Constructions and bounds for subspace codes

Sascha Kurz

sascha.kurz@uni-bayreuth.de

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Abstract Subspace codes are the q-analog of binary block codes in the Hamming metric. Here the codewords are vector spaces over a finite field. They have e.g. applications in random linear network coding [148], distributed storage [193, 194], and cryptography [92]. In this chapter we survey known constructions and upper bounds for subspace codes.

Contents

1 Introduction

An important and classical family of error-correcting codes are so-called *block codes*. Given a non-empty *alphabet* Σ and a *length* $n \in \mathbb{N}_{>0}$, a block code C is a subset of Σ^n. The elements of C are called *codewords*. For $\mathbf{c}, \mathbf{c}' \in \Sigma^n$ the *Hamming distance* is given by

$$\mathrm{d_H}(\mathbf{c}, \mathbf{c}') = \#\left\{1 \le i \le n \,:\, c_i \neq c'_i\right\}, \tag{1.1}$$

i.e., the number of positions where the two codewords differ. With this, the *minimum Hamming distance* of a block code C is defined as

$$\mathrm{d_H}(C) = \min\{\mathrm{d_H}(\mathbf{c}, \mathbf{c}') \,:\, \mathbf{c}, \mathbf{c}' \in C, \mathbf{c} \neq \mathbf{c}'\}. \tag{1.2}$$

By convention we formally set $\mathrm{d_H}(C) = \infty$ if $\#C < 2$, i.e., $\mathrm{d_H}(C) > m$ for each integer m. If the alphabet Σ is a finite field (or a ring), we can call a block code C *linear* if it is linearly closed. While there is a lot of research on block codes with $\#\Sigma > 2$, we want to consider the binary case $\Sigma = \mathbb{F}_2 = \{0, 1\}$ only. By $A(n, d)$ we denote the maximum possible cardinality of a binary block code C with length n and minimum Hamming distance at least d. The determination of $A(n, d)$ is an important problem that has achieved wide attention but is still widely open. I.e., except for a few special cases only lower and upper bounds for $A(n, d)$ are known, see e.g. [1, 169, 176, 198]. For a vector $\mathbf{c} \in \mathbb{F}_2^n$ the Hamming distance $\mathrm{d_H}(\mathbf{c}, \mathbf{0})$ between $\mathbf{c}$ and the all-zero vector $\mathbf{0} \in \mathbb{F}_2^n$ is called the *Hamming weight* $\mathrm{wt}(\mathbf{c})$ of $\mathbf{c}$, counting the number of non-zero entries. A block code C where each codeword has the same Hamming weight, say w, is called *constant weight* (block) code. The corresponding maximum possible cardinality is denoted by $A(n, d, w)$. For bounds and exact values for $A(n, d, w)$ we refer the reader e.g. to [40, 188, 192] and the citing papers.

The aim of this chapter is the study of so-called *subspace codes*. One way to introduce these codes is to consider them as q-analog of binary block codes, i.e., the codewords are subspaces of the vector space $\mathbb{F}_q^n$.

q-analogs

Many combinatorial structures are based on the subset lattice of some finite set $\mathcal{U}$, which is mostly called "universe". If we replace the subset lattice with the subspace lattice of a $\#\mathcal{U}$-dimensional vector space V over $\mathbb{F}_q$, then we obtain a q-analog, see e.g. [12, 20, 68, 214] for examples. The elements of $\mathcal{U}$ correspond to the 1-dimensional subspaces of V, t-subsets correspond to t-subspaces, and the union of two subsets corresponds to the sum of two subspaces. In Section 2 we will introduce the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ that corresponds to the binomial coefficient $\binom{n}{q}$. See also Section 4 where we mention the q-Pochhammer symbol.

Endowed with a suitable metric, see Section 2 for details, the maximum possible sizes $A_q(n, d)$ of subspace codes in $\mathbb{F}_q^n$ with minimum distance at least d can be studied. If all codewords of a subspace code C have the same dimension, say k, we speak of a *constant dimension code* and denote the corresponding maximum possible cardinality by $A_q(n, d; k)$. More precisely, here we want to survey known lower and upper bounds for $A_q(n,d)$ and $A_q(n,d;k)$ cf. [122]. Besides being a generalization of classical codes, another motivation comes from e.g. random linear network coding, see [24, 105, 148].

The remaining part of this chapter is structured as follows. First we introduce necessary preliminaries in Section 2. Due to their close connection to constant dimension codes rank metric codes are discussed in Section 3. In Section 4 we survey upper bounds for $A_q(n,d;k)$ and lower bounds, i.e. constructions, in Section 5. The special parameters $A_2(7,4;3)$, i.e. the first open case where $A_2(n,d,k)$ has not been determined so far, is treated in Section 6. In Section 7 we summarize the currently best known lower bounds for constant dimension codes for small parameters. Mixed dimension subspace codes are the topic of Section 8. We close with a few remarks on related topics in Section 9.

2 Preliminaries

For a prime power $q > 1$ let $\mathbb{F}_q$ be the finite field with q elements. By $\mathbb{F}_q^n$ we denote the standard vector space of dimension $n \geq 1$ over $\mathbb{F}_q$. The set of all subspaces of $\mathbb{F}_q^n$, ordered by the incidence relation $\subseteq$, is called $(n-1)$-*dimensional (coordinate) projective geometry over* $\mathbb{F}_q$ and denoted by $\mathrm{PG}(n-1, q)$, cf. [213]. It forms a finite modular geometric lattice with *meet* $U \wedge W = U \cap W$ and *join* $U \vee W = U + W$. The graph theoretic distance

$$d_{\mathrm{S}}(U, W) = \dim(U + W) - \dim(U \cap W) \tag{2.1}$$

in this lattice is called the *subspace distance* between U and W. By $\mathcal{P}_q(n)$ we denote the set of all subspaces in $\mathbb{F}_q^n$ and by $\mathcal{G}_q(n, k)$ the subset of those with dimension $0 \leq k \leq n$, i.e., $\dot{\bigcup}_{k=0}^{n} \mathcal{G}_q(n, k) = \mathcal{P}_q(n)$. The elements of $\mathcal{G}_q(n, k)$ are also called k-*spaces* for brevity. Using geometric language, we also call 1-, 2-, 3-, 4-, and $(n-1)$-spaces *points*, *lines*, *planes*, *solids*, and *hyperplanes*, respectively. An $(n-k)$-space is also called a subspace of *codimension* k, i.e., a hyperplane has codimension 1. A *subspace code* C is a subset of $\mathcal{P}_q(n)$, where $n \geq 1$ is a suitable integer. If $C \subseteq \mathcal{G}_q(n, k)$, i.e., all elements $U \in C$ have dimension $\dim(U) = k$, we speak of a *constant dimension code* (CDC). A subspace code C that is not a constant dimension code is also called *mixed dimension (subspace) code* (MDC).

Exercise 2.1. *Verify that the subspace distance d_S is a metric on $\mathcal{P}_q(n)$ and satisfies*

$$\begin{aligned} d_S(U, W) &= \dim(U) + \dim(W) - 2 \cdot \dim(U \cap W) & (2.2)\\ &= 2 \cdot \dim(U + W) - \dim(U) - \dim(W). & (2.3) \end{aligned}$$

The *minimum subspace distance* $d_{\mathrm{S}}(C)$ of a subspace code C is defined as

$$d_{\mathrm{S}}(C) = \min\{d_{\mathrm{S}}(U, W) \,:\, U, W \in C, U \neq W\},$$

where we formally set $d_{\mathrm{S}}(C) = \infty$ if $\#C < 2$, i.e., $d_{\mathrm{S}}(C) > m$ for each integer m. The maximum possible cardinality of a subspace code in $\mathbb{F}_q^n$ with minimum subspace distance at least d is denoted by $A_q(n, d)$. For constant dimension codes with codewords of dimension k we denote the maximum possible cardinality by $A_q(n, d; k)$. Note that the subspace distance between two k-spaces satisfies $d_{\mathrm{S}}(U, W) = 2k - 2 \cdot \dim(U \cap W) = 2 \cdot \dim(U + W) - 2k$, i.e., it is an even non-negative integer. For each subset $T \subseteq \{0, 1, \ldots, n\}$ we denote by $A_q(n, d; T)$ the maximum possible cardinality of a subspace code C in $\mathbb{F}_q^n$ with $d_{\mathrm{S}}(C) \geq d$ and $\dim(U) \in T$ for all $U \in C$, so that e.g. $A_q(n, d; k) = A_q(n, d; \{k\})$. Mostly we omit curly braces for one-element sets. If $C \subseteq \mathcal{G}_q(n, k)$ with $d(C) \geq d$, then we also speak of an $(n, d; k)_q$–CDC. From Equation (2.2) we conclude that the dimension of the intersection of two codewords in C is at most $k - d/2$ and also the minimum subspace distance is determined by the maximum dimension of the intersection of a pair of different codewords.[1]

[1]The same is true for the minimum dimension of the sum of two different codewords. The dimension of the sum of triples of codewords was considered in [18] as another invariant of a CDC.

Exercise 2.2. *Let B be a non-degenerated bilinear form on $\mathbb{F}_q^n$ and*

$$U^\perp = \left\{x \in \mathbb{F}_q^n \,:\, B(x,y) = 0 \,\forall y \in W\right\},$$

i.e., $U^\perp$ is the orthogonal complement of U with respect to B. Show $\dim(U^\perp) = n - \dim(U)$ *and* $d_S(U^\perp, W^\perp) = d_S(U,W)$ *for all $U, W \in \mathcal{P}_q(n)$.*

As an implication we remark

$$A_q(n,d;T) = A_q(n,d;\{n-t \,:\, t \in T\}) \tag{2.4}$$

and

$$A_q(n,d;k) = A_q(n,d;n-k), \tag{2.5}$$

so that we will mostly assume $2k \le n$. Under this assumption the maximum possible subspace distance between two k-spaces is $2k$, i.e., we have $A_q(n,d;k) = 1$ if $d > 2k$ and $0 \le k \le n$. If $n < 0$, $k < 0$, or $k > n$, then we set $A_q(n,d;k) = 0$, which allows us to omit explicit conditions on the parameters n, d, and k in the following. For $A_q(n,d;T)$ we use the same type of conventions. Using geometric language, an $(n,2k;k)_q$–CDC is also called *partial spread* or *partial k-spread*, to be more precise. Note that for a partial k-spread C of cardinality at least 2 we have $n \ge 2k$.

Given a CDC C we also call $C^\perp := \{U^\perp \,:\, U \in C\}$ the *dual code*.

As a representation for a k-space $U \in \mathcal{P}_q(n)$ we use matrices $M \in \mathbb{F}_q^{k\times n}$ whose k rows form a basis of U and write $U = \langle M\rangle$. In this case we say that M is a *generator matrix* of U. If the underlying field is not clear from the context we more precisely write $\langle M\rangle_{\mathbb{F}_q}$ for the row span of M.

Definition 2.3. *Let C be a subspace code in $\mathbb{F}_q^n$. We call a set of matrices $\mathcal{G}$ a* generating set *of C if $\#C = \#\mathcal{G}$ and $C = \{\langle G\rangle \,:\, G \in \mathcal{G}\}$.*

In other words a generating set of a subspace code consist of a corresponding set of generator matrices.

For $U, W \in \mathcal{P}_q(n)$ we have

$$\dim(U+W) = \operatorname{rk}\left(\begin{pmatrix} G_U \\ G_W \end{pmatrix}\right),$$

where $\operatorname{rk}(X)$ denotes the rank of a matrix X and G_U, G_W are generator matrices of U and W, respectively. Inserting into Equation (2.3) gives

$$d_S(U,W) = 2\cdot\operatorname{rk}\left(\begin{pmatrix} G_U \\ G_W \end{pmatrix}\right) - \dim(U) - \dim(W). \tag{2.6}$$

The number of k-spaces in $\mathbb{F}_q^n$ can be easily counted:

Exercise 2.4. *Show that there are exactly $\prod_{i=0}^{k-1}\left(q^n - q^i\right)$ generator matrices (or ordered bases) for a k-space in $\mathbb{F}_q^n$ and that each such k-space admits $\prod_{i=0}^{k-1}\left(q^k - q^i\right)$ different generator matrices, so that*

$$\#\mathcal{G}_q(n,k) = \prod_{i=0}^{k-1}\frac{q^{n-i}-1}{q^{k-i}-1}. \tag{2.7}$$

As further notation we set $\begin{bmatrix} n \\ k \end{bmatrix}_q := \#\mathcal{G}_q(n,k)$, which is called *q-binomial* or *Gaussian binomial coefficient* since they are the q-analog of the binomial coefficient $\binom{n}{k}$ counting the number of k-element subsets of an n-element set.

Exercise 2.5. *Consider* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ *as a function of* q *on* $\mathbb{R}_{>0}$ *using Equation (2.7) and show*

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}$$

for all integers $0 \le k \le n$.

Exercise 2.6. *Show* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$ *and* $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ *whenever the occurring Gaussian binomial coefficients are well defined.*

For lower and upper bounds for $\begin{bmatrix} n \\ k \end{bmatrix}_q$ we refer to the beginning of Section 4, see e.g. Inequality (4.2).

Applying the Gaussian elimination algorithm to a generator matrix G of a k-space U gives a unique generator matrix $E(G)$ in *reduced row echelon form*. Since $E(G) = E(G')$ for any two generator matrices G and G' of U, we will also directly write $E(U)$. By $v(G) \in \mathbb{F}_2^n$ or $v(U) \in \mathbb{F}_2^n$ we denote the characteristic vector of the pivot columns in $E(G)$ or $E(U)$, respectively. These vectors are also called *identifying* or *pivot vectors*. If $U \in \mathcal{G}_q(n,k)$, then $\operatorname{wt}(v(U)) = k$, i.e., the identifying vector of a k-space consists of k ones (and $n-k$) zeroes. Slightly abusing notation we use $\mathcal{G}_1(n,k) := \left\{ v \in \mathbb{F}_2^n \,:\, \operatorname{wt}(v) = k \right\}$.

Example 2.7. *For*

$$U = \left\langle \begin{pmatrix} 1&0&1&1&1&0&1&0&1 \\ 1&0&0&1&1&1&1&1&1 \\ 0&0&0&1&0&0&0&1&0 \\ 0&0&0&0&0&1&1&0&1 \end{pmatrix} \right\rangle \in \mathcal{G}_2(9,4)$$

we have

$$E(U) = \begin{pmatrix} 1&0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&1&1&1 \\ 0&0&0&1&0&0&0&1&0 \\ 0&0&0&0&0&1&1&0&1 \end{pmatrix}$$

and $v(U) = 101101000 \in \mathbb{F}_2^9$.

Consider $M_U = E(U)$ and $M_W = E(W)$ in Equation (2.6). Since the union of the pivot positions in $E(U)$ and $E(W)$ has cardinality

$$\frac{\mathrm{d_H}\big(v((U), v(W)\big) + \dim(U) + \dim(W)}{2},$$

we have have

$$2 \cdot \operatorname{rk}\left(\begin{pmatrix} E(U) \\ E(W) \end{pmatrix} \right) \ge \mathrm{d_H}\big(v(U), v(W)\big) + \dim(U) + \dim(W),$$

so that applying Equation (2.6) gives

$$d_S(U,W) \geq d_H\big(v(U), v(W)\big), \tag{2.8}$$

cf. [76, Lemma 2].

Exercise 2.8. *Let $q > 1$ be a prime power, $\mathbf{u}, \mathbf{w} \in \mathbb{F}_2^n$, and*

$$d_H(\mathbf{u},\mathbf{w}) \leq d \leq \min\left\{\mathrm{wt}(\mathbf{u}) + \mathrm{wt}(\mathbf{w}), n - \frac{\mathrm{wt}(\mathbf{u}) + \mathrm{wt}(\mathbf{w}) - d_H(\mathbf{u},\mathbf{w})}{2}\right\}$$

with $d \equiv 0 \pmod 2$ be arbitrary. Construct subspaces $U \in \mathcal{G}_q\big(n, \mathrm{wt}(\mathbf{u})\big)$ and $W \in \mathcal{G}_q\big(n, \mathrm{wt}(\mathbf{w})\big)$ with $d_S(U,W) = d$.

Note that $v(U)$ depends on the ordering of the positions. By $\mathcal{S}_n$ we denote the symmetric group on $\{1,\ldots,n\}$. Let $\pi \in \mathcal{S}_n$ be a permutation and $M \in \mathbb{F}_q^{k\times n}$ be a matrix. By $\pi M \in \mathbb{F}_q^{k\times n}$ we denote the matrix arising by permuting the columns of M according to π. For a subspace $U \in \mathcal{G}_q(n,k)$ we denote by πU the k-space $\langle \pi E(U)\rangle$. Note that $\langle \pi G\rangle = \langle \pi E(U)\rangle$ for every generator matrix G of U.

Exercise 2.9. *Show* $\dim(U) = \dim(\pi U)$ *and $d_S(U,W) = d_S(\pi U, \pi W)$ for all $U, W \in \mathcal{P}_q(n)$ and $\pi \in \mathcal{S}_n$.*

Example 2.10. *Consider the two 2-spaces*

$$U = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\rangle, \quad W = \left\langle \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \right\rangle$$

in $\mathcal{P}_3(4)$. We have $v(U) = 1100 \in \mathbb{F}_2^4$ and $v(W) = 1100 \in \mathbb{F}_2^4$, so that $d_H\big(v(U), v(W)\big) = 0 < 4 = d_S(U,W)$. For the permutation $\pi = (13)(24)$ we have $v(\pi U) = 0011 \in \mathbb{F}_w^4$ and $v(\pi W) = 1100 \in \mathbb{F}_2^4$, so that $d_H\big(v(\pi U), v(\pi W)\big) = 4 = d_S(U,W)$.

Exercise 2.11. *Let $U, W \in \mathcal{P}_q(n)$ be arbitrary. Show the existence of a permutation $\pi \in \mathcal{S}_n$ with $d_S(U,W) = d_H\big(v(\pi U), v(\pi W)\big)$.*

In other words, we have $d_S(U,W) \geq d_H\big(v(\pi U), v(\pi W)\big)$ for all $\pi \in \mathcal{S}_n$ and there exists a permutation attaining equality.

Definition 2.12. *Let $C \subseteq \mathcal{G}_q(n,k)$ be a CDC. The* pivot structure *of C is the subset $\mathcal{V} := \{v(U) : U \in C\} \subseteq \mathcal{G}_1(n,k)$ of binary vectors that are attained by pivot vectors of the codewords. By $A_q(n,d;k;\mathcal{V})$ we denote the maximum cardinality of a CDC $C \subseteq \mathcal{G}_q(n,k)$ with minimum subspace distance at least d whose pivot structure is a subset of $\mathcal{V}$.*

In order to describe specially structured subsets of $\mathcal{G}_1(n,k)$ we denote by

$$\binom{n_1}{k_1}, \ldots, \binom{n_l}{k_l}$$

the set of binary vectors which contain exactly k_i ones in positions $1 + \sum_{j=1}^{i-1} n_j$ to $\sum_{j=1}^{i} n_j$ for all $1 \leq i \leq l$. The cases of at least k_i ones are denoted by $\binom{n_i}{\geq k_i}$ and the cases of at most k_i ones are denoted by $\binom{n_i}{\le k_i}$. Also in this generalized setting we assume that the described set is a subset of $\mathcal{G}_1(n,k)$, where $n=\sum_{i=1}^{l} n_i$ and $k=\sum_{i=1}^{l} k_i$, i.e.

$$\binom{n_1}{\le k_1}, \binom{n-n_1}{\ge k-k_1} \subseteq \mathcal{G}_1(n,k).$$

For two subsets $\mathcal{V},\mathcal{V}' \subseteq \mathbb{F}_2^n$ we write $\mathrm{d_H}(\mathcal{V},\mathcal{V}')$ for the minimum Hamming distance $\mathrm{d_H}(v,v')$ for arbitrary $v\in\mathcal{V}$ and $v'\in\mathcal{V}'$.

Exercise 2.13. *Let* $\mathcal{V}=\binom{m}{k},\binom{n-m}{0}$ *and* $\mathcal{V}'=\binom{m}{\le k-d/2},\binom{n-m}{\ge d/2}$ *be two subsets of* $\mathcal{G}_1(n,k)$. *Verify* $\mathrm{d}_H(\mathcal{V},\mathcal{V}')=d$.

Our counting formula for k-spaces in Equation (2.7) can be refined to prescribed pivot vectors. To this end, let the *Ferrers tableaux* $T(U)$ of U arise from $E(U)$ by removing the zeroes from each row of $E(U)$ left to the pivots and afterwards removing all pivot columns. If we then replace all remaining entries by dots we obtain the *Ferrers diagram* $\mathcal{F}(U)$ of U which only depends on the identifying vector $v(U)$.

Example 2.14. *For the subspace U from Example 2.7 we have*

$$T(U)=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 0 \\ & & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{F}(U)=\begin{matrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & \bullet & \bullet \end{matrix}.$$

The partially filled $k\times(n-k)$ matrix $T(U)$ contains all essential information to describe the codeword U. The entries in $T(U)$ have no further restrictions besides being contained in $\mathbb{F}_q$, which is reflected by the notation $\mathcal{F}(U)$. Indeed, every different choice gives a different k-dimensional subspace in $\mathbb{F}_q^n$. So, the pivot vector $v(U)$ and the Ferrers diagram $\mathcal{F}(U)$ of U both partition $\mathcal{G}_q(n,k)$ into specific classes. As indicated before, these classes are not preserved by permutations of the coordinates. If n is given, $v(U)$ and $\mathcal{F}(U)$ can be converted into each other.[2] So, we also write $v(\mathcal{F})$ for a given Ferrers diagram and $\mathcal{F}(\mathbf{u})$ for a given vector $\mathbf{u}\in\mathbb{F}_2^n$.

Denoting the number of dots in $\mathcal{F}(\mathbf{u})$ by $\#\mathcal{F}(\mathbf{u})$ we can state that the number of $\mathrm{wt}(\mathbf{u})$-spaces in $\mathbb{F}_q^n$ is given by $q^{\#\mathcal{F}(\mathbf{u})}$.

Exercise 2.15. *Show that for* $\mathbf{u}\in\mathbb{F}_2^n$ *we have* $\#\mathcal{F}(\mathbf{u})=\sum_{i=1}^{n} u_i\cdot\sum_{j=i+1}^{n}\left(1-u_j\right)$.

For two k-spaces with the same pivot vector Equation (2.6) can be used to relate the subspace distance with the rank distance of the corresponding generator matrices:

Lemma 2.16. *([205, Corollary 3]) For* $U,W\in\mathcal{G}_q(n,k)$ *with* $v(U)=v(W)$ *we have* $d_S(U,W)=2d_R(E(U),E(W))$.

[2] The only issue occurs for pivot vectors $v(U)$ starting with a sequence of zeroes corresponding to the same number of leading empty columns in the Ferrers diagram. The latter, or their number, may not be directly visible.

As we will see later on, a different kind of codes is closely related to subspace codes. For two matrices $U, W \in \mathbb{F}_q^{m\times n}$ the *rank distance* is defined as $d_{\mathrm{R}}(U,W) = \operatorname{rk}(U-W)$. As observed e.g. in [87], d_{R} is indeed a metric on the set of $(m \times n)$ matrices over $\mathbb{F}_q$ with values in $\{0, 1, \ldots, \min\{m, n\}\}$. A subset $\mathcal{M} \subseteq \mathbb{F}_q^{m\times n}$ is called a *rank metric code* (RMC) and by $d_{\mathrm{R}}(\mathcal{M}) := \min\{d_{\mathrm{R}}(A,B) : A, B \in \mathcal{M}, A \neq B\}$ we denote the corresponding *minimum rank distance*. As a shorthand, we speak of an $(m \times n, d)_q$–RMC. We call $\mathcal{M}$ *additive* if it is additively closed and *linear* if it forms a subspace of $\mathbb{F}_q^{m\times n}$. In Section 3 we will summarize more details on RMCs that actually are part of the preliminaries and relevant for the later sections.

For the sake of completeness, we mention a few standard notations that we are using in the following. The *sum* of two sets A and B is given by $A + B := \{a + b : a \in A, b \in B\}$. For $a \in A$ we also use the abbreviation $a + B$ for $\{a\} + B$.

Definition 2.17. ***(Packings and partitions)***
A packing $\mathcal{P} = \{P_1, \ldots, P_s\}$ *of a set X is a set of subsets $P_i \subseteq X$ such that $P_i \cap P_j = \emptyset$ for all $1 \le i < j \le s$, i.e., the subsets P_i are pairwise disjoint. The number of elements s is also called the* cardinality $\#\mathcal{P}$ *of $\mathcal{P}$. If additionally $\cup_{i=1}^{s} P_i = X$, then we speak of a* partition.

For packings or partitions of CDCs or RMCs we will need a stronger condition than pairwise disjointness in some applications.

Definition 2.18. ***(d-packings and d-partitions of codes)***
*A packing $\mathcal{P} = \{P_1, \ldots, P_s\}$ of a CDC C is called d-*packing *if $d_S(\mathcal{P}_i) \ge d$ (and $\mathcal{P}_i \subseteq C$) for all $1 \le i \le s$. Similarly, a packing $\mathcal{P} = \{P_1, \ldots, P_s\}$ of a RMC $\mathcal{M}$ is called d-*packing *if $d_R(\mathcal{P}_i) \ge d$ (and $\mathcal{P}_i \subseteq \mathcal{M}$) for all $1 \le i \le s$. If the packings are partitions, then we speak of a d-partition in both cases.*

3 Rank metric codes

Since rank metric codes (RMCs) are closely related to subspace codes, we summarize several facts on ranks of matrices and rank metric codes that will be frequently used later on in this chapter. For a broader overview we refer to e.g. [88] and the references mentioned therein.

Via Equation (2.6) the subspace distance between two spaces $U, W \in \mathbb{F}_q^n$ is linked to the ranks of certain matrices. I.e., if G_U and G_W are generator matrices of U and W, respectively, then we have

$$d_S(U,W) = 2\,\mathrm{rk}\left(\begin{pmatrix} G_U \\ G_W \end{pmatrix}\right) - \mathrm{rk}(G_U) - \mathrm{rk}(G_W). \tag{3.1}$$

So, we summarize a few equations and inequalities for the rank of a matrix. First note that the operations of the Gaussian elimination algorithm do not change the rank of a matrix, which also holds for column permutations.

Exercise 3.1. *Show that for compatible matrices we have*

$$\begin{aligned}
&\mathrm{rk}(M) = \mathrm{rk}(M^\perp);\\
&\mathrm{rk}(M) \le \mathrm{rk}\left(\begin{pmatrix} M & M' \end{pmatrix}\right) \le \mathrm{rk}(M) + \mathrm{rk}(M');\\
&|\,\mathrm{rk}(M) - \mathrm{rk}(M')| \le d_R(M,M') = |\,\mathrm{rk}(M-M')| \le \mathrm{rk}(M) + \mathrm{rk}(M');\\
&\mathrm{rk}\left(\begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,l} \\ \mathbf{0} & M_{2,2} & \dots & M_{2,l} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & M_{l,l} \end{pmatrix}\right) = \sum_{i=1}^{l} \mathrm{rk}(M_{i,i}) \text{ for } l \ge 1.
\end{aligned}$$

Lemma 3.2. ***(Singleton bound for rank metric codes – e.g. [87])***
Let $m, n \ge d$ be positive integers, $q > 1$ a prime power, and $\mathcal{M} \subseteq \mathbb{F}_q^{m\times n}$ be a rank metric code with minimum rank distance d. Then, $\#\mathcal{M} \le q^{\max\{n,m\}\cdot(\min\{n,m\}-d+1)}$.

Codes attaining this upper bound are called *maximum rank distance* (MRD) codes. More precisely, $(m \times n, d)_q$–MRD codes. They exist for all (suitable) choices of parameters, which remains true if we restrict to linear rank metric codes, see e.g. the survey [202]. If $m < d$ or $n < d$, then only $\#\mathcal{M} = 1$ is possible, which can be achieved by a zero matrix and may be summarized to the single upper bound

$$\#\mathcal{M} \le \left\lceil q^{\max\{n,m\}\cdot(\min\{n,m\}-d+1)} \right\rceil =: A_q^R(m \times n, d). \tag{3.2}$$

Delsarte–Gabidulin codes [46, 55, 87, 195]

A *linearized polynomial* (over $\mathbb{F}_{q^n}$) is a polynomial of type $f_0x + f_1x^q + \dots + f_{n-1}x^{q^{n-1}}$ with coefficients $f_i \in \mathbb{F}_{q^n}$. The q-degree of a non-zero linearized polynomial is the maximum i

such that $f_i \neq 0$. A rank metric code can be described as a set of linearized polynomials. By $\mathcal{L}_{k,q,n}$ we denote the set of linearized polynomials of q-degree at most $k-1$ over $\mathbb{F}_{q^n}$. Now $\dim_{\mathbb{F}_q}\left(\mathcal{L}_{k,q,n}\right) = nk$, and since every non-zero element of $\mathcal{L}_{k,q,n}$ has nullity at most $k-1$ it has a rank of at least $n-k+1$. Thus, $\mathcal{L}_{k,q,n}$ gives an $(n \times n, n-k+1)_q$–MRD code. Via puncturing or shortening, see e.g. [202], $(m \times n, d)_q$–MRD codes can be obtained for the cases $m \neq n$. One might say that Delsarte–Gabidulin codes are the rank metric analogue of Reed-Solomon codes.

In [209, Section IV.A] RMCs were related to CDCs via a so-called *lifting construction*, cf. Subsection 5.1. Given a matrix $M \in \mathbb{F}_q^{k\times m}$ its lifting is the k-space $\left\langle \begin{pmatrix} I_k & M \end{pmatrix} \right\rangle \in \mathcal{G}_q(k+m,k)$. By lifting a given RMC $\mathcal{M}$ we understand the CDC C arising as the union of the liftings of the elements of $\mathcal{M}$. If U arises from lifting M and U' arises from lifting M', then we have $d_S(U,U') =$

$$\begin{aligned} & 2\,\mathrm{rk}\left(\begin{pmatrix} I_k & M \\ I_k & M' \end{pmatrix}\right) - \mathrm{rk}\left(\begin{pmatrix} I_k & M \end{pmatrix}\right) - \mathrm{rk}\left(\begin{pmatrix} I_k & M' \end{pmatrix}\right) = 2\,\mathrm{rk}\left(\begin{pmatrix} I_k & M \\ \mathbf{0} & M-M' \end{pmatrix}\right) - 2k \\ = \; & 2\,\mathrm{rk}(I_k) + 2\,\mathrm{rk}(M-M') - 2k = 2d_R(M,M'), \end{aligned}$$

cf. Lemma 2.16, so that $d_S(C) = 2d_R(\mathcal{M})$. A CDC obtained from lifting an MRD code is called *lifted MRD* (LMRD) code yielding:

Theorem 3.3. ***(Lifted MRD code – [209])***

$$A_q(m+k,d;k) \geq A_q^R(k \times m, d/2) = q^{\max\{m,k\}\cdot(\min\{m,k\}-d/2+1)}.$$

In some applications the ranks of the codewords of a RMC have to lie in some set $R \subseteq \mathbb{N}_0$. Each $(m \times n, d)_q$–RMC $\mathcal{M}$, where $\mathrm{rk}(M) \in R$ for each $M \in \mathcal{M}$, is called $(m \times n, d; R)_q$–RMC. The corresponding maximum possible cardinality is denoted by $A_q^R(m \times n, d; R)$. For a non-negative integer l we also use the notations $\leq l$ and $[0,l]$ for the set $R = \{0,\dots,l\}$. More generally, we also write $[a,b]$ for the interval of integers $\{a, a+1, \dots, b-1, b\}$.

The number of matrices of given rank r in $\mathbb{F}_q^{m\times n}$ is well known and its determination can be traced back at least to [160]. Clearly, these numbers yield the exact values of $A_q^R(m \times n, 1; R)$ for minimum rank distance 1.

Proposition 3.4.

$$A_q^R(m \times n, 1; R) = \sum_{r \in R} \begin{bmatrix} m \\ r \end{bmatrix}_q \cdot \prod_{i=0}^{r-1}\left(q^n - q^i\right) = \sum_{r \in R} \begin{bmatrix} n \\ r \end{bmatrix}_q \cdot \prod_{i=0}^{r-1}\left(q^m - q^i\right).$$

Corollary 3.5.

$$A_q^R(m \times n, 1; \leq 1) = \frac{(q^n-1)(q^m-1)}{q-1} + 1.$$

If a MRD code $\mathcal{M}$ is additive, then its rank distribution is completely determined by its parameters:

Theorem 3.6. ***(Rank distribution of additive MRD codes – [55, Theorem 5.6], [202, Theorem 5])***

The number of codewords of rank r in an additive $(m \times n, d)_q$–`MRD` *code is given by* $a_q(m \times n, d; r) :=$

$$\begin{bmatrix}\min\{n,m\}\\ r\end{bmatrix}_q \sum_{s=0}^{r-d} (-1)^s q^{\binom{s}{2}} \cdot \begin{bmatrix} r\\ s\end{bmatrix}_q \cdot \left(q^{\max\{n,m\}\cdot(r-d-s+1)} - 1\right) \qquad (3.3)$$

for all $d \le r \le \min\{n, m\}$*.*

Clearly, there is a unique codeword of rank strictly smaller than d – the zero matrix, which has to be contained in any additive rank metric code. This may be different for non-additive `MRD` codes.

Example 3.7. *For $n = m = 4$ and $d = 2$ the rank distribution of an additive $(4 \times 4, 2)_q$–*`MRD` *is given by*

$$\begin{aligned}
a_q(4\times 4,2;0) &= 1,\\
a_q(4\times 4,2;1) &= 0,\\
a_q(4\times 4,2;2) &= q^8+q^7+2q^6+q^5-q^3-2q^2-q-1\\
&= \left(q^2+q+1\right)\left(q^2+1\right)^2(q+1)(q-1),\\
a_q(4\times 4,2;3) &= q^{11}+q^{10}-q^8-3q^7-3q^6-q^5+q^4+2q^3+2q^2+q\\
&= \left(q^3-q-1\right)\left(q^2+1\right)^2(q+1)^2(q-1)q, \text{ and}\\
a_q(4\times 4,2;4) &= q^{12}-q^{11}-q^{10}+2q^7+q^6-q^4-q^3\\
&= \left(q^5-q^4-q^3+q+1\right)\left(q^2+1\right)(q+1)(q-1)q^3.
\end{aligned}$$

Of course, these five terms add up to $A_q^R(4 \times 4, 2) = q^{12}$*.*

Lemma 3.8. *For each $R \subseteq \mathbb{N}_0$ we have*

$$A_q^R(m \times n, d; R) \ge \sum_{r\in R} a_q(m \times n, d; r).$$

The easy observation in Lemma 3.8 is implicitly contained in e.g. [227].

Example 3.9. *From Example 3.7 and Lemma 3.8 we directly compute*

$$\begin{aligned}
A_q^R(4\times 4,2;0) &\ge 1,\\
A_q^R(4\times 4,2;\le 1) &\ge 1,\\
A_q^R(4\times 4,2;\le 2) &\ge q^8+q^7+2q^6+q^5-q^3-2q^2-q,\\
A_q^R(4\times 4,2;\le 3) &\ge q^{11}+q^{10}-2q^7-q^6+q^4+q^3, \text{ and}\\
A_q^R(4\times 4,2;\le 4) &\ge q^{12},
\end{aligned}$$

i.e., $A_2(4\times4, 2; 0) \ge 1$*,* $A_2(4\times4, 2; \le 1) \ge 1$*,* $A_2(4\times4, 2; \le 2) \ge 526$*,* $A_2(4\times4, 2; \le 3) \ge 2776$*, and* $A_2(4 \times 4, 2; \le 4) \ge 4096$*.*

Exercise 3.10. *Let m, n, d be positive integers and $R \subseteq \mathbb{N}_0$. Show*

(1) $A_q(m \times n, d; 0) = 1$*;*

(2) $A_q(m \times n, d; R) \le 1$ *if* $R \subseteq \left[0, \left\lfloor \frac{d-1}{2} \right\rfloor\right]$*;*

(3) $A_q(m \times n, d; R') \le A_q(m \times n, d; R)$ *if* $R' \subseteq R$*; and*

(4) $A_q(m \times n, d; R) = A_q(m \times n, d)$ *if* $[0, n] \subseteq R$.

In order to exploit the inequality $d_{\mathrm{R}}(M, M') \ge |\operatorname{rk}(M) - \operatorname{rk}(M')|$ we define a metric d on subsets of non-negative integers. Specializing the usual metric on $\mathbb{R}$ we set $d(s, s') = |s - s'|$ for all $s, s' \in \mathbb{N}_0$. With this, we set $d(S) = \min\{d(s, s'),\ s, s' \in S, s \ne s'\}$ and $d(S, S') := \min\{d(s, s') : s \in S, s' \in S\}$ for any two arbitrary subsets $S, S' \subseteq \mathbb{N}_0$. Actually we use the two later constructs for any metric, i.e., we also use the notations $d_{\mathrm{S}}(C, C')$ and $d_{\mathrm{R}}(\mathcal{M}, \mathcal{M}')$ for the minimum subspace distance between two subspaces from two different CDCs and for the minimum rank-distance between two matrices from two different RMCs.

Lemma 3.11. *Let $\mathcal{M}$ be an $(m \times n, d; R)_q$–RMC and $\mathcal{M}'$ be an $(m \times n, d; R')_q$–RMC. If $d(R, R') \ge d \ge 1$, then $\mathcal{M} \cup \mathcal{M}'$ is an $(m \times n, d; R \cup R')_q$–RMC of cardinality $\#\mathcal{M} + \#\mathcal{M}'$.*

Example 3.12. *The union of a $(4 \times 3, 2; \le 1)_q$–RMC and a $(4 \times 3, 2; 3)_q$–RMC is a $(4 \times 3, 2; \le 3)_q$–RMC.*

$(m \times n, d; R)_q$–RMCs with $R = \{r\}$ are also called *constant rank codes* and their relation to constant dimension codes has e.g. been studied in [93, 94].

Lemma 3.13. *[94, Proposition 3]*

$$A_q^R(m \times n, d_1/2 + d_2/2; r) \ge \min\left\{A_q(m, d_1; r), A_q(n, d_2, r)\right\}$$

Example 3.14. *From Lemma 3.13 we can conclude*

$$A_q^R(4 \times 4, 2; \le 1) \ge A_q^R(4 \times 4, 2; 1) \ge A_q(4, 2; 1) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = q^3 + q^2 + q + 1$$

and

$$A_q^R(4 \times 3, 2; 1) \ge \min\left\{A_q(4, 2; 1), A_q(3, 2; 1)\right\} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = q^2 + q + 1.$$

Proposition 3.15. *[94, Corollary 4] If $1 \le r \le \min\{m, n\}$, then we have*

$$A_q^R(m \times n, r + 1; r) = \begin{bmatrix} \min\{m, n\} \\ r \end{bmatrix}_q = A_q(\min\{m, n\}, 2; r).$$

Further lower bounds for $A_q^R(m \times n, d; r)$ can be concluded from the pigeonhole principle. To this end we use the following partitioning result for MRD codes.

Lemma 3.16. ***(Parallel MRD codes – [77, Lemma 5])***
For $d' > d > 0$ there exists an $(n \times m, d)_q$–MRD code $\mathcal{M}$ that can be partitioned in $\alpha := A_q^R(n \times m, d)/A_q^R(n \times m, d')$ RMCs $\mathcal{M}_i$ with $d_R(\mathcal{M}_i) \ge d'$ for $1 \le i \le \alpha$.

Let $\mathcal{M}$ be a linear $(n \times m, d)_q$–$\mathtt{MRD}$ code that contains a linear $(n \times m, d')_q$–$\mathtt{MRD}$ $\mathcal{M}'$ as a subcode. With this, the set $\{M + \mathcal{M}' : M \in \mathcal{M}\}$ is such a partition described in Lemma 3.16, cf. Lemma 5.66. In terms of Definition 2.18 we also speak of a d'-partition of $\mathcal{M}$.

Exercise 3.17. *Prove the following statements in order to deduce Lemma 3.16.*

(1) Let $\mathcal{M}$ be an $(n\times m, d)_q$–$\mathtt{RMC}$. For each matrix $M \in \mathbb{F}_q^{n\times m}$ also $M+\mathcal{M}$ is an $(n\times m, d)_q$–$\mathtt{RMC}$ with the same cardinality $\#\mathcal{M}$.

(2) Let $\mathcal{M}$ be an additive $(n\times m, d)_q$–$\mathtt{RMC}$ and $M, M' \in \mathbb{F}_q^{n\times m}$ be arbitrary matrices. We have $M + \mathcal{M} = M' + \mathcal{M}$ iff $M' - M \in \mathcal{M}$ and $(M + \mathcal{M}) \cap (M' + \mathcal{M}) = \emptyset$ otherwise.

(3) Let $\mathcal{M}$ be an $(n \times m, d)_q$–$\mathtt{RMC}$ that contains an additive $(n \times m, d')_q$–$\mathtt{RMC}$ as a subcode, where $d' \ge d$. Then, $\{M + \mathcal{M}' : M \in \mathcal{M}\}$ is a set of $(n \times m, d')_q$–$\mathtt{RMC}$s $\mathcal{M}_1, \ldots, \mathcal{M}_s$, where $s \ge \#\mathcal{M}/\#\mathcal{M}'$ and $d_R(\mathcal{M}_i, \mathcal{M}_j) \ge d$ for all $1 \le i < j \le s$. Moreover, $\cup_{i=1}^{s}\mathcal{M}_i$ is an $(n \times m, d)_q$–$\mathtt{RMC}$ of cardinality $s \cdot \#\mathcal{M}'$.

(4) Use the Delsarte–Gabidulin $\mathtt{MRD}$-codes to show that for any positive integers m and n there exists a chain of linear $m \times n$–$\mathtt{MRD}$-codes $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \ldots$ such that $\mathcal{M}_i$ has minimum rank distance i for all $1 \le i \le min\{n, m\}$.

Remark 3.18. *Note that there are examples of $\mathtt{MRD}$ codes with minimum rank distance d which cannot be extended to an $\mathtt{MRD}$ code with minimum rank distance $d + 1$, see e.g. [200, Section 1.6] and [42, Example 34]. In [201, Theorem 9] it was shown that every binary additive $\mathtt{MRD}$ code with minimum rank distance $n - 1$ contains a binary additive $\mathtt{MRD}$ code with minimum rank distance n as a subcode.*

Lemma 3.19. *For each $R \subseteq \mathbb{N}_0$ we have*

$$A_q^R(m \times n, d; R) \ge \max_{1\le d'\le d} \frac{A_q^R(m \times n, d)}{A_q^R(m \times n, d')} \cdot \sum_{r\in R} a_q(m \times n, d'; r).$$

Proof. Let $\mathcal{M}'$ be a linear $(n \times m, d')_q$–$\mathtt{MRD}$ code that contains a linear $(n \times m, d)_q$–$\mathtt{MRD}$ $\mathcal{M}$ as a subcode. By $\mathcal{M}_1, \ldots, \mathcal{M}_\alpha$ we denote the $\alpha := A_q^R(m \times n, d')/A_q^R(m \times n, d)$ cosets $M + \mathcal{M}$ of $\mathcal{M}$ in $\mathcal{M}'$. By the pigeonhole principle there exists an index $1 \le i \le \alpha$ such that $\#\{M \in \mathcal{M}_i \ \mathrm{rk}(M) \in R\} \ge \frac{1}{\alpha} \cdot \#\{M \in \mathcal{M}' : \mathrm{rk}(M) \in R\}$. □

Example 3.20. *From Theorem 3.6 we compute $a_q(4\times 4, 1; 1) = q^7+q^6+q^5+q^4-q^3-q^2-q-1$, so that*

$$A_q^R(4 \times 4, 2; 1) \ge \left\lceil \frac{a_q(4 \times 4, 1; 1)}{q^4} \right\rceil = q^3 + q^2 + q^1 + \left\lceil \frac{q^4 - q^3 - q^2 - q}{q^4} \right\rceil = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q.$$

Due to Proposition 3.15 this lower bound is tight. Note that $\mathrm{rk}(M' - M) \le \mathrm{rk}(M) + \mathrm{rk}(M')$ implies $A_q^R(4 \times 4, 2; \le 1) = A_q^R(4 \times 4, 2; 1)$. For $A_q^R(4 \times 4, 2; \le 2)$ and $A_q^R(4 \times 4, 2; \le 3)$ Lemma 3.19 yields a weaker lower bound than Lemma 3.8.

Removing the coset $\mathbf{0} + \mathcal{M} = \mathcal{M}$ from the consideration yields a slightly different variant of Lemma 3.19:

Lemma 3.21. *For each $R \subseteq \mathbb{N}_0$ we have $A_q^R(m \times n, d; R) \ge$*

$$\max_{1 \le d' < d} \frac{1}{A_q^R(m \times n, d')/A_q^R(m \times n, d) - 1} \cdot \sum_{r \in R} \Big(a_q(m \times n, d'; r) - a_q(m \times n, d; r) \Big).$$

Corollary 3.22. *(Cf. [171, Proposition 2.4]) If $m \le n$ and $r < d$, then we have*

$$A_q^R(m \times n, d; \le r) \ge \max_{1 \le d' < d} \frac{1}{q^{d-d'} - 1} \cdot \sum_{1 \le i \le r} a_q(m \times n, d'; i).$$

Example 3.23. *We compute*

$$\begin{array}{rcl}
a_q(5 \times 5, 1; 0) &=& 1, \\
a_q(5 \times 5, 1; 1) &=& q^9 + q^8 + q^7 + q^6 + q^5 - q^4 - q^3 - q^2 - q - 1, \\
a_q(5 \times 5, 1; 2) &=& q^{16} + q^{15} + 2q^{14} + 2q^{13} + q^{12} - q^{11} - 2q^{10} - 4q^9 - 4q^8 \\
&& -2q^7 - q^6 + q^5 + 2q^4 + 2q^3 + q^2 + q, \\
a_q(5 \times 5, 1; 3) &=& q^{21} + q^{20} + 2q^{19} + q^{18} - 3q^{16} - 4q^{15} - 5q^{14} - 3q^{13} + 3q^{11} \\
&& +5q^{10} + 4q^9 + 3q^8 - q^6 - 2q^5 - q^4 - q^3, \\
a_q(5 \times 5, 1; 4) &=& q^{24} + q^{23} - q^{21} - 2q^{20} - 3q^{19} - 2q^{18} + q^{17} + 3q^{16} + 4q^{15} \\
&& +3q^{14} + q^{13} - 2q^{12} - 3q^{11} - 2q^{10} - q^9 + q^7 + q^6, \\
a_q(5 \times 5, 1; 5) &=& q^{25} - q^{24} - q^{23} + q^{20} + q^{19} + q^{18} - q^{17} - q^{16} - q^{15}, \\
&& +q^{12} + q^{11} - q^{10}, \\
a_q(5 \times 5, 2; 0) &=& 1, \\
a_q(5 \times 5, 2; 1) &=& 0,
\end{array}$$

$$\begin{array}{rcl}
a_q(5 \times 5, 2; 2) &=& q^{11} + q^{10} + 2q^9 + 2q^8 + 2q^7 - 2q^4 - 2q^3 - 2q^2 - q - 1, \\
a_q(5 \times 5, 2; 3) &=& q^{16} + q^{15} + 2q^{14} + q^{13} - 3q^{11} - 4q^{10} - 6q^9 - 4q^8 - 2q^7 + q^6 \\
&& +3q^5 + 4q^4 + 3q^3 + 2q^2 + q, \\
a_q(5 \times 5, 2; 4) &=& q^{19} + q^{18} - q^{16} - 2q^{15} - 3q^{14} - 2q^{13} + q^{12} + 3q^{11} + 5q^{10} \\
&& +4q^9 + 2q^8 - q^7 - 2q^6 - 3q^5 - 2q^4 - q^3, \text{ and} \\
a_q(5 \times 5, 2; 5) &=& q^{20} - q^{19} - q^{18} + q^{15} + q^{14} + q^{13} - q^{12} - q^{11} - 2q^{10} + q^7 + q^6.
\end{array}$$

So, choosing $d' = 1$ in Lemma 3.21 gives $A_q^R(5 \times 5, 2; \le 3)$

$$\begin{array}{rl}
\ge & \frac{1}{q^5 - 1} \cdot \sum_{r=1}^{3} \big(a_q(5 \times 5, 1; r) - a_q(5 \times 5, 2; r) \big) \\
= & \Big(q^4 + q^3 + q^2 + q + 1 \Big) \cdot \Big(q^9 + q^7 - q^6 - q^5 - q^4 - q^3 + q^2 + q + 1 \Big) \cdot q^3 \\
= & q^{16} + q^{15} + 2q^{14} + q^{13} - 2q^{11} - 3q^{10} - 3q^9 - q^8 + q^7 + 2q^6 + 3q^5 + 2q^4 + q^3.
\end{array}$$

We remark that Lemma 3.8 gives only

$$A_q^R(5 \times 5, 2; \leq 3) \geq q^{11} + q^{10} + 2q^9 + 2q^8 + 2q^7 - 2q^4 - 2q^3 - 2q^2 - q.$$

Sometimes we want to control the possible ranks of submatrices of the elements in a RMC. By suitably choosing the RMCs $\mathcal{M}_i$ this is e.g. possible via:

Lemma 3.24. ***(Product construction for rank metric codes)***
Let $l \geq 1$ and $\bar{n} = (n_1, \ldots, n_l) \in \mathbb{N}^l$. For $1 \leq i \leq l$ let $\mathcal{M}_i$ be a $(k \times n_i, d)_q$–RMC. With this,

$$\mathcal{M} = \left\{ \begin{pmatrix} M_1 & \ldots & M_l \end{pmatrix} \,:\, M_i \in \mathcal{M}_i \,\forall 1 \leq i \leq l \right\}$$

is a $(k \times n, d)_q$–RMC with cardinality $\#\mathcal{M} = \prod_{i=1}^{l} \#\mathcal{M}_i$, where $n = \sum_{i=1}^{l} n_i$.

Proof. It suffices to show $d_{\mathrm{R}}(\mathcal{M}) \geq d$. To this end let $M = \begin{pmatrix} M_1 \ldots & M_l \end{pmatrix}$ and $M' = \begin{pmatrix} M_1' \ldots & M_l' \end{pmatrix}$ be two different codewords in $\mathcal{M}$. Since $M \neq M'$, there exists an index $1 \leq i \leq l$ with $M_i \neq M_i'$, so that $d_{\mathrm{R}}(M, M') = \operatorname{rk}\left(\begin{pmatrix} M_1 - M_1' & \ldots & M_l - M_l' \end{pmatrix}\right) \geq \operatorname{rk}\left(M_i - M_i'\right) = d_{\mathrm{R}}(M_i, M_i') \geq d_{\mathrm{R}}(\mathcal{M}_i) \geq d$. □

As abbreviation we write $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_l$ for an RMC obtained by the product construction. Another variant can be used to combine several RMCs to an RMC with a larger minimum rank distance.

Lemma 3.25. ***(Diagonal concatenation of rank metric codes)***
Let $\mathcal{M}_1$ be a $(k_1 \times n_1, d_1)_q$–RMC, $\mathcal{M}_2$ be a $(k_2 \times n_2, d_2)_q$–RMC, and $M_1^1, \ldots, M_1^{s_1}$, $M_2^1, \ldots, M_2^{s_2}$ arbitrary enumerations of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Then.

$$\mathcal{M} = \left\{ \begin{pmatrix} M_1^i & \mathbf{0}_{k_1 \times n_2} \\ \mathbf{0}_{k_2 \times n_1} & M_2^i \end{pmatrix} \,:\, 1 \leq i \leq \min\{s_1, s_2\} \right\}$$

is a $((k_1 + k - 2) \times (n_1 + n_2), d_1 + d_2)_q$–RMC with cardinality $\#\mathcal{M} = \min\{\#\mathcal{M}_1,\ \#\mathcal{M}_2\}$.

Proof. Let $G = \begin{pmatrix} M_1 & \mathbf{0} \\ \mathbf{0} & M_2 \end{pmatrix}$ and $G' = \begin{pmatrix} M_1' & \mathbf{0} \\ \mathbf{0} & M_2' \end{pmatrix}$ be two different elements in $\mathcal{M}$. By construction, $G \neq G'$ implies $M_1 \neq M_1'$ and $M_2 \neq M_2'$, so that $d_{\mathrm{R}}(G, G') =$

$$\operatorname{rk}(G - G') = \operatorname{rk}\left(\begin{pmatrix} M_1' - M_1 & \mathbf{0} \\ \mathbf{0} & M_2' - M_2 \end{pmatrix}\right) = \operatorname{rk}(M_1' - M_1) + \operatorname{rk}(M_2' - M_2) \geq d_1 + d_2,$$

i.e., $d_{\mathrm{R}}(\mathcal{M}) \geq d_1 + d_2$. □

We remark that the iterative application of Lemma 3.25 results in a $(k \times n, d)_q$–RMC $\mathcal{M}$ with cardinality $\min\{\#\mathcal{M}_i \,:\, 1 \leq i \leq\}$ given $(k_i \times n_i, d_i)_q$–RMCs $\mathcal{M}_i$ for $1 \leq i \leq l$, where $l \geq 1$, $n = \sum_{i=1}^{l} n_i$, $d = \sum_{i=1}^{l} d_i$, and $k = \sum_{i=1}^{l} k_i$.

Sum-rank metric codes

In the following we want to consider restrictions on the ranks of different submatrices of a rank metric code. It turns out that those restrictions fit into the framework of sum-rank metric codes that were already used for space-time coding, see e.g. [66, 187]. For positive integers t, $m_1, \ldots, m_t, n_1, \ldots, n_t$ consider the product of t matrix spaces

$$\Pi := \bigoplus_{i=1}^{t} \mathbb{F}_q^{m_i \times n_i}$$

and define the *sum-rank* of an element $X = (X_1, \ldots, X_t) \in \Pi$ as

$$\operatorname{srk}(X) := \sum_{i=1}^{t} \operatorname{rk}(X_i). \tag{3.4}$$

Exercise 3.26. *Show that the sum-rank induces a metric on* Π *via* $(X, Y) \mapsto \operatorname{srk}(X - Y)$.

Definition 3.27. *A subset* $\mathcal{M} \subseteq \Pi$ *is called a* sum-rank metric code *(*`SRMC`*) and by* $d_{S\text{-}R}(\mathcal{M}) := \min\{d_{S\text{-}R}(A, B) \,:\, A, B \in \mathcal{M}, A \neq B\}$ *we denote the corresponding* minimum sum-rank distance. *We call* $\mathcal{M}$ additive *if it is additively closed and* linear *if it forms a subspace of* Π*. By* $A_q^r(m_1 \times n_1, \ldots, m_t \times n_t, d)$ *we denote the corresponding maximum possible cardinality for minimum sum-rank distance* d*. If we additionally require that the sum-ranks of the elements in* $\mathcal{M}$ *have to be contained in a set* $R \subset \mathbb{N}_0$*, then we denote the corresponding maximum possible cardinality by* $A_q^r(m_1 \times n_1, \ldots, m_t \times n_t, d; R)$*.*

In the following we will state two explicit construction for `SRMC`s and refer to e.g. [41] for further results.

Lemma 3.28. *Let* $\mathcal{M}_1$ *be an* $(m_1 \times n_1, d; R_1)_q$*–*`RMC` *and* $\mathcal{M}_2$ *be an* $(m_2 \times n_2, d;\; R_2)_q$*–*`RMC`*. Then, there exists an* $(m_1 \times n_1, m_2 \times n_2, d; R_1 + R_2)_q$*–*`SRMC` *with cardinality* $\#\mathcal{M} = \#\mathcal{M}_1 \cdot \#\mathcal{M}_2$*.*

Proof. Let $\mathcal{M} = \{(M_1, M_2) \,:\, M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2\}$, so that $\#\mathcal{M} = \#\mathcal{M}_1 \cdot \#\mathcal{M}_2$. Consider arbitrary elements $(M_1, M_2), \left(M_1', M_2'\right) \in \mathcal{M}$ with $(M_1, M_2) \neq \left(M_1', M_2'\right)$. If $M_1 \neq M_1'$, then we have

$$\begin{aligned} d_{\text{S-R}}\big((M_1, M_2), (M_1', M_2')\big) &= d_{\text{R}}(M_1, M_1') + d_{\text{R}}(M_2, M_2') \\ &\geq d_{\text{R}}(M_1, M_1') \geq d_{\text{R}}(\mathcal{M}_1) \geq d. \end{aligned}$$

If $M_1 = M_1'$, then we have $M_2 \neq M_2'$ and

$$\begin{aligned} d_{\text{S-R}}\big((M_1, M_2), (M_1', M_2')\big) &= d_{\text{R}}(M_1, M_1') + d_{\text{R}}(M_2, M_2') \\ &\geq d_{\text{R}}(M_2, M_2') \geq d_{\text{R}}(\mathcal{M}_2) \geq d. \end{aligned}$$

□

Lemma 3.29. *Let* $\mathcal{M}_1$ *be an* $(m_1 \times n_1, d_1; R_1)_q$*–*`RMC` *and* $\mathcal{M}_2$ *be an* $\left(m_2 \times n_2, d_2;\; R_2\right)_q$*–*`RMC`*. Then, there exists an* $(m_1 \times n_1, m_2 \times n_2, d_1 + d_2; R_1 + R_2)_q$*–*`SRMC` *with cardinality* $\#\mathcal{M} = \min\{\#\mathcal{M}_1, \#\mathcal{M}_2\}$*.*

Proof. Let $M_1^1, \ldots, M_1^s$ be an arbitrary numbering of the elements of $\mathcal{M}_1$ and $M_2^1, \ldots, M_2^r$ be an arbitrary numbering of the elements of $\mathcal{M}_2$. With this we set $\mathcal{M} = \left\{\left(M_1^i, M_2^i\right) : 1 \le i \le \min\{s, r\}\right\}$, so that $\#\mathcal{M} = \min\{\#\mathcal{M}_1, \#\mathcal{M}_2\}$. Let $(M_1, M_2) \in \mathcal{M}$ be an arbitrary element. By construction we have $\operatorname{rk}(M_1) + \operatorname{rk}(M_2) \in R_1 + R_2$. Let $\left(M_1', M_2'\right) \in \mathcal{M}$ be another element with $(M_1, M_2) \neq \left(M_1', M_2'\right)$. Then, we have $M_1 \neq M_1'$ and $M_2 \neq M_2'$, so that

$$\begin{aligned} d_{\text{S-R}}\left((M_1, M_2), (M_1', M_2')\right) &= d_{\text{R}}(M_1, M_1') + d_{\text{R}}(M_2, M_2') \\ &\ge d_{\text{R}}(\mathcal{M}_1) + d_{\text{R}}(\mathcal{M}_2) \ge d_1 + d_2. \end{aligned}$$

□

Lemma 3.30. *For $M_1, M_1' \in \mathbb{F}_q^{m_1 \times n_1}$ and $M_2, M_2' \in \mathbb{F}_q^{m_2 \times n_2}$ we have*

$$d_R(M_1, M_1') + d_R(M_2, M_2') \ge \left|\operatorname{rk}(M_1) - \operatorname{rk}(M_1')\right| + \left|\operatorname{rk}(M_2) - \operatorname{rk}(M_2')\right|.$$

Example 3.31. *Applying Lemma 3.29 to a $(3 \times 3, 1; 0)_q$–RMC and a $(3 \times 3, 2; 0)_q$–RMC yields a $(3\times 3, 3\times 3, 3, 0)_q$–SRMC $\mathcal{M}_1$ of cardinality 1. Applying Lemma 3.29 to a $(3\times 3, 1; 1)_q$–RMC and a $(3 \times 3, 2; 2)_q$–RMC yields a $(3 \times 3, 3 \times 3, 3, 3)_q$–SRMC $\mathcal{M}_2$ of cardinality*

$$\min\left\{A_q^R(3 \times 3, 1; 1), A_q^R(3 \times 3, 2; 2)\right\} \ge \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \cdot \left(q^3 - 1\right) = q^5 + q^4 + q^3 - q^2 - q - 1.$$

Applying Lemma 3.28 to a $(3\times 3, 3; 3)_q$–RMC and a $(3\times 3, 3; 0)_q$–RMC yields a $(3\times 3, 3\times 3, 3, 3)_q$–SRMC $\mathcal{M}_3$ of cardinality $q^3 \cdot 1 = q^3$. From Lemma 3.30 we conclude that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ is a $(3 \times 3, 3 \times 3, 3, \le 3)_q$–SRMC, so that $A_q^R(3 \times 3, 3 \times 3, 3, \le 3) \ge q^5 + q^4 + 2q^3 - q^2 - q$, i.e., $A_2^R(3 \times 3, 3 \times 3, 3, \le 3) \ge 58$ for $q = 2$.

We remark that Example 3.31 will be explicitly used in the construction for a CDC considered in Example 5.85.

4 Upper bounds for constant dimension codes

In this section we want to survey upper bounds for $A_q(n, d; k)$ and variants thereof. Since the codewords of an $(n, d; k)_q$–CDC are contained in $\mathcal{G}_q(N, k)$, we have $A_q(n, d; k) \le \begin{bmatrix} n \\ k \end{bmatrix}_q$. For minimum subspace distance $d = 2$ this upper bound is tight, i.e., $C = \mathcal{G}_q(n, k)$ is an $(n, 2; k)_q$–CDC with cardinality $\begin{bmatrix} n \\ k \end{bmatrix}_q$. In [146, Lemma 4] the bounds $1 < q^{-l(n-k)} \cdot \begin{bmatrix} n \\ k \end{bmatrix}_q < 4$ were shown. The corresponding proof itself and associated remarks actually give a refined upper bound.

q-Pochhammer symbol

The q-analog of the *Pochhammer symbol* is the *q-Pochhammer symbol*

$$(a; q)_n := \prod_{i=0}^{n-1} \left(1 - aq^i\right) \tag{4.1}$$

with $(a; q)_0 = 1$ by definition. In the theory of basic hypergeometric series (or q-hypergeometric series), the q-Pochhammer symbol plays the role that the ordinary Pochhammer symbol plays in the theory of generalized hypergeometric series. It can be extended to an infinite product $(a; q)_\infty = \prod_{i=0}^{\infty} \left(1 - aq^i\right)$. Setting $a = q$ this is an analytic function of q in the interior of the unit disk and can also be considered as a formal power series in q, whose reciprocal is the generating function of integer partitions, see e.g. [221, Chapter 15].

Here we specialize the q-Pochhammer symbol to $(1/q; 1/q)_n = \prod_{i=1}^{n} \left(1 - 1/q^i\right)$ and state the bounds

$$1 \le \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{q^{k(n-k)}} \le \frac{1}{(1/q; 1/q)_k} < \frac{1}{(1/q; 1/q)_\infty} \le \frac{1}{(1/2; 1/2)_\infty} \approx 3.4627, \tag{4.2}$$

see [125, Section 5].

Exercise 4.1. *Show that the sequence $(1/q; 1/q)_\infty$ is monotonically increasing with q and approaches $(q-1)/q$ for large q.*

Exercise 4.2. *Show* $\lim\limits_{a\to\infty} \frac{\begin{bmatrix} a+b \\ b \end{bmatrix}_q}{q^{ab}} = \frac{1}{(1/q;1/q)_b}$ *for each $b \in \mathbb{N}_{\ge 0}$.*

q	2	3	4	5	7	8	9	11	16	32	64	128	256	512
$1/(1/q; 1/q)_\infty$	3.46	1.79	1.45	1.32	1.20	1.16	1.14	1.11	1.07	1.03	1.02	1.01	1.004	1.002

Table 4.1: Approximate values of $1/(1/q; 1/q)_\infty$ for selected field sizes.

Due to $A_q(n, d; k) = A_q(n, d; n-k)$, see Equation (2.5), we assume $2k \le n$ in this section. For d we consider only even values between 4 and $2k$, so that $k \ge 2$ and $n \ge 4$. Since the maximum size of a code with certain parameters is always an integer and some of the latter upper bounds can produce non-integer values, we may always round them down. To ease the notation we will mostly omit the final rounding step. For other surveys on upper bounds for constant dimension codes we refer e.g. to [125, 142]. First we want to study the q-analogs of the classical upper bounds for binary constant weight codes. Then we briefly discuss other approaches from the literature. The special case of the maximum possible minimum subspace distance $d = 2k$, assuming $2k \le n$, is the topic of Subsection 4.1. The latest improvements of upper bounds for $A_q(n, d; k)$ are based on q^{k-1}-divisible (multi-) sets of points. The necessary background and the corresponding upper bounds for CDCs are presented in Subsection 4.2.

Grassmann graph

The vertices of the *Grassmann graph* $J_q(n, k)$, named after Hermann Günther Graßmann, are the $\begin{bmatrix} n \\ k \end{bmatrix}_q$ k-spaces in $\mathbb{F}_q^n$ where two vertices are adjacent when their intersection is $(k-1)$-dimensional. Grassmann graphs are q-analogs of *Johnson graphs* and *distance-regular*[1].

Note that $\dim(U \cap W) \ge k - t$ is equivalent to $d_S(U, W) \le m - k + 2t$. The fact that the Grassmann graph is distance-regular implies a sphere-packing bound. To this end we count k-dimensional subspaces having a "large" intersection with a fixed m-dimensional subspace:

Exercise 4.3. *Show that for integers* $0 \le t \le k \le n$ *and* $k - t \le m \le n$ *we have*

$$\#\left\{U \in \begin{bmatrix} V \\ k \end{bmatrix} \mid \dim(U \cap W) \ge k - t\right\} = \sum_{i=0}^{t} q^{(m+i-k)i} \begin{bmatrix} m \\ k-i \end{bmatrix}_q \begin{bmatrix} n-m \\ i \end{bmatrix}_q,$$

where $V = \mathbb{F}_q^n$, $W \le V$, *and* $\dim(W) = m$.

Theorem 4.4. ***(Sphere-packing bound – [146, Theorem 6])***

$$A_q(n, d; k) \le \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\sum\limits_{i=0}^{\lfloor (d/2-1)/2 \rfloor} q^{i^2} \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} n-k \\ i \end{bmatrix}_q}$$

We remark, that we can obtain the denominator of the formula of Theorem 4.4 by setting $m = k$, $2t = d/2 - 1$ in Exercise 4.3 and applying $\begin{bmatrix} k \\ k-i \end{bmatrix}_q = \begin{bmatrix} k \\ i \end{bmatrix}_q$. The right hand side is symmetric with respect to orthogonal complements, i.e., the mapping $k \mapsto n - k$ leaves it invariant.

By defining a puncturing operation one can decrease the dimension of the ambient space and the codewords. Since the minimum distance decreases by at most two, we can iteratively puncture $d/2-1$ times, so that $A_q(n, d; k) \le \begin{bmatrix} n-d/2+1 \\ k-d/2+1 \end{bmatrix}_q = \begin{bmatrix} n-d/2+1 \\ v-k \end{bmatrix}_q$ since $A_q(v', 2; k') = \begin{bmatrix} v' \\ k' \end{bmatrix}_q$. Considering either the code or its dual code gives:

Theorem 4.5. ***(Singleton bound – [146, Theorem 9])***

$$A_q(n, d; k) \le \begin{bmatrix} n - d/2 + 1 \\ \max\{k, n-k\} \end{bmatrix}_q$$

[1] A distance-regular graph is a regular graph such that for any two vertices v and w, the number of vertices at distance j from v and at distance k from w depends only upon j, k, and the distance i between v and w.

Comparison between the Sphere-packing and the Singleton bound

Referring to [146] the authors of [142] state that the Singleton bound is always stronger than the sphere packing bound for non-trivial codes. However, for $q = 2$, $n = 8$, $d = 6$, and $k = 4$, the sphere-packing bound gives an upper bound of $200787/451 \approx 445.20399$ while the Singleton bound gives an upper bound of $\begin{bmatrix}6\\4\end{bmatrix}_2 = 651$. For $q = 2$, $n = 8$, $d = 4$, and $k = 4$ it is just the other way round, i.e., the Singleton bound gives $\begin{bmatrix}7\\3\end{bmatrix}_2 = 11811$ and the sphere-packing bound gives $\begin{bmatrix}8\\4\end{bmatrix}_2 = 200787$. For $d = 2$ both bounds coincide and for $d = 4$ the Singleton bound is always stronger than the sphere-packing bound since $\begin{bmatrix}n-1\\k\end{bmatrix}_q < \begin{bmatrix}n\\k\end{bmatrix}_q$. The asymptotic bounds [146, Corollaries 7 and 10], using normalized parameters, and [146, Figure 1] suggest that there is only a small range of parameters where the sphere-packing bound can be superior to the Singleton bound.

Exercise 4.6. *Show that the sphere-packing bound is strictly tighter than the Singleton bound iff $q = 2$, $n = 2k$, and $d = 6$.*

For $k \le n-k$ (or $2k \le n$) an `LMRD` code gives the lower bound $A_q(n, d; k) \ge q^{(n-k)\cdot(k-d/2+1)}$, see Theorem 3.3. In [146] it was observed that the Singleton bound implies $A_q(n, d; k) \le 4 \cdot q^{(n-k)\cdot(k-d/2+1)}$, i.e., `LMRD` codes are at most a factor of four (2 bits) distant to optimal codes. We will give a tighter estimate in Proposition 4.11.

Proposition 4.7. *([125, Proposition 7])*
For $k \le n - k$ the ratio of the size of an `LMRD` code divided by the size of the Singleton bound converges for $n \to \infty$ monotonically decreasing to

$$(1/q; 1/q)_{k-d/2+1} > (1/q; 1/q)_\infty \ge (1/2; 1/2)_\infty > 0.288788.$$

Anticode bounds

Given an arbitrary metric space X, an *anticode* of diameter e is a subset whose elements have pairwise distance at most e. For every association scheme, which applies to the q-Johnson scheme in our situation, the anticode bound of Delsarte [54] can be applied. As a standalone argument we go along the lines of [2] and consider bounds for codes on transitive graphs. By double-counting the number of pairs $(a, g) \in A \cdot \mathrm{Aut}(\Gamma)$, where $g(a) \in B$, we obtain:

Lemma 4.8. *([2, Lemma 1], cf. [3, Theorem 1'])*
Let $\Gamma = (V, E)$ be a graph that admits a transitive group of automorphisms $\mathrm{Aut}(\Gamma)$ and let A, B be arbitrary subsets of the vertex set V. Then, there exists a group element $g \in \mathrm{Aut}(\Gamma)$ such that

$$\frac{|g(A) \cap B|}{|B|} \ge \frac{|A|}{|V|}.$$

Corollary 4.9. *([2, Corollary 1], cf. [3, Theorem 1])*
Let $C_D \subseteq \mathcal{G}_q(n, k)$ be a code with (injection or graph) distances from $D = \{d_1, \ldots, d_s\} \subseteq \{1, \ldots, v\}$. Then, for an arbitrary subset $\mathcal{B} \subseteq \mathcal{G}_q(n, k)$ there exists a code $C_D^*(\mathcal{B}) \subseteq \mathcal{B}$ with distances from D such that

$$\frac{\left|C_D^*(\mathcal{B})\right|}{|\mathcal{B}|} \ge \frac{|C_D|}{\begin{bmatrix}n\\k\end{bmatrix}_q}.$$

If $C_D \subseteq \mathcal{G}_q(n,k)$ is a CDC with minimum injection distance d, i.e., $D = \{d, \ldots, v\}$, and $\mathcal{B}$ is an anticode with diameter $d-1$, we have $\#C_D^*(\mathcal{B}) = 1$, so that we obtain Delsarte's anticode bound

$$\#C_D \le \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\#\mathcal{B}}. \tag{4.3}$$

The set of all elements of $\mathcal{G}_q(n,k)$ which contain a fixed $(k-d/2+1)$-dimensional subspace is an anticode of diameter $d-2$ with $\begin{bmatrix} n-k+d/2-1 \\ d/2-1 \end{bmatrix}_q$ elements. By duality, the set of all elements of $\mathcal{G}_q(n,k)$ which are contained in a fixed $(k+d/2-1)$-dimensional subspace is also an anticode of diameter $d-2$ with $\begin{bmatrix} k+d/2-1 \\ k \end{bmatrix}_q = \begin{bmatrix} k+d/2-1 \\ d/2-1 \end{bmatrix}_q$ elements. Frankl and Wilson proved in [85, Theorem 1] that these anticodes have the largest possible size, which implies:

Theorem 4.10. ***(Anticode bound – [223, Theorem 5.2])***

$$A_q(n,d;k) \le \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} \max\{k,n-k\}+d/2-1 \\ d/2-1 \end{bmatrix}_q}$$

Codes whose size attain the anticode bound are called *Steiner structures*. The reduction to Delsarte's anticode bound can e.g. be found in [80, Theorem 1].

Since the sphere underlying the proof of Theorem 4.4 is also an anticode, Theorem 4.4 is implied by Theorem 4.10. For $d = 2$ both bounds coincide. In [226, Section 4] Xia and Fu verified that the anticode bound is always stronger than the Singleton bound for the ranges of parameters considered by us.

Proposition 4.11. *([125, Proposition 8])*
For $k \le n-k$ the ratio of the size of an LMRD *code divided by the size of the anticode bound converges for $n \to \infty$ monotonically decreasing to*

$$\frac{(1/q;1/q)_k}{(1/q;1/q)_{d/2-1}} \ge \frac{q}{q-1} \cdot (1/q;1/q)_k \ge 2 \cdot (1/2;1/2)_\infty > 0.577576.$$

The largest gap of this estimate is attained for $d = 4$ and $k = \lfloor n/2 \rfloor$. If k does not vary with n (or does increase very slowly), then the anticode bound can be asymptotically attained by an optimal code.

Theorem 4.12. ***(Asymptotic value – [84, Theorem 4.1], cf. [34])***

$$\lim_{n\to\infty} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} \max\{k,n-k\}+d/2-1 \\ d/2-1 \end{bmatrix}_q \cdot A_q(n,d;k)} = 1$$

Mimicking a classical bound of Johnson on binary error-correcting codes with respect to the Hamming distance, see [140, Theorem 3] and also [216], the following upper bound was obtained:

Theorem 4.13. ***(Johnson type bound I – [226, Theorem 2])***
If $\left(q^k-1\right)^2 > (q^n-1)\left(q^{k-d/2}-1\right)$, then

$$A_q(n,d;k) \le \frac{\left(q^k - q^{k-d/2}\right)(q^n-1)}{\left(q^k-1\right)^2 - (q^n-1)\left(q^{k-d/2}-1\right)}.$$

However, the required condition of Theorem 4.13 is rather restrictive and can be simplified considerably.

Proposition 4.14. *([125, Proposition 1])*
For $0 \le k < n$, the bound in Theorem 4.13 is applicable iff $d = 2\min\{k, n-k\}$ and $k \ge 1$. Then, it is equivalent to

$$A_q(n, d; k) \le \frac{q^n - 1}{q^{\min\{k,n-k\}} - 1}.$$

In other words, Theorem 4.13 is equivalent to a rather simple upper bound for partial spreads, see Subsection 4.1.

Let C be a CDC in $\mathrm{PG}(n-1, q)$. For each point P and each hyperplane H in $\mathrm{PG}(n-1, q)$ consider the subcodes $C_P := \{U \in C : P \le U\}$ and $C_H := \{U \in C : U \le H\}$. A little counting argument gives:

Theorem 4.15. ***(Johnson type bound II – [226, Theorem 3], [80, Theorem 4,5])***

$$A_q(n,d;k) \le \frac{[n]_q A_q(n-1,d;k-1)}{[k]_q} = \frac{q^n-1}{q^k-1} \cdot A_q(n-1,d;k-1) \tag{4.4}$$

$$A_q(n,d;k) \le \frac{[n]_q A_q(n-1,d;k-1)}{[n-k]_q} = \frac{q^n-1}{q^{n-k}-1} \cdot A_q(n-1,d;k) \tag{4.5}$$

Type II Johnson bounds for binary constant weight codes

In [140, Inequality (5)] the upper bounds $A(n, d; w) \le \lfloor n/w \cdot A(n-1, d; w-1)\rfloor$ and $A(n, d; w) \le \lfloor n/(n-w) \cdot A(n-1, d; w)\rfloor$ for binary constant weight codes were obtained. Of course both bounds can be applied iteratively. However, the optimal choice of the corresponding inequalities is unclear, see e.g. [175, Research Problem 17.1]. The bounds in Theorem 4.15 are the q-analog of the mentioned bounds for constant weight codes.

While e.g. the authors of [80, 142] stated that the optimal choice of Inequality (4.4) or Inequality (4.5) is unclear too, there is now an explicit answer for CDCs:

Proposition 4.16. *([125, Proposition 3]) For $k \le n/2$ we have*

$$\left\lfloor \frac{q^n-1}{q^k-1} A_q(n-1, d; k-1) \right\rfloor \le \left\lfloor \frac{q^n-1}{q^{n-k}-1} A_q(n-1, d; k) \right\rfloor,$$

where equality holds iff $n = 2k$.

Exercise 4.17. *Consider the dual code to show that Inequality (4.4) and Inequality (4.5) are equivalent.*

Knowing the optimal choice between Inequality (4.4) and Inequality (4.5), we can iteratively apply Theorem 4.15 in an ideal way (initially assuming $k \le n/2$):

Corollary 4.18. ***(Implication of the Johnson type bound II)***

$$A_q(n,d;k) \le \left\lfloor \frac{q^n-1}{q^k-1} \left\lfloor \frac{q^{n-1}-1}{q^{k-1}-1} \left\lfloor \ldots \left\lfloor \frac{q^{n-k+d/2+1}-1}{q^{d/2+1}-1} A_q(n-k+d/2, d; d/2) \right\rfloor \ldots \right\rfloor \right\rfloor \right\rfloor$$

We remark that this upper bound is commonly stated in an explicit version, where $A_q(n-k+d/2, d; d/2) \le \left\lfloor \frac{q^{n-k+d/2}-1}{q^{d/2}-1} \right\rfloor$ is inserted, see e.g. [80, Theorem 6], [142, Theorem 7], and [226, Corollary 3]. However, better bounds for partial spreads are available now, see Subsection 4.1.

Comparison of the Johnson bound with the previous bounds

It is shown in [226] that the Johnson bound of Theorem 4.15 improves on the anticode bound in Theorem 4.10, see also [15]. To be more precise, removing the floors in the upper bound of Corollary 4.18 and replacing $A_q(n-k+d/2, d; d/2)$ by $\frac{q^{n-k+d/2}-1}{q^{d/2}-1}$ gives

$$\prod_{i=0}^{k-d/2} \frac{q^{n-i}-1}{q^{k-i}-1} = \frac{\prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}}{\prod_{i=k-d/2+1}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}} = \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} n-k+d/2-1 \\ d/2-1 \end{bmatrix}_q}, \tag{4.6}$$

which is the right hand side of the anticode bound for $k \le n-k$. So, all upper bounds mentioned so far are (weakly) dominated by Corollary 4.18, if we additionally assume $k \le n-k$. We will slightly improve upon Theorem 4.15 in Theorem 4.42 where we replace the possible rounding down by a tighter variant based on divisible multisets of points.

As a possible improvement [2, Theorem 3] was mentioned in [142, Theorem 8], cf. [125, Theorem 8].

Theorem 4.19. ***(Ahlswede and Aydinian bound – [2, Theorem 3])***
For integers $0 \le t < r \le k$, $k-t \le m \le n$, and $t \le n-m$ we have

$$A_q(n, 2r; k) \le \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q A_q(m, 2r-2t; k-t)}{\sum_{i=0}^{t} q^{i(m+i-k)} \begin{bmatrix} m \\ k-i \end{bmatrix}_q \begin{bmatrix} n-m \\ i \end{bmatrix}_q}.$$

As Theorem 4.19 has quite some degrees of freedom, we partially discuss the optimal choice of parameters. For $t=0$ and $m \le v-1$, we obtain $A_q(n, d; k) \le \begin{bmatrix} n \\ k \end{bmatrix}_q / \begin{bmatrix} m \\ k \end{bmatrix}_q \cdot A_q(m, d; k)$, which is the $(n-m)$-fold iteration of Inequality (4.5) of the Johnson bound (without rounding). Thus, $m = n-1$ is the best choice for $t = 0$, yielding a bound that is equivalent to Inequality (4.5). For $t = 1$ and $m = n-1$ the bound can be rewritten to $A_q(n, d; k) \le A_q(n-1, d-2; k-1)$. For $t > n-m$ the bound remains valid but is strictly weaker than for $t = n-m$. Choosing $m = n$ gives the trivial bound $A_q(n, 2r; k) \le A_q(m, 2r-2t; k-t)$. For the range of parameters $2 \le q \le 9$, $4 \le n \le 100$ and $4 \le d \le 2k \le n$, where q is a prime power and d is even, the situation is as follows. If $d \ne 2k$, there are no proper improvements with respect to Theorem 4.15. For the case $d = 2k$ we have some improvements compared to most easy upper bound $A_q(n, 2k; k) \le \lfloor (q^n-1)/(q^k-1) \rfloor$ while the tightest known upper bounds for partial spreads, see Subsection 4.1, are not improved.

Research problem

Verify that the upper bounds of Theorem 4.19 are implied by other known upper bounds or find specific parameters where this is not the case.

Linear programming bounds

Every association scheme gives rise to a linear programming upper bound, see e.g. [54]. For linear codes this relation can be expressed via the so-called MacWilliams identities. General introductions can e.g. be found in [57, 212]. Explicit parametric upper bounds can be commonly obtained via this approach. Examples for linear codes are given in e.g. [32] and [33, Section 15.3]. For binary block and constant weight codes we refer e.g. to [177]. The Delsarte linear programming bound for the q-Johnson scheme was obtained in [56]. However, numerical computations indicate that it is not better than the anticode bound, see [15]. In [229] it was shown that the anticode bound is implied by the Delsarte linear programming bound. In [15] it was shown that a semidefinite programming formulation[2], that is equivalent to the Delsarte linear programming bound, implies the anticode bound of Theorem 4.10, the sphere-packing bound of Theorem 4.4, the Johnson type I bound of Theorem 4.13, and the Johnson type II bound of Theorem 4.15.

Theorem 4.20. ***(Linear programming bound for CDCs – e.g. [229, Proposition 3])*** *For integers $0 \le k \le n$ and $2 \le d \le \min\{k, n-k\}$ such that d is even, we have*

$$A_q(n,d;k) \quad \le \quad \max\left\{1 + \sum_{i=d/2}^{k} x_i \,\middle|\, \sum_{i=d/2}^{k} -Q_j(i)x_i \le u_j \; \forall j = 1,2,\ldots,k \text{ and } x_i \ge 0 \; \forall i = d/2, d/2+1, \ldots, k\right\} \tag{4.7}$$

with

$$u_j = \begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} n \\ j-1 \end{bmatrix}_q, \tag{4.8}$$

$$v_i = q^{i^2} \begin{bmatrix} l \\ i \end{bmatrix}_q - \begin{bmatrix} n-1 \\ i \end{bmatrix}_q, \tag{4.9}$$

$$E_i(j) = \sum_{m=0}^{i} (-1)^{i-m} q^{\binom{i-m}{2}+jm} \begin{bmatrix} k-m \\ k-1 \end{bmatrix}_q \begin{bmatrix} k-j \\ m \end{bmatrix}_q \begin{bmatrix} n-k-j+m \\ m \end{bmatrix}_q \text{ and} \tag{4.10}$$

$$Q_j(i) = \frac{u_j}{v_i} E_i(j). \tag{4.11}$$

Remark 4.21. *Using* `Maple` *and exact arithmetic, we have checked that for all $2 \le q \le 9$, $4 \le n \le 19$, $2 \le k \le n/2$, $4 \le d \le 2k$ the optimal value of the Delsarte linear programming bound is indeed the anticode bound. Given the result from [229] it remains to construct a feasible solution of the Delsarte linear programming formulation whose target value equals the anticode bound. Such a feasible solution can also be constructed recursively. To this end, let $x_0, \ldots, x_{k-1}$ denote a primal solution for the parameters of $A_q(n-1, d; k-1)$, then $z_0, \ldots, z_k$ is a feasible solution for the parameters of $A_q(n, d; k)$ setting $z_i = x_i \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q$ for all $0 \le i \le k-1$ and $z_k = \begin{bmatrix} n \\ k \end{bmatrix}_q / \begin{bmatrix} n-k+d/2-1 \\ d/2-1 \end{bmatrix}_q - z_0 - \cdots - z_{k-1}$. For the mentioned parameter space this conjectured primal solution is feasible with the anticode bound as target value.*

In [225] it was shown that the optimal solution of the linear program in Theorem 4.20 is given by the anticode bound, see Remark 4.21.

[2] Due to the property of the symmetry group of $(\mathbb{F}_q^n, d_S)$, i.e., two-point homogeneous, the symmetry reduced version of the semidefinite programming formulation of the maximum clique problem formulation collapses the Delsarte linear programming bound for the q-Johnson scheme.

The iterated application of the Johnson bound of Theorem 4.15 rounded down to integers in each iteration can improve upon the anticode bound. In Subsection 4.2 and Subsection 4.1 we will present further upper bounds that improve upon the anticode or Johnson bound. Adding corresponding constraints to our linear programming formulation of Theorem 4.20 of course gives tighter bounds.

Research problem

Find additional inequalities for the linear programming approach and improve at least one of the known upper bounds for $A_q(n,d;k)$.

As mentioned in the introduction, semidefinite programming bounds for $A(n,d)$ and $A(n,d;w)$ were quite successful in recent years, see e.g. [218]. The same is true for MDCs, i.e., upper bounds for $A_q(n,d)$, see [15, 121]. For CDCs currently no improvement via semidefinite programming is known, see the blog entry

https://ratiobound.wordpress.com/2018/10/11/.

For related literature into this direction we refer to [62, 165].

Another rather general technique to obtain upper bounds for the maximum clique sizes of a graph is to use p-ranks of adjacency matrices.

Lemma 4.22. *(E.g. [139, Lemma 1.3])*
Let G be a graph with adjacency matrix A and Y be a clique of G, then

$$|Y| \le \begin{cases} \operatorname{rank}_p(A)+1 & \text{if } p \text{ divides } |Y|-1, \\ \operatorname{rank}_p(A) & \text{otherwise.} \end{cases}$$

Some numerical experiments suggest that the resulting upper bounds are rather weak for CDCs. We e.g. have $A_2(4,4;2) \le 5$, $A_2(5,4;2) \le 19$, $A_2(6,4;2) \le 49$, $A_2(6,4;3) \le 223$, and $A_2(6,6;3) \le 19$.

Integer linear programming formulations for $\mathbf{A_q(n,d;k)}$

The exact determination of $A_q(n,d;k)$ can be formulated as an integer linear program (ILP). To this end we introduce binary variables $x_K \in \{0,1\}$ for each k-space $K \in \mathcal{G}_q(n,k)$ and maximize their sum $\sum_{K\in\mathcal{G}_q(n,k)} x_K$ subject to the constraints

$$\sum_{K\in\mathcal{G}_q(n,k)\,:\,S\le K} x_K \le 1 \tag{4.12}$$

for all $S \in \mathcal{G}_q(n,k-d/2+1)$, which guarantee the minimum subspace distance. This ILP can be solved directly for rather small parameters only. However, it was the basis for the determination of $A_2(6,4;3)=77$ and the classification of the corresponding five optimal isomorphism types in [132]. The determination of $A_2(8,6;4)=257$ and the classification of the corresponding two optimal isomorphism types required a tailored approach with relaxations to subconfigurations, see [119] for the details.[3] We remark that the ILP approach can also be used to construct CDC's of large cardinality. To restrict the search space typically a subgroup of the automorphism group of the CDC is prescribed, see e.g. [147].

[3] The intermediate upper bound $A_2(8,6;4) \le 272$ was determined in [127].

If the presence of certain automorphisms is assumed, then for many cases improved upper bounds can be concluded from the LP relaxation. It is also possible to deduce parametric bounds from this approach, see e.g. [115, Section 10].

We close this overview mentioning that CDCs containing a lifted MRD code as subcode allow tighter upper bounds on their cardinality, see [77, 116, 152]. We remark that many of the currently best known constructions for CDCs involve a lifted MRD as a subcode, see Section 5. In [158, Section 4] the underlying techniques have been extended to infer upper bounds for CDCs arising from other specific constructions from the literature.

Research problem

Provide more specialized upper bounds for subcodes appearing in constructions for CDCs in the literature (or Section 5).

4.1 Upper bounds for partial spreads

Assume, as before, $k \le n-k$. An $(n,2k;k)_q$–CDC is also called *partial spread* or *partial k-spread* to be more precise. Those CDCs attain the maximum possible subspace distance, which is equivalent to the geometric description that the pairwise intersection of the k-spaces is trivial, i.e., 0-dimensional. Applying the Johnson bound of Theorem 4.15 to the parameters of a partial spread yields

$$A_q(n,2k;k) \le \frac{[n]_q}{[k]_q} \cdot A_q(n-1,2k;k-1) = \frac{[n]_q}{[k]_q}$$

since $A_q(n-1,2k;k-1) = 1$. An easy direct geometric justification comes from the fact that $\mathrm{PG}(n-1,q)$ contains $[n]_q$ points and each k-space contains $[k]_q$ points. Spelling out the q-factorials and rounding down we obtain

$$A_q(n,2k;k) \le \left\lfloor \frac{q^n-1}{q^k-1} \right\rfloor. \tag{4.13}$$

In the following we review improved classical bounds for partial spreads from the literature. Other surveys can e.g. be found in [134, 213]. In the subsequent Subsection 4.2 we will briefly introduce a contemporary approach based on q^{k-1}-divisible (multi-) sets of points. It will turn out that all upper bounds of this subsection can be obtained from non-existence results for q^{k-1}-divisible sets of points in $\mathrm{PG}(n-1,q)$, where n is assumed to be sufficiently large.

An $(n,2k;k)_q$–CDC of cardinality $[n]_q/[k]_q$ is called a *k-spread* (or just *spread*). A handy existence criterion is known from the work of Segre in 1964.

Theorem 4.23. ***(Existence of spreads – [199, §VI])***
$\mathrm{PG}(n-1,q)$ *contains a k-spread iff k is a divisor of n.*

Exercise 4.24. *Write $n = tk + r$ with $1 \le r \le k-1$ and $t \ge 2$. Verify*

$$A_q(n,2k;k) \le \left\lfloor \frac{q^n-1}{q^k-1} \right\rfloor = \frac{q^{tk+r}-q^r}{q^k-1} + \left\lfloor \frac{q^r-1}{q^k-1} \right\rfloor = \sum_{s=0}^{t-1} q^{sk+r} = q^r \begin{bmatrix} t \\ 1 \end{bmatrix}_{q^k}.$$

Definition 4.25. ***(Deficiency of partial*** k***-spreads in*** $\mathrm{PG}(n-1,q)$ ***– cf. [28])***
The number σ defined by

$$A_q(tk+r;2k;k) = \sum_{s=0}^{t-1} q^{sk+r} - \sigma,$$

where $0 \le r \le k-1$ and $t \ge 2$, is called the deficiency *of the partial k-spreads of maximum possible size in* $\mathrm{PG}(tk+r-1,q)$.[4]

Deficiency of a partial k-spread $\mathcal{P}$ in $\mathrm{PG}(n-1,q)$

If $\mathcal{P}$ is a partial k-spread in $\mathrm{PG}(n-1,q)$, where $n = tk+r$ with $0 \le r \le k-1$ and $t \ge 2$, then the deficiency of $\mathcal{P}$ is defined as $\sum_{s=0}^{t-1} q^{sk+r} - \#\mathcal{P}$ in several papers. I.e. the value σ is just a lower bound for the deficiency of a given partial spread and there is some interest in the possible deficiencies of inclusion-maximal partial spreads.

Theorem 4.26. *([28, 29], cf. [63, Theorem 2.7(a)])*
The deficiency of a maximal k-spread in $\mathrm{PG}(n-1,q)$*, where k does not divide n, is at least* $q-1$.

We remark that we indeed have

$$A_q(tk+r,2k;k) \ge \sum_{s=0}^{t-1} q^{sk+r} - (q^r - 1) \tag{4.14}$$

for all $k,t \ge 2$ and $0 \le r \le k-1$, see e.g. [28] or Exercise 5.32. So, the cases "$r = 0$" and "$r = 1$" are completely resolved.

Theorem 4.27. *([149, Theorem 4.3]) We have*

$$A_2(tk+2,2k;k) \le \sum_{s=0}^{t-1} 2^{sk+2} - \left(2^2 - 1\right) \tag{4.15}$$

for all $k \ge 4$, $t \ge 2$.

Theorem 4.28. **(k *sufficiently large, the asymptotic case – [180, Theorem 5])***
We have

$$A_q(tk+r,2k;k) \le \sum_{s=0}^{t-1} q^{sk+r} - (q^r - 1) \tag{4.16}$$

for all $k > [r]_q$, $t \ge 2$.

Theorem 4.29. *([150, Theorem 2.9],[134, Theorem 9],[134, Corollary 7])*
For integers $r \ge 1$, $t \ge 2$, $u \ge 0$, and $z \ge 0$ with $k = [r]_q + 1 - z + u > r$ we have

$$A_q(tk+r,2k;k) \le \sum_{s=0}^{t-1} q^{sk+r} - (q^r - 1) + z(q-1). \tag{4.17}$$

[4] This makes sense also for $r = 0$: Spreads are assigned deficiency $\sigma = 0$.

Setting $z = 0$ in Theorem 4.29 gives Theorem 4.28.

For a long time the best upper bound for partial spreads was given by Drake and Freeman:

Theorem 4.30. *([60, Corollary 8]) If $n = kt + r$ with $0 < r < k$ and $t \geq 2$, then*

$$A_q(n,2k;k) \leq \sum_{i=0}^{t-1} q^{ik+r} - \lfloor\theta\rfloor - 1 = q^r \cdot \frac{q^{kt}-1}{q^k-1} - \lfloor\theta\rfloor - 1 = \frac{q^n - q^r}{q^k-1} - \lfloor\theta\rfloor - 1,$$

where $2\theta = \sqrt{1+4q^k(q^k-q^r)} - (2q^k - 2q^r + 1)$.

Example 4.31. *If we apply Theorem 4.30 with $q = 5$, $n = 16$, $k = 6$, and $r = 4$, then we obtain $\theta \approx 308.81090$ and $A_5(16,12;6) \leq 9765941$.*

Theorem 4.32. *([134, Theorem 10],[150, Theorem 2.10]) For integers $r \geq 1$, $t \geq 2$, $y \geq \max\{r,2\}$, $z \geq 0$ with $\lambda = q^y$, $y \leq k$, $k = [r]_q + 1 - z > r$, $n = kt + r$, and $l = \frac{q^{n-k}-q^r}{q^k-1}$, we have*

$$A_q(n,2k;k) \leq lq^k + \left\lceil \lambda - \frac{1}{2} - \frac{1}{2}\sqrt{1+4\lambda\left(\lambda-(z+y-1)(q-1)-1\right)} \right\rceil. \tag{4.18}$$

Using Theorem 4.32 with $q = 5$, $k = 6$, $n = 15$, $r = 3$, $z = 17$, and $y = 5$ gives $A_5(15,12;6) \leq 1953186$. Choosing $y = t$ we obtain Theorem 4.30. Theorem 4.32 also covers [179, Theorems 6,7] and yields improvements in a few instances, e.g. $A_3(15,12;6) \leq 19695$.

A few further parametric upper bounds have been mentioned in [150]. For $t \geq 2$ we have

- $2^4 l + 1 \leq A_2(4t+3,8;4) \leq 2^4 l + 4$, where $l = \frac{2^{4t-1}-2^3}{2^4-1}$;
- $2^6 l + 1 \leq A_2(6t+4,12;6) \leq 2^6 l + 8$, where $l = \frac{2^{6t-2}-2^4}{2^6-1}$;
- $2^6 l + 1 \leq A_2(6t+5,12;6) \leq 2^6 l + 18$, where $l = \frac{2^{6t-1}-2^5}{2^6-1}$;
- $3^4 l + 1 \leq A_3(4t+3,8;4) \leq 3^4 l + 14$, where $l = \frac{3^{4t-1}-3^3}{3^4-1}$;
- $3^5 l + 1 \leq A_3(5t+3,10;5) \leq 3^5 l + 13$, where $l = \frac{3^{5t-2}-3^5}{3^3-1}$;
- $3^5 l + 1 \leq A_3(5t+4,10;5) \leq 3^5 l + 44$, where $l = \frac{3^{5t-1}-3^4}{3^5-1}$;
- $3^6 l + 1 \leq A_3(6t+4,12;6) \leq 3^6 l + 41$, where $l = \frac{3^{6t-2}-3^4}{3^6-1}$;
- $3^6 l + 1 \leq A_3(6t+5,12;6) \leq 3^6 l + 133$, where $l = \frac{3^{6t-1}-3^5}{3^6-1}$;
- $3^7 l + 1 \leq A_3(7t+4,14;7) \leq 3^7 l + 40$, where $l = \frac{3^{7t-3}-3^4}{3^7-1}$;
- $4^4 l + 1 \leq A_4(4t+2,8;4) \leq 4^4 l + 6$, where $l = \frac{4^{4t-2}-4^2}{4^4-1}$;
- $4^5 l + 1 \leq A_4(5t+3,10;5) \leq 4^5 l + 32$, where $l = \frac{4^{5t-2}-4^3}{4^5-1}$;
- $4^6 l + 1 \leq A_4(6t+3,12;6) \leq 4^6 l + 30$, where $l = \frac{4^{6t-3}-4^3}{4^6-1}$;

- $4^6 l + 1 \le A_4(6t+5, 12; 6) \le 4^6 l + 548$, where $l = \frac{4^{6t-1}-4^5}{4^6-1}$;
- $4^7 l + 1 \le A_4(7t+4, 14; 7) \le 4^7 l + 128$, where $l = \frac{4^{7t-3}-4^4}{4^7-1}$;
- $5^5 l + 1 \le A_5(5t+2, 10; 5) \le 5^5 l + 7$, where $l = \frac{5^{5t-3}-5^2}{5^5-1}$;
- $5^5 l + 1 \le A_5(5t+4, 10; 5) \le 5^5 l + 329$, where $l = \frac{5^{5t-1}-5^4}{5^5-1}$;
- $5^6 l + 1 \le A_5(6t+3, 8; 4) \le 5^6 l + 61$, where $l = \frac{5^{6t-3}-5^3}{5^6-1}$;
- $5^6 l + 1 \le A_5(6t+4, 8; 4) \le 5^6 l + 316$, where $l = \frac{5^{6t-2}-5^4}{5^6-1}$;
- $7^5 l + 1 \le A_7(5t+4, 10; 5) \le 7^5 l + 1246$, where $l = \frac{7^{5t-1}-7^2}{7^5-1}$;
- $7^6 l + 1 \le A_7(6t+2, 8; 4) \le 7^6 l + 15$, where $l = \frac{7^{6t-4}-7^3}{7^6-1}$;
- $8^4 l + 1 \le A_8(4t+3, 8; 4) \le 8^4 l + 264$, where $l = \frac{8^{4t-1}-8^3}{8^4-1}$;
- $8^5 l + 1 \le A_8(5t+2, 10; 5) \le 8^5 l + 25$, where $l = \frac{8^{5t-3}-8^2}{8^5-1}$;
- $8^6 l + 1 \le A_8(6t+2, 8; 4) \le 8^6 l + 21$, where $l = \frac{8^{6t-4}-8^3}{8^6-1}$;
- $9^3 l + 1 \le A_9(3t+2, 6; 3) \le 9^3 l + 41$, where $l = \frac{9^{3t-1}-9^2}{9^3-1}$;
- $9^5 l + 1 \le A_9(5t+3, 10; 5) \le 9^5 l + 365$, where $l = \frac{9^{5t-2}-9^3}{9^5-1}$.

Actually, each improved upper bound for $A_q(n, 2k; k)$ for specific parameters implies a parametric series of upper bounds.

Lemma 4.33. *([134, Lemma 4])*
For fixed q, k and r the deficiency σ is a non-increasing function of $n = kt + r$.

4.2 Upper bounds based on divisible multisets of points

A *multiset* $\mathcal{M}$ of points in $\mathrm{PG}(n-1, q)$ is a mapping $\mathcal{M}\colon \mathcal{G}_q(n,1) \to \mathbb{N}_0$. For each point $P \in \mathcal{G}_q(n,1)$ the integer $\mathcal{M}(P)$ is called the *multiplicity* of P and it counts how often point P is contained in the multiset. If $\mathcal{M}(P) \in \{0,1\}$ for all $P \in \mathcal{G}_q(n,1)$ we also speak of a set instead of a multiset (of points). We call a multiset of points Δ-divisible iff the corresponding linear code C is Δ-divisible, i.e., if the weights of all codewords in C are divisible by Δ. Note that this is equivalent to

$$\mathcal{M}(H) \equiv \#\mathcal{M} \pmod{\Delta} \tag{4.19}$$

for every hyperplane H, where $\mathcal{M}(H)$ is the sum of the multiplicities of the points contained in H and $\#M$ is the sum of the multiplicities over all points. The set of points of a k-space, the multiset of points of a multiset of k-spaces, and the set of holes of a partial k-spread are q^{k-1}-divisible.

Here we briefly state upper bounds for $A_q(n, d; k)$ that can be concluded from non-existence results of Δ-divisible multisets of points. For an introduction we refer e.g. to [120, 134].

For each integer r and each dimension $1 \le i \le r+1$ the q^{r+1-i}-fold repetition of an i-space in $\mathrm{PG}(v-1, q)$ is a q^r-divisible multiset of points of cardinality $q^{r+1-i} \cdot [i]_q$. So, for a fixed prime power q, a non-negative integer r, and $i \in \{0, \ldots, r\}$, we define

$$s_q(r, i) := q^i \cdot [r-i+1]_q = \frac{q^{r+1} - q^i}{q-1} = \sum_{j=i}^{r} q^j = q^i + q^{i+1} + \ldots + q^r \tag{4.20}$$

and state:

Lemma 4.34. *For each $r \in \mathbb{N}_0$ and each $i \in \{0, \ldots, r\}$ there is a q^r-divisible multiset of points of cardinality $s_q(r, i)$.*

As a consequence of Lemma 4.34 all integers $n = \sum_{i=0}^{r} a_i s_q(r, i)$ with $a_i \in \mathbb{N}_0$ are realizable cardinalities of q^r-divisible multisets of points. Note that the number $s_q(r, i)$ is divisible by q^i, but not by q^{i+1}. This property allows us to create kind of a positional system upon the sequence of base numbers

$$S_q(r) := (s_q(r, 0), s_q(r, 1), \ldots, s_q(r, r)).$$

Exercise 4.35. *Show that each integer n has a unique $S_q(r)$*-adic expansion

$$n = \sum_{i=0}^{r} a_i s_q(r, i) \tag{4.21}$$

with $a_0, \ldots, a_{r-1} \in \{0, \ldots, q-1\}$ and leading coefficient $a_r \in \mathbb{Z}$.

Algorithm

Input: $n \in \mathbb{Z}$, field size q, exponent $r \in \mathbb{N}_0$
Output: representation $n = \sum\limits_{i=0}^{r} a_i s_q(r, i)$ with $a_0, \ldots, a_{r-1} \in \{0, \ldots, q-1\}$ and $a_r \in \mathbb{Z}$

$m \leftarrow n$
For $i \leftarrow 0$ To $r-1$
  $a_i \leftarrow m \bmod q$
  $m \leftarrow \frac{m - a_i \cdot [r-i+1]_q}{q}$
$a_r \leftarrow m$

Here $m \bmod q$ denotes the remainder of the division of m by q.

Example 4.36. *The $S_2(2)$-adic expansion of $n = 11$ is given by $11 = 1 \cdot 7 + 0 \cdot 6 + 1 \cdot 4$ and the $S_2(2)$-adic expansion of $n = 9$ is given by $1 \cdot 7 + 1 \cdot 6 - 1 \cdot 4$, i.e., the leading coefficient is -1.*

Exercise 4.37. *Compute the $S_3(3)$-adic expansion of $n = 137$ and determine the leading coefficient.*

Theorem 4.38. ***(Possible lengths of divisible codes – [144, Theorem 1])***
For $n \in \mathbb{Z}$ and $r \in \mathbb{N}_0$ the following statements are equivalent:

(i) There exists a q^r-divisible multiset of points of cardinality n over $\mathbb{F}_q$.

(ii) There exists a full-length q^r-divisible linear code of length n over $\mathbb{F}_q$.

(iii) The leading coefficient of the $S_q(r)$-adic expansion of n is non-negative.

So, the $S_q(r)$-adic expansion of n provides a certificate not only for the existence, but remarkably also for the non-existence of a q^r-divisible multiset of size n. As computed in Exercise 4.37, the leading coefficient of the $S_3(3)$-adic expansion of $n = 137$ is -2, so that there is no 27-divisible ternary linear code of effective length 137. If $q = p^m$ is a proper prime power then also the possible cardinalities of p^r-divisible multisets of points over $\mathbb{F}_q$ are of interest when r is not divisible by m. To that end [144, Theorem 1]) was completed in [159].

Sharpened rounding

Definition 4.39. *For $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$ let $\llfloor a/b \rrfloor_{q^r}$ be the maximal $n \in \mathbb{Z}$ such that there exists a q^r-divisible $\mathbb{F}_q$-linear code of effective length $a - nb$. If no such code exists for any n, we set $\llfloor a/b \rrfloor_{q^r} = -\infty$. Similarly, let $\llceil a/b \rrceil_{q^r}$ denote the minimal $n \in \mathbb{Z}$ such that there exists a q^r-divisible $\mathbb{F}_q$-linear code of effective length $nb - a$. If no such code exists for any n, we set $\llceil a/b \rrceil_{q^r} = \infty$.*

Note that the symbols $\llfloor a/b \rrfloor_{q^r}$ and $\llceil a/b \rrceil_{q^r}$ encode the four values a, b, q and r. Thus, the fraction a/b is a formal fraction and the power q^r is a formal power, i.e. we assume $1530/14 \neq 765/7$ and $2^2 \neq 4^1$ in this context.

Exercise 4.40. *Compute $\llfloor 765/7 \rrfloor_{2^2}$ and $\llfloor 1530/14 \rrfloor_{4^1}$. Verify*

$$\llfloor 0/b \rrfloor_{q^r} = \llceil 0/b \rrceil_{q^r} = 0$$

and

$$\begin{aligned} \ldots \le \llfloor a/b \rrfloor_{q^2} \le \llfloor a/b \rrfloor_{q^1} \le \llfloor a/b \rrfloor_{q^0} = \left\lfloor \tfrac{a}{b} \right\rfloor \\ \le a/b \le \lceil a/b \rceil = \llceil a/b \rrceil_{q^0} \le \llceil a/b \rrceil_{q^1} \le \llceil a/b \rrceil_{q^2} \le \ldots \end{aligned}$$

Lemma 4.41. *([144, Lemma 13])*
Let $k \in \mathbb{Z}_{\ge 1}$ and $\mathcal{U}$ be a multiset of k-spaces in $\operatorname{PG}(n-1, q)$.

(i) If every point in $\mathcal{P}$ is covered by at most λ elements of $\mathcal{U}$, then

$$\#\mathcal{U} \le \llfloor \lambda [n]_q / [k]_q \rrfloor_{q^{k-1}}.$$

(ii) If every point in $\mathcal{P}$ is covered by at least λ elements in $\mathcal{U}$, then

$$\#\mathcal{U} \ge \llceil \lambda [n]_q / [k]_q \rrceil_{q^{k-1}}.$$

An improvement of the Johnson bound from Theorem 4.15

Instead of rounding down the right hand side of Inequality (4.4) we can use the sharpened rounding from Definition 4.39:

Theorem 4.42. *([144, Theorem 12])*

$$A_q(n,d;k) \le \left\llbracket \frac{[n]_q \cdot A_q(n-1,d;k-1)}{[k]_q} \right\rrbracket_{q^{k-1}}.$$

With $n' = n-k+d/2$, the iterated application of Theorem 4.42 yields

$$A_q(n,d;k) \le \left\llbracket \tfrac{[n]_q}{[k]_q} \cdot \left\llbracket \tfrac{[n-1]_q}{[k-1]_q} \cdot \left\llbracket \cdots \right.\right.\right.$$
$$\left.\left.\left.\left\llbracket \tfrac{[n'+1]_q}{[d/2+1]_q} \cdot A_q(n',d;d/2) \right\rrbracket_{q^{d/2-1}} \cdots \right\rrbracket_{q^{k-3}} \right\rrbracket_{q^{k-2}} \right\rrbracket_{q^{k-1}}.[5]$$

Example 4.43. *So far, the best known upper bound on* $A_2(9,6;4)$ *has been given by the Johnson bound* (4.4)*, using* $A_2(8,6;3) = 34$ *from [67]:*

$$A_2(9,6;4) \le \left\lfloor \frac{[9]_2}{[4]_2} \cdot A_2(8,6;3) \right\rfloor = \left\lfloor \frac{2^9-1}{2^4-1} \cdot 34 \right\rfloor = 1158.$$

To improve that bound by Theorem 4.42, we are looking for the largest integer n such that a q^{k-1}*-divisible multiset of size*

$$M(n) = [9]_2 \cdot A_2(8,6;3) - n \cdot [4]_2 = 17374 - 15n$$

exists.

This question can be investigated with Theorem 4.38. We have $S_2(3) = (15,14,12,8)$*. The* $S_2(3)$*-adic expansion of* $M(1157) = 17374 - 15 \cdot 1157 = 19$ *is* $1 \cdot 15 + 0 \cdot 14 + 1 \cdot 12 + (-1) \cdot 8$*. As the leading coefficient* -1 *is negative, there is no* 8*-divisible multiset of points of size* 19 *by Theorem 4.38. The* $S_2(3)$*-adic expansion of* $M(1156) = 34$ *is* $0 \cdot 15 + 1 \cdot 14 + 1 \cdot 12 + 1 \cdot 8$*. As the leading coefficient* 1 *is non-negative, there exists a* 8*-divisible multiset of points of size* 34*. Thus, we have*

$$A_2(9,6;4) \le \left\llbracket \frac{[9]_2}{[4]_2} \cdot A_2(8,6;3) \right\rrbracket_{2^3} = \llbracket 17374/15 \rrbracket_{2^3} = 1156,$$

which improves the original Johnson bound (4.4) *by* 2.

Lemma 4.44. *([144, Lemma 17]) The improvement of Theorem 4.42 over the original Johnson bound* (4.4) *is at most* $(q-1)(k-1)$*.*

The sharpened rounding in Theorem 4.42 can also be evaluated parametric in the field size q.

Proposition 4.45. *([144, Proposition 2]) For all prime powers* $q \ge 2$ *we have*

$$\begin{aligned} A_q(11,6;4) &\le q^{14} + q^{11} + q^{10} + 2q^7 + q^6 + q^3 + q^2 - 2q + 1 \\ &= (q^2-q+1)(q^{12} + q^{11} + q^8 + q^7 + q^5 + 2q^4 + q^3 - q^2 - q + 1). \end{aligned}$$

[5] Expressions of the form $\llbracket \frac{a}{b} \cdot c \rrbracket_{q^r}$ should be read as $\llbracket \frac{a \cdot c}{b} \rrbracket_{q^r}$.

As a refinement of the sharpened rounding from Definition 4.39 we introduce:

Definition 4.46. *For $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$ let $\llbracket a/b \rrbracket_{q^r,\lambda}$ be the maximal $n \in \mathbb{Z}$ such that there exists a q^r-divisible multisets of points in* $\operatorname{PG}(v-1,q)$ *for suitably large v with maximum point multiplicity at most λ and cardinality $a - nb$. If no such multiset exists for any n, we set $\llbracket a/b \rrbracket_{q^r,\lambda} = -\infty$.*

With this we can sharpen the almost trivial upper bound (4.13) for partial spreads, see e.g. [120, 134] for the details.

Lemma 4.47. *Let $\mathcal{U}$ be a set of k-spaces in* $\operatorname{PG}(v-1,q)$*, where $1 \le k \le v$, with pairwise trivial intersection. Then, we have*

$$\#\mathcal{U} \le \llbracket [v]_q/[k]_q \rrbracket_{q^{k-1},1}. \tag{4.22}$$

So, for $2 \le k \le n/2$ we obtain the upper bound $A_q(n,2k;k) \le \llbracket [n]_q/[k]_q \rrbracket_{q^{k-1},1}$. In contrast to $\llbracket a/b \rrbracket_{q^r}$ there is no known efficient algorithm to evaluate $\llbracket a/b \rrbracket_{q^r,\lambda}$ in general. In other words, the determination of the possible cardinalities of q^r-divisible multisets of points with maximum point multiplicity λ is a hard open problem, see e.g. [137]. For a survey of partial results for $\lambda = 1$ we refer to [120].

Example 4.48. *In e.g. [155] it was shown that no 2^4-divisible set of* 131 *points exists in* $\operatorname{PG}(v-1,2)$. *This implies $A_2(13,10;5) \le 259$ since a partial 5-spread in* $\operatorname{PG}(12,2)$ *of cardinality* 260 *would give a 2^4-divisible set of* 131 *holes (i.e. uncovered points). With this, Theorem 4.42 e.g. yields $A_2(14,10;6) \le 67349$.*

Nevertheless, several parametric bounds for q^r-divisible sets of points (where $\lambda = 1$) are known, see [134]. And indeed, all upper bounds for partial spreads presented in Subsection 4.1 can be deduced from Lemma 4.47. Some partial results for q^r-divisible multisets of points with restricted point multiplicity larger than 1 have been obtained in [**?**].

The tightest known upper bounds for CDCs

Assume $k \le n-k$. All currently known upper bounds for partial k-spreads are implied by $A_q(n,2k;k) \le \llbracket [n]_q/[k]_q \rrbracket_{q^{k-1},1}$, see Lemma 4.47, and non-existence results for q^{k-1}-divisible sets of points. For $d < 2k$ all currently known upper bounds for $A_q(n,d;k)$ are implied by the improved Johnson bound in Theorem 4.42 except $A_2(6,4;3) = 77$ and $A_2(8,6;4) = 257$, which are obtained via extensive ILP computations, see [132] and [119], respectively.

In [118] it was observed that also a combinatorial relaxation of a CDC $C \subset \mathcal{G}_2(8,4)$ with minimum subspace distance 6 has a maximum possible cardinality strictly less than 289, which is the upper bound for $A_2(8,6;4)$ that can be obtained by Theorem 4.42. Possibly the notion of generalized vector space partitions from [118] allows further theoretical insights.

The dominance relation between the upper bounds is just a snapshot

The clear picture on the dominance between the different known upper bounds for CDCs might just reflect our fragmentary knowledge and may change with time. While we currently do not know a single upper bound for $A_q(n,2k;k)$ that cannot be obtained via a non-existence result for q^{k-1}-divisible sets of points, there are indeed known criteria to show that certain q^{k-1}-divisible sets of points cannot coincide with the set of holes of a partial k-spread.

Research problem

Find a computer-free proof of $A_2(6,4;3) < 81$ or $A_2(8,6;4) < 289$.

5 Constructions for constant dimension codes

In this section we want to review lower bounds for $A_q(n, d; k)$, i.e., constructions for constant dimension codes. Our aim will to be to make the underlying ideas as clearly as possible, to show up the relations between different constructions from the literature, and to highlight potential for further improvements. To this end, we introduce a classification scheme to get a quick, rough picture of the different constructions. We will also try to decompose the, sometimes quite involved constructions, into smaller and easier components. While we want to trace the evolution of different constructions and their successive improvements, we will also have a closer look at the underlying distance analysises and possibilities to add further codewords. In some cases we so obtain improvements over the existing literature.

Common components are constant dimension codes (of smaller size), abbreviated by `C`, and rank metric codes, abbreviated by `R`. A *matrix description* of a subspace code $\mathcal{V}$ is a dissection of a rectangle into sub rectangles describing the structure of a generating set for $\mathcal{V}$, i.e., the structure of generator matrices for codewords in $\mathcal{V}$. As an example we consider the following matrix description for $\mathcal{V}$:

C	R

The meaning is that we assume the existence of a CDC C and a RMC $\mathcal{M}$ so that

$$\left\{\begin{pmatrix} A & M \end{pmatrix} \,:\, A \in \mathcal{G}, M \in \mathcal{M}\right\}$$

is a generating set of $\mathcal{V}$, where $\mathcal{G}$ is a generating set of C. Note that we need matrices representing the constant dimension codes in the components, since we want to end up with a generating set of matrices in the end. The fact that the matrices in $\mathcal{G}$ and $\mathcal{M}$ must have the same number of rows is indicated by common vertical border edge between the two cells. However, we do not assume that the rectangle dissection is true to scale. I.e., while the two cells have the same width, we do not assume that the matrices in $\mathcal{G}$ and $\mathcal{M}$ have the same number of columns. Of course the parameters of C and $\mathcal{M}$ determine the parameters of $\mathcal{V}$. E.g. we are interested in a lower bound for the minimum distance and the cardinality of $\mathcal{V}$ as well as whether $\mathcal{V}$ is a CDC. The details then are subject to a theorem. In our example the construction principle is called *Construction D* in [207] and the details can be found in Theorem 5.1.

By `0` we denote a rectangular all-zero matrix and by `I` a unit matrix, which gives us the extra condition that the corresponding rectangle has to be a square in the dissection. Since an identity matrix generates a CDC of cardinality 1, we can specialize our example to:

I	R

This construction is known under the name of *lifted MRD codes* assuming that the involved RMC is of maximum possible size, see Theorem 3.3.

Another, almost trivial, specialization of our initial matrix description is:

C	0

Since we may permute columns arbitrary, it is equivalent to the description:

0	C

Such a subcode will be useful if combined with others only. So, we will also consider the combination of different matrix descriptions by listing them one underneath the other. An example, corresponding to the *linkage construction* in Theorem 5.7, is given by:

C	R
0	C

Here we align the vertical lines such that they reflect the relationship between the matrix sizes involved in the different subcodes. As an example, the *improved linkage construction*, see Theorem 5.12, is described by:

<table>
<tr><td colspan="2">C</td><td colspan="2">R</td></tr>
<tr><td>0</td><td colspan="3">C</td></tr>
</table>

I.e., the length of the second CDC can be strictly larger than the length of the used RMC.

While those matrix descriptions are useful, not all constructions from the literature can be described that way.

For other surveys on constructions for constant dimension codes we refer e.g. to [138, 142].

5.1 Lifting, linkage, and related constructions

In this subsection we briefly survey the so-called linkage construction with its different variants. The starting point is the same as for lifted MRD codes. Instead of a $k \times k$ identity matrix I_k (or $I_{k\times k}$) we can also use any matrix of full row rank k as a prefix for the matrices from a rank metric code.

Theorem 5.1. ***(Lifting construction / Construction D – [207, Theorem 37])***
Let C be an $(n_1, d; k)_q$–CDC and $\mathcal{M}$ be a $(k \times n_2, d/2)$–RMC. Then

$$\mathcal{W} := \left\{ \left\langle \begin{pmatrix} G & M \end{pmatrix} \right\rangle \,:\, G \in \mathcal{G}, M \in \mathcal{M} \right\},$$

where $\mathcal{G}$ is a generating set of C, is an $(n_1 + n_2, d; k)_q$–CDC with cardinality $\#\mathcal{W} = \#C \cdot \#\mathcal{M}$.

Proof. For all $G \in \mathcal{G}$ and all $M \in \mathcal{M}$ we have $k \geq \operatorname{rk}(\begin{pmatrix} G & M \end{pmatrix}) \geq \operatorname{rk}(G) = k$, so that $\dim(W) = k$ for all $W \in \mathcal{W}$, i.e., $\mathcal{W}$ is a CDC with codewords of dimension k.

Now let $G, G' \in \mathcal{G}$, $M, M' \in \mathcal{M}$ be arbitrary, $U = \langle G \rangle$, $U' = \langle G' \rangle$, $W = \left\langle \begin{pmatrix} E(U) & A \end{pmatrix} \right\rangle$, and $W' = \left\langle \begin{pmatrix} E(U)' & A' \end{pmatrix} \right\rangle$. If $G \neq G'$, then we have $U \neq U'$ so that

$$d_{\mathrm{S}}(W, W') = 2 \cdot \mathrm{rk}\left(\begin{pmatrix} G & M \\ G' & M' \end{pmatrix}\right) - 2k \geq 2 \cdot \mathrm{rk}\left(\begin{pmatrix} G \\ G' \end{pmatrix}\right) - 2k = d_{\mathrm{S}}(U, U') \geq d.$$

If $G = G'$, then we have $U = U'$ and $M \neq M'$ so that

$$\begin{aligned} d_{\mathrm{S}}(W, W') &= 2 \cdot \mathrm{rk}\left(\begin{pmatrix} G & M \\ G & M' \end{pmatrix}\right) - 2k = 2 \cdot \mathrm{rk}\left(\begin{pmatrix} G & M \\ \mathbf{0}_{k \times m} & M' - M \end{pmatrix}\right) - 2k \\ &= 2\,\mathrm{rk}(G) + 2\,\mathrm{rk}(M' - M) - 2k = 2 d_{\mathrm{R}}(M, M') \geq d. \end{aligned}$$

□

This generalized lifting idea was called *Construction D* in [207, Theorem 37], cf. [101, Theorem 5.1]. Note that if C contains two codewords U, U' with distance $d_{\mathrm{S}}(U, U') = d$ and $\mathcal{M}$ contains an element M with $\mathrm{rk}(M) \leq d/2$, which is the case if $\#\mathcal{M} > 1$, then we have $d_{\mathrm{S}}(W, W') = d$ for $W = \left\langle (U, M) \right\rangle$, $W' = \left\langle (U', M) \right\rangle$. If $\mathcal{M}$ contains two elements M, M' with distance $d_{\mathrm{R}}(M, M') = d/2$ and C at least one element U, then we have $d_{\mathrm{S}}(W, W') = d$ for $W = \left\langle (U, M) \right\rangle$, $W' = \left\langle (U, M') \right\rangle$. So, the assumptions on the minimum distances of C and $\mathcal{M}$ are tight, i.e., they cannot be further relaxed besides degenerated and uninteresting special cases. Moreover, the parameter m is the only degree of freedom that we have if we want to end up with an $(n, d; k)_q$–CDC in the end, i.e., the formulation is as general as possible (assuming the corresponding matrix description).

Choosing C and $\mathcal{M}$ as large as possible and using the parameterization $m = n_1$ and $n = n_1 + n_2$, we conclude:

Corollary 5.2. *(C.f. [207, Theorem 37])*

$$A_q(n, d; k) \geq A_q(m, d; k) \cdot A_q^R(k \times (n - m), d/2) \tag{5.1}$$

We find it convenient to split [207, Theorem 37] into Theorem 5.1 and Corollary 5.2 since we will use Theorem 5.1 in other contexts where we assume further conditions for $\mathcal{M}$. The matrix description of construction D in Theorem 5.1 is given by

C	R

Directly from the construction we read off:

Lemma 5.3. *The pivot structure of a* CDC *obtained via construction D in Theorem 5.1 is a subset of* $\left(\binom{n_1}{k}, \binom{n_2}{0}\right)$.

Corollary 5.4.

$$A_q\left(n, d; k; \binom{m}{k}, \binom{n-m}{0}\right) \geq A_q(m, d; k) \cdot A_q^R(k \times (n - m)n, d/2) \tag{5.2}$$

Besides being recursive, the lower bound in Corollary 5.4 is very explicit and the only subtlety is a good choice of the free parameter m. Since the parameter space is rather small one may simply loop over all $1 \le m \le n-1$.

In [151] it was analyzed which codewords can be added to a subcode obtained via construction D in Theorem 5.1 without violating the minimum subspace distance.

Lemma 5.5. *Let C be a CDC obtained via construction D in Theorem 5.1 with parameters (n_1, n_2, d, k) and $U \in \mathcal{G}_q(n_1, k)$ with generator matrix G and pivot vector v. We have $d_S(C, U) \ge d$, i.e. $C \cup \{U\}$ is an $(n_1+n_2, d; k)_q$–CDC, if one of the following equivalent conditions is satisfied:*

(a) $\mathrm{d}_H\left(\left(\binom{n_1}{k}, \binom{n_2}{0}\right), v\right) \ge d$;

(b) at least $d/2$ of the k ones in v are contained in the last n_2 positions;

(c) $\mathrm{rk}(G_1) \le k - d/2$, *where* $G_1 \in \mathbb{F}_q^{k\times n_1}$, $G_2 \in \mathbb{F}_q^{k\times n_2}$ *with* $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$; *and.*

(d) $\dim(U \cap E_2) \ge d/2$, *where E_2 is the n_2-space spanned by the unit vectors $\mathbf{e}_i$ with $n_1 + 1 \le i \le n_1 + n_2$.*

While the listed conditions are only sufficient in general, in some sense, they are indeed also necessary if our only information on C is its matrix description or the pivot structure from Lemma 5.3.

Corollary 5.6.

$$A_q(n, d; k) \ge A_q\left(n, d; k; \begin{pmatrix} m \\ k \end{pmatrix}, \begin{pmatrix} n-m \\ 0 \end{pmatrix}\right) + A_q\left(n, d; k; \begin{pmatrix} m \\ \le k - d/2 \end{pmatrix}, \begin{pmatrix} n-m \\ \ge d/2 \end{pmatrix}\right)$$

See e.g. Exercise 2.13 for the corresponding distance analysis.

While the lower bound in Corollary 5.6 is very handy and indeed an essential ingredient for many good constructions in the literature, the second summand gives no hint how to construct corresponding subcodes.

Theorem 5.7. ***(Linkage construction – [207, Corollary 39], [102, Theorem 2.3])*** *Let C_1 be an $(n_1, d; k)_q$–CDC, C_2 be an $(n_2, d; k)_q$–CDC, and $\mathcal{M}$ be a $(k \times n_2, d/2)$–RMC. Then, $\mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1 + n_2, d; k)$–CDC of cardinality $\#C_1 \cdot \#\mathcal{M} + \#C_2$, where*

$$\left\{\begin{pmatrix} G & M \end{pmatrix} \,:\, G \in \mathcal{G}_1, M \in \mathcal{M}\right\}$$

is a generating set of $\mathcal{W}_1$,

$$\left\{\begin{pmatrix} 0_{k\times n_1} & G' \end{pmatrix} \,:\, G' \in \mathcal{G}_2\right\}$$

is a generating set of $\mathcal{W}_2$, and $\mathcal{G}_1, \mathcal{G}_2$ are generating sets of C_1, C_2, respectively.

The matrix description of the linkage construction is given by:

C	R
0	C

The properties of the subcodes $\mathcal{W}_1$ and $\mathcal{W}_2$ may be directly concluded from Theorem 5.1. The "linkage property" $d_S(\mathcal{W}_1, \mathcal{W}_2) \geq d$ follows e.g. from Lemma 5.5.(d) and $d \leq 2k$. The latter also implies the observation

$$A_q\left(n, d; k; \binom{m}{\leq k - d/2}, \binom{n-m}{\geq d/2}\right) \geq A_q(n-m, d; k).$$

Example 5.8. *For $n_1 = 4$, $n_2 = 4$, $d = 6$, and $k = 4$ choose $C_1 = C_2 = \{\langle I_4 \rangle\}$, and $\mathcal{M}$ as a $(4 \times 4, 3)_q$–MRD code in Theorem 5.7. Since $\#C_1 = \#C_2 = 1$ and $\#\mathcal{M} = q^8$ we have $\#\mathcal{W}_1 = q^8$, $\#\mathcal{W}_2 = 1$, and $\#\mathcal{W} = q^8 + 1$, so that $A_q(8, 6; 4) \geq q^8 + 1$. We remark that this is still the best known lower bound for all field sizes q and that $A_2(8, 6; 4) = 2^8 + 1 = 257$ was shown in [119].*

We remark that the verbal comparison of [102, Theorem 2.3]), [207, Corollary 39], and other similar variants in the literature with Theorem 5.7 are a bit involved due to different parameterizations and additional conditions that exclude cases where other constructions with competing code sizes are known.

Exercise 5.9. *Show:*

(a) if $n_1 < k$, then $\#\mathcal{W}_1 = 0$; if $n_2 < k$, then $\#\mathcal{W}_2 = 0$;

(b) if $2k \leq n_1 + n_2 \leq 3k - 1$, then the optimal choice is $n_1 = k$, so that $\mathcal{W}_1$ is an LMRD code, cf. the additional condition $3k \leq n_1 + n_2$ in [207, Corollary 39] noting that for $2k > n_1 + n_2$ one may consider the orthogonal code;

(c) if C_1, C_2, and $\mathcal{M}$ have minimum distance d_1, d_2, and d_r, respectively, then we have $d_1 \geq d$, $d_2 \geq d$, and $d_r \geq d/2$ for $d = \min\{d_1, d_2, 2d_r\}$, cf. [102, Theorem 2.3].

Corollary 5.10.

$$A_q(n, d; k) \geq A_q(m, d; k) \cdot A_q^R(k \times (n-m); d/2) + A_q(n-m, d; k)$$

Since the matrix descriptions of two subcodes in Theorem 5.7 are just column permutations of

C	R

we can use Lemma 5.5.(d) to directly conclude a sufficient condition for the addition of further codewords to a CDC constructed via the linkage construction:

Lemma 5.11. *Let C be a CDC obtained via the linkage construction in Theorem 5.7 with parameters (n_1, n_2, d, k), E_2 be the n_2-space spanned by the unit vectors $\mathbf{e}_i$ with $n_1 + 1 \leq i \leq n_1 + n_2$, and E_1 be the n_1-space spanned by the unit vectors $\mathbf{e}_i$ with $1 \leq i \leq n_1$. If $\dim(U \cap E_1) \geq d/2$ and $\dim(U \cap E_2) \geq d/2$ for $U \in \mathcal{G}_q(n_1 + n_2, k)$, then $C \cup \{U\}$ is an $(n_1 + n_2, d; k)_q$–CDC.*

Since we actually have $d_S(\mathcal{W}_1, \mathcal{W}_2) \geq 2k$ in Theorem 5.7 it can be easily improved if $d < 2k$:

Theorem 5.12. ***(Improved linkage construction – [125, Theorem 18])***
Let C_1 be an $(n_1, d; k)_q$–CDC, C_2 be an $(n_2+k-d/2, d; k)_q$–CDC, and $\mathcal{M}$ be a $(k \times n_2, d/2)$–RMC. Then, $\mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1+n_2, d; k)$–CDC of cardinality $\#C_1 \cdot \#\mathcal{M} + \#C_2$, where

$$\left\{\begin{pmatrix} G_1 & M \end{pmatrix} : G_1 \in \mathcal{G}_1, M \in \mathcal{M}\right\}$$

is a generating set of $\mathcal{W}_1$,

$$\left\{\begin{pmatrix} 0_{k\times(n_1-k+d/2)} & G_2 \end{pmatrix} : G_2 \in \mathcal{G}_2\right\}$$

is a generating set of $\mathcal{W}_2$, and $\mathcal{G}_1, \mathcal{G}_2$ are generating sets of C_1, C_2, respectively.

The matrix description of the improved linkage construction is given by:

C	R
0	C

The "linkage property" $d_S(\mathcal{W}_1, \mathcal{W}_2) \geq d$ follows e.g. from Lemma 5.5.(b).

Corollary 5.13.

$$A_q(n, d; k) \geq A_q(m, d; k) \cdot A_q^R(k \times (n-m); d/2) + A_q(n-m+k-d/2, d; k)$$

Clearly, the lower bounds that can be obtained with Theorem 5.12 are at least as large as those from Theorem 5.7.

Also using Lemma 5.5.(d), we can adjust Lemma 5.11 to the improved linkage construction:

Lemma 5.14. *Let C be a CDC obtained via the improved linkage construction in Theorem 5.12 with parameters (n_1, n_2, d, k), E_2 be the n_2-space spanned by the unit vectors $\mathbf{e}_i$ with $n_1+1 \leq i \leq n_1+n_2$, and E_1 be the $n_1-k+d/2$-space spanned by the unit vectors $\mathbf{e}_i$ with $1 \leq i \leq n_1-k+d/2$. If $\dim(U \cap E_1) \geq d/2$ and $\dim(U \cap E_2) \geq d/2$ for $U \in \mathcal{G}_q(n, k)$, then $C \cup \{U\}$ is an $(n_1+n_2, d; k)_q$–CDC.*

Exercise 5.15. *Let $\mathcal{W}$ be a $(12, 6; 4)_q$–CDC constructed via the improved linkage construction in Theorem 5.12 with $m = 6$. Determine all $\mathbf{v} \in \mathcal{G}_1(12, 6)$ such that for every $U \in \mathcal{G}_q(12, 6)$ with pivot vector $\mathbf{v}$ we have $d_S(\mathcal{W}, U) \geq 4$.*

A different variant of the linkage construction exploits Lemma 5.5.(c), i.e., we ensure that the generator matrices of the additional codewords have rank at most $k-d/2$ in their first n_1 columns to deduce the "linkage property" $d_S(\mathcal{W}_1, \mathcal{W}_2) \geq d$:

Theorem 5.16. ***(Generalized linkage construction – [47, Lemma 4.1 with l = 2])***
Let C_1 be an $(n_1, d; k)_q$–CDC, C_2 be an $(n_2, d; k)_q$–CDC, $\mathcal{M}_1$ be a $(k \times n_2, d/2)$–RMC, and $\mathcal{M}_2$ be a $(k \times n_1, d/2; \leq k-d/2)$–RMC. Then, $\mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1+n_2, d; k)$–CDC of cardinality $\#C_1 \cdot \#\mathcal{M}_1 + \#C_2 \cdot \mathcal{M}_2$, where

$$\left\{\begin{pmatrix} G_1 & M_1 \end{pmatrix} : G_1 \in \mathcal{G}_1, M_1 \in \mathcal{M}_1\right\}$$

is a generating set of $\mathcal{W}_1$,

$$\left\{\begin{pmatrix} M_2 & G_2 \end{pmatrix} : G_2 \in \mathcal{G}_2, M_2 \in \mathcal{M}_2\right\}$$

is a generating set of $\mathcal{W}_2$, and $\mathcal{G}_1, \mathcal{G}_2$ are generating sets of C_1, C_2, respectively.

The matrix description of the generalized linkage construction is given by

C	R
R	C

so that the linkage construction is contained as a special subcase. See also [109, Theorem 2].

Corollary 5.17. *We have* $A_q(n, d; k) \geq$

$$A_q(m, d; k) \cdot A_q^R(k \times (n-m); d/2) + A_q(n-m, d; k) \cdot A_q^R(k \times m, d/2; \leq k - d/2).$$

The right hand side can be attained as the cardinality of an $(n, k; d)_q$*–*CDC $\mathcal{W}$ *constructed by the generalized linkage construction in Theorem 5.16.*

Using Lemma 5.5.(d) we can directly conclude a sufficient condition for the addition of further codewords to a CDC constructed via the generalized linkage construction:

Lemma 5.18. *Let* C *be a* CDC *obtained via the generalized linkage construction in Theorem 5.16 with parameters* (n_1, n_2, d, k)*,* E_2 *be the* n_2*-space spanned by the unit vectors* $\mathbf{e}_i$ *with* $n_1 + 1 \leq i \leq n_1 + n_2$*, and* E_1 *be the* n_1*-space spanned by the unit vectors* $\mathbf{e}_i$ *with* $1 \leq i \leq n_1$*. If* $\dim(U \cap E_1) \geq d/2$ *and* $\dim(U \cap E_2) \geq d/2$ *for* $U \in \mathcal{G}_q(n_1 + n_2, k)$*, then* $C \cup \{U\}$ *is an* $(n_1 + n_2, d; k)_q$*–*CDC*.*

Theorem 5.16 has a lot of predecessors in the literature that cover special subcases and also alternative proofs. As indicated, Theorem 5.16 is just a special case of [47, Lemma 4.1]. In Subsection 5.1.1 we will consider variants and generalizations of Theorem 5.16. However, for none of these an explicit strict improvement over Theorem 5.16 is known. See also e.g. [45, 164] for further variations of the linkage construction.

5.1.1 Variants of the generalized linkage construction

In its original formulation of the generalized linkage construction in [47, Lemma 4.1], the approach was extended to $l \geq 2$ subcodes $\mathcal{W}_i$. Here we decompose the result into a few sub statements. Combining Construction D (Theorem 5.1) with the product construction for rank metric codes (Lemma 3.24) yields:

Lemma 5.19. *Let* $l \geq 2$ *and* $\bar{n} = (n_1, \ldots, n_l) \in \mathbb{N}^l$*. For* $2 \leq i \leq l$ *let* $\mathcal{M}_i$ *be a* $(k \times n_i, d)_q$*–*RMC *and* C *be an* $(n_1, d; k)_q$*–*CDC *with representation set* $\mathcal{G}$*. With this, let*

$$\left\{\begin{pmatrix} G & M_2 & \ldots & M_l \end{pmatrix} : G \in C, M_i \in \mathcal{M}_i \forall 2 \leq i \leq l\right\}$$

a generating set and $\mathcal{W}$ *be the generated subspace code. Then,* $\mathcal{W}$ *is an* $(n, d; k)_q$*–*CDC *with cardinality* $\#\mathcal{W} = \#C \cdot \prod_{i=2}^{l} \#\mathcal{M}_i$*, where* $n = \sum_{i=1}^{l} n_i$*.*

The corresponding matrix description is given by

C	R	...	R

where the unique CDC-component may be permuted to each of the $l \ge 2$ positions.

Theorem 5.20. *Let $l \ge 2$ and $\bar{n} = (n_1, \ldots, n_l) \in \mathbb{N}^l$. For $1 \le i \le l$ let C_i be an $(n_i, d; k)_q$–CDC and $\mathcal{G}_i$ a corresponding representation set. For $1 \le j < i \le l$ let $\mathcal{M}_i^j$ be a $(k \times n_i, d; \le k - d/2)_q$–RMC and for $1 \le i < j \le l$ let $\mathcal{M}_i^j$ be a $(k \times n_i, d)_q$–RMC. With this, let*

$$\left\{ \begin{pmatrix} M_i^1 & \ldots & M_i^{i-1} & G_i & M_i^{i+1} & \ldots M_i^l \end{pmatrix} : G_i \in \mathcal{G}_i, M_i^j \in \mathcal{M}_i^j \, \forall 1 \le j \le l, j \ne i \right\}$$

be a generating set for the subcode $\mathcal{W}_i$, where $1 \le i \le l$. Then, $\mathcal{W} = \cup_{i=1}^l \mathcal{W}_i$ is an $(n, k; d)_q$–CDC, where $n = \sum_{i=1}^l n_i$.

Proof. For $1 \le i \le l$ the subcode $\mathcal{W}_i$ is an $(n, d; k)_q$–CDC with cardinality

$$\prod_{j=1}^{i-1} \#\mathcal{M}_i^j \cdot \#C_i \cdot \prod_{j=i+1}^{l} \mathcal{M}_i^j$$

by Lemma 5.19. Let

$$H = \begin{pmatrix} M_1 & \ldots M_{i-1} & G & M_{i+1} & \ldots M_l \end{pmatrix}$$

be an arbitrary element in the generating set of the subcode $\mathcal{W}_i$ and

$$H' = \begin{pmatrix} M'_1 & \ldots M'_{i-1} & G' & M'_{i+1} & \ldots M'_l \end{pmatrix}$$

be an arbitrary element in the generating set of the subcode $\mathcal{W}_j$, where $1 \le i < j \le l$ are arbitrary. Set $\bar{H} = \begin{pmatrix} G & M_j \end{pmatrix}$ and $\bar{H}' = \begin{pmatrix} M'_i & G' \end{pmatrix}$ and note $\mathrm{rk}(H) = \mathrm{rk}(H') = \mathrm{rk}(\bar{H}) = \mathrm{rk}(\bar{H}') = k$, so that $d_S(\langle H \rangle, \langle H' \rangle) \ge d_S(\langle \bar{H} \rangle, \langle \bar{H}' \rangle)$. Since $\mathrm{rk}(M'_i) \le k - d/2$ we can apply Lemma 5.5.(c) to deduce $d_S(\mathcal{W}_i, \mathcal{W}_j) \ge d$, so that $d_S(\mathcal{W}) \ge d$. □

The corresponding matrix description is given by

C	R	R	...	R
R	C	R	...	R
⋮	⋱	⋱	⋱	⋮
R	...	R	C	R
R	...	R	R	C

Corollary 5.21.

$$A_q(n, d; k) \ge \sum_{i=1}^{l} \left(\prod_{j=1}^{i-1} A_q^R(k \times n_j, \tfrac{d}{2}; k - \tfrac{d}{2}) \right) \cdot A_q(n_i, d; k) \cdot \left(\prod_{j=i+1}^{l} A_q^R(k \times n_j, \tfrac{d}{2}) \right)$$

We remark that in the original formulation of [47, Lemma 4.1] the rank metric codes $\mathcal{M}_i^j$; where $1 \le j \le l$ and $j \ne i$, are assumed to be subcodes of a $(k \times n_i, d/2)_q$–RMC $\mathcal{M}_i$, which is not necessary and may make a difference if $l \ge 3$ only. However, currently none of the best known codes uses Theorem 5.20 or [47, Lemma 4.1] with $l \ge 3$. Actually, the parameter l in Theorem 5.20 can be recursively reduced to 2, so that we finally end up with Theorem 5.7:

Exercise 5.22. *Let $\mathcal{W}$ be an $(n,d;k)_q$–CDC constructed via Theorem 5.20 with $l \ge 3$. Set*

- $\widehat{n}_i = n_i$ *for all* $1 \le i \le l-2$, $\widehat{n}_{l-1} = n_{l-1} + n_l$*;*
- $\widehat{C}_i = C_i$ *for all* $1 \le i \le l-2$*;*
- $\widehat{\mathcal{M}}_i^j = \mathcal{M}_i^j$ *for all* $1 \le i, j \le l-2$, $i \ne j$*;*
- $\widehat{\mathcal{M}}_i^{l-1} = \mathcal{M}_i^{l-1} \times \mathcal{M}_i^l$ *for all* $1 \le i \le l-2$*;*
- $\widehat{C}_{l-1}$ *to the CDC obtained from the generalized linkage construction in Theorem 5.16 using* C_{l-1}, C_l, $\mathcal{M}_l^{l-1}$, *and* $\mathcal{M}_{l-1}^l$*; and*
- $\widehat{\mathcal{M}}_{l-1}^j = \mathcal{M}_h^j$ *for all* $1 \le j \le l-2$, *where* $h \in \{l-1, l\}$ *maximizes* $\#\mathcal{M}_h^1 \times \cdots \times \mathcal{M}_h^{l-2}$.

Show that we can apply Theorem 5.20 with the above components to obtain a CDC $\widehat{\mathcal{W}}$ with $\#\widehat{\mathcal{W}} \ge \#\mathcal{W}$.

In principle it is not necessary that the matrix description of the generalized linkage construction has a grid-like structure.

Theorem 5.23. *([117, Theorem 26]) Let C_1 be an $(n_1,d;k)_q$–CDC, C_2 be an $(n_2+t,d;k)_q$–CDC, $\mathcal{M}_1$ be a $(k \times n_2, d/2)$–RMC, and $\mathcal{M}_2$ be a $(k \times (n_1 - t), d/2; \le k - d/2 - t)$–RMC. Then, $\mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1+n_2,d;k)$–CDC of cardinality $\#C_1 \cdot \#\mathcal{M} + \#C_2 \cdot \#\mathcal{M}_2$, where*

$$\left\{\begin{pmatrix} G_1 & M_1 \end{pmatrix} : G_1 \in \mathcal{G}_1, M_1 \in \mathcal{M}\_1\right\}$$

is a generating set of $\mathcal{W}_1$,

$$\left\{\begin{pmatrix} M_2 & G_2 \end{pmatrix} : G_2 \in \mathcal{G}_2, M_2 \in \mathcal{M}_2\right\}$$

is a generating set of $\mathcal{W}_2$, and $\mathcal{G}_1, \mathcal{G}_2$ are generating sets of C_1, C_2, respectively.

The corresponding matrix description is given by

C	R

R	C

so that Theorem 5.23 generalizes the improved linkage construction in Theorem 5.12. However, currently no single case where Theorem 5.23 yields strictly larger codes than Theorem 5.12 and Theorem 5.16 is known.

Corollary 5.24.

$$\begin{aligned} A_q(n,d;k) \quad \ge \quad & A_q(m,d;k) \cdot A_q^R(k \times (n-m), d/2) \\ & + A_q(n-m+t,d;k) \cdot A_q^R(k \times (m-t), d/2; \le k - d/2) \end{aligned}$$

5.2 The Echelon–Ferrers construction and their variants

The basis for the *Echelon–Ferrers* or *multilevel construction* from [76] is Inequality (2.8), i.e. $d_S(U,W) \ge \mathrm{d_H}\big(v(U), v(W)\big)$.

Theorem 5.25. ***(Multilevel construction – [76, Theorem 3])***
Let $\mathcal{S} \subseteq \mathcal{G}_1(n,k)$ with $\mathrm{d}_H(\mathcal{S}) \ge d$. If $C_v \subseteq \mathcal{G}_q(n,k)$ is an $(n,d;k)_q$–CDC whose codewords have pivot vector v for each $v \in \mathcal{S}$, then $C = \cup_{v\in\mathcal{S}} C_v$ is an $(n,d;k)_q$–CDC with cardinality $\sum_{v\in\mathcal{S}} \#C_v$.

Suitable choices for the C_v are also discussed in e.g. [76] and we will do so in a moment, see Example 5.29. The set $\mathcal{S}$ is a binary code with minimum Hamming distance d and sometimes called *skeleton code*. By $A_q(n,d;k;v)$ we denote the maximum possible cardinality M of an $(n,d;k)_q$-CDC where all codewords have pivot vector v, so that Theorem 5.25 gives the lower bound

$$A_q(n,d;k) \ge \sum_{v\in\mathcal{S}} A_q(n,d;k;v), \tag{5.3}$$

where $\mathrm{d_H}(\mathcal{S}) \ge d$. Actually the notion $A_q(n,d;k;v)$ is a special case of our notion $A_q(n,d;k;\mathcal{V})$ for arbitrary subsets $\mathcal{V} \subseteq \mathcal{G}_1(n,k)$. And so also Theorem 5.25 can be generalized:

Theorem 5.26. *([158, Theorem 2.3])*
Let $\mathcal{V}_1,\dots,\mathcal{V}_s$ be subsets of $\mathcal{G}_1(n,k)$ with $\mathrm{d}_H(\mathcal{V}_i,\mathcal{V}_j) \ge d$ for all $1 \le i < j \le s$. If $C_{\mathcal{V}_i} \subseteq \mathcal{G}_q(n,k)$ is an $(n,d;k)_q$–CDC with pivot structure $\mathcal{V}_i$ for each $1 \le i \le s$, then $C = \cup_{1\le i\le s} C_{\mathcal{V}_i}$ is an $(n,d;k)_q$–CDC with cardinality $\sum_{1\le i\le s} \#C_{\mathcal{V}_i}$.

We call $S = \{\mathcal{V}_1,\dots,\mathcal{V}_s\}$ a *generalized skeleton code*, see [158]. For constructions that fit into the context of Theorem 5.26 we refer e.g. to [110, 158].

Given a Ferrers diagram $\mathcal{F}$ with m dots in the rightmost column and l dots in the top row, we call a rank-metric code $C_{\mathcal{F}}$ a *Ferrers diagram rank-metric* (FDRM) code if for any codeword $M \in \mathbb{F}_q^{m\times l}$ of $C_{\mathcal{F}}$ all entries not in $\mathcal{F}$ are zero. By $d_{\mathrm{R}}(C_{\mathcal{F}})$ we denote the minimum rank distance, i.e., the minimum of the rank distance between pairs of different codewords.

Definition 5.27. *([207])*
Let $\mathcal{F}$ be a Ferrers diagram and $C_{\mathcal{F}} \subseteq \mathbb{F}_q^{k\times(n-k)}$ be an FDRM code. The corresponding lifted FDRM code *$C_{\mathcal{F}}$ is given by*

$$C_{\mathcal{F}} = \big\{U \in \mathcal{G}_q(n,k) \,:\, \mathcal{F}(U) = \mathcal{F}, T(U) \in C_{\mathcal{F}}\big\}.$$

Lemma 5.28. *([76, Lemma 4])*
Let $C_{\mathcal{F}} \subseteq \mathbb{F}_q^{k\times(n-k)}$ be an FDRM code with minimum rank distance δ, then the lifted FDRM code $C_{\mathcal{F}} \subseteq \mathcal{G}_q(n,k)$ is an $(n,2\delta;k)_q$–CDC of cardinality $\#C_{\mathcal{F}}$.

Example 5.29. *For the Ferrers diagram*

$$\mathcal{F} = \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

over $\mathbb{F}_2$ *a linear* `FDRM` *code with minimum rank distance* $d_R = 3$ *and cardinality* 16 *is given by*

$$C_{\mathcal{F}} = \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \right\rangle \subseteq \mathbb{F}_2^{3\times 4}.$$

Via lifting we obtain a `CDC` *with pivot structure* $\{(1,1,1,0,0,0,0)\}$ *showing*

$$A_2(7,6;3;(1,1,1,0,0,0,0)) \ge 16.$$

Since $\mathrm{d}_H\big((1,1,1,0,0,0,0),(0,0,0,1,1,0,1)\big) = 6$ *we have*

$$A_2(7,6;3) \ge A_2(7,6;3;(1,1,1,0,0,0,0)) + A_2(7,6;3;(0,0,0,1,1,0,1)).$$

The Ferrers diagram for pivot vector $(0,0,0,1,1,0,1)$ *is* $\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}$ *with e.g.* $\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$ *as a possible* `FDRM` *code. The corresponding lifted codeword has generator matrix*

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $A_2(7,6;3) = 17$*, see e.g. the partial spread bound in Theorem 4.26, we have*

$$A_2(7,6;3;(1,1,1,0,0,0,0)) = 16$$

and $A_2(7,6;3;(0,0,0,1,1,0,1)) = 1$.

Lifted `FDRM` codes $C_{\mathcal{F}}$ are exactly the subcodes C_v needed in the Echelon-Ferrers construction in Theorem 5.25. In [76, Theorem 1] a general upper bound for (linear) `FDRM` codes was given. Since the bound is also true for non-linear `FDRM` codes, as observed by several authors, denoting the pivot vector corresponding to a given Ferrers diagram $\mathcal{F}$ by $v(\mathcal{F})$ and using Lemma 5.28, we can rewrite the upper bound to:

Theorem 5.30.

$$A_q(n,d;k;v(\mathcal{F})) \le q^{\min\{\nu_i \,:\, 0 \le i \le d/2-1\}},$$

where ν_i *is the number of dots in* $\mathcal{F}$*, which are neither contained in the first* i *rows nor contained in the last* $\frac{d}{2} - 1 - i$ *columns.*

If we choose a minimum subspace distance of $d = 6$, then we obtain

$$A_2(9,6;4;101101000) \le 2^7$$

due to

$$\begin{array}{ccccc} \circ & \circ & \circ & \bullet & \bullet \\ & \circ & \circ & \bullet & \bullet \\ & \circ & \circ & \bullet & \bullet \\ & & \circ & \bullet & \bullet \end{array} \qquad \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ & \circ & \circ & \circ & \bullet \\ & \circ & \circ & \circ & \bullet \\ & & \circ & \circ & \bullet \end{array} \qquad \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \bullet \\ & \circ & \circ & \circ & \circ \\ & & \circ & \circ & \circ \end{array}.$$

where the non-solid dots are those that are neither contained in the first i rows nor contained in the last $\frac{d}{2} - 1 - i$ columns for $1 \le i \le 3$.

While it is conjectured that the upper bound from Theorem 5.30 (and the corresponding bound for FDRM codes) can always be attained, this problem is currently solved for specific instances like e.g. rank-distances $\delta = 2$ only. For more results see e.g. [14, 13, 71, 170] and the references mentioned therein.

Example 5.31. *We choose a generalized skeleton code $\mathcal{S}$ with vertices* $\left(\binom{4}{0},\binom{7}{4}\right)$, 00010000111, 00010100011, 00011000011, 00011000110, 00100001011, 00100001101, 00100001110, 00100100101, 00100100110, 00100101001, 00101000101, 00110000110, 00110101000, 01100010001, 10000101100, 10001001001, 10011100000 10100000011, *and* 10100110000, *so that*

$$A_q(11,4;4) \ge q^{21} + q^{17} + 2q^{15} + 3q^{14} + 4q^{13} + q^{12} + q^{11} + q^9 + 2q^7 + 2q^6 + q^5 + A_q(7,4;4),$$

see [158, Proposition 3.1].

While the upper bound from Theorem 5.30 can always be attained for minimum subspace distance $d = 4$, the determination of a "good" (generalized) skeleton code is still a tough discrete optimization problem.[1] In [154] several new (generalized) skeleton codes improving the previously best known lower bounds for $A_q(n,d;k)$ are given. We remark that it is also possible to compute upper bounds for the cardinalities of CDCs that can be obtained by the Echelon–Ferrers construction and to perform those computations parametric in the field size q, see [82]. There are many other papers with explicitly determine (generalized) skeleton codes and heuristic algorithms to compute them, see the citations of [76]. For greedy-type approaches we refer to e.g. [112, 203, 204].

For the case of partial spreads, i.e. for $d = 2k \le n$, the determination of a good skeleton code for the Echelon–Ferrers construction is rather easy. Note that the condition $\mathrm{d_H}(v, v') \ge d = 2k$ for $v, v' \in \mathcal{G}_1(n,k)$ means that the ones of v and those of v' have to be disjoint, so that $A(n, 2k; k) \le \lfloor n/k \rfloor$. By choosing $v^i \in \mathcal{G}_1(n,k)$ such that the k ones are in positions $(i-1)k+1, \dots, ik$ for $1 \le i \le \lfloor n/k \rfloor$ the upper bound can be attained and all corresponding Ferrers diagrams are rectangular, so that we can use MRD codes.

Exercise 5.32. *Show* $A_q(n, 2k; k) \ge \frac{q^n - q^k(q^{(n \bmod k)} - 1) - 1}{q^k - 1}$ *for* $2k \le n$.

We remark that a more general construction, along similar lines and including explicit formulas for the respective cardinalities, has been presented in [211], see also [91]. For another approach how to select the skeleton codes via so-called lexicodes see [205].

Consider the following three Ferrers diagrams

$$\begin{array}{cccccc} \circ & \circ & \circ & \bullet & \bullet & \bullet \\ & & \circ & \bullet & \bullet & \bullet \\ & & & \bullet & \bullet & \bullet \end{array}, \quad \begin{array}{cccc} \circ & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet \end{array}, \quad \text{and} \quad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array},$$

where we have marked a few special dots by non-solid circles. For minimum rank distance $d_{\mathrm{R}} = 3$ corresponding FDRM or lifted FDRM codes can have a cardinality of at most q^3 in all three cases (and this upper bound can indeed be attained). So, we can remove the non-solid circles

[1] Note that it generalizes the computation of $A(n,d;k)$.

from the diagrams without decreasing the upper bound. Or, framed differently, we can used this free extra positions to add a few more codewords. The single non-solid circle in the middle diagram is called a *pending dot*, see [77] for the details. This notion was generalized to so-called *pending blocks* and the four non-solid circles in the leftmost diagram form such a pending block. For details we refer to [206, 207, 217].

Explicit series of constructions using pending dots are e.g. given by the following two theorems.

Theorem 5.33. ***(Construction 1 – [77, Chapter IV, Theorem 16])***

$$A_q(n, 2(k-1); k) \geq q^{2(n-k)} + A_q(n-k, 2(k-2); k-1)$$

if $q^2+q+1 \geq s$ with $s = n-4$ if n is odd and $s = n-3$ else.

Theorem 5.34. ***(Construction 2 –[77, Chapter IV, Theorem 17])***

$$A_q(n, 4; 3) \geq q^{2(n-3)} + \sum_{i=1}^{\alpha} q^{2(n-3-(q^2+q+2)i)}$$

if $q^2+q+1 < s$ with $s = n-4$ if n is odd and $s = n-3$ else and $\alpha = \left\lfloor \frac{n-3}{q^2+q+2} \right\rfloor$

Explicit series of constructions using pending blocks are e.g. given by the following two theorems.

Theorem 5.35. ***(Construction A – [207, Chapter III, Theorem 19, Corollary 20])***
Let $n \geq \frac{k^2+3k-2}{2}$ and $q^2+q+1 \geq \ell$, where $\ell = n - \frac{k^2+k-6}{2}$ for odd $n - \frac{k^2+k-6}{2}$ (or $\ell = n - \frac{k^2+k-4}{2}$ for even $n - \frac{k^2+k-6}{2}$). Then $A_q(n, 2k-2; k) \geq q^{2(n-k)} + \sum_{j=3}^{k-1} q^{2(n-\sum_{i=j}^{k} i)} + \left[{n-\frac{k^2+k-6}{2} \atop 2}\right]_q$.

Theorem 5.36. ***(Construction B – [207, Chapter IV, Theorem 26, Corollary 27])***
Let $n \geq 2k+2$. Then we have $A_q(n, 4; k) \geq$

$$\sum_{i=1}^{\lfloor \frac{n-2}{k} \rfloor - 1} \left(q^{(k-1)(n-ik)} + \frac{(q^{2(k-2)}-1)(q^{2(n-ik-1)}-1)}{(q^4-1)^2} q^{(k-3)(n-ik-2)+4} \right).$$

5.3 The coset construction

The starting point for the so-called *coset construction* introduce in [126] was [77, Construction III] leading to the lower bound $A_2(8, 4; 4) \geq 4797$. The corresponding generator matrices have the form

$$\begin{pmatrix} G_1 & \varphi_H(M) \\ \mathbf{0} & G_2 \end{pmatrix}$$

where $G_1 \in \mathbb{F}_q^{k_1 \times n_1}$ and $G_2 \in \mathbb{F}_q^{k_2 \times n_2}$ are generator matrices of $(n_1, d; k_1)_q$- and $(n_2, d; k_2)_q$–CDCs, respectively. The matrix $M \in \mathbb{F}_q^{k_1 \times (n_2-k_2)}$ is an element of a $(k_1 \times (n_2-k_2), d/2)_q$–RMC and the function φ_{G_2} maps M into $\mathbb{F}_q^{k_1 \times n_2}$ by inserting k_2 additional zero columns at a set S of positions where corresponding submatrix of G_2 has rank k_2.

Definition 5.37. *Let $M \in \mathbb{F}_q^{k\times n}$ be arbitrary and S a subset of $\{1,\dots,n\}$. By $M|_S$ we denote the restriction of M to the columns of M with indices in S.*

For one-element subsets we also use the abbreviation $M|_i = M|_{\{i\}}$.

Example 5.38. *For $M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{F}_2^{2\times 5}$ and $S = \{1,3,5\}$ we have $M|_S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.*

Definition 5.39. *Let $G \in \mathbb{F}_q^{k_2\times n}$ of rank k_2 and $M \in \mathbb{F}_q^{k_1\times(n-k_2)}$ be arbitrary. We call function $\varphi\colon \mathbb{F}_q^{k_1\times(n-k_2)} \to \mathbb{F}_q^{k_1\times n}$ an* embedding function compatible with *G if there exists a subset $S \subseteq \{1,\dots,n\}$ of cardinality k_2 such that $\varphi(M)|_S = \mathbf{0}_{k_1\times k_2}$ and $\operatorname{rk}(G|_S) = \operatorname{rk}(G) = k_2$.*

In order to indicate the dependence on H we typically denote embedding functions compatible with G by φ_G. As an abbreviation for the function value $\varphi_G(M)$ we also write $M\!\uparrow_G$ or $M\!\uparrow$, whenever G is clear from the context or secondary. A feasible and typical choice for φ_G is to choose the index set S as the set of the pivot positions in $E(G)$.

Example 5.40. *For $G = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ we have*

$$E(G) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so that $v(G) = 010101$ and $S := \{1 \le i \le 6 : v(G)|_i = 1\} = \{2,4,6\}$. For the embedding function φ_G compatible with H defined via the index set S we have

$$\varphi_G(M) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Lemma 5.41. *Let $G \in \mathbb{F}_q^{k_2\times n}$ with $\operatorname{rk}(G) = k_2$ and $\varphi_G\colon \mathbb{F}_q^{k_1\times(n-k_2)} \to \mathbb{F}_q^{k_1\times n}$ an embedding function compatible with G. Then, we have*

$$\operatorname{rk}\left(\begin{pmatrix} \varphi_G(M) \\ G \end{pmatrix}\right) = \operatorname{rk}(G) + \operatorname{rk}(M) = k_2 + \operatorname{rk}(M) \tag{5.4}$$

for all $M \in \mathbb{F}_q^{k_1\times(n-k_2)}$ and

$$\begin{aligned} \operatorname{rk}\left(\begin{pmatrix} \sum_{i=1}^{l} \lambda_i \cdot \varphi_G(M_i) \\ G \end{pmatrix}\right) &= \operatorname{rk}(G) + \operatorname{rk}\left(\sum_{i=1}^{l} \lambda_i \cdot M_i\right) \\ &= k_2 + \operatorname{rk}\left(\sum_{i=1}^{l} \lambda_i \cdot M_i\right) \end{aligned} \tag{5.5}$$

for all $l \in \mathbb{N}$, and $\lambda_i \in \mathbb{F}_q$, $M_i \in \mathbb{F}_q^{k_1\times(n-k_2)}$ with $1 \le i \le l$.

Proof. Let $S \subseteq \{1,\dots,n\}$ be the subset in Definition 5.39 corresponding to φ_G and $[n]\backslash S = \{1,\dots,n\}\backslash S$. Note that we have $\varphi_G(M)|_S = \mathbf{0}_{k_1\times k_2}$ and $\varphi_G(M)|_{[n]\backslash S} = M$ for all $M \in \mathbb{F}_q^{k_1\times(n-k_2)}$. Since $\operatorname{rk}(G|_S) = \operatorname{rk}(G) = k_2$ we have

$$\operatorname{rk}\left(\begin{pmatrix}\varphi_G(M)\\ G\end{pmatrix}\right) = \operatorname{rk}\left(\begin{pmatrix}\mathbf{0}_{k_1\times k_2} & M\\ G|_S & G|_{[n]\backslash S}\end{pmatrix}\right) = \operatorname{rk}(M) + \operatorname{rk}(G|_S) = \operatorname{rk}(G) + \operatorname{rk}(M),$$

i.e., the first equation is valid (using $\operatorname{rk}(G) = k_2$).

Set $M = \sum_{i=1}^{l} \lambda_i M_i \in \mathbb{F}_q^{k_1\times(n-k_2}$ and $M' = \sum_{i=1}^{l} \varphi_G(M_i) \in \mathbb{F}_q^{k_1\times n}$. Since $\varphi_G(M) = M'$ the second equation directly follows from the first. □

Lemma 5.42. ***(Product construction for constant dimension codes)*** *Let $\mathcal{C}_1$ be an $(n_1, d; k_1)_q$–CDC, $\mathcal{C}_2$ be an $(n_2, d; k_2)_q$–CDC, $\mathcal{M}$ be a $(k_1 \times (n_2 - k_2), d/2)_q$–RMC, and $\mathcal{G}_1$, $\mathcal{G}_2$ be generating sets of $\mathcal{C}_1$, $\mathcal{C}_2$, respectively. For each $G_2 \in \mathcal{G}_2$ we denote by φ_{G_2} an embedding function $\mathbb{F}_q^{k_1\times(n_2-k_2)} \to \mathbb{F}_q^{k_1\times n_2}$ compatible with G_2. With this,*

$$\left\{\begin{pmatrix}G_1 & \varphi_{G_2}(M)\\ \mathbf{0}_{k_2\times n_1} & G_2\end{pmatrix} : G_1 \in \mathcal{G}_1, M \in \mathcal{M}, G_2 \in \mathcal{G}_2\right\}$$

is the generating set of an $(n_1+n_2, d; k_1+k_2)_q$–CDC $\mathcal{W}$ with cardinality $\#\mathcal{C}_1 \cdot \#\mathcal{M} \cdot \#\mathcal{C}_2$.

Proof. Let $W \in \mathcal{W}$ be an arbitrary codeword with generator matrix

$$H = \begin{pmatrix}G_1 & \varphi_{G_2}(M)\\ \mathbf{0} & G_2\end{pmatrix}.$$

Since $\operatorname{rk}(H) = \operatorname{rk}(G_1) + rk(G_2) = k_1 + k_2$ we have $\dim(W) = k_1 + k_2$. Let $W' \in \mathcal{W}$ be another codeword with $W' \neq W$ with generator matrix $H' = \begin{pmatrix}G_1' & \varphi_{G_2'}(M')\\ \mathbf{0} & G_2'\end{pmatrix}$. Set

$$R := \operatorname{rk}\left(\begin{pmatrix}G_1 & \varphi_{G_2}(M)\\ \mathbf{0} & G_2\\ G_1' & \varphi_{G_2'}(M')\\ \mathbf{0} & G_2'\end{pmatrix}\right) = \operatorname{rk}\left(\begin{pmatrix}G_1 & \varphi_{G_2}(M)\\ G_1' - G_1 & \varphi_{G_2'}(M') - \varphi_{G_2}(M)\\ \mathbf{0} & G_2\\ \mathbf{0} & G_2' - G_2\end{pmatrix}\right)$$

and note that

$$\operatorname{rk}\left(\begin{pmatrix}G_1\\ G_1' - G_1\end{pmatrix}\right) = \frac{d_S(\langle G_1\rangle, \langle G_1'\rangle)}{2} + k_1 \geq \frac{d}{2} + k_1$$

$$\operatorname{rk}\left(\begin{pmatrix}G_2\\ G_2' - G_2\end{pmatrix}\right) = \frac{d_S(\langle G_2\rangle, \langle G_2'\rangle)}{2} + k_2 \geq \frac{d}{2} + k_2.$$

Since $d_S(W, W') = 2 \cdot (R - k_1 - k_2)$ it suffices to show $R \geq k_1 + k_2 + \frac{d}{2}$ in order to deduce $d_S(W, W')$.

If $G_1 \neq G_1'$ we have

$$R \geq \operatorname{rk}\left(\begin{pmatrix}G_1 & \star\\ G_1' - G_1 & \star\\ \mathbf{0} & G_2\end{pmatrix}\right) = \operatorname{rk}\left(\begin{pmatrix}G_1\\ G_1' - G_1\end{pmatrix}\right) + \operatorname{rk}(G_2) \geq d/2 + k_1 + k_2.$$

If $G_1 = G_1'$ and $G_2 \neq G_2'$ we have

$$R \geq \operatorname{rk}\left(\begin{pmatrix} G_1 & \star \\ \mathbf{0} & G_2 \\ \mathbf{0} & G_2' - G_2 \end{pmatrix}\right) = \operatorname{rk}(G_1) + \operatorname{rk}\left(\begin{pmatrix} G_2 \\ G_2' - G_2 \end{pmatrix}\right) \geq d/2 + k_1 + k_2.$$

If $G_1 = G_1'$ and $G_2 = G_2'$ then we have $M \neq M'$ so that $\operatorname{rk}(M - M') = d_{\mathrm{R}}(M, M') \geq d/2$ and

$$\begin{aligned} R &\geq \operatorname{rk}\left(\begin{pmatrix} G_1 & \star \\ \mathbf{0} & \varphi_{G_2}(M') - \varphi_{G_2}(M) \\ \mathbf{0} & G_2 \end{pmatrix}\right) = \operatorname{rk}(G_1) + \operatorname{rk}\left(\begin{pmatrix} \varphi_{G_2}(M') - \varphi_{G_2}(M) \\ G_2 \end{pmatrix}\right) \\ &= k_1 + k_2 + \operatorname{rk}(M - M') \geq k_1 + k_2 + d/2. \end{aligned}$$

Thus we have $d_{\mathrm{S}}(\mathcal{W}) \geq d$ and the stated cardinality follows from the distance analysis. □

The corresponding matrix description is denoted by

C	R↑
𝟘	C

where R↑ indicates a RMC whose length is increased by addition additional zero columns according to a CDC sharing the same positions of the final code.

While the conditions on the components C_1, C_2, and $\mathcal{M}$ in the product construction in Lemma 5.42 are rather demanding, one advantage is that the three code sizes are multiplied. The other is that we can combine several such subcodes to a larger CDC:

Theorem 5.43. ***(Coset construction – [126, Lemma 3, Lemma 4])***
Let C_1 be an $(n_1, d_1; k_1)_q$–CDC, C_2 be an $(n_2, d_2; k_2)_q$–CDC, and $\mathcal{M}$ be a $(k_1 \times (n_2 - k_2), d/2)_q$–RMC, where $d = d_1 + d_2$. For a positive integer s let $C_1^1, \ldots, C_1^s$ be a d-packing of C_1 and $C_2^1, \ldots, C_2^s$ be a d-packing of C_2. For $j \in \{1, 2\}$ and $1 \leq i \leq s$ let $\mathcal{G}_j^i$ be a generating set of C_j^i and $\mathcal{G}_j = \cup_{i=1}^s \mathcal{G}_j^i$, where $j \in \{1, 2\}$. For each $G \in \mathcal{G}_2$ let φ_G be an embedding function $\mathbb{F}_q^{k_1 \times (n_2 - k_2)} \to \mathbb{F}_q^{k_1 \times n_2}$ compatible with G. With this let

$$\left\{\begin{pmatrix} G_1 & \varphi_{G_2}(M) \\ \mathbf{0}_{k_2 \times n_1} & G_2 \end{pmatrix} : G_1 \in \mathcal{G}_1^i, M \in \mathcal{M}, G_2 \in \mathcal{G}_2^i\right\}$$

be a generating set of a subcode $\mathcal{W}^i$ for $1 \leq i \leq s$. Then, $\mathcal{W} = \cup_{i=1}^s \mathcal{W}^i$ is an $(n_1 + n_2, d_1 + d_2; k_1 + k_2)_q$–CDC with cardinality

$$\#\mathcal{W} = \sum_{i=1}^{s} \#\mathcal{W}^i = \#\mathcal{M} \cdot \sum_{i=1}^{s} \#C_1^i \cdot \#C_2^i. \tag{5.6}$$

Proof. The subcodes $\mathcal{W}^i$ are $(n_1 + n_2, d_1 + d_2; k_1 + k_2)_q$–CDCs for all $1 \leq i \leq s$ by Lemma 5.42, which also yields the stated cardinality of $\mathcal{W}$. For arbitrary $G_1, G_1' \in \mathcal{G}_1$, $G_2, G_2' \in \mathcal{G}_2$, and $M, M' \in \mathcal{M}$ let

$$H = \begin{pmatrix} G_1 & \varphi_{G_2}(M) \\ \mathbf{0} & G_2 \end{pmatrix} \quad \text{and} \quad H' = \begin{pmatrix} G_1' & \varphi_{G_2'}(M) \\ \mathbf{0} & G_2' \end{pmatrix}$$

i.e., $W = \langle H \rangle$, $W' = \langle H' \rangle$ are arbitrary codewords in $\mathcal{W}$.

If $G_1 = G_1'$ or $G_2 = G_2'$ then there exists an index $1 \le i \le s$ so that $W, W' \in \mathcal{W}^i$ and either $W = W'$ or $d_S(W, W') \ge d_S(\mathcal{W}^i) \ge d_1 + d_2$.

If $G_1 \ne G_1'$ and $G_2 \ne G_2'$, then we set $U_1 = \langle G_1 \rangle$, $U_1' = \langle G_1' \rangle$, $U_2 = \langle G_2 \rangle$, $U_2' = \langle G_2' \rangle$, so that

$$\operatorname{rk}\left(\begin{pmatrix} G_1 \\ G_1' - G_1 \end{pmatrix}\right) = \frac{d_S(U_1, U_1')}{2} + k_1 \ge \frac{d_S(C_1)}{2} + k_1 \ge \frac{d_1}{2} + k_1$$

and

$$\operatorname{rk}\left(\begin{pmatrix} G_2 \\ G_2' - G_2 \end{pmatrix}\right) = \frac{d_S(U_2, U_2')}{2} + k_2 \ge \frac{d_S(C_2)}{2} + k_2 \ge \frac{d_2}{2} + k_2.$$

Since

$$\begin{aligned} R &:= \operatorname{rk}\left(\begin{pmatrix} G_1 & \varphi_{G_2}(M) \\ \mathbf{0} & G_2 \\ G_1' & \varphi_{G_2'}(M) \\ \mathbf{0} & G_2' \end{pmatrix}\right) = \operatorname{rk}\left(\begin{pmatrix} G_1 & \star \\ G_1' - G_1 & \star \\ \mathbf{0} & G_2 \\ \mathbf{0} & G_2' - G_2 \end{pmatrix}\right) \\ &= \operatorname{rk}\left(\begin{pmatrix} G_1 \\ G_1' - G_1 \end{pmatrix}\right) + \operatorname{rk}\left(\begin{pmatrix} G_2 \\ G_2' - G_2 \end{pmatrix}\right) \ge \frac{d_1 + d_2}{2} + k_1 + k_2 \end{aligned}$$

we have $d_S(W, W') = 2 \cdot (R - k_1 - k_2) \ge d_1 + d_2$. □

The corresponding matrix description is denoted by

$\mathtt{C}^i$	R↑
𝟘	$\mathtt{C}^i$

where $\mathtt{C}^i$ indicates that the have a sequence of CDCs and using the same superscript i indicates how the components have to be arranged.

We remark that we may also use different RMCs $\mathcal{M}^i$ for the construction of the subcodes $\mathcal{W}^i$ instead a single RMC $\mathcal{M}$ for all. However, since there is no obvious benefit of such a generalization we prefer the simplicity of the stated formulation and Equation (5.6) for the cardinality of the resulting code.

Definition 5.44. *By $C_q(n_1, n_2, d; k_1, k_2)$ we denote that maximum possible cardinality of a* CDC *$\mathcal{W}$ obtained via the coset construction in Theorem 5.43 with* RMC *$\mathcal{M} = \left\{\mathbf{0}_{k_1 \times (n_2 - k_2)}\right\}$, where d_1, d_2 are arbitrary besides satisfying $d_1 + d_2 = d$.*

In other words, $C_q(n_1, n_2, d; k_1, k_2)$ is a shorthand for the maximum possible value of $\sum_{i=1}^{s} \#C_1^i \cdot \#C_2^i$ in Equation (5.6).

Exercise 5.45. *Show $C_q(n_1, n_2, d; k_1, k_2) = C_q(n_2, n_1, d; k_2, k_1)$ and $C_q(n_1, n_2, d; k_1, k_2) = C_q(n_1, n_2, d; k_1, n_2 - k_2)$.*

Since the optimal choice for the RMC $\mathcal{M}$ in the coset construction for a CDC $\mathcal{W}$ is an MRD code, $C_q(n_1, n_2, d; k_1, k_2)$ is indeed the essential quantity to express the maximum possible cardinality $\#\mathcal{W}$:

Lemma 5.46. *Let $\mathcal{W}$ be a* CDC *constructed via the coset construction in Theorem 5.43 with parameters $(n_1, n_2, d; k_1, k_2)$ of maximum possible cardinality. Then, we have*

$$\begin{aligned}
\#\mathcal{W} &= A_q^R(k_1 \times (n_2 - k_2), d/2) \cdot C_q(n_1, n_2, d; k_1, k_2) \\
&= \left\lceil q^{\max\{k_1, n_2-k_2\} \cdot (\min\{k_1, n_2-k_2\}-d+1)} \right\rceil \cdot C_q(n_1, n_2, d; k_1, k_2). \qquad (5.7)
\end{aligned}$$

When estimating lower bounds for constant dimension codes we may also replace the term $C_q(n_1, n_2, d; k_1, k_2)$ by some lower bound. The matrix description underlying Definition 5.44 can be written as

$\mathtt{C}^i$	$\mathbb{0}$
$\mathbb{0}$	$\mathtt{C}^i$

We remark that [47, Lemma 4.4] for $l = 2$ can be seen as a special case of this construction.

Before we state an example for the coset construction we introduce another notion from geometry.

Definition 5.47. *(Parallelisms)*
A parallelism *in $\mathcal{G}_q(n, k)$ is a $2k$-partition of the $(n, 2; k)_q$–*CDC *$\mathcal{G}_q(n, k)$. A $2k$-packing of $\mathcal{G}_q(n, k)$ is called* partial parallelism *in $\mathcal{G}_q(n, k)$.*

In other words, a parallelism is a partition of the k-spaces in $\mathbb{F}_q^n$ into k-spreads. The size of a spread in $\mathcal{G}_q(n, k)$ (or a k-spread in $\mathbb{F}_q^n$) is given by $A_q(n, 2k; k) = \begin{bmatrix} n \\ 1 \end{bmatrix}_q / \begin{bmatrix} k \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q^k-1}$.

Proposition 5.48. *Parallelisms in $\mathcal{G}_q(n, k)$ are known to exist for:*

(a) $k = 2$, $q = 2$, and n even [16, 17];

(b) $k = 2$, all q and $n = 2^m$ for $m \geq 2$ [27];

(c) $k = 2$, $q = 3$, and $n = 6$ [81];

(d) $k = 3$, $q = 2$, and $n = 6$ [130, 197].

See e.g. [78, Section 4.9] for more details. For lower bounds for partial parallelisms we refer to [30, 70, 228].

Example 5.49. *Consider the coset construction for parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (4, 4, 2, 2, 2, 2)$. To this end, let $C_1 = C_2 = \mathcal{G}_q(4, 2)$ and $\mathcal{M}$ be a $(2 \times 2, 2)_q$–*MRD *code. For $s = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q / A_q(4, 4; 2) = q^2 + q + 1$ let $\{C_1^1, \ldots, C_1^s\}$ and $\{C_2^1, \ldots, C_2^s\}$ be parallelisms in $\mathcal{G}_q(4, 2)$. With this we can apply the coset construction in Theorem 5.43 to construct an $(8, 4; 4)_q$–*CDC *$\mathcal{W}_2$. Since $\#\mathcal{M} = q^2$ and $\#C_j^i = q^2 + 1$ for all $j \in \{1, 2\}$ and all $1 \leq i \leq s$ we have*

$$\#\mathcal{W}_2 = q^2 \cdot \left(q^2 + q + 1\right) \cdot \left(q^2 + 1\right)^2 = q^8 + q^7 + 3q^6 + 2q^5 + 3q^4 + q^3 + q^2.$$

For the chosen parameters n_i, k_i, and d_i the other choices are indeed optimal for the coset construction. I.e., starting from Equation (5.6) we note $\#\mathcal{M} \leq A_q^R(k_1 \times (n_2 - k_2), (d_1 + d_2)/2)$ and:

Lemma 5.50. *([126, Corollary 1])*

$$C_q(n_1,n_2,d;k_1,k_2) \le \min\left\{ \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \cdot A_q(n_2,d;k_2), \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q \cdot A_q(n_1,d;k_1) \right\}$$

Via orthogonality the existence question for a 4-partition of $\mathcal{G}_q(6,4)$ translates to the existence question for a parallelism in $\mathcal{G}_q(6,2)$, which is known for $q \in \{2,3\}$, see Proposition 5.48.

Example 5.51. *Consider the coset construction for parameters $(n_1,n_2,d_1,d_2,k_1,k_2) = (6,6,2,2,4,2)$ and assume $q \in \{2,3\}$. To this end, let $C_1 = \mathcal{G}_q(6,4)$, $C_2 = \mathcal{G}_q(6,2)$, and $\mathcal{M}$ be a $(4{\times}4,2)_q$–MRD code. For $s = \left[\begin{smallmatrix}6\\2\end{smallmatrix}\right]_q / A_q(6,4;2) = \left[\begin{smallmatrix}5\\1\end{smallmatrix}\right]_q$ let $\left\{C_2^1,\dots,C_2^s\right\}$ be a parallelism in $\mathcal{G}_q(6,2)$ and set $C_1^i = \left(C_2^i\right)^\perp$ for $1 \le i \le s$. Since $A_q(6,4;2) = q^4+q^2+1$ we have*

$$C_q(6,6,4;4,2) \ge \sum_{i=1}^{s} \#C_1^i \cdot \#C_2^i = \begin{bmatrix} 6 \\ 2 \end{bmatrix}_q \cdot \left(q^4+q^2+1\right),$$

i.e., the upper bound from Lemma 5.50 is attained with equality. Since $\#\mathcal{M} = q^{12}$, the CDC $\mathcal{W}$ resulting from the corresponding coset construction has cardinality 55 996 416 if $q = 2$ and 532 504 413 441 if $q = 3$.

As conjectured in [77], Example 5.49 is just an instance of a more general result:

Proposition 5.52. *([126, Theorem 9]) If parallelisms in $\mathcal{G}_q(n_1,k_1)$, $\mathcal{G}_q(n_2,k_2)$ exist and $d_1 = d_2 = 2$, then we have*

$$C_q(n_1,n_2,4;k_1,k_2) = \min\left\{ \begin{bmatrix} n_1 \\ k_1 \end{bmatrix}_q \cdot A_q(n_2,d;k_2), \begin{bmatrix} n_2 \\ k_2 \end{bmatrix}_q \cdot A_q(n_1,d;k_1) \right\}.$$

Example 5.53. *Consider a CDC $\mathcal{W}$ obtained by the coset construction in Theorem 5.43 with parameters $(n_1,n_2,d_1,d_2,k_1,k_2) = (4,6,2,4,1,3)$. For the components we do not have too many choices. Since $C_1 \subseteq \mathcal{G}_q(4,1)$ we have $s \le \left[\begin{smallmatrix}4\\1\end{smallmatrix}\right]_q = q^3+q^2+q+1$. The fact that $2k_1 < d_1+d_2$ implies $\#C_1^i = 1$ for all $1 \le i \le s$. Similarly, the $(1\times 1,3)_q$–RMC $\mathcal{M}$ has to be of cardinality 1. The ambient code C_2 has to be a $(6,3;4)_q$–CDC and the C_2^i have to be $(6,3;6)_q$–CDCs, i.e. partial spreads, for all $1 \le i \le s$. From Equation (5.6) we conclude*

$$\#\mathcal{W} = \#\mathcal{M} \cdot \sum_{i=1}^{s} \#C_1^i \cdot \#C_2^i = \sum_{i=1}^{s} C_2^i \le \#C_2 \le A_q(6,4;3).$$

For $q = 2$ we have $s \le 15$ and $A_2(6,4;3) = 77$. In [126] a 6-partition with cardinality 15 of a $(6,4;3)_2$–CDC of cardinality 76 was obtained via ILP computations and its optimality was shown, i.e., $C_2(4,6,6;1,3) = 76$. Here indeed the maximum cardinality of $\left[\begin{smallmatrix}4\\1\end{smallmatrix}\right]_2 = 15$ is indeed a limiting factor.

The packing problem of a given ambient CDC into CDCs of larger minimum subspace distance is a hard but interesting algorithmical problem. For ambient CDCs with a specific structure we give preliminary parametric constructions in a moment. First we consider the compatibility with other subcode constructions and the extenability problem.

Directly from the construction we conclude:

Lemma 5.54. *The pivot structure of a* CDC $\mathcal{W}$ *obtained via the coset construction in Theorem 5.43 is a subset of* $\left(\binom{n_1}{k_1}, \binom{n_2}{k_2}\right)$.

So we can directly apply the generalized Echelon–Ferrers construction:

Example 5.55. *(Sequel of Example 5.49)*
Let $\mathcal{W}_2$ *as in Example 5.49, so that its pivot structure is contained in* $\binom{4}{2}, \binom{4}{2}$. *Let* $\mathcal{W}_1$ *be the* $(8,4;4)_q$*–*LMRD *code of cardinality* q^{12} *and* $\mathcal{W}_3 = \left\{\left\langle\begin{pmatrix}\mathbf{0}_{4\times 4} & I_4\end{pmatrix}\right\rangle\right\}$ *be an* $(8,4;4)_q$*–*CDC *of cardinality* 1. *The pivot structures of these two codes are given by the unique vectors* 11110000 *and* 00001111. *Due to* $\mathrm{d}_H(\binom{4}{2}, \binom{4}{2}, \{11110000, 00001111\}) = 4$ *and* $\mathrm{d}_H(11110000, 00001111) \geq 4$ *we have*

$$d_S(\mathcal{W}_1, \mathcal{W}_2), d_S(\mathcal{W}_1, \mathcal{W}_3), d_S(\mathcal{W}_2, \mathcal{W}_3) \geq 4,$$

so that $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$ *is an* $(8,4;4)_q$*–*CDC *of cardinality* $q^{12} + (q^2+q+1)\cdot(q^2+1)^2 + 1$.

We remark that corresponding lower bound

$$A_q(8,4;4) \geq q^{12} + \left(q^2+q+1\right)\cdot\left(q^2+1\right)^2 + 1 \tag{5.8}$$

is still unsurpassed for all $q \geq 3$. For $q = 2$ the corresponding code size of 4797 was surpassed by CDCs of sizes 4801 and 4802, see [39] and [230], respectively.

Exercise 5.56. *Show that* $\left\langle\begin{pmatrix}\mathbf{0}_{4\times 4} & I_4\end{pmatrix}\right\rangle \in \mathcal{G}_q(8,4)$ *is the unique codeword that can be added to the* $(8,4;4)_q$*–*CDC $\mathcal{W}_1 + \mathcal{W}_2$ *in Example 5.55 without violating the minimum subspace distance.*

From Lemma 5.54 and Lemma 5.5.(b)) we conclude:

Lemma 5.57. ***(Construction D + coset construction)***
Let $\mathcal{W}_1$ *be a* CDC *constructed via construction D in Theorem 5.1 with parameters* (n_1, n_2, d, k) *and* $\mathcal{W}_2$ *be a* CDC *constructed via the coset construction in Theorem 5.43 with parameters* $(n_1, n_2, d_1, d_2, k_1, k_2)$*, where* $k_1 + k_2 = k$ *and* $d_1 + d_2 = d$*. If* $k_2 \geq d/2$*, then* $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ *is an* $(n_1+n_2, d; k)_q$*–*CDC *with cardinality* $\#\mathcal{W}_1 + \#\mathcal{W}_2$.

The corresponding matrix description is given by:

C	R
$\mathtt{C}^i$	R↑
$\mathbb{0}$	$\mathtt{C}^i$

Example 5.58. *(Sequel of Example 5.53)*
Let $\mathcal{W}_1$ *be constructed via construction D in Theorem 5.1 with parameters* $(n_1, n_2, d, k) = (4,6,6,4)$ *and* $\mathcal{W}_2$ *be constructed via the coset construction in Theorem 5.43 with parameters* $(n_1, n_2, d_1, d_2, k_1, k_2) = (4,6,2,4,1,3)$*. Since the "linkage condition"* $k_2 \geq d/2$ *in Lemma 5.57 is satisfied,* $\mathcal{W}_1 \cup \mathcal{W}_2$ *is a* $(10,6;4)_q$*–*CDC *of cardinality* $\#\mathcal{W}_1 + \#\mathcal{W}_2$*, so that*

$$A_q(10,6;4) \geq A_q(4,6;4)\cdot A_q^R(4\times 6, 3) + C_q(4,6,6,1,3) = q^{12} + C_q(4,6,6,1,3).$$

For $q = 2$*,* $C_2(4,6,6,1,3) = 76$ *was mentioned Example 5.53, so that* $\#\mathcal{W}_1 + \mathcal{W}_2 = 4172$ *can be attained. In [126] it was observed by an exhaustive computer search that an additional codeword can be added to* $\mathcal{W}$*, so that* $A_2(10,6;4) \geq 4173$*. This is still the best known lower bound.*

We remark that Construction 1 in Theorem 5.33 yields the same lower bound.

Also different subcodes constructed via the coset construction can be combined to yield larger codes. Here the distance analysis in the Hamming metric combined with Lemma 5.54 gives:

Lemma 5.59. ***(Coset construction + coset construction – cf. [126, Lemma 6])*** *Let $\mathcal{W}_1$ be a* CDC *constructed via the coset construction in Theorem 5.43 with parameters $(n_1, n_2, d_1, d_2, k_1, k_2)$ and $\mathcal{W}_2$ be a* CDC *constructed via the coset construction in Theorem 5.43 with parameters $(n_1, n_2, d'_1, d'_2, k'_1, k'_2)$. If $k := k_1 + k_2 = k'_1 + k'_2$, $d := d_1 + d_2 = d'_1 + d'_2$ and $\left|k_1 - k'_1\right| + \left|k_2 - k'_2\right| \ge d$, then $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1 + n_2, d; k)_q$-*CDC *with cardinality $\#\mathcal{W}_1 + \#\mathcal{W}_2$.*

The corresponding matrix description is given by:

$\mathtt{C}^i$	$\mathtt{R}\uparrow$
$\mathtt{0}$	$\mathtt{C}^i$

$\mathtt{C}^i$	$\mathtt{R}\uparrow$
$\mathtt{0}$	$\mathtt{C}^i$

Example 5.60. *Let $\mathcal{W}_2$ and $\mathcal{W}_3$ be constructed via the coset construction in Theorem 5.43 with parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (6, 6, 2, 2, 4, 2)$ and $\left(n_1, n_2, d'_1, d'_2, k'_1, k'_2\right) = (6, 6, 2, 2, 2, 4)$, respectively. Note that the conditions of Lemma 5.59 for the combination of $\mathcal{W}_2$ and $\mathcal{W}_3$ are satisfied and $C_q(6, 6, 4; 4, 2) = C_q(6, 6, 4; 2, 4)$. The maximum size of the* RMC *for $(6, 6, 4, 4, 2)$ is $A^R_q(4 \times 4, 2) = q^{12}$ and $A^R_q(2 \times 2, 2) = q^2$ for $(6, 6, 4, 2, 4)$. Since the conditions of Lemma 5.57 are satisfied for $k_2 \in \{2, 4\}$, we can choose $\mathcal{W}_1$ as the $(6 \times 6, 4)_q$–*LMRD *code of cardinality q^{30}, so that considering the* CDC *$\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$ yields*

$$A_q(12, 4; 6) \ge q^{30} + C_q(6, 6, 4; 4, 2) \cdot \left(q^{12} + q^2\right).$$

For $q \in \{2, 3\}$ we can use the exact value of $C_q(6, 6, 4; 4, 2)$ determined in Example 5.51 to conclude

$$A_2(12, 4; 6) \ge 1\,129\,792\,924 \quad \textit{and} \quad A_3(12, 4; 6) \ge 206\,423\,645\,526\,099.$$

Mirrored coset construction

Of course one can easily adjust the coset construction in Theorem 5.43 so that its matrix description is given by

$\mathtt{C}^i$	$\mathtt{0}$
$\mathtt{R}\uparrow$	$\mathtt{C}^i$

instead of

$\mathtt{C}^i$	$\mathtt{R}\uparrow$
$\mathtt{0}$	$\mathtt{C}^i$

and call it mirrored coset construction. In Lemma 5.57 we then have to replace the condition $k_2 \ge d/2$ by $k_2 - \mathrm{rk}(M) \ge d/2$ for all $M \in \mathcal{M}$ if we use a subcode obtained by the mirrored coset construction and $\mathcal{M}$ is its utilized RMC.

In Example 5.60 the advantage of choosing the mirrored coset construction for $\mathcal{W}_3$ with parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (6, 6, 2, 2, 2, 4)$ is that we can choose a RMC of size $A_q^R(4 \times 4, 2; \le 2) > A_q^R(2 \times 2, 2)$. However, in a modified version of Lemma 5.59 considering the combination of a subcode from the coset construction with a subcode from the mirrored coset construction we have to replace the condition $\left|k_1 - k_1'\right| + \left|k_2 - k_2'\right| \ge d$. The following example shows that the ranks of the elements in the involved RMCs have to be taken into account. The generator matrix

$$H = \begin{pmatrix} 100000 & 000000 \\ 010000 & 000000 \\ 001000 & 001000 \\ 000100 & 000100 \\ 000000 & 100000 \\ 000000 & 010000 \end{pmatrix} = \begin{pmatrix} G_1 & M\!\uparrow_{G_2} \\ \mathbf{0}_{2\times 6} & G_2 \end{pmatrix}$$

with $G_1 \in \mathbb{F}_q^{4\times 6}$, $\operatorname{rk}(G_1) = 4$, $G_2 \in \mathbb{F}_q^{2\times 6}$, $\operatorname{rk}(G_2) = 2$, $M \in \mathbb{F}_q^{4\times 4}$, and $\operatorname{rk}(M) \le 2$ fits into the shape of the coset construction with parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (6, 6, 2, 2, 4, 2)$. Similarly, the generator matrix

$$H' = \begin{pmatrix} 100000 & 000000 \\ 010000 & 000000 \\ 000000 & 100000 \\ 000000 & 010000 \\ 001000 & 001000 \\ 000100 & 000100 \end{pmatrix} = \begin{pmatrix} G_1' & \mathbf{0}_{2\times 6} \\ M'\!\uparrow_{G_1'} & G_2' \end{pmatrix}$$

with $G_1' \in \mathbb{F}_q^{2\times 6}$, $\operatorname{rk}(G_1') = 2$, $G_2' \in \mathbb{F}_q^{4\times 6}$, $\operatorname{rk}(G_2') = 4$, $M' \in \mathbb{F}_q^{4\times 4}$, and $\operatorname{rk}(M') \le 2$ fits into the shape of the mirrored coset construction with parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (6, 6, 2, 2, 2, 4)$. However, as H' arises from H by swapping row three with row five and row four with row six, we have $\langle H\rangle = \langle H'\rangle$, i.e., $d_S(\langle H\rangle, \langle H'\rangle) = 0$.

While it is possible to suitably modify the condition in Lemma 5.59 we are not aware of a construction of a CDC leading to the best known lower bound that involves both a subcode obtained from the coset construction and a subcode obtained from the mirrored coset construction. So, we refrain from going into more details.

If we want to combine the generalized linkage construction with the coset construction, then we eventually have the restrict the maximum occurring ranks in the RMC of the coset part, as it is the case if we combine construction D with the mirrored coset construction.

Lemma 5.61. ***(Generalized linkage construction + coset construction)***
Let $\mathcal{W}_1$ be a CDC constructed via the generalized linkage construction in Theorem 5.16 with parameters (n_1, n_2, d, k) and $\mathcal{W}_2$ be a CDC constructed via the coset construction in Theorem 5.43 with parameters $(n_1, n_2, d_1, d_2, k_1, k_2)$ and RMC $\mathcal{M}$. If $k_1 + k_2 = k$, $d_1 + d_2 = d$, $k_2 \ge d/2$ and $k_1 - \operatorname{rk}(M) \ge d/2$ for all $M \in \mathcal{M}$, then $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1+n_2, d; k)_q$–CDC with cardinality $\mathcal{W}_1 + \mathcal{W}_2$.

Proof. Let E_1 and E_2 be as in Lemma 5.18 for $\mathcal{W}_1$. For each codeword $U \in \mathcal{W}_2$ we have $\dim(U \cap E_2) \geq k_2 \geq d/2$ and $\dim(U \cap E_1) \geq k_1 - \mathrm{rk}(M) \geq d/2$, where $M \in \mathcal{M}$ is the matrix used in the generator matrix of U. □

The corresponding matrix description is given by:

C	R
R	C
C^i	R↑
0	C^i

Example 5.62. *Let $\mathcal{W}_1$ arise from the generalized linkage construction with parameters $(n_1, n_2, d, k) = (5, 5, 4, 5)$, so that we can assume $\#\mathcal{W}_1 = q^{20} + A_q^R(5 \times 5, 2; \leq 3)$. Let $\mathcal{W}_2$ arise from the coset construction with parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (5, 5, 2, 2, 3, 2)$, so that we can assume $\#\mathcal{W}_2 = A_q^R(3 \times 3, 2; \leq 1) \cdot C_q(5, 5, 4, 3, 2)$. Due to Lemma 5.61 we can consider the* CDC *$\mathcal{W}_1 \cup \mathcal{W}_2$ to conclude*

$$A_q(10, 4; 5) \geq q^{20} + A_q^R(5 \times 5, 2; \leq 3) + A_q^R(3 \times 3, 2; \leq 1) \cdot C_q(5, 5, 4, 3, 2),$$

which can be refined to

$$A_q(10, 4; 5) \geq q^{20} + A_q^R(5 \times 5, 2; \leq 3) + \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \cdot C_q(5, 5, 4, 3, 2)$$

using Proposition 3.15. For a lower bound for $A_q^R(5 \times 5, 2; \leq 3)$ we refer to Example 3.23 and for a lower bound for $C_q(5, 5, 4, 3, 2)$ we refer to Proposition 5.68 and Exercise 5.45 noting the computer result $C_q(5, 5, 4, 3, 2) \geq 1313$ mentioned in Subsection 5.4. Plugging in these lower bounds gives

$$\begin{aligned} A_q(10, 4; 5) \quad \geq \quad & q^{20} + q^{16} + q^{15} + 2q^{14} + q^{13} - q^{11} - 2q^{10} - q^9 + 2q^8 \\ & +5q^7 + 4q^6 + 7q^5 + 11q^4 + 15q^3 + 12q^2 + 6q + 2 \end{aligned} \tag{5.9}$$

and

$$A_2(10, 4; 5) \geq 1048576 + 130696 + 7 \cdot 1313 = 1\,188\,463. \tag{5.10}$$

Flawed bound in the literature

The construction for a lower bound for $A_q(10, 4; 5)$ from [47] was flawed. Applying Lemma 5.59 with $(k_1, k_2) = (3, 2)$ and $(k_1', k_2') = (2, 3)$ is possible for minimum subspace distance 2 only. However, the lower bound from Example 5.62 is better anyway.

Example 5.63. *Consider the construction from Example 5.60 again, e.g. we choose the parameters $(n_1, n_2, d, k) = (6, 6, 4, 6)$. This time we let $\mathcal{W}_1$ arise from the generalized linkage construction, so that we can assume $\#\mathcal{W}_1 = q^{30} + A_q^R(6 \times 6, 2; \leq 4)$. For the* CDCs *$\mathcal{W}_2$ and $\mathcal{W}_3$, obtained from the coset construction, we have to adjust the corresponding* RMC *$\mathcal{M}$ so that the condition $k_1 - \mathrm{rk}(M) \geq d/2$ from Lemma 5.61 is satisfied for all $M \in \mathcal{M}$. For $\mathcal{W}_2$ with parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (6, 6, 2, 2, 4, 2)$ we can choose $\mathcal{M}$ as a $(4 \times 4, 2; \leq 2)_q$–*RMC*. For $\mathcal{W}_3$*

with parameters $(n_1, n_2, d_1, d_2, k_1, k_2) = (6, 6, 2, 2, 2, 4)$ *we have to use a* $(2 \times 2, 2; \leq 0)_q$*–*RMC*, i.e., we can just use the one-element* RMC *consisting of* $\mathbf{0}_{2\times 2}$*. Considering the* $(12, 4; 6)_q$*–*CDC $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$ *yields*

$$A_q(12, 4; 6) \geq q^{30} + A_q^R(6 \times 6, 2; \leq 4) + C_q(6, 6, 4; 4, 2) \cdot \left(A_q^R(4 \times 4, 2; \leq 2) + 1\right).$$

Using Lemma 3.8 and Example 5.51 we conclude

$$A_2(12, 4; 6) \geq 1\,212\,418\,496 \;+\; 7\,204\,617 = 1\,219\,623\,113$$

and

$$A_3(12, 4; 6) \geq 209\,943\,770\,460\,426 \;+\; 10\,422\,814\,402 = 209\,954\,193\,274\,828.$$

We remark that the stated construction constitutes the best known lower bound for $(12, 4; 6)_q$–CDCs where $q \in \{2, 3\}$. For $q > 3$ the existence a parallelism in $\mathcal{G}_q(6, 2)$ is unknown, so that we cannot apply the construction in Example 5.60 for $C_q(6, 6, 4; 4, 2)$ directly. In the subsequent Subsection 5.4 we study general constructions for d-packings of CDCs and take up the construction in Example 5.63 again.

Exercise 5.64. *Compute a parametric lower bound for* $A_q(12, 4; 6)$*, where* $q \geq 4$*, based on the construction in Example 5.63 and the parametric lower bound for* $C_q(6, 6, 4; 4, 2)$ *determined in Subsection 5.4.*

What are sufficient conditions for a symmetric version of the coset construction?

Given the nice symmetry of the matrix description of the generalized linkage construction, the question arises if a generalized version of the coset construction with matrix description

C^i	$\mathrm{R}\!\uparrow$
$\mathrm{R}\!\uparrow$	C^i

exists?

The following example for subspace distance $d = 4$ shows that we need further, possibly quite restrictive, conditions at the very least. The generator matrix

$$H = \begin{pmatrix} 1000 & 0001 \\ 0100 & 0000 \\ 0010 & 0100 \\ 0000 & 0010 \end{pmatrix} = \begin{pmatrix} G_1 & M_1\!\uparrow_{G_2} \\ M_2\!\uparrow_{G_1} & G_2 \end{pmatrix}$$

with $G_1 \in \mathbb{F}_q^{2\times 4}$, $\operatorname{rk}(G_1) = 2$, $G_2 \in \mathbb{F}_q^{2\times 4}$, $\operatorname{rk}(G_2) = 2$, $M_1 \in \mathbb{F}_q^{2\times 2}$, $\operatorname{rk}(M_1) \leq 1$, $M_2 \in \mathbb{F}_q^{2\times 2}$, and $\operatorname{rk}(M_2) \leq 1$ as well as the generator matrix

$$H' = \begin{pmatrix} 0100 & 0000 \\ 0010 & 1000 \\ 0000 & 0100 \\ 0001 & 0010 \end{pmatrix} = \begin{pmatrix} G'_1 & M'_1\!\uparrow_{G'_2} \\ M'_2\!\uparrow_{G'_1} & G'_2 \end{pmatrix}$$

with $G'_1 \in \mathbb{F}_q^{2\times 4}$, $\operatorname{rk}(G'_1) = 2$, $G'_2 \in \mathbb{F}_q^{2\times 4}$, $\operatorname{rk}(G'_2) = 2$, $M'_1 \in \mathbb{F}_q^{2\times 2}$, $\operatorname{rk}(M'_1) \leq 1$, $M'_2 \in \mathbb{F}_q^{2\times 2}$, and $\operatorname{rk}(M'_2) \leq 1$ fit into the shape of the desired matrix description. Setting $U_1 = \langle G_1 \rangle$,

$U_2 = \langle G_2 \rangle$, $U_1' = \langle G_1' \rangle$, $U_2' = \langle G_2' \rangle$ we observe $d_S(U_1, U_1') = 2$ and $d_S(U_2, U_2') = 2$, so that $d_S(U_1, U_1') + d_S(U_2, U_2') = 4 \geq d$. For

$$M_1 = \begin{pmatrix} 01 \\ 00 \end{pmatrix}, \quad M_1' = \begin{pmatrix} 00 \\ 10 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 10 \\ 00 \end{pmatrix}, \text{ and } M_2' = \begin{pmatrix} 00 \\ 01 \end{pmatrix}$$

we have $d_R(M_1, M_1') = 2 \geq d/2$ and $d_R(M_2, M_2') = 2 \geq d/2$ (using the natural choice for $\uparrow$). However, both $W := \langle H \rangle$ and $W' := \langle H' \rangle$ contain the 3-space generated by

$$\begin{pmatrix} 0100 & 0000 \\ 0010 & 1000 \\ 0000 & 0100 \end{pmatrix}$$

as a subspace, so that $d_S(W, W') \leq 2 < d$. Restricting the ranks of M_1, M_1' to be smaller than 1 or the ranks of M_2, M_2' to be smaller than 1, we end up with the original coset or the mirrored coset construction, respectively.

We leave it as an open research problem to generalize the coset construction and refer to Theorem 5.74 for a possible first step into that direction.

5.4 Constructions for d-packings of CDCs and RMCs

As already mentioned, we can separate the problem of the choice of the RMC in the coset construction and the problem of a coset construction with matrix description

$\mathtt{C}^i$	$\mathtt{0}$
$\mathtt{0}$	$\mathtt{C}^i$

where the parts $\mathtt{C}^i$ correspond to d-packings of CDCs. If parallelisms are not available or the desired minimum subspace distance is larger than 4 then we need different techniques for the construction of the needed d-packings.

Without the relation to the coset construction the following result was obtain in [47] in the context of the extension problem for the generalized linkage construction.

Proposition 5.65. *(Cf. [47, Corollary 4.5 with $l = 2$])*

$$C_q(n_1, n_2, d; k_1, k_2) \geq \min\{\alpha_1, \alpha_2\} \cdot \prod_{i=1}^{2} A_q^R(k_i \times (n_i - k_i), d/2),$$

where $\alpha_i = A_q^R(k_i \times (n_i - k_i), d_i/2) / A_q^R(k_i \times (n_i - k_i), d/2)$ for $i = 1, 2$ and $d_1, d_2 \in 2\mathbb{N}$ with $d_1 + d_2 = d$.

The underlying idea can be briefly indicated by the matrix description

$\mathtt{I}$	$\mathtt{R}^i$	$\mathtt{0}$	$\mathtt{0}$
$\mathtt{0}$	$\mathtt{0}$	$\mathtt{I}$	$\mathtt{R}^i$

and Lemma 3.16 mimicking parallelisms for LMRD codes, cf. [78, Section 4.9].

Lemma 5.66. ***(Parallel*** `FDRM` ***codes – C.f. [162, Lemma 2.5], [47, proof Corollary 4.5])***
Let $\mathcal{F}$ be a Ferrers diagram and $\mathcal{M}$ be a corresponding additive `FDRM` *code with minimum rank distance d. If $\mathcal{M}$ is a subcode of a an additive* `FDRM` *code $\mathcal{M}'$ with minimum rank distance $d' < d$ and Ferrers diagram $\mathcal{F}$, then there exist* `FDRM` *codes $\mathcal{M}_i$ with Ferrers diagram $\mathcal{F}$ for $1 \le i \le \alpha := \#\mathcal{M}'/\#\mathcal{M}$ satisfying*

(1) $d_R(\mathcal{M}_i) \ge d$ for all $1 \le i \le \alpha$;

(2) $d_R(\mathcal{M}_i, \mathcal{M}_j) \ge d'$ for all $1 \le i < j \le \alpha$; and

(3) $\mathcal{M}_1, \dots, \mathcal{M}_\alpha$ is a partition of $\mathcal{M}'$.

Proof. For each $M' \in \mathcal{M}'$ the code $\mathcal{M}+M' := \{M+M' \,:\, M \in \mathcal{M}\}$ is an `FDRM` code with Ferrers diagram $\mathcal{F}$ and minimum rank distance d. For $M', M'' \in \mathcal{M}'$ we have $M' + \mathcal{M} = M'' + \mathcal{M}$ iff $M' - M'' \in \mathcal{M}$ and $M' + \mathcal{M} \cap M'' + \mathcal{M} = \emptyset$ otherwise. Now let $\mathcal{M}_1, \dots, \mathcal{M}_\alpha$ be the $\alpha = \#\mathcal{M}'/\#\mathcal{M}$ different codes $M + \mathcal{M}$, which are cosets of $\mathcal{M}$ in $\mathcal{M}'$ and partition $\mathcal{M}'$. Since all elements of $\mathcal{M}_i$ and $\mathcal{M}_j$ are different elements of $\mathcal{M}'$ we have $d_{\mathrm{R}}(\mathcal{M}_i, \mathcal{M}_j) \ge d'$ for all $1 \le i < j \le \alpha$. □

Choosing $\mathcal{F}$ as $a \times b$ rectangular Ferrers diagram, we end up with [162, Lemma 2.5], see also Exercise 3.17. Note that we have to choose Delsarte–Gabidulin (or some other specific class of) `MRD` codes in order to ensure that an `MRD` code for minimum rank distance d contains an `MRD` code with minimum rank distance $d+1$ as a subcode. In the proof of [47, Corollary 4.5] this lemma is indirectly applied with $a = a_i$ and $b = n_i - a_i$. Note that for minimum rank distance $\delta = 2$ the upper bound from [76, Theorem 1], cf. Theorem 5.30, can always be attained by linear rank metric codes. Moreover, the only choice for δ' then is $\delta' = 1$ and $\mathcal{M}'$ consists of all matrices with Ferrers diagram $\mathcal{F}$. Thus, $\mathcal{M}'$ is automatically linear and contains $\mathcal{M}$ as a subcode.

Research problem

Study the existence of “large” linear `FDRM` codes that contain `FDRM` codes of larger minimum rank distance as a subcode.

A first approach might be to start from a linear Delsarte–Gabidulin `MRD` code and to consider linear subcodes going in line with the support restrictions of a given Ferrers diagram $\mathcal{F}$.

Corollary 5.67.

$$C_q(5,5,4;2,2) \ge q^9 + q^7 + q^6 + 7q^5 + 5q^4 + 3q^3 + 2q^2 + q + 1$$

I.e., we have $C_2(5,5,4;2,2) \ge 1043$. Proposition 5.65 yields $C_q(5,5,4;2,2) \ge q^9$, i.e., $C_2(5,5,4;2,2) \ge 512$. Proposition 5.73 gives $C_q(5,5,4;2,2) \ge q^9+q^7+q^6+q^5+q^4+q^3+2q^2+q+1$, i.e., $C_2(5,5,4;2,2) \ge 771$. In [158] the lower bound $C_2(5,5,4;2,2) \ge 1313$ was shown by a heuristic computer search. By an easy argument the upper bound $C_2(5,5,4;2,2) \le 1381$ was shown.

We can also use more geometric ideas.

Proposition 5.68.

$$C_q(5,5,4;2,2) \ge q^9 + q^7 + 2q^6 + q^5 - q^4 + 4q^3 + 6q^2 + 4q + 2$$

pivot vector	size $m(q,\mathcal{F},2)$	# of cosets $m(q,\mathcal{F},1)/m(q,\mathcal{F},2)$
11000	q^3	q^3
10100	q^2	q^3
10010	q	q^3
10001	1	q^3
01100	q^2	q^2
01010	q	q^2
01001	1	q^2
00110	1	q^2
00101	1	q
00011	1	1

Table 5.1: Data for Lemma 5.66 with $\mathcal{F} \in \mathcal{G}_1(5,2)$.

skeleton code	size	# of used cosets
$\{11000, 00110\}$	q^3+1	q^2
$\{11000, 00101\}$	q^3+1	q
$\{11000, 00011\}$	q^3+1	1
$\{11000\}$	q^3	q^3-q^2-q-1
$\{10100, 01010\}$	q^2+q	q^2
$\{10100, 01001\}$	q^2+1	q^2
$\{10100\}$	q^2	q^3-2q^2
$\{01100, 10010\}$	q^2+q	q^2
$\{10010\}$	q	q^3-q^2
$\{10001\}$	1	q^3

Table 5.2: 4-packing scheme for $\mathcal{G}_q(5,2)$.

Proof. Let π and π' be two planes in $\mathbb{F}_q^5$ intersecting in a point P. Let C be an `LMRD` code disjoint to π that can be partitioned into q^3 partial line spreads C_i of cardinality q^3. Similarly, let C' be an `LMRD` code disjoint to π' that can be partitioned into q^3 partial line spreads C_i of cardinality q^3. For $1 \le i \le \begin{bmatrix}3\\2\end{bmatrix}_q = q^2+q+1$ we add one of the $\begin{bmatrix}3\\2\end{bmatrix}_q$ different lines contained in π to C_i. To ensure that no line occurs twice we only keep those lines in C_i' that intersect π in exactly a point. Let us now determine the resulting sizes $\#C_i'$. To this end, let $\mathcal{L}$ be the set of the q^2 lines in π that do not contain P. Since the elements of $\mathcal{L}$ are pairwise intersecting in a point, there are exactly q^2 partial line spreads C_i' that contain one element from $\mathcal{L}$. For these, exactly $\begin{bmatrix}3\\1\end{bmatrix}_q - \begin{bmatrix}2\\1\end{bmatrix}_q - 1 = q^2-1$ elements intersect in exactly one point. For the other q^3-q^2 partial line spreads, $\begin{bmatrix}3\\1\end{bmatrix}_q - 1 = q^2+q$ of its elements intersect π in exactly a point. Since π' contains $\begin{bmatrix}2\\1\end{bmatrix}_q = q+1$ lines intersecting π in a point, we can add a further line to $q+1$ of the latter partial line spreads C_i' each. This gives

$$\begin{aligned}
&\sum_{i=1}^{q^3} (\#C_i)^2 + \sum_{i=1}^{q^3} (\#C_i')^2 \\
=\; & \left(q^2+q+1\right)\cdot\left(q^3+1\right)^2 + \left(q^3-q^2-q-1\right)\cdot\left(q^3\right)^2 \\
& + q^2\cdot\left(q^2-1\right)^2 + (q+1)\cdot\left(q^2+q+1\right)^2 + \left(q^3-q^2-q-1\right)\cdot\left(q^2+q\right)^2 \\
=\; & q^9+q^7+2q^6+q^5-q^4+4q^3+6q^2+4q+2.
\end{aligned}$$

□

Exercise 5.69. *Improve the lower bound of Proposition 5.68 by taking the unused lines into account. Conclude a similar bound assuming that the planes π and π' intersect in a line.*

Corollary 5.70.

$$C_q(6,6,4;2,2) \ge q^{12}+q^{10}+q^9+7q^8+5q^7+6q^6+5q^5+4q^4+2q^3+7q^2+q+1$$

I.e., we have $C_2(6,6,4;2,2) \ge 8719$. Proposition 5.65 yields $C_q(6,6,4;2,2) \ge q^{12}$, i.e., $C_2(6,6,4;2,2) \ge 4096$. The upper bound from Lemma 5.50 is given by

$$q^{12}+q^{11}+3q^{10}+3q^9+6q^8+5q^7+7q^6+5q^5+6q^4+3q^3+3q^2+q+1,$$

i.e., $C_2(6,6,4;2,2) \le 13671$. Due to he existence of parallelisms in $\mathcal{G}_q(6,2)$ for $q \in \{2,3\}$ the upper bound is indeed attained. So our packing constructions are very far from being optimal. (For $q=2$ the polynomial in Proposition 5.72 would result in 8839.)

Exercise 5.71. *Improve the stated packing scheme for $\mathcal{G}_q(6,2)$ for $q>2$.*

Proposition 5.72. *For $q \ge 3$ we have*

$$C_q(6,6,4;2,2) \ge q^{12}+q^{10}+q^9+7q^8+5q^7+8q^6+4q^5+6q^4+3q^3+3q^2+q+1.$$

pivot vector	size $m(q,\mathcal{F},2)$	# of cosets $m(q,\mathcal{F},1)/m(q,\mathcal{F},2)$
110000	q^4	q^4
101000	q^3	q^4
100100	q^2	q^4
100010	q	q^4
100001	1	q^4
011000	q^3	q^3
010100	q^2	q^3
010010	q	q^3
010001	1	q^3
001100	q^2	q^2
001010	q	q^2
001001	1	q^2
000110	1	q^2
000101	1	q
000011	1	1

Table 5.3: Data for Lemma 5.66 with $\mathcal{F} \in \mathcal{G}_1(6,2)$.

skeleton code	size	# of used cosets
$\{110000, 001100, 000011\}$	q^4+q^2+1	1
$\{110000, 001100\}$	q^4+q^2	q^2-1
$\{110000, 001010, 000101\}$	q^4+q+1	q
$\{110000, 001010\}$	q^4+q	q^2-q
$\{110000, 000110, 001001\}$	q^4+q+1	q
$\{110000, 001001\}$	q^4+1	q^2-q
$\{110000\}$	q^4	q^4-3q^2
$\{101000, 010100\}$	q^3+q^2	q^3
$\{101000, 010010\}$	q^3+q	q^3
$\{101000\}$	q^3	q^4-2q^3
$\{011000, 100100\}$	q^3+q^2	q^3
$\{100100, 010001\}$	q^2+1	q^3
$\{100100\}$	q^2	q^4-2q^3
$\{100010\}$	q	q^4
$\{100001\}$	1	q^4

Table 5.4: 4-packing scheme for $\mathcal{G}_q(6,2)$.

Proof. Let S be the solid with pivot vector 001111 in $\mathbb{F}_q^6$ and C be an LMRD code disjoint to S that can be partitioned into q^4 partial line spreads C_i of cardinality q^4. Since $S \cong \mathbb{F}_q^4$ there exists a parallelism of S, so that we can add q^2+1 additional lines to q^2+q+1 of the partial line spreads C_i. So, we have

$$\begin{aligned}\sum_{i=1}^{q^4} (\#C_i)^2 &= \left(q^2+q+1\right)\cdot\left(q^4+q^2+1\right)^2+\left(q^4-q^2-q-1\right)\cdot\left(q^4\right)^2\\ &= q^{12}+2q^8+2q^7+5q^6+3q^5+5q^4+2q^3+3q^2+q+1.\end{aligned}$$

The lines used so far, all lines being either disjoint to S or contained in S, i.e., the have pivot vector 110000 or their pivot vector is contained in $\left(\binom{2}{0},\binom{4}{2}\right)$. For the remaining pivot vectors we consider the packing scheme

skeleton code	size	# of used cosets
$\{101000, 010100\}$	q^3+q^2	q^3
$\{101000, 010010\}$	q^3+q	q^3
$\{101000, 010001\}$	q^3+1	q^3
$\{100100, 011000\}$	q^3+q^2	q^3
$\{101000\}$	q^3	q^4-3q^3
$\{100100\}$	q^2	q^4-q^3
$\{100010\}$	q	q^4
$\{100001\}$	1	q^4

yielding an additional contribution of

$$q^{10}+q^9+5q^8+3q^7+3q^6+q^5+q^4+q^3.$$

□

Proposition 5.73. *([158, Proposition 3.5])*

$$C_q(n,n,4;k,k) \geq \sum_{v\in\mathcal{G}_1(n,k)} A_q(n,1;k;v)\cdot A_q(n,2;k;v)$$

5.5 Inserting constructions

We have seen in Subsection 5.1 that the generalized linkage construction yields CDCs with competitive cardinalities. In Lemma 5.61 we have summarized sufficient conditions for the combination with subcodes obtained via the coset construction. In these subsection we want to study further variants of subcodes that can be used to improve the generalized linkage construction. In e.g. [161, 184, 185] the authors speak of inserting constructions cf. also [111].

Packings of RMCs constructed in Subsection 5.4 can be exploited as follows:

Theorem 5.74. ***(Block inserting construction I – [161, Theorem 4])***
Let C_1 be an $(n_1,d;k)_q$–CDC, C_2 be an $(n_3,d;k)_q$–CDC, $\mathcal{M}_3$ be a $(k_1\times n_4, d/2; k_1-d/2)_q$–RMC, $\mathcal{M}_4$ be a $(k_2\times n_2, d/2; k_2-d/2)_q$–RMC, $\mathcal{M}_1$ be a $(k_1\times n_2, d_1/2)_q$–RMC, and $\mathcal{M}_2$ be a

$(k_2 \times n_3, d_2/2)_q$–RMC, where $d_1 + d_2 = d$. Let $\mathcal{M}_1^1, \ldots, \mathcal{M}_1^s$ and $\mathcal{M}_2^1, \ldots, \mathcal{M}_2^s$ be $\frac{t}{2}$-packings of cardinality s of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. With this let

$$\left\{ \begin{pmatrix} G_1 & M_1 & \mathbf{0}_{k_1 \times n_3} & M_3 \\ \mathbf{0}_{k_2 \times n_1} & M_4 & G_2 & M_2 \end{pmatrix} \; : \; G_1 \in \mathcal{G}_1, M_1 \in \mathcal{M}_1^i, M_3 \in \mathcal{M}_3, M_4 \in \mathcal{M}_4, G_2 \in \mathcal{G}_2, M_2 \in \mathcal{M}_2^i \right\}$$

be a generating set of a subcode $\mathcal{W}^i$ for $1 \le i \le s$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are generating sets of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then, $\mathcal{W} = \cup_{i=1}^s \mathcal{W}^i$ is an $(n_1 + n_2 + n_3 + n_4, d; k_1 + k_2)_q$–CDC with cardinality

$$\#\mathcal{W} = \sum_{i=1}^{s} \#\mathcal{W}^i = \#\mathcal{C}_1 \cdot \#\mathcal{C}_2 \cdot \#\mathcal{M}_3 \cdot \#\mathcal{M}_4 \cdot \sum_{i=1}^{s} \mathcal{M}_1^i \cdot \#\mathcal{M}_2^i.$$

Proof. Let

$$H = \begin{pmatrix} G_1 & M_1 & \mathbf{0} & M_3 \\ \mathbf{0} & M_4 & G_2 & M_2 \end{pmatrix}$$

be the generator matrix of an arbitrary codeword $W \in \mathcal{W}$. Since $k_1 + k_2 \ge \operatorname{rk}(H) \ge \operatorname{rk}(G_1) + \operatorname{rk}(G_2) = k_1 + k_2$, every codeword is a $(k_1 + k_2)$-space.

Let

$$H' = \begin{pmatrix} G_1' & M_1' & \mathbf{0} & M_3' \\ \mathbf{0} & M_4' & G_2' & M_2' \end{pmatrix}$$

be the generator matrix of another codeword $W' \in \mathcal{W}$,

$$R := \operatorname{rk}\left(\begin{pmatrix} G_1 & M_1 & \mathbf{0} & M_3 \\ \mathbf{0} & M_4 & G_2 & M_2 \\ G_1' & M_1' & \mathbf{0} & M_3' \\ \mathbf{0} & M_4' & G_2' & M_2' \end{pmatrix}\right) = \operatorname{rk}\left(\begin{pmatrix} G_1 & \mathbf{0} & M_1 & M_3 \\ G_1' - G_1 & \mathbf{0} & M_1' - M_1 & M_3' - M_3 \\ \mathbf{0} & G_2 & M_4 & M_2 \\ \mathbf{0} & G_2' - G_2 & M_4' - M_4 & M_2' - M_2 \end{pmatrix}\right),$$

and $U_1 := \langle G_1 \rangle$, $U_2 := \langle G_2 \rangle$, $U_1' := \langle G_1' \rangle$, $U_2' := \langle G_2' \rangle$.

If $G_1 \ne G_1'$ or $G_2 \ne G_2'$, then we have $U_1 \ne U_1'$ or $U_2 \ne U_2'$, so that

$$\begin{aligned} d_S(W, W') &= 2(R - k_1 - k_2) \ge 2 \cdot \operatorname{rk}\left(\begin{pmatrix} G_1 & \mathbf{0} \\ G_1' - G_1 & \mathbf{0} \\ \mathbf{0} & G_2 \\ \mathbf{0} & G_2' - G_2 \end{pmatrix}\right) - 2k_1 - 2k_2 \\ &= d_S(U_1, U_1') + d_S(U_2, U_2') \ge d. \end{aligned}$$

If $G_1 = G_1'$ and $G_2 = G_2'$, then we have

$$R = \operatorname{rk}\left(\begin{pmatrix} G_1 & \mathbf{0} & M_1 & M_3 \\ \mathbf{0} & G_2 & M_4 & M_2 \\ \mathbf{0} & \mathbf{0} & M_1' - M_1 & M_3' - M_3 \\ \mathbf{0} & \mathbf{0} & M_4' - M_4 & M_2' - M_2 \end{pmatrix}\right) = k_1 + k_2 + \operatorname{rk}\Bigg(\overbrace{\begin{pmatrix} M_1' - M_1 & M_3' - M_3 \\ M_4' - M_4 & M_2' - M_2 \end{pmatrix}}^{\widetilde{M} :=}\Bigg),$$

so that it suffices to show $\mathrm{rk}(\widetilde{M}) \ge d/2$ in order to deduce $d_S(W, W') \ge d$.

If $M_3 \neq M_3'$ or $M_4 \neq M_4'$, then we have $\mathrm{rk}(\widetilde{M}) \ge \mathrm{rk}(M_3 - M_3') + \mathrm{rk}(M_4 - M_4') \ge \min\{d_R(\mathcal{M}_3), d_R(\mathcal{M}_4)\} \ge d/2$.

If $M_3 = M_3'$ and $M_4 = M_4'$, then we have $\mathrm{rk}(\widetilde{M}) = \mathrm{rk}(M_1 - M_1') + \mathrm{rk}(M_2 - M_2') = d_R(M_1, M_1') + d_R(M_2, M_2')$. If $M_1 = M_1'$, then their exists an index $1 \le i \le s$ with $M_2, M_2' \in \mathcal{M}_2^i$ and we have $M_2 \neq M_2'$, so that $\mathrm{rk}(\widetilde{M}) \ge d_R(M_2, M_2') \ge d_R(\mathcal{M}_2^i) \ge d/2$. Similarly, if $M_2 = M_2'$, then their exists an index $1 \le i \le s$ with $M_1, M_1' \in \mathcal{M}_1^i$ and we have $M_1 \neq M_1'$, so that $\mathrm{rk}(\widetilde{M}) \ge d_R(M_1, M_1') \ge d_R(\mathcal{M}_1^i) \ge d/2$. If $M_1 \neq M_1'$ and $M_2 \neq M_2'$, then we have $\mathrm{rk}(\widetilde{M}) \ge d_R(M_1, M_1') + d_R(M_2, M_2') \ge d_R(\mathcal{M}_1) + d_R(\mathcal{M}_2) \ge d_1/2 + d_2/2 = d/2$. □

The matrix description of the block inserting construction I is given by

C	R^i	0	R
0	R	C	R^i

Corollary 5.75. *Let $\mathcal{W}$ be a CDC constructed via the block inserting construction in Theorem 5.74 with parameters $(n_1, n_2, n_3, n_4, d, k_1, k_2)$, where d_1, d_2 with $d_1+d_2 = d$ are arbitrary, of maximum possible cardinality. Then, we have*

$$\begin{aligned} \#\mathcal{W} \quad \ge \quad & A_q(n_1, d; k_1) \cdot A_q(n_3, d; k_2) \cdot A_q^R(k_1 \times n_4, \tfrac{d}{2}; \le k_1 - \tfrac{d}{2}) \cdot \\ & A_q^R(k_2 \times n_2, \tfrac{d}{2}; k_2 - \tfrac{d}{2}) \cdot A_q^R(k_1 \times n_2, d/2) \cdot A_q^R(k_2 \times n_4, d/2) \cdot \alpha, \end{aligned}$$

where

$$\alpha = \max_{d_1, d_2 \,:\, d_1 + d_2 = d} \min \left\{ \frac{A_q^R(k_1 \times n_2, d_1/2)}{A_q^R(k_1 \times n_2, d/2)}, \frac{A_q^R(k_2 \times n_4, d_2/2)}{A_q^R(k_2 \times n_4, d/2)} \right\}.$$

Example 5.76. *Let $\mathcal{W}$ be a CDC constructed via the block inserting construction in Theorem 5.74 with parameters $(n_1, n_2, n_3, n_4, d_1, d_2, k_1, k_2) = (3, 3, 3, 3, 2, 2, 3, 3)$ of maximum possible cardinality. Then, we have*

$$\#\mathcal{W} \ge q^{12} \cdot A_q^R(4 \times 4, 2; \le 2) \ge q^{20} + q^{19} + 2q^{18} + q^{17} - q^{15} - 2q^{14} - q^{13}$$

using Lemma 3.8.

Note that the upper rank bounds for the matrices in $\mathcal{M}_3$ and $\mathcal{M}_4$ are not necessary in the proof of Theorem 5.74

Lemma 5.77. ***(Generalized linkage construction + block inserting construction)***
Let $\mathcal{W}_1$ be a CDC constructed via the generalized linkage construction in Theorem 5.16 with parameters (n_1', n_2', d, k) and $\mathcal{W}_2$ be a CDC constructed via the block inserting construction in Theorem 5.74 with parameters $(n_1, n_2, n_3, n_4, d_1, d_2, k_1, k_2)$. If $n_1' = n_1 + n_2$, $n_2' = n_3 + n_4$, $d = d_1 + d_2$, and $k = k_1 + k_2$, then $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1' + n_2', d; k)_q$–CDC with cardinality $\#\mathcal{W} = \#\mathcal{W}_1 + \#\mathcal{W}_2$.

Proof. Let

$$H = \begin{pmatrix} G_1 & M_1 & \mathbf{0} & M_3 \\ \mathbf{0} & M_4 & G_2 & M_2 \end{pmatrix} =: \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

be the generator matrix of an arbitrary codeword $W_2 \in \mathcal{W}_2$, $U_1 := \langle P_1 \rangle$, $U_2 := \langle P_2 \rangle$, and E_1, E_2 be the special subspaces for $\mathcal{W}_1$ as in Lemma 5.18. Since $\dim(W_2 \cap E_1) \geq \dim(U_1 \cap E_1) \geq \operatorname{rk}(G_1) - \operatorname{rk}(M_3) \geq d/2$ and $\dim(W_2 \cap E_2) \geq \dim(U_2 \cap E_2) \geq \operatorname{rk}(G_2) - \operatorname{rk}(M_4) \geq d/2$ we have $d_S(\mathcal{W}_1, W_2) \geq d$ by Lemma 5.18. □

Example 5.78. *The CDC obtained from the block inserting construction I in Example 3.8 is compatible with a CDC obtained from the generalized linkage construction with parameters $(n_1, n; 2, d, k) = (6, 6, 4, 6)$, so that*

$$A_q(12,4;6) \geq A_q(12,4;6) \geq q^{30} + A_q^R(6 \times 6, 2; \leq 4) + q^{12} \cdot A_q^R(4 \times 4, 2; \leq 2).$$

However, as mentioned after Example 5.76, the effort for the more complicated coset construction pays off, see Example 5.63.

As a special case of the block inserting construction in Theorem 5.74 we mention:

Proposition 5.79. *([162, Proposition 2.1])*
Let $\mathcal{M}_3$ be a $(k_1 \times n_4, d/2; k_1 - d/2)_q$–RMC, $\mathcal{M}_4$ be a $(k_2 \times n_2, d/2; k_2 - d/2)_q$–RMC, $\mathcal{M}_1$ be a $(k_1 \times n_2, d_1/2)_q$–RMC, and $\mathcal{M}_2$ be a $(k_2 \times n_3, d_2/2)_q$–RMC, where $d_1 + d_2 = d$. Let $\mathcal{M}_1^1, \ldots, \mathcal{M}_1^s$ and $\mathcal{M}_2^1, \ldots, \mathcal{M}_2^s$ be $\frac{t}{2}$-packings of cardinality s of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. With this let

$$\left\{ \begin{pmatrix} I_{k_1} & M_1 & \mathbf{0}_{k_1 \times n_3} & M_3 \\ \mathbf{0}_{k_2 \times n_1} & M_4 & I_{k_2} & M_2 \end{pmatrix} : M_1 \in \mathcal{M}_1^i, M_3 \in \mathcal{M}_3, M_4 \in \mathcal{M}_4, M_2 \in \mathcal{M}_2^i \right\}$$

be a generating set of a subcode $\mathcal{W}^i$ for $1 \leq i \leq s$. Then, $\mathcal{W} = \cup_{i=1}^s \mathcal{W}^i$ is an $(n_1 + n_2 + n_3 + n_4, d; k_1 + k_2)_q$–CDC with cardinality

$$\#\mathcal{W} = \sum_{i=1}^{s} \#\mathcal{W}^i = \#\mathcal{M}_3 \cdot \#\mathcal{M}_4 \cdot \sum_{i=1}^{s} \#\mathcal{M}_1^i \cdot \#\mathcal{M}_2^i.$$

Corollary 5.80. *Let $\mathcal{W}$ be a CDC constructed via Proposition 5.79 with parameters $(n_1, n_2, n_3, n_4, d, k_1, k_2)$, where d_1, d_2 with $d_1 + d_2 = d$ are arbitrary, of maximum possible cardinality. Then, we have*

$$\begin{aligned} \#\mathcal{W} \quad \geq \quad & A_q^R(k_1 \times n_4, \tfrac{d}{2}; \leq k_1 - \tfrac{d}{2}) \cdot A_q^R(k_2 \times n_2, \tfrac{d}{2}; k_2 - \tfrac{d}{2}) \cdot \\ & A_q^R(k_1 \times n_2, d/2) \cdot A_q^R(k_2 \times n_4, d/2) \cdot \alpha, \end{aligned}$$

where

$$\alpha = \max_{d_1, d_2 \,:\, d_1 + d_2 = d} \min \left\{ \frac{A_q^R(k_1 \times n_2, d_1/2)}{A_q^R(k_1 \times n_2, d/2)}, \frac{A_q^R(k_2 \times n_4, d_2/2)}{A_q^R(k_2 \times n_4, d/2)} \right\}.$$

Example 5.81. *Let $\mathcal{W}$ be a $(12, 4; 6)_q$–CDC obtained via the block inserting construction in Theorem 5.74 with parameters $(n_1, n_2, n_3, n_4, d_1, d_2, k_1, k_2) = (4, 2, 2, 4, 2, 2, 4, 2)$. Let $\mathcal{M}_4 = \langle \mathbf{0}_{2 \times 2} \rangle$, so that we can assume*

$$\#\mathcal{W} \geq q^{12} \cdot A_q^R(4 \times 4; 2 \leq 2) \geq q^{20} + q^{19} + 2q^{18} + q^{17} - q^{15} - 2q^{14} - q^{13}.$$

Example 5.82. *Let $\mathcal{W}$ be a $(12,6;6)_q$–CDC obtained via the block inserting construction in Theorem 5.74 with parameters $(n_1,n_2,n_3,n_4,d_1,d_2,k_1,k_2) = (3,3,3,3,2,4,3,3)$. Let $\mathcal{M}_3 = M_4 = \langle \mathbf{0}_{3\times 3}\rangle$ and choose $\mathcal{M}_1 = \mathcal{M}_2$ as $(3\times 3,2)_q$–MRD codes, so that we can assume $\#\mathcal{W} \geq q^9$.*

In [161, Theorem 5] another inserting construction being compatible with the generalized linkage construction and the block inserting construction I was proposed as *block inserting construction II*. We give a slight generalization under the same name.

Theorem 5.83. ***(Block inserting construction II – cf. [161, Theorem 5], [162, Theorem 2.7])*** *Let $\mathcal{M}$ be a $(k_1\times n_1, k_2\times n_3, d/2; \leq k_1+k_2-d/2)_q$–SRMC, C_1 be an $(n_2,d;k_1)_q$–CDC, and be a C_2 be an $(n_4,d;k_2)_q$–CDC. With this, let*

$$\left\{\begin{pmatrix} M_1 & G_1 & \mathbf{0}_{k_1\times n_3} & \mathbf{0}_{k_1\times n_4} \\ \mathbf{0}_{k_2\times n_1} & \mathbf{0}_{k_2\times n_2} & M_2 & G_2 \end{pmatrix} : G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2, (M_1,M_2) \in \mathcal{M}\right\}$$

be a generating set of a subspace code $\mathcal{W}$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ be generating sets of C_1 and C_2, respectively. Then, $\mathcal{W}$ is an $(n_1+n_2+n_3+n_4,d;k_1+k_2)_q$–CDC with cardinality $\#C_1\cdot\#C_2\cdot\#\mathcal{M}$.

Proof. Let

$$H = \begin{pmatrix} M_1 & G_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_2 & G_2 \end{pmatrix}$$

be the generator matrix of an arbitrary codeword $W \in \mathcal{W}$. Since $k_1+k_2 \geq \operatorname{rk}(H) \geq \operatorname{rk}(G_1) + \operatorname{rk}(G_2) = k_1+k_2$, every codeword is a (k_1+k_2)-space. Let

$$H' = \begin{pmatrix} M_1' & G_1' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_2' & G_2' \end{pmatrix}$$

be the generator matrix of another codeword $W' \in \mathcal{W}$,

$$R := \operatorname{rk}\left(\begin{pmatrix} M_1 & G_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_2 & G_2 \\ M_1' & G_1' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_2' & G_2' \end{pmatrix}\right) = \operatorname{rk}\left(\begin{pmatrix} G_1 & \mathbf{0} & M_1 & \mathbf{0} \\ G_1'-G_1 & \mathbf{0} & M_1'-M_1 & \mathbf{0} \\ \mathbf{0} & G_2 & \mathbf{0} & M_2 \\ \mathbf{0} & G_2'-G_2 & \mathbf{0} & M_2'-M_2 \end{pmatrix}\right),$$

and $U_1 := \langle G_1\rangle$, $U_2 := \langle G_2\rangle$, $U_1' := \langle G_1'\rangle$, $U_2' := \langle G_2'\rangle$.

If $G_1 \neq G_1'$ or $G_2 \neq G_2'$, then we have $U_1 \neq U_1'$ or $U_2 \neq U_2'$, so that

$$\begin{aligned} d_S(W,W') &= 2(R-k_1-k_2) \geq 2\cdot\operatorname{rk}\left(\begin{pmatrix} G_1 & \mathbf{0} \\ G_1'-G_1 & \mathbf{0} \\ \mathbf{0} & G_2 \\ \mathbf{0} & G_2'-G_2 \end{pmatrix}\right) - 2k_1 - 2k_2 \\ &= d_S(U_1,U_1') + d_S(U_2,U_2') \geq d. \end{aligned}$$

If $G_1 = G_1'$ and $G_2 = G_2'$, then we have

$$R = \operatorname{rk}\left(\begin{pmatrix} G_1 & \mathbf{0} & M_1 & \mathbf{0} \\ \mathbf{0} & G_2 & \mathbf{0} & M_2 \\ \mathbf{0} & \mathbf{0} & M_1'-M_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & M_2'-M_2 \end{pmatrix}\right) = k_1+k_2+\operatorname{rk}(M_1'-M_1)+\operatorname{rk}(M_2'-M_2),$$

so that $\operatorname{rk}(M_1' - M_1) + \operatorname{rk}(M_2' - M_2) \ge d_{\mathrm{R}}(M_1, M_1') + d_{\mathrm{R}}(M_2, M_2') \ge d/2$ implies $d_{\mathrm{S}}(W, W') \ge d$. □

Corollary 5.84. *Let $\mathcal{W}$ be a CDC constructed via the block inserting construction II in Theorem 5.83 with parameters $(n_1, n_2, n_3, n_4, d, k_1, k_2)$ of maximum possible cardinality. Then, we have*

$$\#\mathcal{W} \ge A_q(n_2, d; k_1) \cdot A_q(n_4, d; k_2) \cdot A_q^R(k_1 \times n_1, k_2 \times n_3, \le k_1 + k_2 - d/2).$$

Example 5.85. *Let $\mathcal{W}$ be the $(12, 6; 6)_q$–CDC obtained via the block inserting construction II in Theorem 5.83 with parameters $(n_1, n_2, n_3, n_4, d, k_1, k_2) = (3, 3, 3, 3, 6, 3, 3)$ of maximum possible cardinality. Since $A_q(3, 6; 3) = 1$ we can assume $\#\mathcal{M} \ge A_q(3 \times 3, 3 \times 3, 3, \le 3) \ge q^5 + q^4 + 2q^3 - q^2 - q$ using Example 3.31 for the later estimation.*

We remark that the variant of the block inserting construction II in [161, Theorem 5] gives a subcode of cardinality $q^5 + q^4 + q^3 - q^2 - q$, i.e., q^3 codewords less.

Note that the upper rank bounds for the matrices in $\mathcal{M}$ are not necessary in the proof of Theorem 5.83

Lemma 5.86. ***(Generalized linkage constr. + block inserting construction I,II)***
Let $\mathcal{W}_1$ be a CDC constructed via the generalized linkage construction in Theorem 5.16 with parameters (n_1', n_2', d, k), $\mathcal{W}_2$ be a CDC constructed via the block inserting construction I in Theorem 5.74 with parameters $(n_1, n_2, n_3, n_4, d_1, d_2, k_1, k_2)$, and $\mathcal{W}_3$ be a CDC constructed via the block inserting construction II in Theorem 5.83 with parameters $(n_1, n_2, n_3, n_4, d, k_1, k_2)$. If $n_1' = n_1 + n_2$, $n_2' = n_3 + n_4$, $d = d_1 + d_2$, $k = k_1 + k_2$, $k_1 \ge d/2$, and $k_2 \ge d/2$, then $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$ is an $(n_1' + n_2', d; k)_q$–CDC with cardinality $\#\mathcal{W} = \#\mathcal{W}_1 + \#\mathcal{W}_2 + \mathcal{W}_3$.

Proof. From Lemma 5.77 we conclude that $\mathcal{W}' := \mathcal{W}_1 \cup \mathcal{W}_2$ is an $(n_1' + n_2', d; k)_q$–CDC with cardinality $\#\mathcal{W}' = \#\mathcal{W}_1 + \#\mathcal{W}_2$. So, let

$$H_3 = \begin{pmatrix} M_1 & G_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_2 & G_2 \end{pmatrix} =: \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

be the generator matrix of an arbitrary codeword $W_3 \in \mathcal{W}_3$, $U_1 := \langle P_1 \rangle$, $U_2 := \langle P_2 \rangle$, and E_1, E_2 be the special subspaces for $\mathcal{W}_1$ as in Lemma 5.18. Since $\dim(W_3 \cap E_1) \ge \dim(U_1 \cap E_1) = \operatorname{rk}(G_1) = k_1 \ge d/2$ and $\dim(W_2 \cap E_2) \ge \dim(U_2 \cap E_2) = \operatorname{rk}(G_2) = k_2 \ge d/2$ we have $d_{\mathrm{S}}(\mathcal{W}_1, W_3) \ge d$ by Lemma 5.18.

Now let

$$H_2 = \begin{pmatrix} G_1' & M_1' & \mathbf{0} & M_3' \\ \mathbf{0} & M_4' & G_2' & M_2' \end{pmatrix}$$

be the generator matrix of an arbitrary codeword $W_2 \in \mathcal{W}_2$. Observe that the pivot vector $v(H_3)$ of H_3 is contained in $\left(\binom{n_1}{k_1}, \binom{n_2}{0}, \binom{n_3}{k_2}, \binom{n_4}{0}\right)$. Since $\operatorname{rk}(M_1) + \operatorname{rk}(M_2) \le k_1 + k_2 - d/2$ we have $\mathrm{d_H}(v(H_3), v(H_2)) \ge d$, so that $d_{\mathrm{S}}(W_3, W_2) \ge d$. □

Example 5.87. *Let $\mathcal{W}_1$ be a CDC constructed via the generalized linkage construction in Theorem 5.16 with parameters $(n_1, n_2, d, k) = (6, 6, 6, 6)$, $\mathcal{W}_2$ be a CDC constructed via the* *block inserting construction I in Theorem 5.74 with parameters* $(n_1, n_2, n_3, n_4, d_1, d_2, k_1, k_2) = (3, 3, 3, 3, 2, 4, 3, 3)$*, and* $\mathcal{W}_3$ *be a* CDC *constructed via the block inserting construction II in Theorem 5.83 with parameters* $(n_1, n_2, n_3, n_4, d, k_1, k_2) = (3, 3, 3, 3, 6, 3, 3)$*. Then, considering the* $(12, 6; 6)_q$*–*CDC *yields*

$$\begin{array}{rcl} A_q(12,6;6) & \geq & q^{24} + A_q^R(6\times 6,3;\leq 3) + q^9 + A_q(3\times 3, 3\times 3, 3; \leq 3) \\ & \geq & q^{24} + q^{15} + q^{14} + 2q^{13} + 3q^{12} + 3q^{11} + 3q^{10} + 3q^9 + q^8 - q^7 - 2q^6 \\ & & -2q^5 - 2q^4 - q^3 - 3q^2 - 2q \end{array}$$

using $A_q^R(6\times 6, 3; \leq 3) \geq \begin{bmatrix}6\\3\end{bmatrix}_q \cdot (q^6-1)+1 = q^{15}+q^{14}+2q^{13}+3q^{12}+3q^{11}+3q^{10}+2q^9+q^8-q^7-2q^6-3q^5-3q^4-3q^3-2q^2-q$ *from Lemma 3.8 and the lower bound for* $A_q(3\times 3, 3\times 3, 3; \leq 3)$ *from Example 3.31. For* $q = 2$ *we e.g. have* $A_2(12, 6; 6) \geq 16865672$*.*

5.6 Combining constant dimension codes geometrically

So far we have combined generating sets of CDCs and matrices of RMCs in order to obtain generating sets of CDCs. Now we want to describe a different possibility how smaller CDCs can be combined to larger CDCs. In [48] the authors combined several $(6, 4; 3)_q$-CDCs to show $A_q(9, 4; 3) \geq q^{12} + 2q^8 + 2q^7 + q^6 + q^5 + q^4 + 1$, which improves upon the previously best known lower bound $A_q(9, 4; 3) \geq q^{12} + 2q^8 + 2q^7 + q^6 + 1$, which was obtained from the improved linkage construction. In [156] the mentioned lower bound was further improved to $A_q(9, 4; 3) \geq q^{12} + 2q^8 + 2q^7 + q^6 + 2q^5 + 2q^4 - 2q^2 - 2q + 1$. Here we want to present the generalization of this approach as introduced in [47]. The idea is to use a CDC $C \subseteq \mathcal{G}_q(k+t, k)$ and an s-space S outside of $\mathrm{PG}(k+t-1, k)$, i.e., we want to use $\mathrm{PG}(k+t-1, q)\times S \cong \mathrm{PG}(k+s+t-1, q)$ as ambient space of the resulting CDC. For each codeword $U \in C$ we consider the $(k + s)$-space $D := U \times S \cong \mathrm{PG}(k + s - 1, q)$. In D we can choose an $(k + s, d; k)_q$–CDC that contains U as a specific codeword and whose codewords intersect S in at most a certain dimension. More precisely, we assume that we have a list of choices for the chosen CDC in D.

Definition 5.88. *An* (n, d, k)-sequence *of* CDC*s is a list* $(\mathcal{D}_0, \ldots, \mathcal{D}_r)$ *of* $(n, d; k)_q$*-*CDC*s such that for each index* $0 \leq i \leq r$ *there exists a codeword* $U \in \mathcal{D}_i$ *and a disjoint* $(n-k)$*-subspace* S *such that* $\dim(U' \cap S) \leq i$ *for all* $U' \in \mathcal{D}_i$*, where* $r = k - \frac{d}{2}$*.*

We remark that an LMRD code gives an example for $\mathcal{D}_0$ and for $\mathcal{D}_i$, with $i \geq 1$, we can take $\mathcal{D}_0$. Another possibility is to start with an arbitrary $(n, d; k)_q$-CDC, pick the special subspace S, and remove all codewords whose dimension of the intersection with S is too large.

Assume that U and U' are two different codewords of C and $D = U \times S$ and $D' = U' \times S$ are the corresponding $(k + s)$-spaces into which we insert codewords from a $(k + 2, d; k)_q$–CDC. If U and U' have a relatively large dimension of their intersection, so have D and D'. In order to guarantee a minimum subspace distance of at least d between a codeword in D and a codeword in D', we can reduce the allowed dimension of the intersection of the codewords with S. To this end we introduce:

Definition 5.89. *A list* $(C_0, \ldots, C_r)$ *is called a* distance-partition *of an* $(n, d; k)_q$*–*CDC C*, where* $r = k - \frac{d}{2}$*, if* $C_0, \ldots, C_r$ *is a partition of* C *and* $\bigcup_{j=0}^{i} C_j$ *is an* $(n, 2k-2i; k)_q$*–*CDC *for all* $0 \leq i \leq r$*.*

A trivial distance-partition of an $(n, d; k)_q$–CDC C is given by $(\emptyset, \ldots, \emptyset, C)$. A subcode $C' \subseteq C$ with maximal subspace distance $d = 2k$ is called a *partial-spread subcode*. Given such a partial-spread subcode C', if $d < 2k$, then $(C', \emptyset, \ldots, \emptyset, C \backslash C')$ is a distance-partition of C.

Lemma 5.90. *([47, Lemma 5.3]) Let $(C_0, \ldots, C_r)$ be a distance-partition of a $(k+t, d; k)_q$–CDC C and $(\mathcal{D}_0, \ldots, \mathcal{D}_r)$ be a $(k+s, d, k)$-sequence, where $r = k - \frac{d}{2}$. If $\mathcal{A}$ is an $(s, d; k)_q$–CDC, then there exists a $(k+s+t, d; k)_q$–CDC C' with cardinality*

$$\#C' = \#\mathcal{A} + \sum_{i=0}^{r} \#C_i \cdot \#\mathcal{D}_{r-i}.$$

Here $\mathcal{A}$ is a CDC that we can insert into the special subspace S and the combination of codewords in C_i with CDC $\mathcal{D}_{r-i}$ ensures that the subspace distance between a codeword of the resulting CDC in D and a codeword in D', using the notation from above, has a subspace distance of at least d. For more details we refer to the proof of [47, Lemma 5.3].

As examples we describe the application of Lemma 5.90 for the construction of CDCs reaching the lower bound for $A_q(9, 4; 3)$ and $A_q(10, 4; 3)$ presented in [156]. First we construct a $(6, 4, 3)$-sequence $(\mathcal{D}_0, \mathcal{D}_1)$. Here we choose $\mathcal{D}_0$ as an LMRD code of cardinality q^6 and $\mathcal{D}_1$ as a $(6, 4; 3)_q$-CDC with cardinality $q^6 + 2q^2 + 2q$. The latter needs a bit more explanation. Choose a $(6, 4; 3)_q$–CDC $\mathcal{D}'_1$ of cardinality $q^6 + 2q^2 + 2q + 1$, see [49, 132], and assume that U and S are two disjoint codewords. Here U and S have the same meanings as above, i.e., U is a special codeword and S is the special subspace used in the construction of the $(s+k)$-space $D = U \times S$. With this let $\mathcal{D}_1$ arise from $\mathcal{D}'_1$ by removing the codeword S. Since $\mathcal{D}'_1$, as well as $\mathcal{D}_1$, is a $(6, 4; 3)_q$–CDC every codeword of $\mathcal{D}_1$ has an intersection of dimension at most 1 with S, which is what we need according to Definition 5.88.

For $A_q(9, 4; 3)$, we choose the CDC C needed in Lemma 5.90 as a $(6, 4; 3)_q$–CDC with cardinality $q^6 + 2q^2 + 2q + 1$, see [49, 132]. In order to determine a distance-partition (C_0, C_1) of C, we need to find a large partial-spread subcode of C. In [48, Theorem 3.12], it is shown that we can choose C_0 of cardinality $q^3 - 1$ if we choose C as constructed in [49]. However, as shown in [156], the same can be done if we choose C as constructed in [132].[2] As subcode $\mathcal{A}$ we choose a single 3-space, so that we obtain

$$\begin{aligned} A_q(9,4;3) &\geq 1 + \#C_0 \cdot \#\mathcal{D}_1 + \#C_1 \cdot \#\mathcal{D}_0 \\ &= 1 + \left(q^3 - 1\right) \cdot \left(q^6 + 2q^2 + 2q\right) + \left(q^6 - q^3 + 2q^2 + 2q + 2\right) \cdot q^6 \\ &= q^{12} + 2q^8 + 2q^7 + q^6 + 2q^5 + 2q^4 - 2q^2 - 2q + 1. \end{aligned}$$

For $A_q(10, 4; 3)$ we choose C as the $(7, 4; 3)_q$–CDC of cardinality $q^8 + q^5 + q^4 + q^2 - q$ constructed in [131, Theorem 4]. Again we need to find a large partial-spread subcode C_0 of C.

[2] This can be made more precise in the language of linearized polynomials. For [132, Lemma 12, Example 4] the representation $\mathbb{F}_q^6 \cong \mathbb{F}_{q^3} \times \mathbb{F}_{q^3}$ is used and the planes removed from the lifted MRD code correspond to $ux^q - u^q x$ for $u \in \mathbb{F}_{q^3}$, so that the monomials ax for $a \in \mathbb{F}_{q^3} \backslash \{\mathbf{0}\}$ correspond to a partial-spread subcode of cardinality $q^3 - 1$.

Here $\#C_0 = q^4$ can be achieved, see [156]. Thus, we obtain

$$\begin{aligned} A_q(10,4;3) &\geq 1 + \#C_0 \cdot \#\mathcal{D}_1 + \#C_1 \cdot \#\mathcal{D}_0 \\ &= 1 + q^4 \cdot \left(q^6 + 2q^2 + 2q\right) + \left(q^8 + q^5 + q^2 - q\right) \cdot q^6 \\ &= q^{14} + q^{11} + q^{10} + q^8 - q^7 + 2q^6 + 2q^5 + 1. \end{aligned}$$

The determination of a large partial-spread subcode is mostly the hardest part in the analytic evaluation of the construction of Lemma 5.90. However, if C contains an $(n, d; k)$–CDC that contains an LMRD code as a subcode, then it contains an $(n, 2k; k)$–CDC as a subcode that is again an LMRD code, i.e., a partial-spread subcode.

Research problem

Determine large partial-spread subcodes for constructions of CDCs from the literature.

We remark that Lemma 5.90 was used in [47] to construct lower bounds for $A_q(3k, 4; k)$, where $k \geq 3$, for $A_q(16, 4; 4)$, and for $A_q(6k, 2k; 2k)$, where $k \geq 4$ is even.

Research problem

Use Lemma 5.90 for the construction of large CDCs for further parameters or improve the known constructions.

5.7 Other constructions for constant dimension codes

The list of constructions for CDCs presented in the previous subsections is far from being exhaustive. There are several constructions based on geometric concepts, see e.g. [52] for an overview and e.g. [50, 51]. As examples we mention two explicit and rather general parametric lower bounds.

Theorem 5.91. *([51, Theorem 3.8])*
If $n \geq 4$ is even, then $A_q(2n, 4; n) \geq$

$$q^{n^2-n} + \sum_{r=2}^{n-2} \begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{j=2}^{r} (-1)^{(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{\binom{r-j}{2}} (q^{n(j-1)} - 1) + \begin{bmatrix} \frac{n}{2} \\ 1 \end{bmatrix}_{q^2} \left(\begin{bmatrix} \frac{n}{2} \\ 1 \end{bmatrix}_{q^2} - 1 \right)$$
$$+ (q+1) \left(\prod_{i=1}^{n-1} (q^i + 1) - 2q^{\frac{n(n-1)}{2}} + q^{\frac{n(n-2)}{4}} \prod_{i=1}^{\frac{n}{2}} (q^{2i-1} - 1) \right) - q \cdot |G| + 1$$

using

$$|G| = 2 \prod_{i=1}^{n/2-1} (q^{2i} + 1) - 2q^{(n(n-2)/4)}$$

if $n/2$ is odd and

$$|G| = 2 \prod_{i=1}^{n/2-1} (q^{2i} + 1) - 2q^{(n(n-2)/4)} + q^{n(n-4)/8} \prod_{i=1}^{n/4} (q^{4i-2} - 1)$$

if $n/2$ is even.

Theorem 5.92. *([51, Theorem 3.11])*
If $n \geq 5$ is odd, then $A_q(2n,4;n) \geq$

$$q^{n^2-n} + \sum_{r=2}^{n-2} \begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{j=2}^{r} (-1)^{(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{\binom{r-j}{2}} (q^{n(j-1)} - 1) \; + \; y(y-1) + 1$$
$$+ \prod_{i=1}^{n-1} (q^i + 1) - q^{\frac{n(n-1)}{2}} - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \left(q^{\frac{(n-1)(n-2)}{2}} - q^{\frac{(n-1)(n-3)}{4}} \prod_{i=1}^{\frac{n-1}{2}} (q^{2i-1} - 1) \right),$$

using $y := q^{n-2} + q^{n-4} + \cdots + q^3 + 1$.

Riemann–Roch spaces can be used to construct CDCs, see [22, 108]. Removing and replacing codewords of lifted MRD codes was the basis of a few specific constructions, see [131, 132]. An entire theoretic framework for such approaches was introduced in [4]. For MRD codes linearity plays an important and natural role. A variant of the concept is considered in [38], see also [191]. Another well studied class are so-called cyclic subspace codes, see e.g. [26, 44, 163, 186, 189, 190, 196]. In principle one can start with any construction of a CDC and check if it can be extended by further codewords. This approach was e.g. successful for the $(7,4;3)_q$–CDC of cardinality 6977 constructed in [131]. Here, an extension by an additional codeword was possible, so that $A_3(7,4;3) \geq 6978$, see [122]. However, even for moderate parameters $\begin{bmatrix} n \\ k \end{bmatrix}_q$ gets huge rather soon, so that one faces algorithmical problems. In [230] the extension problem is restricted to the set C_1, C_2 of codewords of two CDCs. More precisely, the problem of the determination of the largest CDC with codewords in $C_1 \cup C_2$ was formulated as a minimum point-covering problem for a bipartite graph that can be solved in polynomial time. As example the improved lower bounds $A_2(8,4;3) \geq 1331$ and $A_2(8,4;4) \geq 4802$ were obtained in [230].

6 On the existence of a binary q-analog of the Fano plane

For the binary case $q = 2$ the smallest unknown value $A_q(n, d; k)$ is $A_2(7, 4; 3)$. Inequality (4.5) of the Johnson bound gives

$$A_2(7,4;3) \le \left\lfloor \frac{127 \cdot A_2(6,4;2)}{7} \right\rfloor = 381$$

since $A_2(6,4;2) = 21 = 3 \cdot 7$ due to the existence of a 2-spread in $\mathrm{PG}(6,2)$. Also the improved Johnson bound in Theorem 4.42 cannot give a tighter bound since in a $(7,4;3)_2$–$\mathtt{CDC}$ C_{381} of cardinality 381 every point is contained in exactly 21 codewords. Also the anticode bound yields the upper bound $A_2(7,4;3) \le \begin{bmatrix}7\\2\end{bmatrix}_2 / \begin{bmatrix}3\\2\end{bmatrix}_2 = 381$, so that any line is contained in exactly one codeword of C_{381}. If C_{381} exists, then it is a so-called *q-design* and called *binary q-analog of the Fano plane*.

Exercise 6.1. *Show* $\#\{U \in C_{381} \,:\, U \le K\} = 5$ *and* $\#\{H \in C_{381} \,:\, U \le K\} = 45$ *for each* $K \in \mathcal{G}_2(7,5)$ *and each hyperplane* $H \in \mathcal{G}_2(7,6)$. *For each point P and each hyperplane H with $P \le H$ show that* $\#\{U \in C_{381} \,:\, P \le U \le H\} = 5$.

Theorem 6.2. *([145, Theorem 1])*
The automorphism group of a binary q-analog of the Fano plane is either trivial or of order 2. *In the latter case, up to conjugacy in* $\mathrm{GL}(7,2)$ *the automorphism group is represented by*

$$\left\langle \begin{pmatrix} 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \right\rangle.$$

For each solid $S \in \mathcal{G}_2(7,4)$ we have $\#\{U \in C_{381} \,:\, U \le S\} \in \{0,1\}$. For the group G of order two in Theorem 6.2 there are exactly 15 fixpoints, i.e. points P such that the $P^g = P$ for all $g \in G$, where P^g denotes the application of the group element g to P. These 15 fixpoints form a special solid $\bar{S} = \langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_5 + \mathbf{e}_6, \mathbf{e}_7 \rangle$. The $\begin{bmatrix}4\\2\end{bmatrix}_2 = 35$ lines in $\bar{S}$ clearly are fixed by G. The other 56 fixed lines are given by $L = \langle P, P^g \rangle$, where P is a arbitrary point outside of $\bar{S}$, so that L intersects $\bar{S}$ in a point. Let $\mathcal{B}_2$ denote the 91 fixed lines . It is a bit more tedious to check that there are exactly 211 planes that are fixed by G. Let $\mathcal{B}_3$ denote these fixed planes. Note that in C_{381} each fixed line must be contained in a codeword U that is fixed by G, i.e. $U \in \mathcal{B}_3$.

Exercise 6.3. *Verify*

$$\frac{1}{7} \cdot \sum_{L \in \mathcal{B}_2 \,:\, L \le \bar{S}} \; \sum_{U \in \mathcal{B}_3 \,:\, L \le U} x_U + \frac{3}{7} \cdot \sum_{L \in \mathcal{B}_2 \,:\, L \not\le \bar{S}} \; \sum_{U \in \mathcal{B}_3 \,:\, L \le U} x_U = \sum_{U \in \mathcal{B}_3} x_U.$$

Since through each line there is at most one codeword, we have $\#\left(C_{381} \cap \mathcal{B}_3\right) \le \frac{1}{7}\cdot 35 + \frac{3}{7}\cdot 56 = 29$. On the other hand the 35 lines in $\bar{S}$ each have to be contained in a codeword from $\mathcal{B}_3$, so that there exists a codeword in $\mathcal{B}_3$ that is contained in $\bar{S}$ and there are 28 codewords in $\mathcal{B}_3$ that intersect $\bar{S}$ in a line each. Of course, this little insight does not exclude the existence of a CDC C_{381} with G as automorphism group.

Exercise 6.4. *Assume that G is a subgroup of the automorphism group of C_{381}. Show that the set $\mathcal{F}$ of fixed points with respect to G is a subspace. Determine restrictions for the possible dimension of $\mathcal{F}$ for $\#G \in \{2, 3, 5, 7, 31, 127\}$.*

Research problem

Decide whether there exist 240 planes in $\operatorname{PG}(6, 2)$ and an automorphism π of order 5 such that all planes are disjoint to the 3-space $\mathcal{F}$ of points fixed by π, no two planes intersect in a line, and each point outside of $\mathcal{F}$ is covered 15 times.

We remark that the "complementary set", admitting π as automorphism, consisting of $\mathcal{F}$ and 140 further planes intersecting $\mathcal{F}$ in a point, such that no line is covered twice indeed exists.

In [123] $A_2(7, 4; 3) \ge 333$ was shown. The constructed code has an automorphism group of order 4 isomorphic to the Klein four-group. We remark that the corresponding code contains a subcode of cardinality 329 that admits an automorphism group of order 16.

Theorem 6.5. *([123, Theorem 1])*
Let C be a set of planes in $\operatorname{PG}(6, 2)$ mutually intersecting in at most a point. If $\#C \ge 329$, then the automorphism group of C is conjugate to one of the 33 subgroups of $\operatorname{GL}(7, 2)$ given in [123, Appendix B]. The orders of these groups are $1^1 2^1 3^2 4^7 5^1 6^3 7^2 8^{11} 9^2 12^1 14^1 16^1$ denoting the number of cases as exponent. Moreover, if $\#C \ge 330$ then $\#\operatorname{Aut}(C) \le 14$ and if $\#C \ge 334$ then $\#\operatorname{Aut}(C) \le 12$.

Interestingly enough, it is not necessary to generate all subgroups of $\operatorname{GL}(7, 2)$ of order at most 16 up to conjugacy to obtain the stated results, see [123] for the algorithmic details. In [115, Section 10] parametric upper bounds for CDCs that admit certain automorphisms are concluded. The group of order 12 mention in Theorem 6.5, that might allow a larger $(7, 4; 3)_2$–CDC, is given by:

$$G_{12,1} = \left\langle \begin{pmatrix} 1&0&0&0&0&1&1\\ 0&0&0&1&1&0&1\\ 1&1&1&1&1&0&0\\ 1&1&0&0&1&1&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&1&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0\\ 1&1&0&0&0&1&1\\ 1&0&1&0&1&0&1\\ 1&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&1&1\\ 0&1&0&1&1&1&1\\ 1&0&1&1&1&0&0\\ 1&1&0&0&0&1&1\\ 1&0&0&0&1&0&0\\ 1&0&0&0&0&1&0\\ 0&0&0&0&0&1&0 \end{pmatrix} \right\rangle \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_4.$$

In [178] it was shown that each hypothetical $(7, 4; 3)_2$–CDC of cardinality 380 can be extended to a CDC of cardinality 381. Using divisible codes it was shown that either $A_2(7, 4; 3) \le 378$ or $A_2(7, 4; 3) = 381$.

For each point $P \in \mathcal{G}_2(7, 1)$ the subcode $C_P := \{U \in C_{381} : P \le U\}$ gives rise to a 2-spread $C_P/P := \{U/P : U \in C_P\}$ in $\operatorname{PG}(6, 2)/P \cong \operatorname{PG}(5, 2)$. In our situation it is called *geometric* if for any two spread lines L and L', the restriction of the 2-spread to the 4-space $\langle L, L'\rangle$ is a 2-spread itself, i.e., 5 lines are contained. Call every point P such that C_P/P is geometric an *α-point*. In [215] it was shown that, even for general field sizes q, there always exists a non-α point $\bar{P}$ in a q-analog of the Fano plane. For a binary q-analog of the Fano plane the result was tightened to the existence of at least one non-α point in every hyperplane [113]. Recently this result was generalized to all prime or even field sizes q in [143]. Here we want to consider a relaxation. Let $C \subseteq \mathcal{G}_2(6,3)$ such that

- every 5-space contains exactly five elements of C;
- every point is contained in exactly five elements of C;
- each line is contained in at most one element of C; and
- each solid contains at most one element of C.

Such sets of 5-spaces indeed exist and have cardinality $\#C = 45$, cf. [72] for general field sizes and the existence of induced substructures of a q-analog of the Fano plane. We call a point P an α' point if the five elements of C that are incident with P span a 5-space (and not the entire 6-dimensional ambient space). Using an ILP formulation of the problem one can computationally show that the maximum number of α' points in a fixed 5-space lies between 15 and 22. The total number of α' points lies between 19 and 44.

Research problem

Determine the maximum number of α' points.

For certain infinite fields a "q-analog of the Fano plane" indeed exists, see [219]. In $\mathrm{PG}(6,q)$ the existence question or the maximum possible size $A_q(7,4;3)$ of a CDC with these parameters is still widely open.

From the improved Johnson bound we conclude

$$A_q(8,4;4) \le \left\lfloor\!\left\lfloor \frac{\frac{[8]_q}{[4]_q} \cdot A_q(7,4;3)}{[4]_q} \right\rfloor\!\right\rfloor_{q^3}.$$

If we cannot improve upon $A_q(7,4;3) \le \begin{bmatrix}7\\2\end{bmatrix}_q / \begin{bmatrix}3\\2\end{bmatrix}_q$, then this upper bound is equivalent to $A_q(8,4;4) \le \begin{bmatrix}8\\3\end{bmatrix}_q / \begin{bmatrix}4\\3\end{bmatrix}_q$, i.e., the anticode bound. For $q = 2$ we obtain $A_2(8,4;4) \le 6477$. However, if such a code C of cardinality 6477 exists, then for each point P the set of codewords of C that contain P would be a binary q-analog of the Fano plane.

7 Lower bounds for constant dimension codes

In this section we summarize the currently best known lower bounds for constant dimension codes. For subspace distance $d = 2$ we can choose $C = \mathcal{G}_q(n,k)$, so that $A_q(n,d;k) = \begin{bmatrix} n \\ k \end{bmatrix}_q$. In general we have $A_q(n,d;k) = A_q(n,d;n-k)$. Thus we assume $4 \le d \le 2k$, $d \equiv 0 \pmod 2$, and $2 \le k \le n/2$. For the dimension of the ambient space we restrict our consideration to $4 \le n \le 9$ and a few selected triples (n,d,k). We also treat the case $d = 2k$, i.e. the case of (partial) k-spreads separately. If $n \equiv 0 \pmod k$, then k-spreads indeed exist and we have $A_q(n,2k;k) = [n]_q/[k]_q$, see Theorem 4.23. For the cases where $n \not\equiv 0 \pmod k$ we have used the Echelon–Ferrers construction to conclude a general lower bound in Exercise 5.32, see also Inequality (4.14):

$$A_q(tk+r,2k;k) \ge \sum_{s=0}^{t-1} q^{sk+r} - (q^r - 1),$$

where $k,t \ge 2$ and $0 \le r \le k-1$. The only known improvement is

$$A_2(3t+2,6;3) \ge \sum_{s=0}^{t-1} 2^{3s+2} - (2^2 - 1) + 1,$$

for arbitrary $t \ge 2$, see [67]. For upper bound for partial spreads much more can be said, see Subsection 4.1. For small parameters the known lower and upper bounds coincide. E.g. we have $A_q(4,4;2) = q^2+1$, $A_q(5,4;2) = q^3+1$, $A_q(6,4;2) = q^4+q^2+1$, $A_q(6,6;3) = q^3+1$, $A_q(7,4;2) = q^5+q^3+1$, and $A_q(7,6;3) = q^4+1$. For $A_q(8,6;3)$ the exact value is known for $q = 2$ only. In the following we will discard the partial spread case and assume $d < 2k$.

For the smallest parameters we have

$$A_q(6,4;3) \ge q^6 + 2q^2 + 2q + 1, \tag{7.1}$$

see [132, 49] for constructions. We remark that the lower bound is tight for $q = 2$ [132]. For $A_q(7,4;3)$ a lower bound for general q was given in [131, Theorem 4]. For $q = 2$ an improved lower bound was found via extensive ILP computations in [124] and for $q = 3$ it was observed that a theoretical construction can be extended by one further codeword, so that we have

$$A_q(7,4;3) \ge q^8 + q^5 + q^4 + q^2 - q, \quad A_2(7,4;3) \ge 333, A_3(7,4;3) \ge 6978. \tag{7.2}$$

The constructions for $A_q(6,4;3)$ and $A_q(7,4;3)$ from [132] and [131] can be described within the framework of the so-called *expurgation-augmentation method*, see [4], where specially selected codewords are removed from a lifted `MRD` code in order to allow the augmentation with more codewords than removed before.

Construction 1, see Theorem 5.33 or [77, Chapter IV, Theorem 16] gives

$$A_q(8,4;3) \ge q^{10} + \begin{bmatrix}5\\2\end{bmatrix}_q = q^{10} + q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1. \tag{7.3}$$

For $q = 2$ the improved lower bound $A_2(8,4;3) \ge 1326$ was found via the prescription of automorphisms.

The lower bound

$$\begin{aligned} A_q(8,4;4) &\ge q^{12} + \left(q^2+q+1\right)\cdot\left(q^2+1\right)^2 + 1 \\ &= q^{12} + q^8 + q^7 + 3q^6 + 2q^5 + 3q^4 + q^3 + q^2 + 1 \end{aligned} \tag{7.4}$$

is attained by several constructions. One example is the coset construction of Theorem 5.43, see Example 5.55 for the details. We remark that the stated lower bound is tight if we additionally assume that a lifted MRD is contained as a subcode, see e.g. [77]. For $q = 2$ this bound gives 4797 as the maximum possible size under this extra assumption. Nevertheless a construction showing $A_2(8,4;4) \ge 4802$ is known [230]. It is obtained by extending an $(8,4;4)_2$–CDC with cardinality 4801, found in [39] via the prescription of automorphisms, by a single codeword.

For the skeleton code $\{1111000, 00001111\}$ the Echelon–Ferrers construction give the lower bound

$$A_q(8,6;4) \ge q^8 + 1. \tag{7.5}$$

In other words, a corresponding code consists of a lifted MRD code and another codeword. For $q = 2$ it was shown in [119] that the lower bound is indeed tight and that there are exactly two isomorphism types of CDCs attaining the maximum possible cardinality 257.

The geometric combination of CDCs described in Subsection 5.6 yields the lower bound

$$A_q(9,4;3) \ge q^{12} + 2q^8 + 2q^7 + q^6 + 2q^5 + 2q^4 - 2q^2 - 2q + 1, \tag{7.6}$$

see also [47]. For $q = 2$ the tighter bound $A_2(9,4;3) \ge 5986$ was obtained in [39] via the prescription of automorphisms.

The pending dots construction gives $A_2(9,4;4) \ge 37265$ and

$$A_q(9,4;4) \ge q^{15} + q^{11} + q^9 + 4q^8 + 5q^7 + 3q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1 \tag{7.7}$$

for $q \ge 3$. Interestingly enough, for $q \ge 5$ we get a tighter lower bound by reverting the Johnson upper bound from Theorem 4.15, cf. [227],

$$A_q(n,d;k) \ge \left\lceil \frac{\left(q^{k+1}-1\right) A_q(n+1,;k+1)}{q^{n+1}-1} \right\rceil. \tag{7.8}$$

Research problem

Improve the tightest known lower bound for $A_q(9,4;4)$ (and $q \ge 5$) in a constructive manner.

For $A_q(10,4;5)$ an improved lower bound is described in Example 5.62. In Example 5.31, see also [158, Proposition 3.1], an improved lower bound for $A_q(11,4;4)$ is presented. For $A_q(12,4;6)$ improved lower bounds are obtained in Example 5.60, Example 5.63, and Exercise 5.64. For $A_q(12,6;6)$ and especially $A_2(12,6;6) \ge 16865672$ we refer to Example 5.87.

8 Constructions and bounds for mixed dimension subspace codes

Most parts of this chapter are devoted to lower and upper bounds for CDCs. The analog questions for MDC are also of interest while so far less intensively studied. Here we restrict ourselves to the subspace distance and refer to e.g. [141, 208] for the injection metric. In the classical situation of block codes in the Hamming metric there are back and forth relations between constant weight codes and their unrestricted versions, i.e., inequalities involving both $A(n,d;k)$ and $A(n,d)$ are known. A few, very easy and natural, observations on the relation between $A_q(n,d;k)$ and $A_q(n,d)$ (or $A_q(n,d;T)$ in general) are already known, see e.g. [133]. The inequality $A_q(n,d;T) \le A_q(n,d;T')$ for $T \subseteq T'$, mentioned in the preliminaries in Section 2, e.g. directly implies $A_q(n,d;k) \le A_q(n,d)$. In the other direction we can choose $T \subseteq \{0,1,\dots,n\}$ such that the differences between the occurring dimensions are sufficiently large with respect to a given minimum subspace distance d.

Theorem 8.1. ***(Dimension layers – [133, Theorem 2.5])***

$$\sum_{\substack{k=0 \\ k \equiv \lfloor n/2 \rfloor \bmod d}}^{n} A_q\big(n, 2\lceil d/2 \rceil; k\big) \le A_q(n,d) \le 2 + \sum_{k=\lceil d/2 \rceil}^{n-\lceil d/2 \rceil} A_q\big(n, 2\lceil d/2 \rceil; k\big)$$

We remark that this constitute the best bound for $A_q(n,d)$ that does not depend on information about the cross-distance distribution between different "dimension layers" $\begin{bmatrix} V \\ k \end{bmatrix}$ and $\begin{bmatrix} V \\ l \end{bmatrix}$.

Lemma 8.2. *([133, Lemma 2.4])*
For $1 \le \delta \le k \le \lfloor n/2 \rfloor$ the inequality

$$\frac{A_q(n,2\delta;k)}{A_q(n,2\delta;k-1)} > q^{n-2k+\delta} \cdot C(q,\delta)$$

holds with $C(q,1) = 1$ and $C(q,\delta) = 1 - 1/q$ for $\delta \ge 2$; in particular, $A_q(n,2\delta;k) > q \cdot A_q(n,2\delta;k-1)$. As a consequence, the numbers $A_q(n,2\delta;k)$, $k \in [\delta, v-\delta]$, form a strictly unimodal sequence.

The bounds of Theorem 8.1 coincide for $d = 1$ where we have

$$A_q(n,1) = \sum_{k=0}^{n} A_q(n,2;k) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q. \tag{8.1}$$

For minimum subspace distance $d = n$ we have $A_q(n,n) = 2$ for odd n and $A_q(n,n) = A_q(n,n;k) = q^k + 1$ for $n = 2k$, see [133, Theorem 3.1] and also [90, Section 5] or [89]. In the latter case of an even dimension of the ambient space the maximum number of codewords $q^k + 1$ can only be attained if all codewords have dimension k, i.e., the codes are k-spreads.

Theorem 8.3. ***(Dimension layers are optimal for*** **d = 2** ***– [133, Theorem 3.4])***

(i) If $n = 2k$ is even then

$$A_q(n,2) = \sum_{\substack{0 \le i \le n \\ i \equiv 0 \bmod 2}} \begin{bmatrix} n \\ i \end{bmatrix}_q .$$

The unique (as a set of subspaces) optimal code in $\mathrm{PG}(n-1,q)$ *consists of all subspaces X of $\mathbb{F}_q^n$ with* $\dim(X) \equiv k \bmod 2$*, and thus of all even-dimensional subspaces for $n \equiv 0 \bmod 4$ and of all odd-dimensional subspaces for $n \equiv 2 \bmod 4$.*

(ii) $n = 2k+1$ is odd then

$$A_q(n,2) = \sum_{\substack{0 \le i \le n \\ i \equiv 0 \bmod 2}} \begin{bmatrix} n \\ i \end{bmatrix}_q = \sum_{\substack{0 \le i \le n \\ i \equiv 1 \bmod 2}} \begin{bmatrix} n \\ i \end{bmatrix}_q , \tag{8.2}$$

and there are precisely two distinct optimal codes in $\mathrm{PG}(n-1,q)$*, containing all even-dimensional and all odd-dimensional subspaces of $\mathbb{F}_q^n$, respectively. Moreover these two codes are isomorphic.*

If $n = 2k$ is even then $A_q(n,n-1) = A_q(n,n;k) = q^k + 1$ and $A_q(n,n-1) = A_q(n,n-1;k) = q^{k+1} + 1$ if $n = 2k+1 \ge 5$ is odd, see [133, Theorem 3.2]. Note that we have to exclude the case $A_q(3,2) = q^2 + q + 2$, see Theorem 8.3. The case of subspace distance $d = n-2 \ge 3$ is much more involved and only partial results are known:

Theorem 8.4. *([133, Theorem 3.3])*

(i) If $n = 2k \ge 8$ is even then $A_q(n,n-2) = A_q(n,n-2;k)$, and the known bound $q^{2k} + 1 \le A_q(n,n-2;k) \le (q^k+1)^2$ applies. Moreover, $A_q(4,2) = q^4 + q^3 + 2q^2 + q + 3$ for all q, $A_2(6,4) = 77$ and $q^6 + 2q^2 + 2q + 1 \le A_q(6,4) \le (q^3+1)^2$ for all $q \ge 3$.

(ii) If $n = 2k+1 \ge 5$ is odd then $A_q(n,n-2) \in \{2q^{k+1}+1, 2q^{k+1}+2\}$. Moreover, $A_q(5,3) = 2q^3 + 2$ for all q and $A_2(7,5) = 2 \cdot 2^4 + 2 = 34$.

Note that the bounds for $A_2(n,n-2)$ with odd n were already established in [68, Theorem 5] and $A_2(5,3) = 18$ in [80, Theorem 14]. Further constructions for $A_q(5,3) = 2q^3+2$ are discussed in [53, 99, 100]. The subspace codes attaining the upper bound $A_2(7,5) = 34$ were classified up to isomorphism in [135]. For $k \ge 3$ it was shown in [133] that subspace codes attaining the upper bound $A_q(n,n-2) \in \{2q^{k+1}+1, 2q^{k+1}+2\}$ for $n = 2k+1$ have to consist of $q^{k+1}+1$ codewords of dimension k and also $q^{k+1}+1$ codewords of dimension $k+1$. For dimension k the codewords form a partial k-spread of maximum cardinality $A_q(2k+1,2k;k) = q^{k+1}+1$ and for dimension $k+1$ the codewords form the dual of such a maximum partial k-spread in $\mathrm{PG}(2k,q)$. Some authors also speak of a *doubling construction*.

Research problem

Does a doubling construction exist for $k \ge 4$ or for $k = 3$ and $q \ge 3$?

Also the proven non-existence of a doubling construction is of interest, since it would yields an improve upper bound for $A_q(2k,2k-2;k)$.

The previous results imply that $A_q(n,d)$ is determined for all $n \le 5$:

$$\begin{aligned}
A_q(3,2) &= q^2+q+2, && (8.3)\\
A_q(3,3) &= 2, && (8.4)\\
A_q(4,2) &= q^4+q^3+2q^2+q+3, && (8.5)\\
A_q(4,3) &= q^2+1, && (8.6)\\
A_q(4,4) &= q^2+1, && (8.7)\\
A_q(5,2) &= q^6+q^5+3q^4+3q^3+3q^2+2q+3, && (8.8)\\
A_q(5,3) &= 2q^3+2, && (8.9)\\
A_q(5,4) &= q^3+1, \text{ and} && (8.10)\\
A_q(5,5) &= 2. && (8.11)
\end{aligned}$$

ILP formulations for the exact determination of $A_q(n,d)$ and bounds for $A_2(n,d)$, where $n \le 8$, are provided in [128]. In [79] an LP upper bound for $A_q(n,3)$ was presented. Another LP upper bound for the general case $A_q(n,d)$ can be found in [2]. For upper bounds based on semidefinite programming we refer to [15, 121]. The Johnson upper bound for `CDC`s from Theorem 4.15 was adjusted to `MDC`s in [136]. There also the refinement using results for divisible codes is discussed. A few general lower bounds for `MDC`s are surveyed in [142].

n/d	1	2	3	4	5	6	7	8
1	2							
2	5	3						
3	16	8	2					
4	67	37	5	5				
5	374	187	18	9	2			
6	2825	1521	108–117	77	9	9		
7	29212	14606	614–776	334–388	34	17	2	
8	417199	222379	5687–9191	4803–6479	263–326	257	17	17

Table 8.1: Exact values and bounds for $A_2(n,d)$.

Research problem

Improve a few lower or upper bounds for $A_2(n,d)$, see Table 8.1.

9 Variants of subspace codes

In this section we want to briefly discuss topics that are closely related to the concept of subspace codes. For block codes the (Hamming) weights of codewords as well as the minimum Hamming distance are important invariants. For linear codes one may also consider the cardinality of the support of the 2-dimensional subcode spanned by two codewords (which have to be linearly independent). This idea can of course be generalized and leads to the notion of *generalized Hamming weights* for linear codes, see e.g. [114, 129, 224]. For networks and subspace codes the notion was generalized in [182] and [18], respectively. The latter considered the dimension of the span of triples of codewords.

Research problem

Study the distribution of combinations of the span and the intersection for triples and quadruples of codewords in CDCs.

Having a minimum subspace distance of at least d for a given CDC $C \subseteq \mathcal{G}_q(n,k)$ is equivalent to the property that the dimension of the intersection of two different codewords is at most $k - d/2$. In other words, every $(k - d/2 + 1)$-space is contained in at most one codeword. A natural generalization of CDCs is to ask for subsets $C \subseteq \mathcal{G}_q(n,k)$ such that every t-space is covered at most λ times, see e.g. [73, 74]. One may also ask for subsets $C \subseteq \mathcal{G}_q(n,k)$ such that every t-space is covered at least once (or at least λ times), see e.g. [69].

Instead of $\mathrm{PG}(n-1,q)$ as ambient space we can also consider subspace codes over different over different geometries over finite fields, see e.g. [222]. For first results into this direction we refer to e.g. [95, 96, 97, 98, 106]. For affine spaces we refer to [183].

Research problem

For $A_q(n,d;k)$ with $d < 2k$ and $2k \le n$ almost all of the tightest known upper bounds are implied by the improved Johnson bound in Theorem 4.42, which is based on divisible codes. Develop a similar theory of divisible codes and generalize the approach of the improved Johnson bound to the settings of the paper mentioned above.

In Subsection 9.1 we briefly consider equidistant subspace codes and flag codes in Subsection 9.2.

9.1 Equidistant subspace codes

Partial k-spreads or CDCs minimum subspace distance $d = 2k$, where $n \ge 2k$, are a special class of so-called *equidistant subspace codes*. These are subspace codes where any two different codewords have the same distance. Another special class of equidistant codes are so-called *sunflowers* where all codewords pairwise intersect in the same subspace, say of dimension t. For the classical set case "$q = 1$", i.e. equidistant block codes in the Hamming metric, we refer the interested reader e.g. to[58, 86, 107, 210, 220]. Of course, geometers have already studied the case $q \geq 2$, see e.g. [19, 31, 64, 65].

By $B_q(n,t;k)$ we denote the maximum number of k-spaces in $\mathrm{PG}(n-1,q)$ such that the intersection of each pair of different k-spaces has dimension exactly t. We also speak of *t-intersecting equidistant codes of k-spaces in* $\mathrm{PG}(n-1,q)$.

Exercise 9.1. *Show that $B_q(n,t;k) = 1$ for $t < 2k-n$ and that the maximum cardinality of a sunflower is $A_q(n-t, 2(k-t); k-t)$ if $t \geq 2k-n$.*

Theorem 9.2. *([75, Theorem 1])*
If C in $\mathcal{G}_q(n,k)$ is a t-intersecting equidistant code with

$$\#C \geq \left(\frac{q^k - q^t}{q-1}\right)^2 + \frac{q^k - q^t}{q-1} + 1,$$

then C is a sunflower.

For $2k > n$ we obtain $B_q(n,t;k) = B_q(n, n-2k+t; n-k)$ by duality. So, optimal codes can also be duals of sunflowers and it remains to restrict to the cases where $2k \leq n$.

Exercise 9.3. *Show $B_q(n,1;2) = [n-1]_q$, $B_2(3,1;3) = 1$, $B_2(4,1;3) = 1$, $B_2(5,1;3) = 9$, $B_2(n,1;3) = A_2(n-1,4;2)$ for $n \geq 7$, and that all values are attained by sunflowers or the dual of a sunflower.*

Sunflower codes and their properties have e.g. been investigated in [21, 35, 59, 103, 173, 174]. In general it seems to be easier to determine $B_q(v,t;k)$ if q gets larger, see e.g. [31], so that we here focus on the binary case $q = 2$. Cf. the remark in the third paragraph of the first section in [36] on the "unusual property" of $\mathbb{F}_2$ in our context. In [23] $B_2(6,1;3) = 20 > 9$ was proven, i.e., the optimal equidistant codes for these parameters are not given by sunflowers or their dual codes.

An $m \times n$ equidistant rank metric code over $\mathbb{F}_q$ with rank distance d is a set $\mathcal{M}$ of $m \times n$ matrices over $\mathbb{F}_q$ such that for each pair of different $M, M' \in \mathcal{M}$ we have $d_{\mathrm{R}}(M, M') = d$. As an example, the five matrices

$$\begin{pmatrix} 0&0&0&0\\ 1&0&1&1\\ 1&1&1&1\\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&1\\ 1&1&1&0\\ 0&0&0&0\\ 0&0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&1&1\\ 1&0&1&1\\ 0&1&0&1\\ 0&1&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&1\\ 1&0&1&1\\ 1&1&0&1\\ 1&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&1&1&0\\ 0&1&0&1\\ 1&0&0&1\\ 1&0&1&0 \end{pmatrix}$$

span a linear 4×4 equidistant rank metric code over $\mathbb{F}_2$ with rank distance 3. By [61, Theorem 6] there cannot be six such matrices. By prepending a suitable unit matrix, i.e. by lifting, we obtain an equidistant subspace code in general. So, our example gives $B_2(8,1;4) \geq 32$. We remark that several linear 4×4 equidistant rank metric codes over $\mathbb{F}_2$ with rank distance 3 and cardinality 2^5 exist and that their lifted versions allow the addition of further codewords. By a computer search up to 8 additional codewords can be found easily, so that $B_2(8,1;4) \geq 40$.

Research problem

Determine the exact value of $B_2(8,1;4)$.

Another example is given by the four matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

which span a linear 3×3 equidistant rank metric code over $\mathbb{F}_2$ with rank distance 2. By [61, Theorem 6] there cannot be five such matrices. Note that this gives $B_2(6, 1; 3) \geq 16$. In [75] an equidistant code with these parameters was stated by explicitly listing 16 codewords. There it was mentioned as a counter example to a conjecture attributed to Deza, i.e., if a t-intersection equidistant code of k-subspaces in $\mathrm{PG}(v-1, q)$ has more than $\begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q$ codewords, then it is a sunflower. In [36] the author determined, using an exhaustive MAGMA search, that there are exactly 1176 binary linear 3×3 equidistant rank metric codes over $\mathbb{F}_2$ with rank distance 2 and dimension 4. Under conjugation by $\mathrm{GL}(3, 2)$ they fall into 12 orbits, which are explicitly listed. An example of a binary linear 4×4 equidistant rank metric code over $\mathbb{F}_2$ with rank distance 3 and dimension 5 as well as a linear 5×5 equidistant rank metric code over $\mathbb{F}_2$ with rank distance 4 and dimension 6, found by a heuristic search using MAGMA, is also stated there. By [61, Theorem 6] the dimension is extremal in both cases. However, the resulting lower bounds $B_2(8, 1; 4) \geq 32$ and $B_2(10, 1; 5) \geq 64$ have not found their way into the literature on equidistant subspace codes. With respect to the two latter bounds we mention the example

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

which shows $B_2(9, 1; 4) \geq 64$, see also [61, Example 1]. By [61, Theorem 6] there cannot be seven such matrices.

According to [25] the problem of determining lower and upper bounds for rank-k-spaces in $\mathbb{F}_q m \times n$ has been studied by matrix theorists, group theorists, and algebraic geometers, see his list of references and [37, 104].

We remark $B_2(11, 1; 5), B_2(12, 1; 6) \geq 64$ since corresponding linear equidistant rank metric codes can be found easily. However, [61, Theorem 6] might allow even linear equidistant rank metric codes of cardinality 2^7.

Research problem

Study linear equidistant rank metric codes and their extendability to equidistant subspace codes.

Instead of restricting the dimension of the pairwise intersection of codewords to a single dimension one might also allow e.g. two possible intersection dimensions, see [172].

9.2 Flag codes

A *full flag* in $\mathrm{PG}(n-1,q)$ is a sequence of nested subspaces with dimensions from 1 to $n-1$. If not all of these dimensions need to occur, we speak of a *flag*. (Full) *flag codes* are collections of flags. The use of flag codes for network coding was proposed in [167]. In [166] the author argues that subspace coding with flags can be ranged between random linear network coding, using constant dimension codes, and optimized routing solutions, whose computation is time-consuming. The interested reader can find more details on this e.g. in [83, 166, 167, 168]. For special multicast networks network coding solutions also lead to hard combinatorial problems, see e.g. [43, 74] for so-called generalized combination networks.

The set of all subspaces in $\mathrm{PG}(n-1,q)$ is turned into a metric space via the *injection distance*

$$\begin{aligned} d_{\mathrm{I}}(U,W) &= \dim(U+W)-\min\{\dim(U),\dim(W)\} \\ &= \max\{\dim(U),\dim(W)\}-\dim(U\cap W) \end{aligned} \tag{9.1}$$

as it is the case for the subspace distance. Note that for $U,W\in\mathcal{G}_q(n,k)$ we have $d_{\mathrm{I}}(U,W)=\dim(U+W)-k=k-\dim(U\cap W)$.

Definition 9.4. *A* flag *is a list of subspaces* $\Lambda=(W_1,\dots,W_m)$ *of* $\mathrm{PG}(n-1,q)$ *with*

$$\{0\}<W_1<\cdots<W_m<\mathbb{F}_q^n.$$

The type *of* $\Lambda=(W_1,\dots,W_m)$ *is the set of dimensions*

$$\mathrm{type}(\Lambda):=\{\dim(W_i)\mid 1\le i\le m\}\subseteq\{1,\dots,n-1\}.$$

Let

$$\mathcal{F}(n,q):=\{\Lambda\mid\Lambda\textit{ is a flag in }\mathrm{PG}(n-1,q)\}$$

denote the set of all flags in $\mathrm{PG}(n-1,q)$ *and for* $T\subseteq\{1,\dots,n-1\}$ *let*

$$\mathcal{F}_T(n,q):=\{\Lambda\in\mathcal{F}(n,q)\mid\mathrm{tpye}(\Lambda)=T\}$$

be the set of all flags of $\mathrm{PG}(n-1,q)$ *of type* T*.*

As noted in [167], the intersection of two flags is again a flag and the set of all flags in $\mathrm{PG}(n-1,q)$ forms a simplicial complex (with respect to inclusion).

Definition 9.5. *Let* $\Lambda=(W_1,\dots,W_m)$ *and* $\Lambda'=\left(W_1',\dots,W_m'\right)$ *be two flags of* $\mathrm{PG}(n-1,q)$ *of the same type* $T=\{k_1,\dots,k_m\}$ *with* $k_i=\dim(W_i)=\dim(W_i')$ *for all* $1\le i\le m$*. Then, the* Grassmann distance *is defined as*

$$d_G(\Lambda,\Lambda'):=\sum_{i=1}^m d_I(W_i,W_i')=\sum_{i=1}^m\left(k_i-\dim(W_i\cap W_i')\right).$$

So, for $m=1$ the Grassmann distance corresponds to the injection distance, i.e., half the subspace distance, between W_1 and W_1'. For $U,W\in\mathcal{G}_q(n,k)$ we have $0\le d_{\mathrm{I}}(U,W)\le\min\{k,v-k\}$, so that we set

$$m(n,T)=(\min\{k_1,n-k_1\},\dots,\min\{k_m,n-k_m\}),$$

where $T = \{k_1, \dots, k_m\} \subseteq \{1, \dots, n-1\}$ with $k_1 < \cdots < k_m$. If $T = \{1, \dots, n-1\}$ we just write $m(n)$ instead of $m(n,T)$. Denoting by x_i the ith component for each vector $x \in \mathbb{R}^n$ we state

$$d_{\mathrm{G}}(\Lambda, \Lambda') \le \sum_i m(n,T)_i$$

for all $\Lambda, \Lambda' \in \mathcal{F}_T(n,q)$. As mentioned in [167, Remark 4.5] we have $1 \le d_{\mathrm{G}}(\Lambda, \Lambda') \le \lfloor (n/2)^2 \rfloor$ for two distinct flags in $\mathrm{PG}(n-1,q)$. A *flag code* C of type T is a collection of flags in $\mathrm{PG}(n-1,q)$ of type T. The minimum distance $d_{\mathrm{G}}(C)$ is the minimum of $d_{\mathrm{G}}(\Lambda, \Lambda')$ over all pairs of distinct elements $\Lambda, \Lambda' \in C$. By $A_q^f(n,d;T)$ we denote the maximum possible cardinality of a flag code C of type T in $\mathrm{PG}(n-1,q)$ that has minimum Grassmann distance at least d. The case of full flags, i.e. $T = \{1, \dots, n-1\}$, is abbreviated as $A_q^f(n,d)$. The *dual* of a flag $\Lambda = (W_1, \dots, W_m)$ in $\mathrm{PG}(n-1,q)$ of type $T \subseteq \{1, \dots, n-1\}$, denoted by $\Lambda^\top$, is given by $\left(W_m^\top, \dots, W_1^\top\right)$. Since we have $d_{\mathrm{I}}(U,W) = d_{\mathrm{I}}(U^\top, W^\top)$ for each $U, W \in \mathcal{G}_q(n,k)$, for some arbitrary integer k, the minimum Grassmann distance $d(C)$ of a flag code of type T in $\mathrm{PG}(n-1,q)$ is the same as $d(C^\top)$, where $C^\top := \{\Lambda^\top \mid \Lambda \in C\}$. Moreover, we have

$$\mathrm{type}\left(C^\top\right) = \{n - t \mid t \in \mathrm{type}(C)\},$$

so that $A_q^f(n,d;T) = A_q^f(n,d;n-t)$.

The arguably easiest case for the determination of $A_q^f(n,d;T)$ is minimum Grassmann distance $d = 1$, where $A_q^f(n,1;T) = \#\mathcal{F}_T(n,q)$. If $T = \{k_1, \dots, k_m\}$ with $0 < k_1 < \cdots < k_m < n$, then we have

$$A_q^f(n,1;T) = \begin{bmatrix} n \\ k_1 \end{bmatrix}_q \cdot \prod_{i=2}^{m} \begin{bmatrix} n - k_{i-1} \\ k_i - k_{i-1} \end{bmatrix}_q \tag{9.2}$$

and

$$A_q^f(n,1) = \prod_{i=2}^{n} \frac{q^i - 1}{q - 1}. \tag{9.3}$$

For the maximum possible minimum Grassmann distance $d = \lfloor (n/2)^2 \rfloor$ we have:

Proposition 9.6. *([157, Proposition 2.4])*
For each integer $k \ge 1$ we have

$$A_q^f(2k, k^2) = q^k + 1$$

and for each integer $k \ge 2$ we have

$$A_q^f(2k+1, k^2+k) = q^{k+1} + 1.$$

We remark that the case $n = 2k$ of Proposition 9.6 was also proven in [7], where the authors also give a decoding algorithm and further details. In [157, Proposition 2.6] the exact value

$$A_q^f(4,3) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = q^3 + q^2 + q + 1 \tag{9.4}$$

was determined. In Table 9.1 and Table 9.2 we present the current knowledge on $A_2^f(n,d)$ from [157]. Research on bounds and constructions for flag codes currently is quite an active research field, see e.g. [5, 6, 7, 8, 9, 10, 11, 157, 181].

Research problem

Find improved lower and upper bounds for $A_q^f(n,d)$.

n/d	1	2	3	4	5	6
2	3					
3	21	7				
4	315	105	15	5		
5	9765	3120–3255	465	155	31	9

Table 9.1: Bounds and exact values for $A_2^f(n, d)$ for $n \leq 5$.

n/d	1	2	3	4	5	6	7	8	9	10	11	12
6	**615195**	205065	29295	9765	1953	567	**63**	**21**	**9**			
7	**78129765**	26043255	3720465	1240155	248031	72009	8001	2667	1143	127	41	**17**

Table 9.2: Upper bounds for $A_2^f(6, d)$ and $A_2^f(7, d)$ (tight bounds in bold).

Bibliography

[1] Erik Agrell, Alexander Vardy, and Kenneth Zeger. A table of upper bounds for binary codes. *IEEE Transactions on Information Theory*, 47(7):3004–3006, 2001.

[2] Rudolf Ahlswede and Harout (Haratyun) K. Aydinian. On error control codes for random network coding. In *2009 Workshop on Network Coding, Theory, and Applications (NetCod 2009), Lausanne, Switzerland*, pages 68–73. IEEE, 2009.

[3] Rudolf Ahlswede, Harout (Haratyun) K. Aydinian, and Levon H. Khachatrian. On perfect codes and related concepts. *Designs, Codes and Cryptography*, 22(3):221–237, 2001.

[4] Jingmei Ai, Thomas Honold, and Haiteng Liu. The expurgation-augmentation method for constructing good plane subspace codes. *arXiv preprint 1601.01502*, 2016.

[5] Clementa Alonso-González and Miguel Ángel Navarro-Pérez. Consistent flag codes. *Mathematics*, 8(12):19 pp., 2020.

[6] Clementa Alonso-González and Miguel Ángel Navarro-Pérez. Cyclic orbit flag codes. *Designs, Codes and Cryptography*, 89(19):2331–2356, 2021.

[7] Clementa Alonso-González, Miguel Ángel Navarro-Pérez, and Xaro Soler-Escrivà. Flag codes from planar spreads in network coding. *Finite Fields and Their Applications*, 68:19 pp., 2020.

[8] Clementa Alonso-González, Miguel Ángel Navarro-Pérez, and Xaro Soler-Escrivà. Distance and bounds for flag codes. *arXiv preprint 2111.00910*, 2021.

[9] Clementa Alonso-González, Miguel Ángel Navarro-Pérez, and Xaro Soler-Escrivà. Flag codes: Distance vectors and cardinality bounds. *Linear Algebra and its Applications*, 656:27–62, 2023.

[10] Clementa Alonso-González, Miguel Ángel Navarro-Pérez, and Xaro Soler-Escrivà. An orbital construction of optimum distance flag codes. *Finite Fields and Their Applications*, 73:21 pp., 2021.

[11] Clementa Alonso-González, Miguel Ángel Navarro-Pérez, and Xaro Soler-Escrivà. Optimum distance flag codes from spreads via perfect matchings in graphs. *Journal of Algebraic Combinatorics*, 54(4):1279–1297, 2021.

[12] George Eyre Andrews. q-analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher. *Discrete Mathematics*, 204(1-3):15–25, 1999.

[13] Jared Antrobus and Heide Gluesing-Luerssen. Maximal Ferrers diagram codes: constructions and genericity considerations. *IEEE Transactions on Information Theory*, 65(10):6204–6223, 2019.

[14] Jared E. Antrobus. *The state of Lexicodes and Ferrers diagram rank-metric codes*. PhD thesis, University of Kentucky, 2019.

[15] Christine Bachoc, Alberto Passuello, and Frank Vallentin. Bounds for projective codes from semidefinite programming. *Advances in Mathematics of Communications*, 7(2):127–145, 2013.

[16] Ronald Dee Baker. Partitioning the planes of $\mathrm{AG}_{2m}(2)$ into 2-designs. *Discrete Mathematics*, 15(3):205–211, 1976.

[17] Ronald Dee Baker, Jacobus Hendricus Van Lint, and Richard Michael Wilson. On the Preparata and Goethals codes. *IEEE Transactions on Information Theory*, 29(3):342–345, 1983.

[18] Edoardo Ballico. Higher distances for constant dimensions codes: the case of osculating spaces to a Veronese variety. *Afrika Matematika*, 27(5):1003–1020, 2016.

[19] Edoardo Ballico, Nadia Chiarli, and Silvio Greco. Families of projective planes meeting in codimension two. *Results in Mathematics*, 46(1-2):16–23, 2004.

[20] Elena Barcucci, Alberto Del Lungo, Elisa Pergola, and Renzo Pinzani. Some combinatorial interpretations of q-analogs of schröder numbers. *Annals of Combinatorics*, 3(2):171–190, 1999.

[21] Roland D. Barrolleta, Emilio Suárez-Canedo, Leo Storme, and Peter Vandendriessche. On primitive constant dimension codes and a geometrical sunflower bound. *Advances in Mathematics of Communications*, 11(4), 2017.

[22] Daniele Bartoli, Matteo Bonini, and Massimo Giulietti. Constant dimension codes from Riemann-Roch spaces. *arXiv preprint 1508.01727*, 2015.

[23] Daniele Bartoli and Francesco Pavese. A note on equidistant subspace codes. *Discrete Applied Mathematics*, 198:291–296, 2016.

[24] Riccardo Bassoli, Hugo Marques, Jonathan Rodriguez, Kenneth W. Shum, and Rahim Tafazolli. Network coding theory: A survey. *IEEE Communications Surveys & Tutorials*, 15(4):1950–1978, 2013.

[25] LeRoy B. Beasley. Spaces of rank-2 matrices over $GF(2)$. *Electronic Journal of Linear Algebra*, 5:11–18, 1999.

[26] Eli Ben-Sasson, Tuvi Etzion, Ariel Gabizon, and Netanel Raviv. Subspace polynomials and cyclic subspace codes. *IEEE Transactions on Information Theory*, 62(3):1157–1165, 2016.

[27] Albrecht Beutelspacher. On parallelisms in finite projective spaces. *Geometriae Dedicata*, 3(1):35–40, 1974.

[28] Albrecht Beutelspacher. Partial spreads in finite projective spaces and partial designs. *Mathematische Zeitschrift*, 145(3):211–229, 1975.

[29] Albrecht Beutelspacher. Correction to "partial spreads in finite projective spaces and partial designs". *Mathematische Zeitschrift*, 147(3):303–303, 1976.

[30] Albrecht Beutelspacher. Partial parallelisms in finite projective spaces. *Geometriae Dedicata*, 36(2-3):273–278, 1990.

[31] Albrecht Beutelspacher, Jörg Eisfeld, and Jörg Müller. On sets of planes in projective spaces intersecting mutually in one point. *Geometriae Dedicata*, 78(2):143–159, 1999.

[32] Jürgen Bierbrauer. A direct approach to linear programming bounds for codes and tms-nets. *Designs, Codes and Cryptography*, 42(2):127–143, 2007.

[33] Jürgen Bierbrauer. *Introduction to Coding Theory*. Chapman and Hall/CRC, 2016.

[34] Simon R. Blackburn and Tuvi Etzion. The asymptotic behavior of Grassmannian codes. *IEEE Transactions on Information Theory*, 58(10):6605–6609, 2012.

[35] Aart Blokhuis, Maarten De Boeck, and Jozefien D'Haeseleer. On the sunflower bound for k-spaces, pairwise intersecting in a point. *Designs, Codes and Cryptography*, page 11 pp., to appear.

[36] Nigel Boston. Spaces of constant rank matrices over $GF(2)$. *Electronic Journal of Linear Algebra*, 20(1):1, 2010.

[37] Nigel Boston. Spaces of constant rank matrices over $GF(2)$. *Electronic Journal of Linear Algebra*, 27(1):624, 2014.

[38] Michael Braun, Tuvi Etzion, and Alexander Vardy. Linearity and complements in projective space. *Linear Algebra and its Applications*, 438(1):57–70, 2013.

[39] Michael Braun, Patric RJ Östergård, and Alfred Wassermann. New lower bounds for binary constant-dimension subspace codes. *Experimental Mathematics*, 27(2):179–183, 2018.

[40] Andries Evert Brouwer, James Bergheim Shearer, Neil James Alexander Sloane, and Warren Douglas Smith. A new table of constant weight codes. *IEEE Transactions on Information Theory*, 36(6):1334–1380, 2006.

[41] Eimear Byrne, Heide Gluesing-Luerssen, and Alberto Ravagnani. Fundamental properties of sum-rank-metric codes. *IEEE Transactions on Information Theory*, 67(10):6456–6475, 2021.

[42] Eimear Byrne and Alberto Ravagnani. Covering radius of matrix codes endowed with the rank metric. *SIAM Journal on Discrete Mathematics*, 31(2):927–944, 2017.

[43] Han Cai, Tuvi Etzion, Moshe Schwartz, and Antonia Wachter-Zeh. Network coding solutions for the combination network and its subgraphs. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pages 862–866. IEEE, 2019.

[44] Bocong Chen and Hongwei Liu. Constructions of cyclic constant dimension codes. *Designs, Codes and Cryptography*, 86(6):1267–1279, 2018.

[45] Hao Chen, Xianmang He, Jian Weng, and Liqing Xu. New constructions of subspace codes using subsets of MRD codes in several blocks. *IEEE Transactions on Information Theory*, 66(9):5317–5321, 2020.

[46] Bruce Nathan Cooperstein. External flats to varieties in $\mathrm{PG}(\wedge^2(v))$ over finite fields. *Geometriae Dedicata*, 69(3):223–235, 1998.

[47] Antonio Cossidente, Sascha Kurz, Giuseppe Marino, and Francesco Pavese. Combining subspace codes. *Advances in Mathematics of Communications*, 17(3):1–15, 2023.

[48] Antonio Cossidente, Giuseppe Marino, and Francesco Pavese. Subspace code constructions. *Ricerche di Matematica*, 71(1):1–16, 2022.

[49] Antonio Cossidente and Francesco Pavese. On subspace codes. *Designs, Codes and Cryptography*, 78(2):527–531, 2016.

[50] Antonio Cossidente and Francesco Pavese. Veronese subspace codes. *Designs, Codes and Cryptography*, 81(3):445–457, 2016.

[51] Antonio Cossidente and Francesco Pavese. Subspace codes in $\mathrm{PG}(2n-1,q)$. *Combinatorica*, 37(6):1073–1095, 2017.

[52] Antonio Cossidente, Francesco Pavese, and Leo Storme. Geometrical aspects of subspace codes. In Marcus Greferath, Mario Osvin Pavčević, Natalia Silberstein, and María Ángeles Vázquez-Castro, editors, *Network Coding and Subspace Designs*, pages 107–129. Springer, 2018.

[53] Antonio Cossidente, Francesco Pavese, and Leo Storme. Optimal subspace codes in $\mathrm{PG}(4,q)$. *Advances in Mathematics of Communications*, 13(3):393–404, 2019.

[54] Philippe Delsarte. *An algebraic approach to the association schemes of coding theory*. PhD thesis, Université Catholique de Louvain, Eindhoven, 6 1973. Philips Research Reports Supplements, No. 10.

[55] Philippe Delsarte. Bilinear forms over a finite field, with applications to coding theory. *Journal of Combinatorial Theory, Series A*, 25(3):226–241, 1978.

[56] Philippe Delsarte. Hahn polynomials, discrete harmonics, and t-designs. *SIAM Journal on Applied Mathematics*, 34(1):157–166, 1978.

[57] Philippe Delsarte and Vladimir Iosifovich Levenshtein. Association schemes and coding theory. *IEEE Transactions on Information Theory*, 44(6):2477–2504, 1998.

[58] Michel Deza and Peter Frankl. Every large set of equidistant $(0, +1, -1)$-vectors forms a sunflower. *Combinatorica*, 1(3):225–231, 1981.

[59] Jozefien D'Haeseleer. *Families of intersecting subspaces*. PhD thesis, Ghent University, 2021.

[60] David Allyn Drake and J.W. Freeman. Partial t-spreads and group constructible (s, r, μ)-nets. *Journal of Geometry*, 13(2):210–216, 1979.

[61] Jean-Guillaume Dumas, Rod Gow, Gary McGuire, and John Sheekey. Subspaces of matrices with special rank properties. *Linear algebra and its applications*, 433(1):191–202, 2010.

[62] Charles F. Dunkl. An addition theorem for some q-Hahn polynomials. *Monatshefte für Mathematik*, 85(1):5–37, 1978.

[63] J. Eisfeld and L. Storme. t-spreads and minimal t-covers in finite projective spaces. *Lecture notes, Universiteit Gent, 29 pages*, 2000.

[64] Jörg Eisfeld. On sets of sets mutually intersecting in exactly one element. *Journal of Geometry*, 67(1-2):96–104, 2000.

[65] Jörg Eisfeld. On sets of n-dimensional subspaces of projective spaces intersecting mutually in an $(n-2)$-dimensional subspace. *Discrete Mathematics*, 255(1-3):81–85, 2002.

[66] Hesham El Gamal and Arthur Roger Hammons. On the design of algebraic space-time codes for MIMO block-fading channels. *IEEE Transactions on Information Theory*, 49(1):151–163, 2003.

[67] Saad Ibrahim El-Zanati, Heather Jordon, George Francis Seelinger, Papa Amar Sissokho, and Lawrence Edward Spence. The maximum size of a partial 3-spread in a finite vector space over $GF(2)$. *Designs, Codes and Cryptography*, 54(2):101–107, 2010.

[68] Tuvi Etzion. Problems on q-analogs in coding theory. *arXiv preprint 1305.6126*, 2013.

[69] Tuvi Etzion. Covering of subspaces by subspaces. *Designs, Codes and Cryptography*, 72(2):405–421, 2014.

[70] Tuvi Etzion. Partial-parallelisms in finite projective spaces. *Journal of Combinatorial Designs*, 23(3):101–114, 2015.

[71] Tuvi Etzion, Elisa Gorla, Alberto Ravagnani, and Antonia Wachter-Zeh. Optimal Ferrers diagram rank-metric codes. *IEEE Transactions on Information Theory*, 62(4):1616–1630, 2016.

[72] Tuvi Etzion and Niv Hooker. Residual q-Fano planes and related structures. *arXiv preprint 1704.07714*, 2017.

[73] Tuvi Etzion, Sascha Kurz, Kamil Otal, and Ferruh Özbudak. Subspace packings. In *The Eleventh International Workshop on Coding and Cryptography 2019 : WCC Proceedings*. IEEE, Saint-Jacut-de-la-Mer, 2019.

[74] Tuvi Etzion, Sascha Kurz, Kamil Otal, and Ferruh Özbudak. Subspace packings: constructions and bounds. *Designs, Codes and Cryptography*, 88:1781–1810, 2020.

[75] Tuvi Etzion and Netanel Raviv. Equidistant codes in the Grassmannian. *Discrete Applied Mathematics*, 186:87–97, 2015.

[76] Tuvi Etzion and Natalia Silberstein. Error-correcting codes in projective spaces via rank-metric codes and Ferrers diagrams. *IEEE Transactions on Information Theory*, 55(7):2909–2919, 2009.

[77] Tuvi Etzion and Natalia Silberstein. Codes and designs related to lifted MRD codes. *IEEE Transactions on Information Theory*, 59(2):1004–1017, 2013.

[78] Tuvi Etzion and Leo Storme. Galois geometries and coding theory. *Designs, Codes and Cryptography*, 78(1):311–350, 2016.

[79] Tuvi Etzion and Alexander Vardy. Error-correcting codes in projective space. In *Proceedings. International Symposium on Information Theory, 2008. ISIT 2008.*, pages 871–875. IEEE, 2008.

[80] Tuvi Etzion and Alexander Vardy. Error-correcting codes in projective space. *IEEE Transactions on Information Theory*, 57(2):1165–1173, 2011.

[81] Tuvi Etzion and Alexander Vardy. Automorphisms of codes in the Grassmann scheme. *arXiv preprint 1210.5724*, 2012.

[82] Tao Feng, Sascha Kurz, and Shuangqing Liu. Bounds for the multilevel construction. *arXiv preprint 2011.06937*, 2020.

[83] Ghislain Fourier and Gabriele Nebe. Degenerate flag varieties in network coding. *Advances in Mathematics of Communications*, 17(4):888–899, 2023.

[84] Péter Frankl and Vojtěch Rödl. Near perfect coverings in graphs and hypergraphs. *European Journal of Combinatorics*, 6(4):317–326, 1985.

[85] Péter Frankl and Richard Michael Wilson. The Erdős-Ko-Rado theorem for vector spaces. *Journal of Combinatorial Theory, Series A*, 43(2):228–236, 1986.

[86] Fang-Wei Fu, Torleiv Kløve, Yuan Luo, and Victor K Wei. On equidistant constant weight codes. *Discrete Applied Mathematics*, 128(1):157–164, 2003.

[87] Ernst Muhamedovich Gabidulin. Theory of codes with maximum rank distance. *Problemy Peredachi Informatsii*, 21(1):3–16, 1985.

[88] Ernst Muhamedovich Gabidulin. *Rank codes*. TUM. University Press, 2021. translated by Vladimir Sidorenko.

[89] Ernst Muhamedovich Gabidulin and Martin Bossert. Codes for network coding. In *Proceedings. International Symposium on Information Theory, 2008. ISIT 2008.*, pages 867–870. IEEE, 2008.

[90] Ernst Muhamedovich Gabidulin and Martin Bossert. Algebraic codes for network coding. *Problems of Information Transmission*, 45(4):343–356, 2009.

[91] Ernst Muhamedovich Gabidulin, Nina Ivanovna Pilipchuk, and Oksana Viacheslavovna Trushina. Bounds on the cardinality of subspace codes with non-maximum code distance. *Problems of Information Transmission*, 57(3):241–247, 2021.

[92] Philippe Gaborit and Jean-Christophe Deneuville. Code-based cryptography. In William Cary Huffman, Jon-Lark Kim, and Patrick Solé, editors, *Concise Encyclopedia of Coding Theory*, pages 799–822. Chapman and Hall/CRC, 2021.

[93] Maximilien Gadouleau and Zhiyuan Yan. Constant rank codes. In *Proceedings. International Symposium on Information Theory, 2008. ISIT 2008.*, pages 876–880. IEEE, 2008.

[94] Maximilien Gadouleau and Zhiyuan Yan. Constant-rank codes and their connection to constant-dimension codes. *IEEE Transactions on Information Theory*, 56(7):3207–3216, 2010.

[95] You Gao, Jinru Gao, and Gang Wang. Bounds on subspace codes based on subspaces of type $(s, 0, 0, 0)$ in pseudo-sympletic spaces and singular pseudo-symplectic spaces. *Applied Mathematics and Computation*, 407:11 pp., 2021.

[96] You Gao and Gang Wang. Bounds on subspace codes based on subspaces of type in singular linear space. *Journal of Applied Mathematics*, page 9 pp., 2014.

[97] You Gao and Gang Wang. Error-correcting codes in attenuated space over finite fields. *Finite Fields and Their Applications*, 33:103–117, 2015.

[98] You Gao, Liyum Zhao, and Gang Wang. Bounds on subspace codes based on totally isotropic subspaces in unitary spaces. *Discrete Mathematics, Algorithms and Applications*, 8(04):1650056, 2016.

[99] Anirban Ghatak. Optimal binary $(5, 3)$ projective space codes from maximal partial spreads. *arXiv preprint 1701.07245*, 2017.

[100] Anirban Ghatak and Sumanta Mukherjee. Intersection patterns in optimal binary $(5, 3)$ doubling subspace codes. *arXiv preprint 2105.01584*, 2021.

[101] Heide Gluesing-Luerssen, Katherine Morrison, and Carolyn Troha. Cyclic orbit codes and stabilizer subfields. *Advances in Mathematics of Communications*, 9(2):177–197, 2015.

[102] Heide Gluesing-Luerssen and Carolyn Troha. Construction of subspace codes through linkage. *Advances in Mathematics of Communications*, 10(3):525–540, 2016.

[103] Elisa Gorla and Alberto Ravagnani. Equidistant subspace codes. *Linear Algebra and its Applications*, 490:48–65, 2016.

[104] Rod Gow. Dimension bounds for constant rank subspaces of symmetric bilinear forms over a finite field. *arXiv preprint 1502.05547*, 2015.

[105] Marcus Greferath, Mario Osvin Pavčević, Natalia Silberstein, and María Ángeles Vázquez-Castro. *Network coding and subspace designs*. Springer, 2018.

[106] Mahdieh Hakimi Poroch. Bounds on subspace codes based on totally isotropic subspace in symplectic spaces and extended symplectic spaces. *Asian-European Journal of Mathematics*, 12(05):1950069, 2019.

[107] Jonathan I. Hall. Bounds for equidistant codes and partial projective planes. *Discrete Mathematics*, 17(1):85–94, 1977.

[108] Johan P. Hansen. Riemann-Roch spaces and linear network codes. *Computer Science*, 10(1):1–11, 2015.

[109] Xianmang He. Construction of constant dimension codes from two parallel versions of linkage construction. *IEEE Communications Letters*, 24(11):2392–2395, 2020.

[110] Xianmang He, Yindong Chen, and Zusheng Zhang. Improving the linkage construction with Echelon-Ferrers for constant-dimension codes. *IEEE Communications Letters*, 24(9):1875–1879, 2020.

[111] Xianmang He, Yindong Chen, Zusheng Zhang, and Kunxiao Zhou. New construction for constant dimension subspace codes via a composite structure. *IEEE Communications Letters*, 25(5):1422–1426, 2021.

[112] Xianmang He, Yindong Chen, Kunxiao Zhou, and Jianguang Deng. A hierarchical-based greedy algorithm for echelon-Ferrers construction. *arXiv preprint 1911.00508*, 2019.

[113] Olof Heden and Papa Amar Sissokho. On the existence of a $(2,3)$-spread in $v(7,2)$. *Ars Combinatoria*, 124:161–164, 2016.

[114] Petra Heijnen and Ruud Pellikaan. Generalized hamming weights of q-ary reed-muller codes. *IEEE Transactions on Information Theory*, 44(1):181–196, 1998.

[115] Daniel Heinlein. *Integer linear programming techniques for constant dimension codes and related structures*. PhD thesis, Universität Bayreuth (Germany), 2018.

[116] Daniel Heinlein. New LMRD code bounds for constant dimension codes and improved constructions. *IEEE Transactions on Information Theory*, 65(8):4822–4830, 2019.

[117] Daniel Heinlein. Generalized linkage construction for constant-dimension codes. *IEEE Transactions on Information Theory*, 67(2):705–715, 2020.

[118] Daniel Heinlein, Thomas Honold, Michael Kiermaier, and Sascha Kurz. Generalized vector space partitions. *Australasian Journal of Combinatorics*, 73(1):162–178, 2019.

[119] Daniel Heinlein, Thomas Honold, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. Classifying optimal binary subspace codes of length 8, constant dimension 4 and minimum distance 6. *Designs, Codes and Cryptography*, 87(2-3):375–391, 2019.

[120] Daniel Heinlein, Thomas Honold, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. On projective q^r-divisible codes. *arXiv preprint 1912.10147*, 2019.

[121] Daniel Heinlein and Ferdinand Ihringer. New and updated semidefinite programming bounds for subspace codes. *Advances in Mathematics of Communications*, 14(4):613, 2020.

[122] Daniel Heinlein, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. Tables of subspace codes. *arXiv preprint 1601.02864*, 2016.

[123] Daniel Heinlein, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. A subspace code of size 333 in the setting of a binary q-analog of the Fano plane. *Advances in Mathematics of Communications*, 13(3):457–475, 2019.

[124] Daniel Heinlein, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. A subspace code of size 333 in the setting of a binary q-analog of the Fano plane. *Advances in Mathematics of Communications*, 13(3):457–475, 2019.

[125] Daniel Heinlein and Sascha Kurz. Asymptotic bounds for the sizes of constant dimension codes and an improved lower bound. In Ángela Isabel Barbero, Vitaly Skachek, and Øyvind Ytrehus, editors, *Coding Theory and Applications: 5th International Castle Meeting, ICMCTA 2017, Vihula, Estonia, August 28-31, 2017, Proceedings*, volume 10495 of *Lecture Notes in Computer Science*, pages 163–191, Cham, 2017. Springer International Publishing. arXiv preprint 1703.08712.

[126] Daniel Heinlein and Sascha Kurz. Coset construction for subspace codes. *IEEE Transactions on Information Theory*, 63(12):7651–7660, 2017.

[127] Daniel Heinlein and Sascha Kurz. An upper bound for binary subspace codes of length 8, constant dimension 4 and minimum distance 6. In *The Tenth International Workshop on Coding and Cryptography*, 2017. arXiv preprint 1705.03835.

[128] Daniel Heinlein and Sascha Kurz. Binary subspace codes in small ambient spaces. *Advances in Mathematics of Communications*, 12(4):817–839, 2018.

[129] Tor Helleseth, Torleiv Klove, and Øyvind Ytrehus. Generalized hamming weights of linear codes. *IEEE Transactions on Information Theory*, 38(3):1133–1140, 1992.

[130] Takaaki Hishida and Masakazu Jimbo. Cyclic resolutions of the BIB design in $\mathrm{PG}(5,2)$. *Australasian Journal of Combinatorics*, 22:73–80, 2000.

[131] Thomas Honold and Michael Kiermaier. On putative q-analogues of the Fano plane and related combinatorial structures. In *Dynamical Systems, Number Theory and Applications: A Festschrift in Honor of Armin Leutbecher's 80th Birthday*, pages 141–175. World Scientific, 2016.

[132] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Optimal binary subspace codes of length 6, constant dimension 3 and minimum distance 4. *Contemporary Mathematics*, 632:157–176, 2015.

[133] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Constructions and bounds for mixed-dimension subspace codes. *Advances in Mathematics of Communications*, 10(3):649–682, 2016.

[134] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Partial spreads and vector space partitions. In Marcus Greferath, Mario Osvin Pavčević, Natalia Silberstein, and María Ángeles Vázquez-Castro, editors, *Network Coding and Subspace Designs*, pages 131–170. Springer, 2018.

[135] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Classification of large partial plane spreads in $\mathrm{PG}(6, 2)$ and related combinatorial objects. *Journal of Geometry*, 110(1):1–31, 2019.

[136] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Johnson type bounds for mixed dimension subspace codes. *The Electronic Journal of Combinatorics*, 26(3):21 pp., 2019.

[137] Thomas Honold, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. The lengths of projective triply-even binary codes. *IEEE Transactions on Information Theory*, 66(5):2713–2716, 2019.

[138] Anna-Lena Horlemann-Trautmann and Joachim Rosenthal. Constructions of constant dimension codes. In Marcus Greferath, Mario Osvin Pavčević, Natalia Silberstein, and María Ángeles Vázquez-Castro, editors, *Network Coding and Subspace Designs*, pages 25–42. Springer, 2018.

[139] Ferdinand Ihringer, Peter Sin, and Qing Xiang. New bounds for partial spreads of $h(2d-1, q^2)$ and partial ovoids of the Ree–Tits octagon. *Journal of Combinatorial Theory, Series A*, 153:46–53, 2018.

[140] Selmer Johnson. A new upper bound for error-correcting codes. *IRE Transactions on Information Theory*, 8(3):203–207, 1962.

[141] Azadeh Khaleghi and Frank Robert Kschischang. Projective space codes for the injection metric. In *2009 11th Canadian Workshop on Information Theory*, pages 9–12. IEEE, 2009.

[142] Azadeh Khaleghi, Danilo Silva, and Frank Robert Kschischang. Subspace codes. In Matthew Geoffrey Parker, editor, *IMA International Conference on Cryptography and Coding*, pages 1–21. Springer, 2009.

[143] Michael Kiermaier. On α-points of q-analogs of the Fano plane. *Designs, Codes and Cryptography*, 90(6):1335–1345, 2022.

[144] Michael Kiermaier and Sascha Kurz. On the lengths of divisible codes. *IEEE Transactions on Information Theory*, 66(7):4051–4060, 2020.

[145] Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. The order of the automorphism group of a binary q-analog of the Fano plane is at most two. *Designs, Codes and Cryptography*, 86(2):239–250, 2018.

[146] Ralf Koetter and Frank Robert Kschischang. Coding for errors and erasures in random network coding. *IEEE Transactions on Information Theory*, 54(8):3579–3591, 2008.

[147] Axel Kohnert and Sascha Kurz. Construction of large constant dimension codes with a prescribed minimum distance. In *Mathematical methods in computer science*, volume 5393 of *Lecture Notes in Computer Science*, pages 31–42. Springer, Berlin, 2008.

[148] Frank Robert Kschischang. Network codes. In William Cary Huffman, Jon-Lark Kim, and Patrick Solé, editors, *Concise Encyclopedia of Coding Theory*, pages 685–714. Chapman and Hall/CRC, 2021.

[149] Sascha Kurz. Improved upper bounds for partial spreads. *Designs, Codes and Cryptography*, 85(1):97–106, 2017.

[150] Sascha Kurz. Packing vector spaces into vector spaces. *Australasian Journal of Combinatorics*, 68:122–130, 2017.

[151] Sascha Kurz. A note on the linkage construction for constant dimension codes. *arXiv preprint 1906.09780*, 2019.

[152] Sascha Kurz. Generalized LMRD code bounds for constant dimension codes. *IEEE Communications Letters*, 24(10):2100–2103, 2020.

[153] Sascha Kurz. Lengths of divisible codes - -the missing cases. *arXiv preprint 2311.01947*, 2023.

[154] Sascha Kurz. Lifted codes and the multilevel construction for constant dimension codes. *arXiv preprint 2004.14241*, 2020.

[155] Sascha Kurz. No projective 16-divisible binary linear code of length 131 exists. *IEEE Communications Letters*, 25(1):38–40, 2020.

[156] Sascha Kurz. Subspaces intersecting in at most a point. *Designs, Codes and Cryptography*, 88(3):595–599, 2020.

[157] Sascha Kurz. Bounds for flag codes. *Designs, Codes and Cryptography*, 89(12):2759–2785, 2021.

[158] Sascha Kurz. The interplay of different metrics for the construction of constant dimension codes. *Advances in Mathematics of Communications*, 17(1):152–171, 2023.

[159] Sascha Kurz and Theresa Körner. Lengths of divisible codes with restricted column multiplicities. *arXiv preprint 2303.17172*, 2023.

[160] Georg Landsberg. Ueber eine Anzahlbestimmung und eine damit zusammenhängende Reihe. *Journal für die reine und angewandte Mathematik*, 1893(111):87–88, 1893.

[161] Huimin Lao, Hao Chen, and Xiaoqing Tan. New constant dimension subspace codes from block inserting constructions. *Cryptography and Communications*, pages 1–13, 2021.

[162] Huimin Lao, Hao Chen, Jian Weng, and Xiaoqing Tan. Parameter-controlled inserting constructions of constant dimension subspace codes. *arXiv preprint 2008.09944*, 2020.

[163] Hunter Ryan Lehmann. *Weight Distributions, Automorphisms, and Isometries of Cyclic Orbit Codes*. PhD thesis, University of Kentucky, 2021.

[164] Fagang Li. Construction of constant dimension subspace codes by modifying linkage construction. *IEEE Transactions on Information Theory*, 66(5):2760–2764, 2019.

[165] Xiaoye Liang, Tatsuro Ito, and Yuta Watanabe. The Terwilliger algebra of the Grassmann scheme $j_q(n, d)$ revisited from the viewpoint of the quantum affine algebra $u_q(\widehat{\mathrm{sl}}_2)$. *Linear Algebra and its Applications*, 596:117–144, 2020.

[166] Dirk Liebhold. *Flag codes with application to network coding*. PhD thesis, RWTH Aachen, 2019.

[167] Dirk Liebhold, Gabriele Nebe, and Angeles Vazquez-Castro. Network coding with flags. *Designs, Codes and Cryptography*, 86(2):269–284, 2018.

[168] Dirk Liebhold, Gabriele Nebe, and María Ángeles Vázquez-Castro. Generalizing subspace codes to flag codes using group actions. In Marcus Greferath, Mario Osvin Pavčević, Natalia Silberstein, and María Ángeles Vázquez-Castro, editors, *Network Coding and Subspace Designs*, pages 67–89. Springer, 2018.

[169] Simon Litsyn. An update table of the best binary codes known. In Vera Pless, Richard Anthony Brualdi, and William Cary Huffman, editors, *Handbook of Coding Theory*, pages 463–498. Elsevier, 1998.

[170] Shuangqing Liu, Yanxun Chang, and Tao Feng. Constructions for optimal Ferrers diagram rank-metric codes. *IEEE Transactions on Information Theory*, 65(7):4115–4130, 2019.

[171] Shuangqing Liu, Yanxun Chang, and Tao Feng. Parallel multilevel constructions for constant dimension codes. *IEEE Transactions on Information Theory*, 66(11):6884–6897, 2020.

[172] Giovanni Longobardi, Leo Storme, and Rocco Trombetti. On sets of subspaces with two intersection dimensions and a geometrical junta bound. *Designs, Codes and Cryptography*, 90:2081–2099, 2022.

[173] Lisa Hernandez Lucas. Properties of sets of subspaces with constant intersection dimension. *Advances in Mathematics of Communications*, 15(1):191–206, 2021.

[174] Lisa Hernandez Lucas, Ivan Landjev, Leo Storme, and Peter Vandendriessche. On the geometrical sunflower bound. In *Eighth International Workshop on Optimal Codes and Related Topics, Sofia, Bulgaria*, pages 93–97, 2017.

[175] Florence Jessie MacWilliams and Neil James Alexander Sloane. *The theory of error correcting codes*, volume 16. Elsevier, 1977.

[176] Beniamin Mounits, Tuvi Etzion, and Simon Litsyn. Improved upper bounds on sizes of codes. *IEEE Transactions on Information Theory*, 48(4):880–886, 2002.

[177] Beniamin Mounits, Tuvi Etzion, and Simon Litsyn. New upper bounds on codes via association schemes and linear programming. *Advances in Mathematics of Communications*, 1(2):173, 2007.

[178] Anamari Nakić and Leo Storme. On the extendability of particular classes of constant dimension codes. *Designs, Codes and Cryptography*, 79(3):407–422, 2016.

[179] Esmeralda Năstase and Papa Amar Sissokho. The maximum size of a partial spread II: Upper bounds. *Discrete Mathematics*, 340(7):1481–1487, 2017.

[180] Esmeralda Năstase and Papa Amar Sissokho. The maximum size of a partial spread in a finite projective space. *Journal of Combinatorial Theory, Series A*, 152:353–362, 2017.

[181] Miguel Ángel Navarro-Pérez and Xaro Soler Escrivà. Flag codes of maximum distance and constructions using Singer groups. *Finite Fields and Their Applications*, 80:102011, 2022.

[182] Chi-Kin Ngai, Raymond W. Yeung, and Zhixue Zhang. Network generalized hamming weight. *IEEE Transactions on Information Theory*, 57(2):1136–1143, 2011.

[183] Min-Yao Niu, Gang Wang, You Gao, and Fang-Wei Fu. Subspace code based on flats in affine space over finite fields. *Discrete Mathematics, Algorithms and Applications*, 10(06):1850078, 2018.

[184] Yongfeng Niu, Qin Yue, and Daitao Huang. New constant dimension subspace codes from generalized inserting construction. *IEEE Communications Letters*, 25(4):1066–1069, 2020.

[185] Yongfeng Niu, Qin Yue, and Daitao Huang. Construction of constant dimension codes via improved inserting construction. *Applicable Algebra in Engineering, Communication and Computing*, page 18 pp., to appear.

[186] Yongfeng Niu, Qin Yue, and Yansheng Wu. Several kinds of large cyclic subspace codes via Sidon spaces. *Discrete Mathematics*, 343(5):111788, 2020.

[187] Frédérique Oggier. Space-time coding. In William Cary Huffman, Jon-Lark Kim, and Patrick Solé, editors, *Concise Encyclopedia of Coding Theory*, pages 673–684. Chapman and Hall/CRC, 2021.

[188] Patric R. J. Ostergard. Classification of binary constant weight codes. *IEEE Transactions on Information Theory*, 56(8):3779–3785, 2010.

[189] Kamil Otal and Ferruh Özbudak. Cyclic subspace codes via subspace polynomials. *Designs, Codes and Cryptography*, 85(2):191–204, 2017.

[190] Kamil Otal and Ferruh Özbudak. Constructions of cyclic subspace codes and maximum rank distance codes. In Marcus Greferath, Mario Osvin Pavčević, Natalia Silberstein, and María Ángeles Vázquez-Castro, editors, *Network Coding and Subspace Designs*, pages 43–66. Springer, 2018.

[191] B. Srikanth Pai and B. Sundar Rajan. On the bounds of certain maximal linear codes in a projective space. *IEEE Transactions on Information Theory*, 61(9):4923–4927, 2015.

[192] Sven Carel Polak. Semidefinite programming bounds for constant-weight codes. *IEEE Transactions on Information Theory*, 65(1):28–38, 2018.

[193] Vinayak Ramkumar, Myna Vajha, Srinivasan Babu Balaji, M. Nikhil Krishnan, Birenjith Sasidharan, and P. Vijay Kumar. Codes for distributed storage. In William Cary Huffman, Jon-Lark Kim, and Patrick Solé, editors, *Concise Encyclopedia of Coding Theory*, pages 735–762. Chapman and Hall/CRC, 2021.

[194] Charlene Raviv. *Subspace codes and distributed storage codes*. PhD thesis, Computer Science Department, Technion, 2017.

[195] Ron M. Roth. Maximum-rank array codes and their application to crisscross error correction. *IEEE Transactions on Information Theory*, 37(2):328–336, 1991.

[196] Ron M. Roth, Netanel Raviv, and Itzhak Tamo. Construction of Sidon spaces with applications to coding. *IEEE Transactions on Information Theory*, 64(6):4412–4422, 2017.

[197] Jumela F. Sarmiento. On point-cyclic resolutions of the $2-(63,7,15)$ design associated with $\mathrm{PG}(5,2)$. *Graphs and Combinatorics*, 18(3):621–632, 2002.

[198] Alexander Schrijver. New code upper bounds from the terwilliger algebra and semidefinite programming. *IEEE Transactions on Information Theory*, 51(8):2859–2866, 2005.

[199] Beniamino Segre. Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane. *Annali di Matematica Pura ed Applicata*, 64(1):1–76, 1964.

[200] John Sheekey. A new family of linear maximum rank distance codes. *Advances in Mathematics of Communications*, 10(3):475–488, 2016.

[201] John Sheekey. Binary additive MRD codes with minimum distance $n-1$ must contain a semifield spread set. *Designs, Codes and Cryptography*, 87(11):2571–2583, 2019.

[202] John Sheekey. MRD codes: Constructions and connections. In *Combinatorics and Finite Fields: Difference Sets, Polynomials, Pseudorandomness and Applications*, volume 23 of *Radon Series on Computational and Applied Mathematics*. De Gruyter, Berlin, 2019.

[203] Alexander Shishkin. New multicomponent network subspace codes: Construction and decoding. In *2016 International Conference on Engineering and Telecommunication (EnT)*, pages 123–127. IEEE, 2016.

[204] Alexander Shishkin, Ernst Muhamedovich Gabidulin, and Nina Ivanovna Pilipchuk. On cardinality of network subspace codes. In *Proceeding of the Fourteenth International Workshop on Algebraic and Combinatorial Coding Theory (ACCT-XIV)*, volume 7, pages 300–306, 2014.

[205] Natalia Silberstein and Tuvi Etzion. Large constant dimension codes and lexicodes. *Advances in Mathematics of Communications*, 5(2):177–189, 2011.

[206] Natalia Silberstein and Anna-Lena Trautmann. New lower bounds for constant dimension codes. In *2013 IEEE International Symposium on Information Theory (ISIT 2013)*, pages 514–518. IEEE, 2013.

[207] Natalia Silberstein and Anna-Lena Trautmann. Subspace codes based on graph matchings, Ferrers diagrams, and pending blocks. *IEEE Transactions on Information Theory*, 61(7):3937–3953, 2015.

[208] Danilo Silva and Frank Robert Kschischang. On metrics for error correction in network coding. *IEEE Transactions on Information Theory*, 55(12):5479–5490, 2009.

[209] Danilo Silva, Frank Robert Kschischang, and Ralf Koetter. A rank-metric approach to error control in random network coding. *IEEE Transactions on Information Theory*, 54(9):3951–3967, 2008.

[210] Kishore Sinha, Z. Wang, and Dianhua Wu. Good equidistant codes constructed from certain combinatorial designs. *Discrete Mathematics*, 308(18):4205–4211, 2008.

[211] Vitaly Skachek. Recursive code construction for random networks. *IEEE Transactions on Information Theory*, 56(3):1378–1382, 2010.

[212] Neil James Alexander Sloane. An introduction to association schemes and coding theory. In Richard Allen Askey, editor, *Theory and application of special functions*, pages 225–260. Elsevier, 1975. Proceedings of an Advanced Seminar Sponsored by the Mathematics Research Center, the University of Wisconsin–Madison, March 31–April 2, 1975.

[213] Leo Storme. Coding theory and galois geometries. In William Cary Huffman, Jon-Lark Kim, and Patrick Solé, editors, *Concise Encyclopedia of Coding Theory*, pages 285–305. Chapman and Hall/CRC, 2021.

[214] Simon Thomas. Designs over finite fields. *Geometriae Dedicata*, 24(2):237–242, 1987.

[215] Simon Thomas. Designs and partial geometries over finite fields. *Geometriae Dedicata*, 63(3):247–253, 1996.

[216] Vladimir D. Tonchev. Codes and designs. In Vera Pless, Richard Anthony Brualdi, and William Cary Huffman, editors, *Handbook of Coding Theory*, volume 2, pages 1229–1267. Elsevier, 1998.

[217] Anna-Lena Trautmann. *Constructions, decoding and automorphisms of subspace codes*. PhD thesis, University of Zurich, 2013.

[218] Frank Vallentin. Semidefinite programming bounds for error-correcting codes. In William Cary Huffman, Jon-Lark Kim, and Patrick Solé, editors, *Concise Encyclopedia of Coding Theory*, pages 267–282. Chapman and Hall/CRC, 2021.

[219] Vincent van der Noort. Cayley algebras give rise to q-Fano planes over certain infinite fields and q-covering designs over others. *arXiv preprint 2006.01268*, 2020.

[220] Jacobus Hendricus van Lint. A theorem on equidistant codes. *Discrete Mathematics*, 6(4):353–358, 1973.

[221] Jacobus Hendricus Van Lint and Richard Michael Wilson. *A course in combinatorics*. Cambridge University Press, 2 edition, 2001.

[222] Zhe-xian Wan. Geometry of classical groups over finite fields and its applications. *Discrete Mathematics*, 174(1-3):365–381, 1997.

[223] Huaxiong Wang, Chaoping Xing, and Reihaneh Safavi-Naini. Linear authentication codes: bounds and constructions. *IEEE Transactions on Information Theory*, 49(4):866–872, 2003.

[224] Victor K. Wei. Generalized Hamming weights for linear codes. *IEEE Transactions on Information Theory*, 37(5):1412–1418, 1991.

[225] Netanel Weiß. *Linear programming bounds in classical association schemes*. PhD thesis, Universität Paderborn, 2023.

[226] Shu-Tao Xia and Fang-Wei Fu. Johnson type bounds on constant dimension codes. *Designs, Codes and Cryptography*, 50(2):163–172, 2009.

[227] Liqing Xu and Hao Chen. New constant-dimension subspace codes from maximum rank distance codes. *IEEE Transactions on Information Theory*, 64(9):6315–6319, 2018.

[228] Tao Zhang and Yue Zhou. New lower bounds for partial k-parallelisms. *Journal of Combinatorial Designs*, 28(1):75–84, 2020.

[229] Zong-Ying Zhang, Yong Jiang, and Shu-Tao Xia. On the linear programming bounds for constant dimension codes. In *2011 International Symposium on Networking Coding (NetCod 2011)*, pages 1–4. IEEE, 2011.

[230] Kunxiao Zhou, Yindong Chen, Zusheng Zhang, Feng Shi, and Xianmang He. A construction for constant dimension codes from the known codes. In *International Conference on Wireless Algorithms, Systems, and Applications*, pages 253–262. Springer, 2021.