin{aligned} \begin{bmatrix} x \\ r_1,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} &= \tau_1^{s_k}\begin{bmatrix} x-1 \\ r_1,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} + \tau_2^{x-m_1}\begin{bmatrix} x-1 \\ r_1-1,r_2,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \\ &+ \tau_2^{x-m_2}\begin{bmatrix} x-1 \\ r_1,r_2-1,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} + \cdots \\ &+ \tau_2^{x-m_k}\begin{bmatrix} x \\ r_1,r_2,\cdots,r_k-1 \end{bmatrix}_{\mathcal{R}(p,q)} \end{aligned} \tag{2.2}$$

Or, equivalently,

$$\begin{aligned} \begin{bmatrix} x \\ r_1,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} &= \tau_2^{s_k}\begin{bmatrix} x-1 \\ r_1,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} + \tau_1^{x-m_1}\begin{bmatrix} x-1 \\ r_1-1,r_2,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} \\ &+ \tau_1^{x-m_2}\tau_2^{s_1}\begin{bmatrix} x-1 \\ r_1,r_2-1,\cdots,r_k \end{bmatrix}_{\mathcal{R}(p,q)} + \cdots \\ &+ \tau_1^{x-m_k}\tau_2^{s_{k-1}}\begin{bmatrix} x-1 \\ r_1,r_2,\cdots,r_k-1 \end{bmatrix}_{\mathcal{R}(p,q)} \end{aligned} \tag{2.3}$$

where $r_j \in \mathbb{N}$ and $j \in \{1,2,\cdots,k\}$, with $m_j = \sum_{i=j}^k r_i$ and $s_j = \sum_{i=1}^j r_i$.

Proof. Since

$$[x]_{s_k,\mathcal{R}(p,q)} = [x]_{\mathcal{R}(p,q)}\,[x-1]_{s_k-1,\mathcal{R}(p,q)},$$
$$[x-1]_{s_k,\mathcal{R}(p,q)} = [x-1]_{s_k-1,\mathcal{R}(p,q)}\,[x-s_k]_{\mathcal{R}(p,q)}$$

and

$$[x]_{\mathcal{R}(p,q)} = \tau_1^{s_k}\,[x-s_k]_{\mathcal{R}(p,q)} + \tau_2^{x-s_k}\,[s_k]_{\mathcal{R}(p,q)}.$$

Then, the $\mathcal{R}(p,q)$-deformed factorials of x of order $s_k = \sum_{i=1}^{k} r_k$ satisfies the recurrance relation

$$[x]_{s_k,\mathcal{R}(p,q)} = \tau_1^{s_k}\,[x-1]_{s_k,\mathcal{R}(p,q)} + \sum_{j=1}^{k}\tau_2^{x-m_j}\,[r_j]_{\mathcal{R}(p,q)}\,[x-1]_{s_k-1,\mathcal{R}(p,q)}. \tag{2.4}$$

Multiplying both sides of the relation (2.4) by $1/[r_1]_{\mathcal{R}(p,q)}![r_2]_{\mathcal{R}(p,q)}!\cdots[r_k]_{\mathcal{R}(p,q)}!$ and using the $\mathcal{R}(p,q)$-deformed multinomial coefficient (2.1), we obtain relation (2.2). Similarly, the $\mathcal{R}(p,q)$-deformed number can be expressed as

$$[x]_{\mathcal{R}(p,q)} = \tau_2^{s_k}\,[x-s_k]_{\mathcal{R}(p,q)} + \sum_{j=1}^{k}\tau_1^{x-m_j}\,\tau_2^{s_{j-1}}\,[r_j]_{\mathcal{R}(p,q)}$$

and the $\mathcal{R}(p,q)$-deformed factorial of x of order s_k satisfies the recursion relation

$$[x]_{s_k,\mathcal{R}(p,q)} = \tau_2^{s_k}\,[x-1]_{s_k,\mathcal{R}(p,q)} + \sum_{j=1}^{k}\tau_1^{x-m_j}\,\tau_2^{s_{j-1}}\,[r_j]_{\mathcal{R}(p,q)}\,[x-1]_{s_k-1,\mathcal{R}(p,q)}, \tag{2.5}$$

with $s_0 = 0$.

Dividing the both sides of the relation (2.5) by $[r_1]_{\mathcal{R}(p,q)}![r_2]_{\mathcal{R}(p,q)}!\cdots[r_k]_{\mathcal{R}(p,q)}!$ and using (2.1), the relation (2.3) is readily derived and the proof is achieved. □

Remark 2.2.

(i) From the relations (1.5), (1.6) and (1.7), we obtain the $\mathcal{R}(p^{-1},q^{-1})$- deformed multinomial coefficients in the simpler form:

$$\left[{x \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p^{-1},q^{-1})} = (\tau_1\tau_2)^{-\sum_{j=1}^{k} r_j(x-m_j)}\left[{x \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} \tag{2.6}$$

or

$$\left[{x \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p^{-1},q^{-1})} = (\tau_1\tau_2)^{-\sum_{j=1}^{k} r_j(x-s_j)}\left[{x \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} \tag{2.7}$$

where $s_j = \sum_{i=1}^{j} r_i$, and $m_j = \sum_{i=j}^{k} r_i$, for $r_j \in \mathbb{N}$, $j \in \{1,2,\cdots,k\}$ and $k \in \mathbb{N}$.

Indeed, by replacing $\mathcal{R}(p,q)$ with $\mathcal{R}(p^{-1},q^{-1})$ in relation (2.1), and using the formulae

$$[x]_{r,\mathcal{R}(p^{-1},q^{-1})} = \big(\tau_1\tau_2\big)^{-xr+\binom{r+1}{2}}\,[x]_{r,\mathcal{R}(p,q)}$$

and

$$[r]_{\mathcal{R}(p^{-1},q^{-1})}! = \big(\tau_1\tau_2\big)^{\binom{r}{2}}\,[r]_{\mathcal{R}(p,q)}!,$$

we obtain,

$$\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{\mathcal{R}(p^{-1},q^{-1})} = \frac{\big(\tau_1\tau_2\big)^{-xs_k+\binom{s_k+1}{2}}[x]_{r_1+r_2+\cdots+r_k,\mathcal{R}(p,q)}}{\big(\tau_1\tau_2\big)^{-\sum_{j=1}^{k}\binom{r_j}{2}}[r_1]_{\mathcal{R}(p,q)}![r_2]_{\mathcal{R}(p,q)}!\cdots[r_k]_{\mathcal{R}(p,q)}!}$$
$$= \big(\tau_1\tau_2\big)^{-xs_k+\binom{s_k+1}{2}+\sum_{j=1}^{k}\binom{r_j}{2}}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{\mathcal{R}(p,q)}.$$

Moreover,

$$-xs_k+\binom{s_k+1}{2}+\sum_{j=1}^{k}\binom{r_j}{2} = -xs_k+\sum_{j=1}^{k-1}r_j\,m_{j+1}+\sum_{j=1}^{k}\left(\binom{r_j+1}{2}+\binom{r_j}{2}\right)$$
$$= -\sum_{j=1}^{k}r_j(x-m_j) = -\sum_{j=1}^{k}r_j(x-s_j).$$

Relations (2.6) and (2.7) follow.

(ii) Another recurrence relations can be obtained by replacing $\mathcal{R}(p,q)$ by $\mathcal{R}(p^{-1},q^{-1})$, and using the expression (2.6), respectively. Thus, the recursion relations (2.2) and (2.3) take the following forms:

$$\begin{aligned}\left[{x \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} &= \tau_2^{m_1}\left[{x-1 \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} + \tau_2^{m_2}\left[{x-1 \atop r_1-1,r_2,\cdots,r_k}\right]_{\mathcal{R}(p,q)}\\ &+ \tau_2^{m_3}\left[{x-1 \atop r_1,r_2-1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} + \cdots\\ &+ \tau_1^{x}\left[{x-1 \atop r_1,r_2,\cdots,r_{k-1}}\right]_{\mathcal{R}(p,q)}\end{aligned} \qquad (2.8)$$

and

$$\begin{aligned}\left[{x \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} &= \tau_1^{x}\left[{x-1 \atop r_1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} + \tau_2^{x-s_1}\left[{x-1 \atop r_1-1,r_2,\cdots,r_k}\right]_{\mathcal{R}(p,q)}\\ &+ \tau_2^{x-s_2}\left[{x-1 \atop r_1,r_2-1,\cdots,r_k}\right]_{\mathcal{R}(p,q)} + \cdots\\ &+ \tau_2^{x-s_k}\left[{x-1 \atop r_1,\cdots,r_{k-1}}\right]_{\mathcal{R}(p,q)}.\end{aligned} \qquad (2.9)$$

(iii) The q-multinomial coefficients and formula given in [3, eq (2.1)] can be recovered by taking $\mathcal{R}(x,1)=(1-q)^{-1}(1-x)$ involving $\tau_1=1$ and $\tau_2=q$.

(iv) Taking $k=1$, we obtained the $\mathcal{R}(p,q)$-deformed binomial coefficients and related relations of [5, p.3].

Let us generalize the multinomial formulas to the general framework of the $\mathcal{R}(p,q)$-deformed quantum algebras.

Theorem 2.3. *For n a positive integers, $x,p,$ and q real numbers, the following* *relation holds:*

$$\prod_{j=1}^{k}(1\oplus x_j)^n_{\mathcal{R}(p,q)} = \sum\left[\begin{matrix} n \\ r_1,\cdots,r_k\end{matrix}\right]_{\mathcal{R}(p,q)}\prod_{j=1}^{k} x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}}$$

$$\times\Big(\tau_1^{n-s_{j-1}}\oplus x_j\tau_2^{n-s_{j-1}}\Big)^{s_{j-1}}_{\mathcal{R}(p,q)}, \tag{2.10}$$

where $r_j\in\{0,\cdots,n\}$, $j\in\{1,\cdots,k\}$, *with* $\sum_{i=1}^{k} r_i\leq n$ *and* $s_j=\sum_{i=1}^{j} r_i$, $s_0=0$.

Proof. Setting

$$s_n(x_1,\cdots,x_k;p,q) = \sum\left[\begin{matrix} n \\ r_1,\cdots,r_k\end{matrix}\right]_{\mathcal{R}(p,q)}\prod_{j=1}^{k} x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}} \times \Big(\tau_1^{n-s_{j-1}}\oplus x_j\tau_2^{n-s_{j-1}}\Big)^{s_{j-1}}_{\mathcal{R}(p,q)}$$

and using

$$\left[\begin{matrix} n \\ r_1\end{matrix}\right]_{\mathcal{R}(p,q)}\left[\begin{matrix} n-s_1 \\ r_2\end{matrix}\right]_{\mathcal{R}(p,q)}\cdots\left[\begin{matrix} n-s_{k-1} \\ r_k\end{matrix}\right]_{\mathcal{R}(p,q)} = \left[\begin{matrix} n \\ r_1,\cdots,r_k\end{matrix}\right]_{\mathcal{R}(p,q)}, \tag{2.11}$$

we get:

$$s_n(x_1,\cdots,x_k;p,q) = \prod_{j=1}^{k}\left(\sum_{r_j=0}^{n-s_{j-1}}\left[\begin{matrix} n-s_{j-1} \\ r_j\end{matrix}\right]_{\mathcal{R}(p,q)} x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}}\right) \times \Big(\tau_1^{n-s_{j-1}}\oplus x_j\tau_2^{n-s_{j-1}}\Big)^{s_{j-1}}_{\mathcal{R}(p,q)}.$$

From the $\mathcal{R}(p,q)$-deformed binomial formula, the j^{th}-sum is

$$\Big(1\oplus x_j\Big)^{n-s_{j-1}}_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n-s_{j-1}}\left[\begin{matrix} n-s_{j-1} \\ r_j\end{matrix}\right]_{\mathcal{R}(p,q)} x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}},$$

where $j\in\{1,2,\cdots,k\}$. Moreover,

$$\Big(1\oplus x_j\Big)^{n-s_{j-1}}_{\mathcal{R}(p,q)}\Big(\tau_1^{n-s_{j-1}}\oplus x_j\,\tau_2^{n-s_{j-1}}\Big)^{s_{j-1}}_{\mathcal{R}(p,q)} = \Big(1\oplus x_j\Big)^{n}_{\mathcal{R}(p,q)},$$

with $j\in\{1,2,\cdots,k\}$. Thus

$$s_n(x_1,\cdots,x_k;p,q) = \prod_{j=1}^{k}(1\oplus x_j)^n_{\mathcal{R}(p,q)}.$$

□

Theorem 2.4. *Let n be a positive integers, p and q real numbers. Then,*

$$\prod_{j=1}^{k}(1\oplus x_j)^n_{\mathcal{R}(p,q)}=\sum\left[\begin{array}{c}n+s_k-1\\ r_1,r_2,\cdots,r_k\end{array}\right]_{\mathcal{R}(p,q)}\prod_{j=1}^{k}\frac{x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}}}{\left(\tau_1^n\oplus x_j\tau_2^n\right)^{s_k-s_{j-1}}_{\mathcal{R}(p,q)}}.$$

Equivalently, $\prod_{j=1}^{k}(1\oplus x_j)^n_{\mathcal{R}(p,q)}=$

$$\sum_{r_j\in\mathbb{N}}\left[\begin{array}{c}n+s_k-1\\ r_1,\cdots,r_k\end{array}\right]_{\mathcal{R}(p,q)}\prod_{j=1}^{k}\frac{x_j^{n+s_k-s_{j-1}}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{n+s_k-s_{j-1}}{2}+r_j}}{\left(\tau_1^n\oplus x_j\tau_2^n\right)^{s_k-s_{j-1}}_{\mathcal{R}(p,q)}},$$

where $j\in\{1,2,\cdots,k\}$, with $s_j=\sum_{i=1}^{j}r_i$ and $s_0=0$.

Proof. Consider the multiple sum defined as follows:

$$s_n(x_1,\cdots,x_k;p,q)=\sum_{r_j=0}^{\infty}\left[\begin{array}{c}n+s_k-1\\ r_1,r_2,\cdots,r_k\end{array}\right]_{\mathcal{R}(p,q)}\prod_{j=1}^{k}\frac{x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}}}{\left(\tau_1^n\oplus x_j\tau_2^n\right)^{s_k-s_{j-1}}_{\mathcal{R}(p,q)}}$$

and using the relation (2.11), with $n+s_k-1$ instead of n, we obtain:

$$s_n(x_1,\cdots,x_k;p,q)=\prod_{j=1}^{k}\left(\sum_{r_j=0}^{\infty}\left[\begin{array}{c}n-s_k-s_{j-1}\\ r_j\end{array}\right]_{\mathcal{R}(p,q)}\frac{x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}}}{\left(\tau_1^n\oplus x_j\tau_2^n\right)^{s_k-s_{j-1}}_{\mathcal{R}(p,q)}}\right).$$

From the negative $\mathcal{R}(p,q)$-deformed binomial formula:

$$\prod_{i=1}^{n}\left(\tau_1^{i-1}+x\,\tau_2^{i-1}\right)^{-1}=\sum_{k=0}^{\infty}\left[\begin{array}{c}n+k-1\\ k\end{array}\right]_{\mathcal{R}(p,q)}\frac{\tau_1^{\binom{n-1}{2}}\tau_2^{\binom{k}{2}}x^k}{\left(\tau_1^n\oplus\tau_2^n\right)^k_{\mathcal{R}(p,q)}},$$

we get:

$$(2.12)\quad\sum_{r_j=0}^{\infty}\left[\begin{array}{c}n-s_k-s_{j-1}\\ r_j\end{array}\right]_{\mathcal{R}(p,q)}\frac{x_j^{r_j}\tau_1^{\binom{n-r_j}{2}}\tau_2^{\binom{r_j}{2}}}{\left(\tau_1^n\oplus x_j\tau_2^n\right)^{s_k-s_{j-1}}_{\mathcal{R}(p,q)}}=\left(1\oplus x_j\right)^n_{\mathcal{R}(p,q)}$$

and so

$$s_n(x_1,\cdots,x_k;p,q)=\prod_{j=1}^{k}\left(1\oplus x_j\right)^n_{\mathcal{R}(p,q)}.$$

An equivalent formula can be derived by putting $p=p^{-1},q=q^{-1},x_j=x_j^{-1}$ and $\mathcal{R}(p,q)=\mathcal{R}(p^{-1},q^{-1})$. □

Theorem 2.5. *Let $x_j, j \in \{1,2,\cdots,k+1\}, p,$ and q real numbers. For n positive integer, the following result holds.*

$$(1\ominus\Lambda_k)^n_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n}\left[\begin{array}{c} n \\ r_1, r_2, \cdots, r_k \end{array}\right]_{\mathcal{R}(p,q)} \times \prod_{j=1}^{k} x_j^{n-s_j}\big(1\ominus x_j\big)^{r_j}_{\mathcal{R}(p,q)}\big(1\ominus x_{k+1}\big)^{n-s_k}_{\mathcal{R}(p,q)}, \tag{2.13}$$

where $r_j \in \{0,\cdots,n\}$, $j \in \{1,\cdots,k\}$, with $\sum_{i=1}^k r_i \le n$ and $s_j = \sum_{i=1}^j r_i$, $s_0 = 0$, and $\Lambda_k = \prod_{j=1}^{k+1} x_j$.

Proof. From the $\mathcal{R}(p,q)-$ deformed binomial formula, we get

$$\Big(1\ominus\Lambda_k\Big)^n_{\mathcal{R}(p,q)} = \sum_{r=0}^{n}\left[\begin{array}{c} n \\ r \end{array}\right]_{\mathcal{R}(p,q)} \tau_1^{\binom{n-r}{2}}\tau_2^{\binom{r}{2}}\Big(-\Lambda_k\Big)^r.$$

Using the relation

$$\sum_{r_1=0}^{n-r}\left[\begin{array}{c} n-r \\ r_1 \end{array}\right]_{\mathcal{R}(p,q)} x_1^{n-r-r_1}\Big(1\ominus x_1\Big)^{r_1}_{\mathcal{R}(p,q)} = 1$$

and interchanging the order of summation, we obtain:

$$\begin{aligned}(1\ominus\Lambda_k)^n_{\mathcal{R}(p,q)} &= \sum_{r_1=0}^{n}\left[\begin{array}{c} n \\ r_1 \end{array}\right]_{\mathcal{R}(p,q)} x_1^{n-r_1}\big(1\ominus x_1\big)^{r_1}_{\mathcal{R}(p,q)} \\ &\times \sum_{r=0}^{n-r_1}\left[\begin{array}{c} n-r_1 \\ r \end{array}\right]_{\mathcal{R}(p,q)} \tau_1^{\binom{n-r}{2}}\tau_2^{\binom{r}{2}}\big(-\Lambda_k\big)^r.\end{aligned}$$

By applying the $\mathcal{R}(p,q)$-deformed binomial formula, we get:

$$(1\ominus\Lambda_k)^n_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n}\left[\begin{array}{c} n \\ r_1 \end{array}\right]_{\mathcal{R}(p,q)} x_1^{n-r_1}\Big(1\ominus x_1\Big)^{r_1}_{\mathcal{R}(p,q)}\prod_{i=1}^{n-r_1}\Big(\tau_1^{i-1}-\prod_{j=2}^{k+1}x_j\tau_2^{i-1}\Big)$$

and generally,

$$\begin{aligned}\prod_{i=1}^{n-s_{j-1}}\Big(\tau_1^{i-1}-\prod_{\nu=j}^{k+1}x_j\tau_2^{i-1}\Big) &= \sum_{r_j=0}^{n-s_{j-1}}\left[\begin{array}{c} n-s_{j-1} \\ r_j \end{array}\right]_{\mathcal{R}(p,q)} x_j^{n-s_j}\Big(1\ominus x_j\Big)^{r_j}_{\mathcal{R}(p,q)} \\ &\times \prod_{i=1}^{n-s_j}\Big(\tau_1^{i-1}-\prod_{\nu=j+1}^{k+1}x_j\tau_2^{i-1}\Big)\end{aligned}$$

for $j \in \{1,2,\cdots,k\}$ with $s_0 = 0$. Applying the last expression, successively for $j \in \{1,2,\cdots,k\}$ and using the relation (2.11), the result is immediately deduced. □

The results contained in the corollary below are the particular case of the relation (2.13) by taking $x_{k+1} = 0$.

Corollary 2.6. *Let n be a positive integer. Then,*

$$\sum_{r_j=0}^{n} \left[\begin{matrix} n \\ r_1, r_2, \cdots, r_k \end{matrix} \right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k} x_j^{n-s_j} \big(1 \ominus x_j\big)^{r_j}_{\mathcal{R}(p,q)} = \tau_1^{\frac{s_k(1+s_k-2n)}{2}}$$

and

$$\sum_{r_j=0}^{n} \left[\begin{matrix} n \\ r_1, r_2, \cdots, r_k \end{matrix} \right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k} x_j^{r_j} \big(1 \ominus x_j\big)^{n-s_j}_{\mathcal{R}(p,q)} = \tau_1^{\frac{s_k(1+s_k-2n)}{2}},$$

where $j \in \{1, \cdots, k\}$, with $\sum_{i=1}^{k} r_j \le n$ and $s_j = \sum_{i=1}^{j} r_i$, $s_0 = 0$.

The generalization of the multinomial formula given by **Gasper and Rahman** [4] can be determined as follows:

$$\big(1 \ominus \Lambda_k\big)^n_{\mathcal{R}(p,q)} = \sum_{r_j=0}^{n} \left[\begin{matrix} n \\ r_1, r_2, \cdots, r_k \end{matrix} \right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k} x_j^{s_j} \big(1 \ominus x_{j-1}\big)^n_{\mathcal{R}(p,q)} \big(1 \ominus x_k\big)^{n-s_k}_{\mathcal{R}(p,q)},$$

where $j \in \{1, 2, \cdots, k\}$, with $\sum_{i=1}^{k} r_j \le n$ and $s_j = \sum_{i=1}^{j} r_i$.

3. $\mathcal{R}(p,q)$-deformed Multinomial Distribution

In this section, we construct the multinomial and negative multinomial probability distribution of the first and second kind in the framework of the $\mathcal{R}(p,q)$-deformed quantum algebras. Moreover, the $\mathcal{R}(p,q)$-deformed multiple Heine, Euler, negative multiple Heine, and negative Euler are obtained as limit of the above probability distriburion as $n \longrightarrow \infty$. We use the following notations in the sequel: $\Theta = \big(\theta_1, \theta_2, \cdots, \theta_k\big)$.

3.1. $\mathcal{R}(p,q)$-deformed multinomial distribution of the first kind

We consider a sequence of independant Bernoulli trials with chain-composite successes (or failures) and suppose that the odds of success of the j^{th} kind at the i^{th} trial is furnished by:

$$\theta_{j,i} = \theta_j \, \tau_1^{1-i} \, \tau_2^{i-1}, \quad 0 < \theta_j < \infty, \quad (j,i) \in \mathbb{N}.$$

The probability of success of the j^{th} kind at the i^{th} trial is derived as

$$p_{j,i} = \frac{\theta_j\,\tau_2^{i-1}}{\tau_1^{i-1} + \theta_j\,\tau_2^{i-1}}. \tag{3.1.1}$$

Naturally, the probability of failure of the j^{th} kind at the i^{th} trial is deduced as

$$q_{j,i} = \frac{\tau_1^{i-1}}{\tau_1^{i-1} + \theta_j\,\tau_2^{i-1}}. \tag{3.1.2}$$

Note that, taking $\mathcal{R}(x,1) = \frac{x-1}{1-q}$, we recover the following q-deformation of probabilities (3.1.1)and (3.1.2) given in [3, eq. 3.2]:

$$p_{j,i} = \frac{\theta_j\,q^{i-1}}{1+\theta_j\,q^{i-1}} \quad \text{and} \quad q_{j,i} = \frac{1}{1+\theta_j\,q^{i-1}}.$$

We denote by $Y_j, j \in \{1,2,\cdots,k\}$ the number of successes of the j^{th} kind in a sequence of n independent Bernoulli trials with chain-composite failures, with the probability of success of the j^{th} kind at the i^{th} trial given by the relation (3.1.1). The distribution of the random vector $\big(Y_1, Y_2, \cdots, Y_k\big)$ can be called the $\mathcal{R}(p,q)$-deformed multinomial probability distribution of the first kind with parameters $n, \Theta, p,$ and q.

Theorem 3.1.1. *The probability function of the $\mathcal{R}(p,q)$-deformed multinomial probability distribution of the first kind with parameters $n, \Theta, p,$ and q is*

$$P\big(Y_1 = y_1, \cdots, Y_k = y_k\big) = \begin{bmatrix} n \\ y_1, y_2, \cdots, y_k \end{bmatrix}_{\mathcal{R}(p,q)} \prod_{j=1}^{k} \frac{\theta_j^{y_j}\,\tau_1^{\binom{n-y_j}{2}}\,\tau_2^{\binom{y_j}{2}}}{(1\oplus\theta_j)^{n-s_{j-1}}_{\mathcal{R}(p,q)}} \tag{3}$$

and their recurrance relations are

$$P_{y+1} = \Big[n - \sum_{j=1}^{k} y_j\Big]_{k,\mathcal{R}(p,q)} \prod_{j=1}^{k} \frac{\theta_j \tau_1^{n-y_j}\,\tau_2^{y_j}\,P_y}{[y_j+1]_{\mathcal{R}(p,q)}\big(1\oplus\theta_j\big)_{\mathcal{R}(p,q)}}$$

with $P_0 = \prod_{j=1}^{k} \frac{\tau_1^{\binom{n}{2}}}{\big(1\oplus\theta_j\big)^n_{\mathcal{R}(p,q)}}$, *where for* $j \in \{1,2,\cdots,k\}$ *we have* $y_j \in \{0,1,\cdots,n\}$ *and* $\sum_{j=1}^{k} y_j \leq n, s_j = \sum_{i=1}^{j} y_i, 0 < \theta_j < 1$.

Proof. The random variable Y_1 is defined on the sequence of n independent Bernoulli trials with space $\omega = \{s_1, f_1\}$, follows the $\mathcal{R}(p,q)$-deformed binomial distribution of the first kind with probability function:

$$P\big(Y_1 = y_1\big) = \begin{bmatrix} n \\ y_1 \end{bmatrix}_{\mathcal{R}(p,q)} \frac{\theta_1^{y_1}\,\tau_1^{\binom{n-y_1}{2}}\,\tau_2^{\binom{y_1}{2}}}{\big(1\oplus\theta_1\big)^n_{\mathcal{R}(p,q)}}, \quad y_1 \in \{0,1,\cdots,n\}.$$

In the same way, the random variable Y_k is defined on the sequence of $n - s_{k-1}$ independent Bernoulli trials, with conditional space $\omega = \{s_k, f_k\}$, obeys a $\mathcal{R}(p,q)$-deformed binomial distribution of the first kind with probability distribution:

$$P\big(Y_k = y_k \mid Y_1 = y_1, \cdots, Y_{k-1} = y_{k-1}\big) = \begin{bmatrix} n - s_{k-1} \\ y_k \end{bmatrix}_{\mathcal{R}(p,q)} \frac{\theta_k^{y_k} \tau_1^{\binom{n-y_k}{2}} \tau_2^{\binom{y_k}{2}}}{\big(1 \oplus \theta_k\big)^{n-s_{k-1}}_{\mathcal{R}(p,q)}},$$

where $y_k \in \{0, 1, \cdots, n - s_{k-1}\}$. Then, from the relations (2.11) and the multiplicative formula for probabilities, the result follows. Using the $\mathcal{R}(p,q)$-deformed multinomial formula, we get:

$$\sum P\big(Y_1 = y_1, \cdots, Y_k = y_k\big) = 1.$$

The recurrence relation is obtained by simpler computation. □

We consider $T_j, j \in \{1, 2, \cdots, k\}$ the number of successes of the j^{th} kind until the occurrence of the n^{th} failure of the k^{th} kind, in a sequence of Bernoulli trials with chain-composite failures, with the probability of success of the j^{th} kind at the i^{th} trial is given by the relation (3.1.1). The distribution of the random vector $\big(T_1, T_2, \cdots, T_k\big)$ may be called the negative $\mathcal{R}(p,q)$-deformed multinomial probability distribution of the first kind with parameters n, Θ, p, and q.

Theorem 3.1.2. *The probability function of the negative $\mathcal{R}(p,q)$-deformed multinomial distribution of the first kind with parameters n, Θ, p and q is given as*

$$P\big(T_1 = t_1, \cdots, T_k = t_k\big) = \begin{bmatrix} n + s_k - 1 \\ t_1, t_2, \cdots, t_k \end{bmatrix}_{\mathcal{R}(p,q)} \prod_{j=1}^{k} \frac{\theta_j^{t_j} \tau_1^{\binom{n-t_j}{2}} \tau_2^{\binom{t_j}{2}}}{\big(1 \oplus \theta_j\big)^{n+s_k-s_{j-1}}_{\mathcal{R}(p,q)}} \tag{4}$$

and their recurrence relation by

$$P_{t+1} = \Big[n - \sum_{j=1}^{k} t_j\Big]_{k,\mathcal{R}(p,q)} \prod_{j=1}^{k} \frac{\theta_j \tau_1^{n-t_j} \tau_2^{t_j} P_t}{[t_j + 1]_{\mathcal{R}(p,q)} \big(1 \ominus \theta_j\big)_{\mathcal{R}(p,q)}},$$

wiht $P_0 = \prod_{j=1}^{k} \frac{\tau_1^{\binom{n}{2}}}{\big(1 \oplus \theta_j\big)^n_{\mathcal{R}(p,q)}}$, where for $j \in \{1, 2, \cdots, k\}$ we have $t_j \in \mathbb{N}$ and $s_j = \sum_{i=1}^{j} t_i, 0 < \theta_j < 1,$.

Proof. From the multiplicative formula, we have:

$$\begin{aligned} P\big(T_1 = t_1, \cdots, T_k = t_k\big) &= P\big(T_1 = t_1 \mid T_2 = t_2, \cdots, T_k = t_k\big) \\ &\times P\big(T_2 = t_2 \mid T_3 = t_3, \cdots, T_k = t_k\big) \cdots P\big(T_k = t_k\big). \end{aligned}$$

The random variable T_1 is defined on the sequence of $n + s_k$ independent Bernoulli trials with space $\Omega_1 = \{s_1, f_1\}$, obeys the negative $\mathcal{R}(p,q)$-deformed binomial distribution of the first kind with probability function

$$P\big(T_1 = t_1 \mid T_2 = t_2, \cdots, T_k = t_k\big) = \begin{bmatrix} n+s_k-1 \\ t_1 \end{bmatrix}_{\mathcal{R}(p,q)} \frac{\theta_1^{t_1}\tau_1^{\binom{n+s_k-t_1}{2}}\tau_2^{\binom{u_1}{2}}}{\big(1\oplus\theta_1\big)^{n+s_k}_{\mathcal{R}(p,q)}}, \quad t_1 \in \mathbb{N}.$$

Then, given the occurrence of the event $\{T_k = t_k\}$, the random variable T_k is defined on the sequence of $n + s_k - s_{k-1} = n + t_k$ independent Bernoulli trials, with conditional space $\Omega_k = \{s_k, f_k\}$, obeys the negative $\mathcal{R}(p, q)$-deformed binomial distribution of the first kind with probability distribution:

$$P\big(T_k = t_k\big) = \begin{bmatrix} n+s_k-s_{k-1}-1 \\ u_k \end{bmatrix}_{\mathcal{R}(p,q)} \frac{\theta_k^{u_k}\tau_1^{\binom{n+u_k}{2}}\tau_2^{\binom{t_k}{2}}}{\big(1\oplus\theta_k\big)^{n+s_k-s_{k-1}}_{\mathcal{R}(p,q)}},$$

where $t_k \in \mathbb{N}$. Then, multplying all the above probabilities and using the relation

$$\begin{bmatrix} n+s_k-1 \\ t_1, t_2, \ldots, t_k \end{bmatrix}_{\mathcal{R}(p,q)} = \prod_{j=1}^{k} \begin{bmatrix} n+s_k-s_{j-1}-1 \\ t_j \end{bmatrix}_{\mathcal{R}(p,q)}, \quad s_0 = 0$$

the result follows. Using the negative $\mathcal{R}(p, q)$-deformed multinomial formula, we get

$$\sum P\big(T_1 = t_1, \ldots, U_k = t_k\big) = 1.$$

. □

Remark 3.1.3. We denote by $V_j, j \in \{1, 2, \ldots, k\}$, the number of failures of the j^{th} kind until the occurrence of the n^{th} success of the k^{th} kind, in a sequence of independent Bernoulli trials with chain-composite successes and $\big(V_1, V_2, \cdots, V_k\big)$ the random vector. The probability function of the negative $\mathcal{R}(p, q)$-binomial distribution of the first kind is:

$$P\big(V = v\big) = \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_{\mathcal{R}(p,q)} \frac{\theta^n \tau_1^{\binom{v}{2}}\tau_2^{\binom{n}{2}+v}}{\big(1\oplus\theta_1\big)^{n+v}_{\mathcal{R}(p,q)}}, \quad v \in \mathbb{N}. \tag{5}$$

From the relation (5) and the steps used to get the (4), the probability function of the random vector $\big(V_1, V_2, \cdots, V_k\big)$ is given by $P\big(V_1 = v_1, \cdots, V_k = v_k\big) =$

$$\begin{bmatrix} n+s_k-1 \\ v_1, v_2, \cdots, v_k \end{bmatrix}_{\mathcal{R}(p,q)} \prod_{j=1}^{k} \frac{\theta_j^{n+s_k-s_{j-1}}\tau_1^{\binom{v_j}{2}}\tau_2^{\binom{n+s_k-s_{j-1}}{2}+v_j}}{\big(1\oplus\theta_j\big)^{n+s_k-s_{j-1}}_{\mathcal{R}(p,q)}}, \tag{6}$$

where $v_j \in \mathbb{N}, s_j = \sum_{i=1}^{j} v_i, 0 < \theta_j < 1$, and $j \in \{1, 2, \cdots, k\}$.

Remark 3.1.4. The $\mathcal{R}(p,q)$-deformed multinomial distributions (3) and (4) can be approximated by the probability function of the $\mathcal{R}(p,q)$-deformed multiple Heine distributions (7) and (8). In fact, setting $\mu_j = \frac{\theta_j}{\tau_1-\tau_2}$ and using $0<q<p<1$, we have:

$$\lim_{n\longrightarrow\infty}\left[\begin{array}{c} n \\ y_1, y_2, \cdots, y_k \end{array}\right]_{\mathcal{R}(p,q)} = \frac{1}{\prod_{j=1}^{k}\big(\tau_1-\tau_2\big)^{y_j}[y_j]_{\mathcal{R}(p,q)}!}$$

and

$$\lim_{n\longrightarrow\infty}\Big(1\oplus\mu_j(\tau_1-\tau_2)\Big)^{n-s_{j-1}}_{\mathcal{R}(p,q)} = \frac{1}{e_{\mathcal{R}(p,q)}(-\mu_j)}.$$

Thus,

$$(7)\quad \lim_{n\longrightarrow\infty}\left[\begin{array}{c} n \\ y_1, \cdots, y_k \end{array}\right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k}\frac{\theta_j^{x_j}\tau_1^{\binom{n-y_j}{2}}\tau_2^{\binom{y_j}{2}}}{(1\oplus\theta_j)^{n-s_{j-1}}_{\mathcal{R}(p,q)}} = \prod_{j=1}^{k} e_{\mathcal{R}(p,q)}(-\mu_j)\frac{\mu_j^{y_j}\tau_2^{\binom{y_j}{2}}}{[y_j]_{\mathcal{R}(p,q)}!}.$$

Similarly, we get:

(8)

$$\lim_{n\longrightarrow\infty}\left[\begin{array}{c} n+s_k-1 \\ t_1, \cdots, t_k \end{array}\right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k}\frac{\theta_j^{t_j}\tau_1^{\binom{n-t_j}{2}}\tau_2^{\binom{t_j}{2}}}{(1\oplus\theta_j)^{n+s_k-s_{j-1}}_{\mathcal{R}(p,q)}} = \prod_{j=1}^{k} E_{\mathcal{R}(p,q)}(-\mu_j)\frac{\mu_j^{t_j}\tau_2^{\binom{t_j}{2}}}{[t_j]_{\mathcal{R}(p,q)}!}.$$

3.2. $\mathcal{R}(p,q)$-deformed multinomial distribution of the second kind

We consider a sequence of independent Bernoulli trials with chain-composite successes(or failures) and suppose that the conditional probability of success of the j^{th} kind at any trial, given that $i-1$ successes of the j^{th} kind occur in the previous trials, is given by:

$$(3.2.1)\qquad p_{j,i} = 1-\theta_j\,\tau_1^{1-i}\,\tau_2^{i-1},\quad 0<\theta_j<1,\quad (j,i)\in\mathbb{N}.$$

We denote by X_j the number of failures of the j^{th} kind in a sequence of n independent Bernoulli trials with chain-composite successes, where the conditional probability of success of the j^{th} kind at any trial, given that $i-1$ successes of the j^{th} kind occur in the previous trials, is given by (3.2.1).

Theorem 3.2.1. *The probability function of the $\mathcal{R}(p,q)$-deformed multinomial distribution of the second kind with parameters n,Θ,p and q is determined by:*
(3.2.2)

$$P\big(X_1=x_1,\cdots,X_k=x_k\big) = \left[\begin{array}{c} n \\ x_1, x_2, \cdots, x_k \end{array}\right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k}\theta_j^{x_j}\Big(1\ominus\theta_j\Big)^{n-s_j}_{\mathcal{R}(p,q)}.$$

The recurrence relation for the $\mathcal{R}(p,q)$-deformed multinomial distribution of the second kind is given by:

$$P_{x+1} = \Big[n-\sum_{j=1}^{k}x_j\Big]_{k,\mathcal{R}(p,q)}\prod_{j=1}^{k}\frac{\theta_j\big(1\ominus\theta_j\big)_{\mathcal{R}(p,q)}}{[x_j+1]_{\mathcal{R}(p,q)}}P_x,\quad \textit{with}\quad P_0=\prod_{j=1}^{k}\big(1\ominus\theta_j\big)^{n}_{\mathcal{R}(p,q)}.$$

where $x_j \in \{0, 1, \cdots, n\}, \sum_{j=1}^{k} x_j \le n, s_j = \sum_{i=1}^{j} x_i, 0 < \theta_j < 1,$ *and* $j \in \{1, 2, \cdots, k\}$.

Remark 3.2.2.

(i) Taking $k = 1$, we deduced the $\mathcal{R}(p, q)$-deformed binomial distribtuion of the second kind ([5], Definition 3.5 in page 9).

(i) The multinomial probability distribution presented in ([13], eq 1 in page 18) is recovered by putting $\mathcal{R}(p, q) = 1$.

Corollary 3.2.3. *The recursion relation for the q-deformed multinomial distribution of the second kind is deduced as :*

$$P_{x+1} = \Big[n - \sum_{j=1}^{k} x_j\Big]_{k,q} \prod_{j=1}^{k} \frac{\theta_j\,(1-\theta_j)(1-q)}{1-q^{x_j+1}} P_x.$$

Proof. By taking $\mathcal{R}(x, 1) = \frac{1-x}{1-q}$ in the general formalism. □

Remark 3.2.4. We denote by $Y_j, j \in \{1, 2, \cdots, k\}$, the number of usccesses of the j^{th} kind in a sequence of n independent Bernoulli trials with chain-composite failures, where the conditional probability of success of the j^{th} kind at any trial, given that $i-1$ successes of the j^{th} kind occur in the previous trials, is given by the relation (3.2.1).

Using the same procedure to derive the relation (3.2.2), the probability function of the random vector $\big(Y_1, Y_2, \cdots, Y_k\big)$ is obtained as:

$$(3.2.3)\quad P\big(Y_1 = y_1, \cdots, Y_k = y_k\big) = \left[\begin{array}{c} n \\ y_1, y_2, \cdots, y_k \end{array}\right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k} \theta_j^{n-s_j}\,\big(1 \ominus \theta_j\big)^{y_j}_{\mathcal{R}(p,q)},$$

where $y_j \in \{0, 1, \cdots, n\}, \sum_{j=1}^{k} y_j \le n, s_j = \sum_{i=1}^{j} y_i, 0 < \theta_j < 1,$ and $j \in \{1, 2, \cdots, k\}$.

Let $W_j, j \in \{1, 2, \ldots, k\}$ be the number of failures of the j^{th} kind until the occurrence of the n^{th} success of the k^{th} kind, in a sequence of n independent Bernoulli trials with chain-composite successes.

Theorem 3.2.5. *The probability function of the negative* $\mathcal{R}(p, q)$*-deformed multinomial distribution of the second kind with parameters* n, Θ, p *and* q *is furnished by:*

(3.2.4)

$$P_w := P\big(W_1 = w_1, \cdots, W_k = w_k\big) = \left[\begin{array}{c} n + s_k - 1 \\ w_1, \cdots, w_k \end{array}\right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k} \theta_j^{w_j}\,\big(1 \ominus \theta_j\big)^{n+s_k-s_j}_{\mathcal{R}(p,q)},$$

where $w_j \in \mathbb{N}, s_j = \sum_{i=1}^{j} w_i, 0 < \theta_j < 1,$ *and* $j \in \{1, 2, \cdots, k\}$. *Furthermore, their recursion relations are given by:*

$$P_{w+1} = \Big[n - \sum_{j=1}^{k} w_j\Big]_{k,\mathcal{R}(p,q)} \prod_{j=1}^{k} \frac{\theta_j \left(1 \ominus \theta_j\right)_{\mathcal{R}(p,q)}}{[w_j + 1]_{\mathcal{R}(p,q)}} P_w, \text{ with } \quad P_0 = \prod_{j=1}^{k} \left(1 \ominus \theta_j\right)^n_{\mathcal{R}(p,q)}.$$

Remark 3.2.6. The limit of the $\mathcal{R}(p, q)$-deformed multinomial distribution of the second kind (3.2.2), as $n \longrightarrow \infty$ is the $\mathcal{R}(p, q)$-deformed multiple Euler distribution:

$$\lim_{n\longrightarrow\infty} \left[\begin{array}{c} n \\ x_1, x_2, \cdots, x_k \end{array}\right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k} \theta_j^{x_j} \Big(1 \ominus \theta_j\Big)^{n-s_j}_{\mathcal{R}(p,q)} = \prod_{j=1}^{k} E_{\mathcal{R}(p,q)}(-\mu_j) \frac{\mu_j^{x_j}}{[x_j]_{\mathcal{R}(p,q)}}.$$

Moreover, the limit of the $\mathcal{R}(p, q)$-deformed multinomial distribution of the second kind (3.2.4), as $n \longrightarrow \infty$ is the $\mathcal{R}(p, q)$-deformed multiple Euler distribution:

$$\lim_{n\longrightarrow\infty} \left[\begin{array}{c} n + s_k - 1 \\ w_1, w_2, \cdots, w_k \end{array}\right]_{\mathcal{R}(p,q)} \prod_{j=1}^{k} \theta_j^{w_j} \left(1 \ominus \theta_j\right)^{n+s_k-s_j}_{\mathcal{R}(p,q)} = \prod_{j=1}^{k} E_{\mathcal{R}(p,q)}(-\mu_j) \frac{\mu_j^{w_j}}{[w_j]_{\mathcal{R}(p,q)}}.$$

Remark 3.2.7 Several kind of the $\mathcal{R}(p, q)$-deformed multivariate absorption distribution are also attracted our attention. Replacing $\mathcal{R}(p, q)$ by $\mathcal{R}(p^{-1}, q^{-1})$, θ_j by $\tau_1^{-m_j}\tau_2^{m_j}$, for $j \in \{1, 2, \cdots, k\}$ in the relation (3.2.1), the probability of successes is reduced as:

$$p_{j,i} = 1 - \tau_1^{-m_j i+1}\tau_2^{m_j+1-i}, \quad 0 < m_j < \infty, \quad j \in \{1, 2, \ldots, k\}, \quad i \in \{1, 2, \cdots, [m_j]\}.$$

Using the relation (2.6) and the $\mathcal{R}(p, q)$-deformed factorial, the probability function (3.2.2) takes the following form:

$$P(X_1 = x_1, \cdots, X_k = x_k) = \left[\begin{array}{c} n \\ x_1, x_2, \cdots, x_k \end{array}\right]_{\mathcal{R}(p,q)} (\tau_1\,\tau_2)^{-\sum_{j=1}^{k} x_j(m_j - n + s_j)} \times \prod_{j=1}^{k} \left(\tau_1 - \tau_2\right)^{n-s_j} [m_j]_{n-s_j,\mathcal{R}(p,q)}.$$

Furtermore, from (2.6), the probability function (3.2.3) can be rewritten as:

$$P(Y_1 = y_1, \cdots, Y_k = y_k) = \left[\begin{array}{c} n \\ y_1, y_2, \cdots, y_k \end{array}\right]_{\mathcal{R}(p,q)} (\tau_1\,\tau_2)^{-\sum_{j=1}^{k}(m_j - y_j)(n - s_j)} \times \prod_{j=1}^{k} \left(\tau_1 - \tau_2\right)^{y_j} [m_j]_{y_j,\mathcal{R}(p,q)}.$$

4. Particular Cases of Multinomial Distribution

In this section, we derive particular multinomial coefficient and multinomial probability distribution induced by the quantum algebras known in the literature.

(i) Taking $\mathcal{R}(x,y) = \frac{x-y}{p-q}$, we obtain the results from the **Jagannathan-Srinivassa** algebra [10]: the (p,q)-deformed multinomial coefficients

$$\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p,q} = \frac{[x]_{r_1+r_2+\cdots+r_k,p,q}}{[r_1]_{p,q}![r_2]_{p,q}!\cdots[r_k]_{p,q}!}$$

satisfy the recursion relation:

$$\begin{aligned}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p,q} &= p^{s_k}\left[{x-1 \atop r_1,r_2,\cdots,r_k}\right]_{p,q} + q^{x-m_1}\left[{x-1 \atop r_1-1,r_2,\cdots,r_k}\right]_{p,q}\\ &+ q^{x-m_2}\left[{x-1 \atop r_1,r_2-1,\cdots,r_k}\right]_{p,q} + \cdots\\ &+ q^{x-m_k}\left[{x-1 \atop r_1,r_2,\cdots,r_k-1}\right]_{p,q}\end{aligned}$$

and alternately,

$$\begin{aligned}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p,q} &= q^{s_k}\left[{x-1 \atop r_1,r_2,\cdots,r_k}\right]_{p,q} + p^{x-m_1}\left[{x-1 \atop r_1-1,r_2,\cdots,r_k}\right]_{p,q}\\ &+ p^{x-m_2}q^{s_1}\left[{x-1 \atop r_1,r_2-1,\cdots,r_k}\right]_{p,q} + \cdots\\ &+ p^{x-m_k}q^{s_{k-1}}\left[{x-1 \atop r_1,r_2,\cdots,r_k-1}\right]_{p,q}.\end{aligned}$$

Moreover, the (p^{-1},q^{-1})-deformed multinomial coefficients provided by

$$\begin{aligned}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p^{-1},q^{-1}} &= (pq)^{-\sum_{j=1}^k r_j(x-m_j)}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p,q}\\ &= (pq)^{-\sum_{j=1}^k r_j(x-s_j)}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p,q}\end{aligned}$$

obey the recursion relations

$$\begin{aligned}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p,q} &= q^{m_1}\left[{x-1 \atop r_1,r_2,\cdots,r_k}\right]_{p,q} + q^{m_2}\left[{x-1 \atop r_1-1,r_2,\cdots,r_k}\right]_{p,q}\\ &+ q^{m_3}\left[{x-1 \atop r_1,r_2-1,\cdots,r_k}\right]_{p,q} + \cdots + p^{x}\left[{x-1 \atop r_1,r_2,\cdots,r_{k-1}}\right]_{p,q}.\end{aligned}$$

and

$$\begin{aligned}\left[{x \atop r_1,r_2,\cdots,r_k}\right]_{p,q} &= p^{x}\left[{x-1 \atop r_1,r_2,\cdots,r_k}\right]_{p,q} + q^{x-s_1}\left[{x-1 \atop r_1-1,r_2,\cdots,r_k}\right]_{p,q}\\ &+ q^{x-s_2}\left[{x-1 \atop r_1,r_2-1,\cdots,r_k}\right]_{p,q} + \cdots + q^{x-s_k}\left[{x-1 \atop r_1,r_2,\cdots,r_{k-1}}\right]_{p,q},\end{aligned}$$

where $r_j \in \mathbb{N}$ and $j \in \{1,2,\cdots,k\}$, with $m_j = \sum_{i=j}^k r_i$ and $s_j = \sum_{i=1}^j r_i$.

For n a positive integers, $x, p,$ and q real numbers, the following relation holds:

$$\prod_{j=1}^{k}(1\oplus x_j)^n_{p,q} = \sum \left[\begin{array}{c} n \\ r_1, r_2, \cdots, r_k \end{array} \right]_{p,q} \prod_{j=1}^{k} x_j^{r_j} p^{\binom{n-r_j}{2}} q^{\binom{r_j}{2}} \left(p^{n-s_{j-1}} \oplus x_j q^{n-s_{j-1}}\right)^{s_{j-1}}_{p,q},$$

where $r_j \in \{0,\cdots,n\}$, $j \in \{1,\cdots,k\}$, with $\sum_{i=1}^{k} r_i \leq n$ and $s_j = \sum_{i=1}^{j} r_i$, and $s_0 = 0$.

Furthermore, for n be a positive integers, we have

$$\prod_{j=1}^{k}(1\oplus x_j)^n_{p,q} = \sum \left[\begin{array}{c} n+s_k-1 \\ r_1, r_2, \cdots, r_k \end{array} \right]_{p,q} \prod_{j=1}^{k} \frac{x_j^{r_j} p^{\binom{n-r_j}{2}} q^{\binom{r_j}{2}}}{(p^n \oplus x_j q^n)^{s_k - s_{j-1}}_{p,q}}.$$

Equivalently,

$$\prod_{j=1}^{k}(1\oplus x_j)^n_{p,q} = \sum_{r_j\in\mathbb{N}} \left[\begin{array}{c} n+s_k-1 \\ r_1, r_2, \cdots, r_k \end{array} \right]_{p,q} \prod_{j=1}^{k} \frac{x_j^{n+s_k-s_{j-1}} p^{\binom{n-r_j}{2}} q^{\binom{n+s_k-s_{j-1}}{2}+r_j}}{(p^n \oplus x_j q^n)^{s_k - s_{j-1}}_{p,q}},$$

where $j \in \{1, 2, \cdots, k\}$, with $s_j = \sum_{i=1}^{j} r_i, \quad s_0 = 0.$

Let $x_j, j \in \{1, 2, \cdots, k+1\}, p,$ and q real numbers. For n positive integer, the following result holds:

$$(1\ominus\Lambda_k)^n_{p,q} = \sum_{r_j=0}^{n} \left[\begin{array}{c} n \\ r_1, r_2, \cdots, r_k \end{array} \right]_{p,q} \prod_{j=1}^{k} x_j^{n-s_j} (1\ominus x_j)^{r_j}_{p,q} (1\ominus x_{k+1})^{n-s_k}_{p,q},$$

where $r_j \in \{0,\cdots,n\}$, $j \in \{1,\cdots,k\}$, with $\sum_{i=1}^{k} r_i \leq n$ and $s_j = \sum_{i=1}^{j} r_i$, $s_0 = 0$, $\Lambda_k = \prod_{j=1}^{k+1} x_j$.

For n a positive integer, we have:

$$\sum_{r_j=0}^{n} \left[\begin{array}{c} n \\ r_1, r_2, \cdots, r_k \end{array} \right]_{p,q} \prod_{j=1}^{k} x_j^{n-s_j} (1\ominus x_j)^{r_j}_{p,q} = p^{\frac{s_k(1+s_k-2n)}{2}}$$

and

$$\sum_{r_j=0}^{n} \left[\begin{array}{c} n \\ r_1, r_2, \cdots, r_k \end{array} \right]_{p,q} \prod_{j=1}^{k} x_j^{r_j} (1\ominus x_j)^{n-s_j}_{p,q} = p^{\frac{s_k(1+s_k-2n)}{2}},$$

where $j \in \{1, \cdots, k\}$, with $\sum_{i=1}^{k} r_j \leq n$ and $s_j = \sum_{i=1}^{j} r_i$, $s_0 = 0$.

The (p,q)-deformed of the multinomial formula given by **Gasper and Rahman** [4] can be determined as follows:

$$(1 \ominus \Lambda_k)^n_{p,q} = \sum_{r_j=0}^{n} \left[\begin{array}{c} n \\ r_1, r_2, \cdots, r_k \end{array} \right]_{p,q} \prod_{j=1}^{k} x_j^{s_j} (1 \ominus x_{j-1})^n_{p,q} (1 \ominus x_k)^{n-s_k}_{p,q},$$

where $j \in \{1, 2, \cdots, k\}$, with $\sum_{i=1}^{k} r_j \leq n$ and $s_j = \sum_{i=1}^{j} r_i$.

(a) The probability function of the (p,q)-deformed multinomial distribution of the first kind with parameters $n, (\theta_1, \theta_2, \cdots, \theta_k), p$ and q is presented by:

$$P(Y_1 = y_1, \cdots, Y_k = y_k) = \left[\begin{array}{c} n \\ y_1, y_2, \cdots, y_k \end{array} \right]_{p,q} \prod_{j=1}^{k} \frac{\theta_j^{y_j} \, p^{\binom{n-x_j}{2}} \, q^{\binom{y_j}{2}}}{(1 \oplus \theta_j)^{n-s_{j-1}}_{p,q}},$$

and their recursion relations as:

$$P_{y+1} = \left[n - \sum_{j=1}^{k} y_j \right]_{k,p,q} \prod_{j=1}^{k} \frac{\theta_j \, p^{n-y_j} q^{y_j} P_y}{[y_j + 1]_{p,q} (1 \oplus \theta_j)_{p,q}},$$

with $P_0 = \prod_{j=1}^{k} \frac{p^{\binom{n}{2}}}{(1 \oplus \theta_j)^n_{p,q}}$, where for $j \in \{1, 2, \cdots, k\}$ we have $y_j \in \{0, 1, \cdots, n\}$, and $\sum_{j=1}^{k} y_j \leq n, s_j = \sum_{i=1}^{j} y_i$ for $0 < \theta_j < 1$.

(b) The probability function of the negative (p,q)-deformed multinomial distribution of the first kind with parameters $n, (\theta_1, \theta_2, \cdots, \theta_k), p$ and q is given as follows:

$$P(T_1 = t_1, \cdots, T_k = t_k) = \left[\begin{array}{c} n + s_k - 1 \\ t_1, t_2, \cdots, t_k \end{array} \right]_{p,q} \prod_{j=1}^{k} \frac{\theta_j^{u_j} \, p^{\binom{n-u_j}{2}} \, q^{\binom{u_j}{2}}}{(1 \oplus \theta_j)^{n+s_k-s_{j-1}}_{p,q}},$$

and their recurrence relation by:

$$P_{t+1} = \left[n - \sum_{j=1}^{k} t_j \right]_{k,p,q} \prod_{j=1}^{k} \frac{\theta_j \, p^{n-t_j} \, q^{t_j} \, P_t}{[t_j + 1]_{p,q} (1 \ominus \theta_j)_{p,q}}$$

with $P_0 = \prod_{j=1}^{k} \frac{p^{\binom{n}{2}}}{(1 \oplus \theta_j)^n_{p,q}}$ where for $j \in \{1, 2, \cdots, k\}$ we have $t_j \in \mathbb{N}$ and $s_j = \sum_{i=1}^{j} t_i$ for $0 < \theta_j < 1$.

(c) The probability function of the (p,q)-deformed multinomial distribution of the second kind with parameters $n, \big(\theta_1,\theta_2,\cdots,\theta_k\big), p$ and q is determined by:

$$P\big(X_1=x_1,\cdots,X_k=x_k\big)=\begin{bmatrix} n \\ x_1,x_2,\cdots,x_k\end{bmatrix}_{p,q}\prod_{j=1}^{k}\theta_j^{x_j}\big(1\ominus\theta_j\big)_{p,q}^{n-s_j}$$

and the recurrence relation

$$P_{x+1}=\Big[n-\sum_{j=1}^{k}x_j\Big]_{k,p,q}\prod_{j=1}^{k}\frac{\theta_j\big(1\ominus\theta_j\big)_{p,q}}{[x_j+1]_{p,q}}P_x,$$

with $P_0=\prod_{j=1}^{k}\big(1\ominus\theta_j\big)_{p,q}^{n}$ where $x_j\in\{0,1,\cdots,n\}$, $\sum_{j=1}^{k}x_j\le n$ and $s_j=\sum_{i=1}^{j}x_i$.

Another (p,q)-deformed multinomial distribution of the second kind

$$P\big(Y_1=y_1,\cdots,Y_k=y_k\big)=\begin{bmatrix} n \\ y_1,y_2,\cdots,y_k\end{bmatrix}_{p,q}\prod_{j=1}^{k}\theta_j^{n-s_j}\big(1\ominus\theta_j\big)_{p,q}^{y_j},$$

where $y_j\in\{0,1,\cdots,n\},\sum_{j=1}^{k}y_j\le n, s_j=\sum_{i=1}^{j}y_i, 0<\theta_j<1$, and $j\in\{1,2,\cdots,k\}$.

(d) The probability function of the negative (p,q)-deformed multinomial distribution of the second kind with parameters $n, \big(\theta_1,\theta_2,\cdots,\theta_k\big), p$ and q is furnished by:

$$P\big(W_1=w_1,\cdots,W_k=w_k\big)=\begin{bmatrix} n+s_k-1 \\ w_1,w_2,\cdots,w_k\end{bmatrix}_{p,q}\prod_{j=1}^{k}\theta_j^{w_j}\big(1\ominus\theta_j\big)_{p,q}^{n+s_k-s_j}.$$

Furthermore, their recursion relations are given as follows:

$$P_{x+1}=\Big[n-\sum_{j=1}^{k}x_j\Big]_{k,p,q}\prod_{j=1}^{k}\frac{\theta_j\big(1\ominus\theta_j\big)_{p,q}}{[x_j+1]_{p,q}}P_x,\ \text{with}\quad P_0=\prod_{j=1}^{k}\big(1\ominus\theta_j\big)_{p,q}^{n}.$$

where $w_j\in\mathbb{N}, s_j=\sum_{i=1}^{j}w_i, 0<\theta_j<1$, and $j\in\{1,2,\cdots,k\}$.

(ii) Putting $\mathcal{R}(x,y)=\frac{1-xy}{(p^{-1}-q)x}$, we obtain the multinomial distribution and properties corresponding to the **Chakrabarty and Jagannathan** algebra [1].

(iii) The multinomial distribution and properties associated to the **Hounkonnou-Ngompe generalized q-Quesne** algebra [9] can be deduced by putting $\mathcal{R}(x,y) = \frac{xy-1}{(q-p^{-1})y}$.

5. Concluding Remarks

The multinomial coefficients and the multinomial probability distribution and the negative multinomial probability distribution from the $\mathcal{R}(p,q)$-deformed quantum algebras have been examined and discussed. Particular cases have been deduced. The numerical interpretation of these probabilitu distributions is in preparation.

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