Mahler's work on the geometry of numbers

Evertse, J.H.; Baake, M.; Borwein, J.M.; Bugeaud, Y.; Coons, M.

1. Mahler's Work on the Geometry of Numbers

Jan-Hendrik Evertse

Mahler has written many papers on the geometry of numbers. Arguably, his most influential achievements in this area are his compactness theorem for lattices, his work on star bodies and their critical lattices, and his estimates for the successive minima of reciprocal convex bodies and compound convex bodies. We give a, by far not complete, overview of Mahler's work on these topics and their impact.

1 Compactness theorem, star bodies and their critical lattices

Many problems in the geometry of numbers are about whether a particular n-dimensional body contains a non-zero point from a given lattice, and quite often one can show that this is true as long as the determinant of the lattice is below a critical value depending on the given body. Mahler intensively studied such problems for so-called *star bodies*. Before mentioning some of his results, we start with recalling some definitions. We follow [M87].

Let $n \geqslant 2$ be an integer that we fix henceforth. A *distance function* on $\mathbb{R}^n$ is a function $F : \mathbb{R}^n \to \mathbb{R}$ such that:

(i) $F(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $F(\mathbf{x}) > 0$ for at least one $\mathbf{x}$;

(ii) $F(t\mathbf{x}) = |t| \cdot F(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$;

(iii) F is continuous.

A *(symmetric) star body* in $\mathbb{R}^n$ is a set of the shape

$$\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) \leqslant 1\},$$

where F is a distance function. We call $\mathcal{S}$ the star body with distance function F. The boundary of $\mathcal{S}$ is $\{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) = 1\}$, and the interior of $\mathcal{S}$ is $\{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) < 1\}$. The set $\mathcal{S}$ is bounded, if and only if $F(\mathbf{x}) > 0$ whenever $\mathbf{x} \neq 0$. The star bodies contain as a subclass the *symmetric convex bodies*, which correspond to the distance functions F satisfying, in addition to (i), (ii), and (iii), the triangle inequality $F(\mathbf{x} + \mathbf{y}) \leqslant F(\mathbf{x}) + F(\mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}$.

Let $\Lambda = \{\sum_{i=1}^{n} z_i\mathbf{a}_i : z_1, \ldots, z_n \in \mathbb{Z}\}$ be a lattice in $\mathbb{R}^n$ with basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. We define its determinant by $d(\Lambda) := |\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)|$. Let $\mathcal{S}$ be a star body. We call Λ $\mathcal{S}$-*admissible* if $\mathbf{0}$ is the only point of Λ in the interior of $\mathcal{S}$. The star body $\mathcal{S}$ is called of *finite type* if it has admissible lattices, and of infinite type otherwise. Bounded star bodies are necessarily of finite type, but conversely, star bodies of finite type do not have to be bounded. For instance, let $\mathcal{S} := \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_1 \cdots x_n| \leqslant 1\}$. Take a totally real number field K of degree n, denote by O_K its ring of integers, and let $\alpha \mapsto \alpha^{(i)}$ $(i = 1, \ldots, n)$ be the embeddings of K in $\mathbb{R}$. Then $\{(\alpha^{(1)}, \ldots, \alpha^{(n)}) : \alpha \in O_K\}$ is an $\mathcal{S}$-admissible lattice.

Assume henceforth that $\mathcal{S}$ is a star body of finite type. Then we can define its determinant,

$$\Delta(\mathcal{S}) := \inf\{d(\Lambda) : \Lambda \text{ admissible lattice for } \mathcal{S}\}.$$

Thus, if Λ is any lattice in $\mathbb{R}^n$ with $d(\Lambda) < \Delta(\mathcal{S})$, then $\mathcal{S}$ contains a non-zero point from Λ. The quantity $\Delta(\mathcal{S})$ cannot be too small. From the Minkowski–Hlawka theorem (proved by Hlawka [8] and earlier stated without proof by Minkowski) it follows that $\Delta(\mathcal{S}) > (2\zeta(n))^{-1}V(\mathcal{S})$, where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$ and $V(\mathcal{S})$ is the volume (n-dimensional Lebesgue measure) of $\mathcal{S}$.

We call Λ a *critical lattice* for $\mathcal{S}$ if Λ is $\mathcal{S}$-admissible and $d(\Lambda) = \Delta(\mathcal{S})$. In a series of papers [M75, M76, M83, M84, M85] Mahler studied star bodies in $\mathbb{R}^2$, proved that they have critical lattices, and computed their determinant in various instances. Later, Mahler picked up the study of star bodies of arbitrary dimension [M87]. We recall Theorem 8 from this paper, which is Mahler's central result on star bodies.

Theorem 1.1. *Let $\mathcal{S}$ be a star body in $\mathbb{R}^n$ of finite type. Then $\mathcal{S}$ has at least one critical lattice.*

The main tool is a compactness result for lattices, also due to Mahler. We say that a sequence of lattices $\{\Lambda_m\}_{m=1}^{\infty}$ in $\mathbb{R}^n$ converges if we can choose a basis $\mathbf{a}_{m,1}, \ldots, \mathbf{a}_{m,n}$ of Λ_m for $m = 1, 2, \ldots$ such that $\mathbf{a}_j := \lim_{m\to\infty} \mathbf{a}_{m,j}$ exists for $j = 1, \ldots, n$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent. We call the lattice Λ with basis $\mathbf{a}_1, \ldots, \mathbf{a}_n$ the limit of the sequence $\{\Lambda_m\}_{m=1}^{\infty}$; it can be shown that this limit, if it exists, is unique. Denote by $\|\mathbf{x}\|$ the Euclidean norm of $\mathbf{x} \in \mathbb{R}^n$. The following result, which became known as *Mahler's compactness theorem* or *Mahler's selection theorem* and turned out to be a valuable tool at various places other than the geometry of numbers, is Theorem 2 from [M87].

Theorem 1.2. *Let $\rho > 0$, $C > 0$. Then any infinite collection of lattices Λ in $\mathbb{R}^n$ such that $\min\{\|\mathbf{x}\| : \mathbf{x} \in \Lambda \setminus \{0\}\} \geqslant \rho$ and $d(\Lambda) \leqslant C$ has an infinite convergent subsequence.*

We recall the quick deduction of Theorem 1.1.

Proof of Theorem 1.1. By the definition of $\Delta(\mathcal{S})$, there is an infinite sequence $\{\Lambda_m\}_{m=1}^{\infty}$ of $\mathcal{S}$-admissible lattices such that $\Delta(\mathcal{S}) \leqslant d(\Lambda_m) \leqslant \Delta(\mathcal{S}) + 1/m$ for $m = 1, 2, \ldots$. Since $\mathbf{0}$ is an interior point of $\mathcal{S}$,there is $\rho > 0$ such that $\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leqslant \rho\} \subseteq \mathcal{S}$. hence $\|\mathbf{x}\| \geqslant \rho$ for every non-zero $\mathbf{x} \in \Lambda_m$ and every $m \geqslant 1$. Further, the sequence $\{d(\Lambda_m)\}$ is clearly bounded. So by Theorem 1.2, $\{\Lambda_m\}$ has a convergent subsequence. After reindexing, we may write this sequence as $\{\Lambda_m\}_{m=1}^{\infty}$ and denote its limit by Λ. We show that Λ is a critical lattice for $\mathcal{S}$.

Choose bases $\mathbf{a}_{m,1}, \ldots, \mathbf{a}_{m,n}$ of Λ_m for $m = 1, 2, \ldots$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of Λ such that $\mathbf{a}_{m,j} \to \mathbf{a}_j$ for $j = 1, \ldots, n$. Clearly $d(\Lambda) = \lim_{m\to\infty} d(\Lambda_m) = \Delta(\mathcal{S})$. To prove that Λ is $\mathcal{S}$-admissible, take a non-zero $\mathbf{x}_0 \in \Lambda$ and assume it is in the interior of $\mathcal{S}$. Then there is $\epsilon > 0$ such that all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x} - \mathbf{x}_0\| < \epsilon$ are in the interior of $\mathcal{S}$. Write $\mathbf{x}_0 = \sum_{i=1}^{n} z_i \mathbf{a}_i$ with $z_i \in \mathbb{Z}$, and then $\mathbf{x}_m = \sum_{i=1}^{n} z_i \mathbf{a}_{m,i}$ for $m \geqslant 1$, so that $\mathbf{x}_m \in \Lambda_m \setminus \{\mathbf{0}\}$. For m sufficiently large, $\|\mathbf{x}_m - \mathbf{x}_0\| < \epsilon$, hence $\mathbf{x}_m$ is in the interior of $\mathcal{S}$, which is however impossible since Λ_m is $\mathcal{S}$-admissible. This completes the proof. $\square$

In [M87], Mahler made a further study of the critical lattices of n-dimensional star bodies. Among other things he proved [M87, Theorem 11] that if $\mathcal{S}$ is any bounded n-dimensional star body and Λ a critical lattice for $\mathcal{S}$, then there are n linearly independent points of Λ lying on the boundary of $\mathcal{S}$. If $P_1, \ldots, P_n$ are such points, then the $2n$ points $\pm P_1, \ldots, \pm P_n$ lie on the boundary of $\mathcal{S}$. A simple consequence of this is, that any lattice of determinant equal to $\Delta(\mathcal{S})$ has a non-zero point either in the interior or on the boundary of $\mathcal{S}$. Mahler showed further [M87, Corollary on p. 165] that for any integer $m \geqslant n$ there exist an n-dimensional star body $\mathcal{S}$ and a critical lattice Λ of $\mathcal{S}$ having precisely $2m$ points on the boundary of $\mathcal{S}$.

In an other series of papers on n-dimensional star bodies [M88] Mahler introduced the notions of reducible and irreducible star bodies. A star body $\mathcal{S}$ is called *reducible* if there is a star body $\mathcal{S}'$ which is strictly contained in $\mathcal{S}$ and for which $\Delta(\mathcal{S}') = \Delta(\mathcal{S})$, and otherwise irreducible. An unbounded star body $\mathcal{S}$ of finite type is called *boundedly reducible* if there is a bounded star body $\mathcal{S}'$ contained in $\mathcal{S}$ such that $\Delta(\mathcal{S}') = \Delta(\mathcal{S})$. Mahler gave criteria for star bodies being (boundedly) reducible and deduced some Diophantine approximation results. To give a flavour we mention one of these results [M88, Theorem P, p. 628]:

Theorem 1.3. *There is a positive constant γ such that if β_1, β_2 are any real numbers and Q is any number > 1, then there are integers v_1, v_2, v_3, not all 0, such that*

$$|v_1 v_2(\beta_1 v_1 + \beta_2 v_2 + v_3)| \leqslant \tfrac{1}{7},$$
$$|x_1| \leqslant Q,\ |x_2| \leqslant Q,\ |\beta_1 v_1 + \beta_2 v_2 + v_3| \leqslant \gamma Q^{-2}.$$

Idea of proof. Let $\mathcal{S}$ be the set of $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ given by $|x_1 x_2 x_3| \leqslant 1$. By a result of Davenport [3], $\mathcal{S}$ is a finite type star body and has determinant

$\Delta(\mathcal{S}) = 7$. Mahler [M88, Theorem M, p. 527] proved that $\mathcal{S}$ is in fact boundedly reducible, which implies that there is $r > 0$ such that the star body $\mathcal{S}'$ given by $|x_1x_2x_3| \leqslant 1$ and $\max_{1\leqslant i\leqslant 3} |x_i| \leqslant r$ also has determinant 7. Now let Λ be the lattice consisting of the points $\big(rQ^{-1}v_1, rQ^{-1}v_2, 7r^{-2}Q^2(\beta_1v_1+\beta_2v_2+v_3)\big)$ with $v_1, v_2, v_3 \in \mathbb{Z}$. This lattice has determinant 7 and so has a non-zero point in $\mathcal{S}'$. It follows that Theorem 1.3 holds with $\gamma = r^3/7$. □

For further theory on star bodies, we refer to Mahler's papers quoted above and the books of Cassels [2] and Gruber and Lekkerkerker [7].

2 Reciprocal convex bodies

Studies of transference principles such as Khintchine's for systems of Diophantine inequalities (see [M44, M56]) led Mahler to consider *reciprocal lattices* and *reciprocal convex bodies* (also called polar lattices and polar convex bodies). We recall some of his results. Here and below, for any real vectors $\mathbf{x}$, $\mathbf{y}$ of the same dimension, we denote by $\mathbf{x}\cdot\mathbf{y}$ their standard inner product, i.e., for $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ we put $\mathbf{x}\cdot\mathbf{y} := \sum_{i=1}^m x_iy_i$. Then the Euclidean norm of $\mathbf{x} \in \mathbb{R}^m$ is $\|\mathbf{x}\| := \sqrt{\mathbf{x}\cdot\mathbf{x}}$.

Now let n be a fixed integer $\geqslant 2$. Given a lattice Λ in $\mathbb{R}^n$, we define the *reciprocal lattice* of Λ by

$$\Lambda^* := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\cdot\mathbf{y} \in \mathbb{Z} \text{ for all } \mathbf{y} \in \Lambda\}.$$

Then Λ^* is again a lattice of $\mathbb{R}^n$, and $d(\Lambda^*) = d(\Lambda)^{-1}$. Let $\mathcal{C}$ be a symmetric convex body in $\mathbb{R}^n$, i.e., $\mathcal{C}$ is convex, symmetric about $\mathbf{0}$ and compact. The set $\mathcal{C}$ may be described alternatively as $\{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) \leqslant 1\}$, where F is a distance function as above, satisfying also the triangle inequality. We define the reciprocal of $\mathcal{C}$ by

$$\mathcal{C}^* = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}\cdot\mathbf{y} \leqslant 1 \text{ for all } \mathbf{y} \in \mathcal{C}\}.$$

Then $\mathcal{C}^*$ is again a symmetric convex body. Mahler [M57, p. 97, formula (6)] proved the following result for the volumes of $\mathcal{C}$ and $\mathcal{C}^*$.

Theorem 2.1. *There are $c_1(n), c_2(n) > 0$ depending only on n with the following property. If $\mathcal{C}$ is any symmetric convex body in $\mathbb{R}^n$ and $\mathcal{C}^*$ its reciprocal, then $c_1(n) \leqslant V(\mathcal{C})\cdot V(\mathcal{C}^*) \leqslant c_2(n)$.*

Mahler proved this with $c_1(n) = 4^n/(n!)^2$ and $c_2(n) = 4^n$. Santaló [16] improved the upper bound to $c_2(n) = \kappa_n^2$ where κ_n is the volume of the n-dimensional Euclidean unit ball $B_n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leqslant 1\}$; this upper bound is attained for $\mathcal{C} = \mathcal{C}^* = B_n$. Bourgain and Milman [1]. improved the lower bound to $c_1(n) = c^n\kappa_n^2$ with some absolute constant c. This is probably not optimal. Mahler conjectured that the optimal value for $c_1(n)$ is $4^n/n!$, which is attained for $\mathcal{C}$ the unit cube $\max_i |x_i| \leqslant 1$ and $\mathcal{C}^*$ the octahedron $\sum_{i=1}^n |x_i| \leqslant 1$.

Recall that the i-th successive minimum $\lambda_i(\mathcal{C},\Lambda)$ of a symmetric convex body $\mathcal{C}$ in $\mathbb{R}^n$ with respect to a lattice Λ in $\mathbb{R}^n$ is the smallest λ such that $\lambda\mathcal{C}\cap\Lambda$ contains i linearly independent points. Thus, $\mathcal{C}$ has n successive minima, and by Minkowski's theorem on successive minima [14] one has

$$\frac{2^n}{n!}\cdot\frac{d(\Lambda)}{V(\mathcal{C})}\leqslant\lambda_1(\mathcal{C},\Lambda)\cdots\lambda_n(\mathcal{C},\Lambda)\leqslant 2^n\cdot\frac{d(\Lambda)}{V(\mathcal{C})}. \tag{2.1}$$

Mahler [M57, p.100, (A), (B)] proved the following *transference principle* for reciprocal convex bodies.

Theorem 2.2. *There is $c_3(n)>0$ depending only on n with the following property. Let Λ, $\mathcal{C}$ be a lattice and symmetric convex body in $\mathbb{R}^n$, and Λ^*, $\mathcal{C}^*$ their respective reciprocals. Then*

$$1\leqslant\lambda_i(\mathcal{C},\Lambda)\cdot\lambda_{n+1-i}(\mathcal{C}^*,\Lambda^*)\leqslant c_3(n).$$

The lower bounds for the products $\lambda_i(\mathcal{C},\Lambda)\lambda_{n+1-i}(\mathcal{C}^*,\Lambda^*)$ are easy to prove, and then the upper bounds are obtained by combining the lower bound in Theorem 2.1 with the upper bound in (2.1) and the similar one for $\mathcal{C}^*$ and Λ^*. With his bound for $c_1(n)$, Mahler deduced Theorem 2.2 with $c_3(n)=(n!)^2$. Using instead the bound for $c_1(n)$ by Bourgain and Milman, one obtains Theorem 2.2 with $c_3(n)=(c'n)^n$ for some absolute constant c'. Kannan and Lovász [11] obtained $\lambda_1(\mathcal{C},\Lambda)\lambda_n^*(\mathcal{C}^*,\Lambda^*)\leqslant c''n^2$ with some absolute constant c''.

Mahler's results led to various applications, among others to inhomogeneous results. A simple consequence, implicit in Mahler's paper [M57] is the following:

Corollary 2.3. *There is $c_4(n)>0$ with the following property. Let $\mathcal{C}$, Λ, $\mathcal{C}^*$ and Λ^* be as in Theorem 2.2 and suppose that $\mathcal{C}^*$ does not contain a non-zero point from Λ^*. Then for every $\mathbf{a}\in\mathbb{R}^n$ there is $\mathbf{z}\in\Lambda$ such that $\mathbf{a}+\mathbf{z}\in c_4(n)\mathcal{C}$.*

Idea of proof. Using that the distance function associated with $\mathcal{C}$ satisfies the triangle inequality, one easily shows that for every $\mathbf{a}\in\mathbb{R}^n$ there is $\mathbf{z}\in\Lambda$ with $\mathbf{a}+\mathbf{z}\in n\lambda_n(\mathcal{C},\Lambda)\cdot\mathcal{C}$. By assumption we have $\lambda_1(\mathcal{C}^*,\Lambda^*)>1$, and thus, $\lambda_n(\mathcal{C}^*,\Lambda^*)<c_3(n)$. □

The second application we mention is a transference principle for systems of Diophantine inequalities. We define the maximum norm and sum-norm of $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$ by $\|\mathbf{x}\|_\infty:=\max_i|x_i|$ and $\|\mathbf{x}\|_1:=\sum_{i=1}^n|x_i|$, respectively. We denote by A^T the transpose of a matrix A.

Corollary 2.4. *Let m,n be integers with $0<m<n$ and let A be a $(n-m)\times m$-matrix with real entries where m,n are integers with $0<m<n$. Let ω be the supremum of the reals $\eta>0$ such that there are infinitely many non-zero $\mathbf{x}\in\mathbb{Z}^m$ for which there exists $\mathbf{y}\in\mathbb{Z}^{n-m}$ with*

$$\|A\mathbf{x}-\mathbf{y}\|_\infty\leqslant\|\mathbf{x}\|_\infty^{-\frac{m}{n-m}(1+\eta)}. \tag{2.2}$$

Further, let ω^* be the supremum of the reals $\eta^* > 0$ for which there are infinitely many non-zero $\mathbf{u} \in \mathbb{Z}^{n-m}$ for which there exists $\mathbf{v} \in \mathbb{Z}^m$ such that

$$\|A^T\mathbf{u} - \mathbf{v}\|_\infty \leqslant \|\mathbf{u}\|_\infty^{-\frac{n-m}{m}(1+\eta^*)}. \tag{2.3}$$

Then

$$\omega^* \geqslant \frac{\omega}{(m-1)\omega + n - 1}, \quad \omega \geqslant \frac{\omega^*}{(n-m-1)\omega^* + n - 1}. \tag{2.4}$$

These inequalities were proved by Dyson [4]. The special case $m = 1$ was established earlier by Khintchine [12, 13] and became known as *Khintchine's transference principle*. Jarník [9] proved that both inequalities are best possible.

Proof. We prove only the first inequality; then the second follows by symmetry.

Let $Q \geqslant 1$, $0 < \eta < \omega$. Put $\eta^* := \frac{\eta}{(m-1)\eta+n-1}$. Consider the convex body $\mathcal{C}_Q$ consisting of the points $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \oplus \mathbb{R}^{n-m} = \mathbb{R}^n$ with $\|\mathbf{x}\|_\infty \leqslant Q$ and $\|A\mathbf{x} - \mathbf{y}\|_\infty \leqslant Q^{-\frac{m}{n-m}(1+\eta)}$. Denote the successive minima of $\mathcal{C}_Q$, $\mathcal{C}_Q^*$, respectively with respect to $\mathbb{Z}^n$ by $\lambda_i(Q)$, $\lambda_i^*(Q)$, for $i = 1, \dots, n$. By the choice of η, there is a sequence of $Q \to \infty$ such that $\lambda_1(Q) \leqslant 1$. Let Q be from this sequence. The body $\mathcal{C}_Q$ has volume $V(\mathcal{C}_Q) \ll Q^{-m\eta}$. The reciprocal body $\mathcal{C}_Q^*$ of $\mathcal{C}_Q$ is the set of $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n-m} \oplus \mathbb{R}^m$ with $Q\|A^T\mathbf{u} - \mathbf{v}\|_1 + Q^{-\frac{m}{n-m}(1+\eta)}\|\mathbf{u}\|_1 \leqslant 1$. Combining Theorem 2.2 with the lower bound in (2.1), we infer that

$$\lambda_1^*(Q) \ll \lambda_n(Q)^{-1} \ll \big(V(\mathcal{C}_Q) \cdot \lambda_1(Q)\big)^{1/(n-1)} \ll Q^{-m\eta/(n-1)},$$

where the implied constants depend on m and n. The body $\lambda_1^*(Q)\mathcal{C}_Q^*$ contains a non-zero point $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^{n-m} \oplus \mathbb{Z}^m$, and thus,

$$\begin{aligned} \|\mathbf{u}\|_\infty &\ll Q^{\frac{m}{n-m}(1+\eta) - \frac{m\eta}{n-1}} =: Q', \\ \|A^T\mathbf{u} - \mathbf{v}\|_\infty &\ll Q^{-1-\frac{m\eta}{n-1}} = Q'^{-\frac{n-m}{m}(1+\eta^*)}. \end{aligned}$$

If there is a non-zero $\mathbf{u}_0 \in \mathbb{Z}^{n-m}$ with $A^T\mathbf{u}_0 = \mathbf{v}_0$ for some $\mathbf{v} \in \mathbb{Z}^m$ then (2.3) holds with all integer multiples of $(\mathbf{u}_0, \mathbf{v}_0)$. Otherwise, if we let $Q \to \infty$ then $\mathbf{u}$ runs through an infinite set. The first inequality of (2.4) easily follows. $\square$

3 Compound convex bodies

Mahler extended his theory of reciprocal convex bodies to so-called *compound convex bodies*, which are in some sense exterior powers of convex bodies.

Let again $n \geqslant 2$ be an integer and p an integer with $1 \leqslant p \leqslant n-1$. Put $N := \binom{n}{p}$ and denote by $\mathcal{I}_{n,p}$ the collection of N integer tuples $(i_1, \dots, i_p)$ with $1 \leqslant i_1 < \cdots < i_p \leqslant n$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$ (i.e., $\mathbf{e}_i$ has a 1 on the i-th place and zeros elsewhere) and $\{\widehat{\mathbf{e}}_1, \dots, \widehat{\mathbf{e}}_N\}$ the standard basis of $\mathbb{R}^N$. We define exterior products of p vectors by means of the multilinear map $(\mathbf{x}_1, \ldots, \mathbf{x}_p) \mapsto \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p$ from $(\mathbb{R}^n)^p$ to $\mathbb{R}^N$, which is such that $\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_p} = \widehat{\mathbf{e}}_j$ for $j = 1, \ldots, N$ if $(i_1, \ldots, i_p)$ is the j-th tuple of $\mathcal{I}_{n,p}$ in the lexicographic ordering, and such that $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p$ changes sign if two of the vectors are interchanged.

Let $\mathcal{C}$ be a symmetric body in $\mathbb{R}^n$ and Λ a lattice in $\mathbb{R}^n$. Then the *p-th compound* $\mathcal{C}_p$ of $\mathcal{C}$ is defined as the convex hull of the points $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p \in \mathbb{R}^N$ with $\mathbf{x}_1, \ldots, \mathbf{x}_p \in \mathcal{C}$, while the p-th compound Λ_p of Λ is the lattice in $\mathbb{R}^N$ generated by the points $\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_p$ with $\mathbf{x}_1, \ldots, \mathbf{x}_p \in \Lambda$. Then $d(\Lambda_p) = d(\Lambda)^P$ where $P := \binom{n-1}{p-1}$. Mahler [M126, Theorem 1] proved the following analogue for the volume of the p-th compound of a symmetric convex body.

Theorem 3.1. *Let $\mathcal{C}$ be any symmetric convex body in $\mathbb{R}^n$ and p any integer with $1 \leqslant p \leqslant n-1$. Then*

$$c_1(n,p) \leqslant V(\mathcal{C}_p) \cdot V(\mathcal{C})^{-P} \leqslant c_2(n,p),$$

where $c_1(n,p)$, $c_2(n,p)$ are positive numbers depending only on n and p.

Idea of proof. The quotient $V(\mathcal{C}_p) \cdot V(\mathcal{C})^{-P}$ is invariant under linear transformations, so Theorem 3.1 holds for ellipsoids, these are the images of the Euclidean unit ball $B_n := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leqslant 1\}$ under linear transformations. Now the theorem follows for arbitrary symmetric convex bodies $\mathcal{C}$, with different $c_1(n,p)$, $c_2(n,p)$, by invoking John's theorem [10], which asserts that for every symmetric convex body $\mathcal{C}$ in $\mathbb{R}^n$ there is an ellipsoid $\mathcal{E}$ such that $n^{-1/2}\mathcal{E} \subseteq \mathcal{C} \subseteq \mathcal{E}$. □

Mahler [M126, Theorem 3] deduced from this the following result on the successive minima of a compound convex body.

Theorem 3.2. *Let $\mathcal{C}$ be a symmetric convex body and Λ a lattice in $\mathbb{R}^n$, and let p be any integer with $1 \leqslant p \leqslant n-1$. Further, let $\mu_1, \ldots, \mu_N$, where $N = \binom{n}{p}$, be the products $\lambda_{i_1}(\mathcal{C}, \Lambda) \cdots \lambda_{i_p}(\mathcal{C}, \Lambda)$ ($(i_1, \ldots, i_p) \in \mathcal{I}_{n,p}$) in non-decreasing order. Then for the successive minima of $\mathcal{C}_p$ with respect to Λ_p we have*

$$c_3(n,p) \leqslant \frac{\lambda_i(\mathcal{C}_p, \Lambda_p)}{\mu_i} \leqslant c_4(n,p) \quad \textit{for } i = 1, \ldots, N,$$

where $c_3(n,p)$, $c_4(n,p)$ depend on n and p only.

Idea of proof. Constants implied by $\ll$ and $\gg$ will depend on n and p only. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly independent vectors of Λ with $\mathbf{v}_i \in \lambda_i \mathcal{C}$, where $\lambda_i = \lambda_i(\mathcal{C}, \Lambda)$ for $i = 1, \ldots, n$. Then for each tuple $(i_1, \ldots, i_p) \in \mathcal{I}_{n,p}$ we have $\mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_p} \in \lambda_{i_1} \cdots \lambda_{i_p} \mathcal{C}_p$. Since the vectors $\mathbf{v}_{i_1} \wedge \cdots \wedge \mathbf{v}_{i_p}$ are linearly independent elements of Λ_p, it follows that $\lambda_i(\mathcal{C}_p, \Lambda_p) \leqslant \mu_i$ for $i = 1, \ldots, N$. On the other hand, by the lower bound of (2.1) applied to $\mathcal{C}_p$, Λ_p we have $\prod_{i=1}^N \lambda_i(\mathcal{C}_p, \Lambda_p) \gg d(\Lambda_p)/V(\mathcal{C}_p)$ and by the upper bound of (2.1), $\mu_1 \cdots \mu_N = (\lambda_1 \cdots \lambda_n)^P \ll (d(\Lambda)/V(\mathcal{C}))^P$. By combining this with Theorem 3.1, one easily deduces Theorem 3.2. □

Mahler's results on compound convex bodies are in fact generalisations of his results on reciprocal bodies. To make this precise, let $\mathcal{C}$ be a symmetric convex body and Λ a lattice in $\mathbb{R}^n$ and let $\mathcal{C}^*$, Λ^* be their reciprocals. Then $\Lambda^* = d(\Lambda)^{-1}\varphi(\Lambda_{n-1})$ where φ is the linear map given by $(x_1, \ldots, x_n) \mapsto (x_n, -x_{n-1}, \ldots, (-1)^{n-1}x_1)$. Further, by an observation of Mahler [M126, Theorem 4],

$$c_5(n)V(\mathcal{C})^{-1}\varphi(\mathcal{C}_{n-1}) \subseteq \mathcal{C}^* \subseteq c_6(n)V(\mathcal{C})^{-1}\varphi(\mathcal{C}_{n-1})$$

for certain numbers $c_5(n)$, $c_6(n)$ depending only on n. Together with these facts, Theorems 3.1 and 3.2 immediately imply Theorems 2.1 and 2.2 in a slightly weaker form.

As Mahler already observed in [M126], it may be quite difficult to compute the compounds of a given convex body, but often one can give an approximation which for applications is just as good. For instance, let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be linearly independent vectors in $\mathbb{R}^n$ and $A_1, \ldots, A_n$ positive reals, and consider the parallelepiped

$$\Pi := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{a}_i \cdot \mathbf{x}| \leqslant A_i \text{ for } i = 1, \ldots, n\},$$

where $\cdot$ denotes the standard inner product. Let $1 \leqslant p \leqslant n-1$, $N = \binom{n}{p}$ and define for $i = 1, \ldots, N$,

$$\widehat{\mathbf{a}}_i := \mathbf{a}_{i_1} \wedge \cdots \wedge \mathbf{a}_{i_p}, \quad \widehat{A}_i := A_{i_1} \cdots A_{i_p}, \tag{3.1}$$

where $(i_1, \ldots, i_p)$ is the i-th tuple of $\mathcal{I}_{n,p}$ in the lexicographic ordering. Then the *p-th pseudocompound* of Π is given by

$$\widehat{\Pi}_p := \{\widehat{\mathbf{x}} \in \mathbb{R}^N : |\widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{x}}| \leqslant \widehat{A}_i \text{ for } i = 1, \ldots, N\}.$$

One easily shows (see [M126, p. 377]), that there are positive numbers $c_7(n,p)$, $c_8(n,p)$ such that $c_7(n,p)\Pi_p \subseteq \widehat{\Pi}_p \subseteq c_8(n,p)\Pi_p$, where Π_p is the p-th compound of Π. This implies that Theorem 3.2 holds with $\widehat{\Pi}_p$ instead of Π_p, with other constants $c_3(n,p), c_4(n,p)$.

Mahler's results on compound convex bodies turned out to be a very important tool in Diophantine approximation. First, it is a crucial ingredient in Schmidt's proof of his celebrated Subspace Theorem [17, 18], and second it has been used to deduce several transference principles for systems of Diophantine inequalities.

We first give a very brief overview of Schmidt's proof of his Subspace Theorem, focusing on the role of Theorem 3.2. For the complete proof, see [18].

Subspace Theorem. *Let $n \geqslant 2$ and let $L_i(\mathbf{X}) = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n$ ($i = 1, \ldots, n$) be linearly independent linear forms with algebraic coefficients in $\mathbb{C}$. Further, let $\delta > 0$. Then the set of solutions of*

$$|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \leqslant \|\mathbf{x}\|^{-\delta} \quad \textit{in } \mathbf{x} \in \mathbb{Z}^n \tag{3.2}$$

is contained in finitely many proper linear subspaces of $\mathbb{Q}^n$.

Outline of the proof. We can make a reduction to the case that $L_1, \dots, L_n$ all have real algebraic coefficients by replacing each L_i by its real or imaginary part, such that the resulting linear forms are linearly independent. Further, after a normalisation we arrange that these linear forms have determinant 1. So henceforth we assume that the coefficients of $L_1, \dots, L_n$ are real algebraic, with $\det(L_1, \dots, L_n) = 1$. Next, it suffices to consider only $\mathbf{x} \in \mathbb{Z}^n$ with $L_i(\mathbf{x}) \neq 0$ for $i = 1, \dots, n$.

Now let $\mathbf{x} \in \mathbb{Z}^n$ be a solution of (3.2) and put

$$A_i := |L_i(\mathbf{x})|/|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})|^{1/n} \ (i = 1, \dots, n),$$
$$\mathbf{A} := (A_1, \dots, A_n), \quad Q(\mathbf{A}) := \max(A_1, \dots, A_n).$$

With this choice, $A_1 \cdots A_n = 1$. Assuming that $\|\mathbf{x}\|$ is sufficiently large, there is a fixed $D > 0$ independent of $\mathbf{x}$, such that $\|\mathbf{x}\|^{-D} \leqslant |L_i(\mathbf{x})| \leqslant \|\mathbf{x}\|^D$ for $i = 1, \dots, n$. Hence $Q(\mathbf{A}) \leqslant \|\mathbf{x}\|^{2D}$. Write $L_i(\mathbf{X}) = \mathbf{a}_i \cdot \mathbf{X}$ where $\mathbf{a}_i$ is the vector of coefficients of L_i and consider the parallelepiped

$$\Pi(\mathbf{A}) := \{\mathbf{y} \in \mathbb{R}^n : \ |\mathbf{a}_i \cdot \mathbf{y}| \leqslant A_i \text{ for } i = 1, \dots, n\}. \tag{3.3}$$

Since $|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})|^{1/n} \leqslant \|\mathbf{x}\|^{-\delta/n} \leqslant Q(\mathbf{A})^{-\delta_1}$ with $\delta_1 := \delta/2nD$, we have

$$\mathbf{x} \in Q(\mathbf{A})^{-\delta_1}\Pi(\mathbf{A}).$$

Let $T(\mathbf{A})$ denote the vector space generated by $Q(\mathbf{A})^{-\delta_1}\Pi(\mathbf{A}) \cap \mathbb{Z}^n$. So $\mathbf{x} \in T(\mathbf{A})$. It clearly suffices to show the following:

for every $\delta_1 > 0$ there is a finite collection $\{T_1, \dots, T_t\}$ of proper linear subspaces of $\mathbb{Q}^n$ such that for every n-tuple $\mathbf{A}$ of positive reals with $A_1 \cdots A_n = 1$, the vector space $T(\mathbf{A})$ is contained in one of $T_1, \dots, T_t$.

Assume that this assertion is false. Pick many tuples $\mathbf{A}_1, \dots, \mathbf{A}_m$ such that the spaces $T^{(i)} := T(\mathbf{A}_i)$ $(i = 1, \dots, m)$ are all different. Then one can construct a polynomial in m blocks of n variables $\mathbf{X}_1, \dots, \mathbf{X}_m$ with integer coefficients, which is homogeneous in each block and divisible by high powers of $L_i(\mathbf{X}_j)$, for $i = 1, \dots, n$, $j = 1, \dots, m$. All partial derivatives of this polynomial of order up to a certain bound have absolute value < 1, hence are 0, at many integral points of $T^{(1)} \times \cdots \times T^{(m)}$. Then by extrapolation, it follows that this polynomial vanishes with high multiplicity on all of $T^{(1)} \times \cdots \times T^{(m)}$. Now one would like to apply a non-vanishing result implying that this is impossible, but such a result can been proved only if the dimensions of $T^{(1)}, \dots, T^{(m)}$ are equal to $n-1$. So the above argument works only for those tuples $\mathbf{A}$ for which $\dim T(\mathbf{A}) = n-1$, that is, for which the $(n-1)$-th successive minimum of $\Pi(\mathbf{A})$ with respect to $\mathbb{Z}^n$ is at most $Q(\mathbf{A})^{-\delta_1}$.

Now Schmidt could make his proof of the Subspace Theorem work for arbitrary tuples $\mathbf{A}$ by means of an ingenious argument, in which he constructs from the parallelepiped $\Pi(\mathbf{A})$ a new parallelepiped $\widehat{\Pi}(\widehat{\mathbf{B}})$, in general of larger dimension N, with $\widehat{\mathbf{B}} = (\widehat{B}_1, \dots, \widehat{B}_N)$ satisfying $\widehat{B}_1 \cdots \widehat{B}_N = 1$, of which the $(N-1)$-th successive minimum with respect to $\mathbb{Z}^N$ is small. In this construction, Mahler's results on compound convex bodies play a crucial role.

In what follows, constants implies by $\ll$, $\gg$, $\asymp$ will depend only on n, δ_1 and $L_1, \dots, L_n$, while $\delta_2, \delta_3, \dots$ will denote positive numbers depending only on δ_1 and n. Denote the successive minima of $\Pi(\mathbf{A})$ with respect to $\mathbb{Z}^n$ by $\lambda_1, \dots, \lambda_n$. Then clearly,

$$\lambda_1 \leqslant Q(\mathbf{A})^{-\delta_1}.$$

Further, by (2.1),

$$\lambda_1 \cdots \lambda_n \asymp 1. \tag{3.4}$$

Let k be the largest index with $\lambda_k \leqslant Q(\mathbf{A})^{-\delta_1}$. Then (3.4) implies that $\lambda_n \gg Q(\mathbf{A})^{k\delta_1/(n-k)}$. Hence there is p with $k \leqslant n-p \leqslant n-1$ such that $\lambda_{n-p}/\lambda_{n-p+1} \ll Q(\mathbf{A})^{-\delta_2}$. Let $S(\mathbf{A})$ be the vector space generated by $\lambda_{n-p}\Pi(\mathbf{A}) \cap \mathbb{Z}^n$. This space contains $T(\mathbf{A})$. So it suffices to prove that $S(\mathbf{A})$ *runs through a finite collection of proper linear subspaces of* $\mathbb{Q}^n$.

Let $N := \binom{n}{p}$ and consider the p-th pseudocompound

$$\widehat{\Pi}_p(\widehat{\mathbf{A}}) = \{\widehat{\mathbf{y}} \in \mathbb{R}^N : |\widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{y}}| \leqslant \widehat{A}_i \ \text{ for } i = 1, \dots, N\}.$$

Denote by $\widehat{\lambda}_1, \dots, \widehat{\lambda}_N$ the successive minima of $\widehat{\Pi}_p(\widehat{\mathbf{A}})$ with respect to $\mathbb{Z}^n$. Then by Theorem 3.2 we have for the last two minima, $\widehat{\lambda}_{N-1} \asymp \lambda_{n-p}\lambda_{n-p+2}\cdots\lambda_n$, $\widehat{\lambda}_N \asymp \lambda_{n-p+1}\cdots\lambda_n$. Hence

$$\widehat{\lambda}_{N-1}/\widehat{\lambda}_N \ll \lambda_{n-p}/\lambda_{n-p+1} \ll Q(\mathbf{A})^{-\delta_2}. \tag{3.5}$$

Moreover, by (3.4), Theorem 3.2 we have

$$\widehat{\lambda}_1 \cdots \widehat{\lambda}_N \asymp 1. \tag{3.6}$$

We still need one reduction step. By a variation on a result of Davenport, proved by Schmidt (see e.g., [18, p. 89]), for every choice of reals $\rho_1, \dots, \rho_N$ with

$$\rho_1 \geqslant \cdots \geqslant \rho_N > 0, \quad \rho_1\widehat{\lambda}_1 \leqslant \cdots \leqslant \rho_N\widehat{\lambda}_N, \quad \rho_1 \cdots \rho_N = 1,$$

there is a permutation σ of $1, \dots, N$ such that the parallelepiped

$$\widehat{\Pi}_p(\widehat{\mathbf{B}}) = \{\widehat{\mathbf{y}} \in \mathbb{R}^N : |\widehat{\mathbf{a}}_i \cdot \widehat{\mathbf{y}}| \leqslant \widehat{B}_i \text{ for } i = 1, \dots, N\},$$

where $\widehat{B}_i := \rho_{\sigma(i)}^{-1}\widehat{A}_i$ for $i = 1, \dots, N$, has successive minima $\widehat{\lambda}_i' \asymp \rho_i\widehat{\lambda}_i$ for $i = 1, \dots, N$. Now with the choice

$$\rho_i = c/\widehat{\lambda}_i \ (i = 1, \dots, N-1), \quad \rho_N = c/\widehat{\lambda}_{N-1}$$

where

$$c = (\widehat{\lambda_1} \cdots \widehat{\lambda_N})^{1/N}(\widehat{\lambda}_{N-1}/\widehat{\lambda}_N)^{1/N}$$

has been chosen to make $\rho_1\cdots\rho_N=1$, we obtain $\widehat{\lambda}'_{N-1}\ll c\ll Q(\mathbf{A})^{-\delta_3}$ in view of (3.5),(3.6). One can show that

$$Q(\widehat{\mathbf{B}}):=\max(\widehat{B}_1,,\ldots,\widehat{B}_N)\ll Q(\mathbf{A})^d$$

with d depending only on n and p. Thus, $\widehat{\lambda}'_{N-1}\ll Q(\widehat{\mathbf{B}})^{-\delta_4}$. Further,

$$\widehat{B}_1\cdots\widehat{B}_N=\rho_1\cdots\rho_N(A_1\cdots A_n)^{\binom{n-1}{p-1}}=1.$$

Now by means of the argument sketched above, with the construction of the polynomial and the application of the non-vanishing result, one can show that if $\mathbf{A}=(A_1,\ldots,A_n)$ runs through the tuples of positive reals with $A_1\cdots A_n=1$, then the vector space $T(\widehat{\mathbf{B}})$ generated by $\widehat{\lambda}'_{N-1}\widehat{\Pi}_p(\widehat{\mathbf{B}})\cap\mathbb{Z}^N$ runs through a finite collection. One can show that $T(\widehat{\mathbf{B}})$ uniquely determines the space $S(\mathbf{A})$. Hence $S(\mathbf{A})$ runs through a finite collection. This proves the Subspace Theorem. □

We should mention here that Faltings and Wüstholz [6] gave a very different proof of the Subspace Theorem, avoiding geometry of numbers but using instead some involved algebraic geometry.

Mahler's results on compound convex bodies have been applied at various other places, in particular to obtain generalisations of Khintchine's transference principle and Corollary 2.4. Many of these results can be incorporated into the *Parametric Geometry of Numbers*, a recent theory which was initiated by Schmidt and Summerer [19, 20]. The general idea is as follows. Let $\mu_1,\ldots,\mu_n$ be fixed reals which we normalise so that $\mu_1+\cdots+\mu_n=0$ and consider the parametrised class of convex bodies in $\mathbb{R}^n$,

$$\mathcal{C}(q):=\{\mathbf{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n:\ |x_i|\leqslant e^{\mu_i q}\text{ for }i=1,\ldots,n\}\quad(q>0).$$

Further, let Λ be a fixed lattice in $\mathbb{R}^n$ and $\lambda_1(q),\ldots,\lambda_n(q)$ the successive minima of $\mathcal{C}(q)$ with respect to Λ. Then one would like to study these successive minima as functions of q. In particular, one is interested in the quantities

$$\begin{cases}\underline{\varphi}_i=\underline{\varphi}_i(\Lambda,\boldsymbol{\mu}):=\liminf\limits_{q\to\infty}(\log\lambda_i(q))/q,\\ \overline{\varphi}_i=\overline{\varphi}_i(\Lambda,\boldsymbol{\mu}):=\limsup\limits_{q\to\infty}(\log\lambda_i(q))/q\end{cases}\quad(i=1,\ldots,n).\tag{3.7}$$

That is, $\underline{\varphi}_i$ is the infimum of all η such that there are arbitrarily large q for which the system of inequalities

$$|x_1|\leqslant e^{(\mu_1+\eta)q},\ldots,|x_n|\leqslant e^{(\mu_n+\eta)q}\tag{3.8}$$

is satisfied by i linearly independent points from Λ, while $\overline{\varphi}_i$ is the infimum of all η such that for every sufficiently large q, system (3.8) is satisfied by i linearly independent points from Λ. The quantities $\underline{\varphi}_i$, $\overline{\varphi}_i$ are finite, since if

$\mu > \max_j |\mu_j|$, then for every sufficiently large q, the body $e^{\mu q}\mathcal{C}(q)$ contains n linearly independent points from Λ, while $e^{-\mu q}\mathcal{C}(q)$ does not contain a non-zero point of Λ.

In case that Λ is an algebraic lattice, i.e., if it is generated by vectors with algebraic coordinates, then by following the proof of the Subspace Theorem one can show that $\underline{\varphi}_i = \overline{\varphi}_i$ for $i = 1, \ldots, n$, i.e., the limits exist (this is a special case of [5, Theorem 16.1], but very likely this was known before). However, for non-algebraic lattices Λ it may happen that $\underline{\varphi}_i < \overline{\varphi}_i$ for some i.

Many of the Diophantine approximation exponents that have been introduced during the last decades can be expressed in terms of the quantities $\underline{\varphi}_i$, $\overline{\varphi}_i$, and thus, results for these exponents can be translated into results for the $\underline{\varphi}_i$, $\overline{\varphi}_i$. For instance, let A be a real $(n-m) \times m$-matrix with $1 \leqslant m < n$, and take

$$\Lambda = \{(\mathbf{x}, A\mathbf{x} - \mathbf{y}) : \mathbf{x} \in \mathbb{Z}^m, \mathbf{y} \in \mathbb{Z}^{n-m}\},$$
$$\mu_1 = \cdots = \mu_m = n - m, \quad \mu_{m+1} = \cdots = \mu_n = m.$$

Define $\underline{\varphi}_i(A) := \underline{\varphi}_i(\Lambda, \boldsymbol{\mu})$ for this Λ and $\boldsymbol{\mu}$. Then for the quantities ω, ω^* from Corollary 2.4 we have

$$\underline{\varphi}_1(A) = -\frac{(n-m)^2\omega}{n + (n-m)\omega}, \quad \underline{\varphi}_1(A^T) = -\frac{m^2\omega^*}{n + m\omega^*},$$

and the inequalities (2.4) become

$$\underline{\varphi}_1(A^T) \leqslant \tfrac{1}{n-1} \cdot \underline{\varphi}_1(A), \quad \underline{\varphi}_1(A) \leqslant \tfrac{1}{n-1} \cdot \underline{\varphi}_1(A^T).$$

Studying the successive minima functions $\lambda_i(q)$ for arbitrary lattices Λ and reals $\mu_1, \ldots, \mu_n$ is probably much too hard. In their papers [19, 20] Schmidt and Summerer considered the special case

$$\begin{cases} \Lambda = \{(x, \xi_1 x - y_1, \cdots \xi_{n-1}x - y_{n-1}) : x, y_1, \ldots, y_{n-1} \in \mathbb{Z}\}, \\ \mu_1 = n-1, \ \mu_2 = \cdots = \mu_n = -1, \end{cases} \tag{3.9}$$

where $\xi_1, \ldots, \xi_{n-1}$ are reals such that $1, \xi_1, \ldots, \xi_{n-1}$ are linearly independent over $\mathbb{Q}$. That is, they considered the system of inequalities

$$|x| \leqslant e^{(n-1)q}, \quad |\xi_i x - y_i| \leqslant e^{-q} \quad (i = 1, \ldots, n-1).$$

Let $\underline{\varphi}_i, \overline{\varphi}_i$ be the quantities defined in (3.7), with $\Lambda, \boldsymbol{\mu}$ as in (3.9). In [19], Schmidt and Summerer showed among other things that for every $i \in \{1, \ldots, n-1\}$ there are arbitrarily large q such that $\lambda_{i+1}(q) = \lambda_i(q)$. As a consequence, $\underline{\varphi}_{i+1} \geqslant \overline{\varphi}_i$ for $i = 1, \ldots, n-1$. They deduced several other algebraic inequalities for the numbers $\underline{\varphi}_i, \overline{\varphi}_i$.

In [20], Schmidt and Summerer continued their research and studied in more detail the functions

$$L_i(q) := \log \lambda_i(q) \quad (i = 1, \ldots, n).$$

To this end, they introduced a class of n-tuples of continuous, piecewise linear functions on $(0,\infty)$ with certain properties, the so-called (n,γ)-*systems*. The key argument in their proof is, that there is an (n,γ)-system $(P_1(q),\dots,P_n(q))$ such that $|L_i(q)-P_i(q)|\leqslant c(n)$ for $i=1,\dots,n$, $q>0$, where $c(n)$ depends on n only. In the construction of these functions, essential use is made of Mahler's results on compound convex bodies. Indeed, for $p=1,\dots,n-1$ let $\mathcal{C}^{(p)}(q)$ be the p-th pseudocompound of $\mathcal{C}(q)$ and let $e^{M_p(q)}$ be the first minimum of $\mathcal{C}^{(p)}(q)$ with respect to the p-th compound Λ_p of Λ. Further, put $M_0(q)=M_n(q):=0$. Schmidt and Summerer showed that the functions $P_i(q):=M_i(q)-M_{i-1}(q)$ $(i=1,\dots,n)$ form an (n,γ)-system. Theorem 3.2 implies that there is $c(n)>0$ such that $|L_i(q)-P_i(q)|\leqslant c(n)$ for $i=1,\dots,n$, $q>0$. It is important that $P_1(q)+\cdots+P_n(q)=0$, while for the original functions $L_1(q),\dots,L_n(q)$ one knows only that their sum is bounded. It is clear that for $i=1,\dots,n$ we have $\underline{\varphi}_i=\underline{\pi}_i$, $\overline{\varphi}_i=\overline{\pi}_i$ where $\underline{\pi}_i:=\liminf_{q\to\infty}P_i(q)/q$ and $\overline{\pi}_i:=\limsup_{q\to\infty}P_i(q)/q$.

Schmidt and Summerer analysed (n,γ)-systems, which involved basically combinatorics and had no connection with geometry of numbers anymore. As a result of their (fairly difficult) analysis they obtained several algebraic inequalities for $\underline{\pi}_i$, $\overline{\pi}_i$ $(i=1,\dots,n)$. These imply of course the same inequalities for $\underline{\varphi}_i$, $\overline{\varphi}_i$ $(i=1,\dots,n)$. This led to new proofs of older results and also various new results.

For instance, it is an easy consequence of Minkowski's theorem on successive minima that

$$(n-1)\underline{\varphi}_1+\overline{\varphi}_n\leqslant 0,\quad (n-1)\overline{\varphi}_n+\underline{\varphi}_1\geqslant 0.$$

Schmidt and Summerer [20, bottom of p. 55] improved this to

$$(n-1)\underline{\varphi}_1+\overline{\varphi}_n\leqslant \overline{\varphi}_1(n-\underline{\varphi}_1+\overline{\varphi}_n),\quad (n-1)\overline{\varphi}_n+\underline{\varphi}_1\geqslant \underline{\varphi}_n(n-\overline{\varphi}_n+\underline{\varphi}_1).$$

Recently, Roy [15] showed that the functions $L_1(q),\dots,L_n(q)$ considered by Schmidt and Summerer can be approximated very well by piecewise linear functions from a more restrictive class, the $(n,0)$-*systems*. This smaller class may be more easy to analyse than the (n,γ)-systems and may perhaps lead to new insights in the functions $L_i(q)$.

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Jan-Hendrik Evertse
Universiteit Leiden
Mathematisch Instituut
Postbus 9512
2300 RA Leiden
The Netherlands