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Chern–Weil and Hilbert–Samuel Formulae for Singular Hermitian Line Bundles

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Abstract. We show a Chern–Weil type statement and a Hilbert–Samuel formula for a large class of singular plurisubharmonic metrics on a line bundle over a smooth projective complex variety. For this we use the theory of b-divisors and the so-called multiplier ideal volume function. We apply our results to the line bundle of Siegel–Jacobi forms over the universal abelian variety endowed with its canonical invariant metric. This generalizes the results of [15] to higher degrees.

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Contents

1 Introduction

1.1 Motivation and background

Let X be a differentiable manifold and let $\mathcal{E}$ be a vector bundle on X. The *Chern classes* of $\mathcal{E}$ are topological invariants $c_i(\mathcal{E}) \in H^{2i}(X, \mathbb{Z})$ that measure, in some sense, how far $\mathcal{E}$ is from being trivial. *Chern-Weil theory* tells us that these Chern classes can be represented in the de Rham cohomology of X by means of smooth differential forms obtained by applying an invariant polynomial to the curvature matrix associated to a smooth connection on $\mathcal{E}$.

When X is a complex manifold and $\mathcal{E}$ is a holomorphic vector bundle on X equipped with a smooth hermitian metric, there is a unique smooth connection on $\mathcal{E}$ compatible at the same time with the hermitian metric and with the holomorphic structure. We see that for a holomorphic vector bundle $\mathcal{E}$ on a complex manifold, a smooth hermitian metric h uniquely determines *Chern forms* $c_1(\mathcal{E}, h), \ldots, c_i(\mathcal{E}, h), \ldots$ representing the Chern classes $c_1(\mathcal{E}), \ldots, c_i(\mathcal{E}), \ldots$ in de Rham cohomology.

Assume that X is compact, that $\dim X = n$, and let $x_1, \ldots, x_n$ be variables where x_i has weight i. Let $P \in \mathbb{Z}[\ldots, x_i, \ldots]$ be a homogeneous polynomial of weight n. As a particular instance of Chern–Weil theory we find the equality

$$P(\ldots, c_i(\mathcal{E}), \ldots) = \int_X P(\ldots, c_i(\mathcal{E}, h), \ldots) \tag{1.1}$$

in $\mathbb{Z}$, where the expression $P(\ldots, c_i(\mathcal{E}), \ldots)$ on the left hand side is interpreted as an integer upon taking the degree, and where $P(\ldots, c_i(\mathcal{E}, h), \ldots)$ is a closed (n, n)-form on X. As a special case, we note that if $\mathcal{L}$ is a holomorphic *line bundle* on X equipped with a smooth hermitian metric h then we have the equality

$$c_1(\mathcal{L})^n = \int_X c_1(\mathcal{L}, h)^n \tag{1.2}$$

in $\mathbb{Z}$, with $c_1(\mathcal{L})^n \in \mathbb{Z}$ the *degree* of the line bundle $\mathcal{L}$.

Now although smooth hermitian metrics always exist, they can be difficult to write down explicitly. In fact in many situations the *natural* hermitian metric h associated to the problem at hand is only smooth on a dense open subset but singular along say a normal crossings divisor D on X. In this context Mumford [35] has introduced the notion of *good* metrics. The condition of being "good" is phrased in terms of conditions on the asymptotic behavior of h and its first and second derivatives near D; we refer to Example 2.34 for the precise definition. Good metrics are a class of singular metrics that for the purpose of Chern–Weil theory are as good as smooth metrics. More precisely, let X be a compact complex manifold, $\mathcal{E}$ a holomorphic vector bundle on X and $U \subset X$ a dense open subset whose complement is a normal crossings divisor. Let h be a smooth metric on $\mathcal{E}|_U$ such that h is good in the sense of Mumford as a singular metric on $\mathcal{E}$. Then for every polynomial $P \in \mathbb{Z}[\ldots, x_i, \ldots]$ the smooth closed differential form

$$P(\mathcal{E},h) = P(c_1(\mathcal{E},h),\dots,c_i(\mathcal{E},h),\dots)$$

on U is locally integrable as a differential form on X, and thus determines a closed current $[P(\mathcal{E},h)]$ on X. In fact, the closed current $[P(\mathcal{E},h)]$ represents the class $P(\mathcal{E}) = P(\dots,c_i(\mathcal{E}),\dots)$ in de Rham cohomology of X. In particular, if x_i has weight i and $P \in \mathbb{Z}[\dots,x_i,\dots]$ is a homogeneous polynomial of weight n, we have the equality

$$P(\dots,c_i(\mathcal{E}),\dots) = \int_U P(\dots,c_i(\mathcal{E},h),\dots) \tag{1.3}$$

in $\mathbb{Z}$, where again the expression $P(\dots,c_i(\mathcal{E}),\dots)$ on the left hand side is interpreted as an integer upon taking the degree. If $\mathcal{L}$ is a holomorphic *line bundle* on X equipped with a smooth metric h on $\mathcal{L}|_U$ such that h is good as a singular metric on $\mathcal{L}$, then we have the equality

$$c_1(\mathcal{L})^n = \int_U c_1(\mathcal{L},h)^n \tag{1.4}$$

in $\mathbb{Z}$.

As equations (1.3) and (1.4) suggest, good metrics are better understood in the context of *extending* a vector bundle from a non-compact manifold to a compactification. In fact, the notion of good metric appears naturally in the study of compactifications of *locally symmetric domains*.

Let D be a bounded symmetric domain and Γ a neat arithmetic group acting on D by isometries. Then the quotient $U = D/\Gamma$ is a locally symmetric space and has a natural structure of a quasi-projective variety. In [1] the authors introduce a family of toroidal compactifications of U. We have that D is a quotient $D = K\backslash G$, where G is a semisimple adjoint group and K is a maximal compact subgroup. Any unitary representation of K defines a (fully decomposable) holomorphic vector bundle $\mathcal{E}^o$ on U together with a smooth hermitian metric h.

As Mumford proves in [35], when X is a smooth toroidal compactification of $U = D/\Gamma$ in the sense of [1], there exists a unique extension of $\mathcal{E}^o$ to a vector bundle $\mathcal{E}$ on X such that the metric h extends to a singular good hermitian metric. In particular, when $n = \dim X$ and the variables x_i have weight i then for every homogeneous polynomial $P \in \mathbb{Z}[\dots,x_i,\dots]$ of weight n we have the equality (1.3) in $\mathbb{Z}$. As a consequence, we see that the element $P(\dots,c_i(\mathcal{E}),\dots) \in \mathbb{Z}$ does not depend on the choice of toroidal compactification X.

Mumford's result applies notably to the case of *pure Shimura varieties*, as the connected components of their associated analytic spaces are locally symmetric spaces.

It makes sense to ask whether an extension of the Chern–Weil result (1.3) holds in the context of *mixed Shimura varieties*, for example moduli spaces of abelian varieties with marked points. Such spaces again come equipped with natural automorphic line bundles, with natural smooth hermitian metrics on them. Moreover, such spaces have natural smooth toroidal compactifications.

However, as was shown in [15] in an example dealing with certain modular elliptic surfaces, in the case of mixed Shimura varieties new types of metric singularities appear, that are *not* good in the sense of Mumford. In fact, a naive extension of the Chern–Weil result (1.3) turns out to fail already in this simple situation.

Let us give some details. Let U denote the moduli space $E(N)$ of elliptic curves with two marked points and with level N. Let $\mathcal{L}$ be the line bundle of Jacobi forms on U of weight four and index four, and let h denote the natural invariant hermitian metric on $\mathcal{L}$. As is proved in [15], for any smooth toroidal compactification X of U, there exists a subset $S \subset X$ of codimension two as well as an integer $r \geq 1$ such that the smooth hermitian line bundle $\mathcal{L}^{\otimes r}$ over U can be extended to a line bundle $\mathcal{L}_{X\setminus S,r}$ over $X \setminus S$ in such a way that $h^{\otimes r}$ extends to a good singular metric in the sense of Mumford. Since S has codimension two, the line bundle $\mathcal{L}_{X\setminus S,r}$ can be extended uniquely to a line bundle $\mathcal{L}_{X,r}$ over X. However, the metric h is no longer good at some of the points of S and in fact, the intersection number

$$\frac{1}{r^2} c_1(\mathcal{L}_{X,r})^2$$

turns out to depend on the choice of toroidal compactification X. It follows that (1.3) can not be directly extended to the case of mixed Shimura varieties. As is shown in [15], to recover a Chern–Weil type result in the setting of the mixed Shimura variety $U = E(N)$ we should not extend the line bundle $\mathcal{L}$ of Jacobi forms as a line bundle, but as a different object. The result is best stated in the language of *b-divisors*. Loosely speaking, a b-divisor is a collection of divisors on a tower of modifications of a given compactification X of U, compatible with push-forward. The precise definition is given in Section 4.

Let s be a non-zero rational section of the holomorphic line bundle $\mathcal{L}$, and let D be the divisor of s, so that $\mathcal{L} \simeq \mathcal{O}(D)$. In [15], a natural b-divisor $D(\mathcal{L}, h, s)$ on X with rational coefficients extending the divisor D is constructed, which is then shown to have a natural self-intersection number $D(\mathcal{L}, h, s)^2$ in $\mathbb{R}$. Here, the self-intersection number $D(\mathcal{L}, h, s)^2$ is computed as a limit of self-intersections of the "incarnations" of $D(\mathcal{L}, h, s)$ on all models in the tower. Finally, the equality

$$D(\mathcal{L}, h, s)^2 = \int_U c_1(\mathcal{L}, h)^{\wedge 2} \tag{1.5}$$

is shown to hold in $\mathbb{R}$. Note that (1.5) can be seen as a Chern–Weil type result, where the intersection number on the left is taken in the sense of b-divisors on X, and the integral on the right is taken over the open dense subset $U \subset X$.

In [15] the above Chern–Weil result is complemented with a result of Hilbert–Samuel type that we also recall. Let $\mathbb{H}$ be the upper half plane. Jacobi forms of weight k and index m are holomorphic functions on $\mathbb{H} \times \mathbb{C}$ satisfying certain transformation properties with respect to a subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ and the abelian group $\mathbb{Z}^2$ as well as a growth condition when approaching the boundary of $\mathbb{H}$. The basic reference for the theory of Jacobi forms is the book [25]. Let $J_{4r,4r}(\Gamma(N))$ be the space of Jacobi forms of weight $4r$ and index $4r$ with respect to the principal congruence subgroup $\Gamma(N)$. The elements of $J_{4r,4r}(\Gamma(N))$ can be seen as sections of the line bundle $\mathcal{L}^{\otimes r}$ over U satisfying some growth conditions along $X \setminus U$. The second main result of [15] is the formula

$$\lim_{r\to\infty} \frac{\dim J_{4r,4r}(\Gamma(N))}{r^2/2} = D(\mathcal{L}, h, s)^2. \tag{1.6}$$

That is, the asymptotic growth of the dimensions of the spaces of Jacobi forms is not given by the self-intersection of a divisor but of a b-divisor. Therefore to recover a Hilbert–Samuel type of result for the space of Jacobi forms we need to extend $\mathcal{L}$ as a b-divisor and not as an ordinary line bundle.

Note that by combining equations (1.5) and (1.6) we find that the asymptotic growth of the space of Jacobi forms is governed by an integral of a smooth differential form over U.

The proof of (1.5) and (1.6) in [15] consists of an explicit computation of both sides of each equation. The present paper finds its origin in the following two questions: whether there could be a more intrinsic approach to proving the equalities (1.5) and (1.6), and whether these equalities might admit a generalization to the setting of automorphic line bundles on mixed Shimura varieties in *higher dimensions*. We will answer both questions in the affirmative. As we will see in a forthcoming paper, in particular this will give us a formula for the asymptotic growth of the dimensions of spaces of Siegel–Jacobi forms.

1.2 Statement of the main results

Let X be a projective complex manifold and let $\mathcal{L}$ be a holomorphic line bundle on X. We start out by defining a suitable class of singular hermitian metrics on $\mathcal{L}$, namely those with *almost asymptotically algebraic singularities*.

The precise definition of almost asymptotically algebraic singularities is given in Definition 3.2, but in essence it asserts that our metrics be *plurisubharmonic*, with local potentials that are bounded away from a normal crossings divisor D on X and that can be approximated well by logarithms of sums of square norms of holomorphic functions near D (i.e., by metrics with *algebraic singularities*).

Many natural singular hermitian metrics turn out to be almost asymptotically algebraic. For example, we prove, using Demailly's celebrated regularization theorem for plurisubharmonic metrics (see [23, Theorem 4.2], [24, Theorem 3.2]), that a *toroidal plurisubharmonic metric* is almost asymptotically algebraic; see Proposition 3.11. Also, as we will see in Lemma 3.3, a plurisubharmonic metric which is good in the sense of Mumford has almost asymptotically algebraic singularities.

Our main results are generalizations of the Chern–Weil formula (1.5) and of the Hilbert–Samuel formula (1.6) to the setting of plurisubharmonic metrics with almost asymptotically algebraic singularities. In order to formulate our Chern–Weil result, let h be a plurisubharmonic metric on $\mathcal{L}$.

We can make sense of $c_1(\mathcal{L}, h)$ as a closed, positive $(1,1)$-current on X simply by taking $-dd^c \log h(s)$ for a local generating section s of $\mathcal{L}$. Let $n = \dim X$. A generalisation of Bedford-Taylor calculus by Boucksom, Eyssidieux, Guedj and Zeriahi [10] allows us to define the non-pluripolar Monge–Ampère measure $\langle c_1(\mathcal{L}, h)^n \rangle$, a closed, positive (n,n)-current on X.

The right hand side of our Chern–Weil formula will be the *non-pluripolar volume*

$$\int_X \langle c_1(\mathcal{L}, h)^n \rangle \tag{1.7}$$

of the plurisubharmonic line bundle $(\mathcal{L}, h)$. We note that, if h has bounded local potentials on a Zariski dense open set of U of X, then the equality

$$\int_X \langle c_1(\mathcal{L}, h)^n \rangle = \int_U c_1(\mathcal{L}, h)^n$$

holds in $\mathbb{R}_{\geq 0}$. In this case we say that h has *Zariski unbounded locus*. In particular such a metric has "small unbounded locus" in the sense of [10]. Most examples of singular metrics arising from algebraic geometry have Zariski unbounded locus; for example, plurisubharmonic metrics with almost asymptotically algebraic singularities have Zariski unbounded locus.

Fix a smooth reference metric h_0 on $\mathcal{L}$. Let $\theta = c_1(\mathcal{L}, h_0)$ be the first Chern form of h_0 and set $\varphi = -\log(h(s)/h_0(s))$, where s is any non-zero rational section of $\mathcal{L}$. Let $\mathcal{J}(\varphi)$ denote the multiplier ideal associated to φ (see Definition 2.15). The non-pluripolar volume (1.7) turns out to be intimately related to the *multiplier ideal volume* of $(\mathcal{L}, h)$, given by the limit

$$\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h) := \lim_{k \to \infty} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{J}(k\varphi))}{k^n/n!}\,.$$

Note that the right hand side is indeed independent of the choice of h_0. The quantity $\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h)$ is called *arithmetic volume* in [18].

More precisely, Darvas and Xia show in [18] that if $\mathcal{L}$ is ample we have the lower bound

$$\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h) \geq \int_X \langle c_1(\mathcal{L}, h)^n \rangle\ . \tag{1.8}$$

Moreover, they show that if one assumes that the non-pluripolar volume is strictly positive, then equality holds in (1.8) if and only if h is in some precise sense very well approximable by algebraic singularities, see [18, Theorem 5.5] for the precise conditions. This can be seen as an analytic Hilbert–Samuel type statement. We note that these results were very recently extended and generalized to the case where $\mathcal{L}$ is only assumed to be pseudoeffective, see [19, Theorem 1.1].

Our notion of almost asymptotically algebraic singularities is probably stronger than the list of equivalent conditions on singularity type that is mentioned in

[18, Theorem 5.5] (see Corollary 3.14 and the discussion afterwards), however our notion of almost asymptotically algebraic singularities has the advantage of being easy to verify in concrete cases.
As a first illustration of our viewpoint, we give a direct proof of equality in (1.8) in the case of almost asymptotically algebraic singularities, without the assumption that $\mathcal{L}$ is ample.

Theorem A. *(Theorem 3.13) Assume that the plurisubharmonic metric h has almost asymptotically algebraic singularities. Then the analytic Hilbert–Samuel type formula*

$$\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h) = \int_X \langle c_1(\mathcal{L}, h)^n \rangle$$

holds in $\mathbb{R}_{\geq 0}$.

In the spirit of [15], the left hand side of our Chern–Weil formula will be an intersection product of b-divisors. Assume that the plurisubharmonic metric h has Zariski unbounded locus. Given a non-zero rational section s of $\mathcal{L}$, we construct an associated $\mathbb{R}$-b-divisor $D(\mathcal{L}, h, s)$ on X using the *Lelong numbers* of the current $c_1(\mathcal{L}, h)$ at all prime divisors on all modifications of X (see Definition 5.4). This extends a construction due to Boucksom, Favre and Jonsson in the local case in [11].
Unfortunately, the set of all $\mathbb{R}$-b-divisors does not admit a natural intersection product. Dang and Favre [17] have shown though that the set of so-called *approximable nef* b-divisors (see Definition 4.8) does admit a natural intersection product with values in $\mathbb{R}$.

Theorem B. *(Theorem 5.18) Assume that the plurisubharmonic metric h has Zariski unbounded locus. Then the associated $\mathbb{R}$-b-divisor $D(\mathcal{L}, h, s)$ on X is approximable nef.*

In particular, the degree $D(\mathcal{L}, h, s)^n \in \mathbb{R}$ is well-defined. This allows us to state the following Chern–Weil type result.

Theorem C. *(Theorem 5.20) Assume that the plurisubharmonic metric h has almost asymptotically algebraic singularities. Then the equality*

$$D(\mathcal{L}, h, s)^n = \int_X \langle c_1(\mathcal{L}, h)^n \rangle$$

holds in $\mathbb{R}_{\geq 0}$.

Combining Theorem A and Theorem C we obtain the equality

$$\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h) = D(\mathcal{L}, h, s)^n \tag{1.9}$$

for plurisubharmonic metrics with almost asymptotically algebraic singularities, which can be seen as a b-divisorial version of the classical Hilbert–Samuel formula for nef line bundles.

It is shown in [5] and [6] that in the toric and toroidal settings the b-divisorial degree of a nef b-divisor can be computed by combinatorial means in terms of a Monge–Ampère measure associated to a weakly concave function on a suitable polyhedral space. Also we have that the *volume* (see (A.1) for a precise definition) of a toric or toroidal b-divisor agrees with its degree (see [5, Theorem 5.11] and [6, Theorem 5.13]). Such results seem to be special to the toric and toroidal cases: in the Appendix we give an example to show that in general the volume function is *not* continuous on the space of big and approximable nef b-divisors (whereas the degree function is continuous).

We see that in the case of an almost asymptotically algebraic psh metric whose associated b-divisor is toroidal, the multiplier ideal volume agrees with the volume of the associated b-divisor, as both agree with the degree of the b-divisor. It would be interesting to know whether the equality between the multiplier ideal volume of a psh metric and the volume of the associated b-divisor continues to be true in the general case of almost asymptotically algebraic psh metrics (see Remark A.3.3).

To illustrate our general results we shall consider the following example dealing with universal abelian varieties. This example generalizes the set-up of [15].

Let $g \in \mathbb{Z}_{\geq 1}$ and $N \in \mathbb{Z}_{\geq 3}$ and let $A_{g,N}$ denote the fine moduli space of principally polarized complex abelian varieties of dimension g and level N. Let $\pi\colon U_{g,N} \to A_{g,N}$ be the universal abelian variety, and $\overline{U}_{g,N}$ and $\overline{A}_{g,N}$ be any projective smooth toroidal compactifications of $\overline{U}_{g,N}$ and $\overline{A}_{g,N}$, respectively, together with a map $\overline{\pi}\colon \overline{U}_{g,N} \to \overline{A}_{g,N}$ extending π, as discussed for example in [26, Chapter XI]. Let $k, m \in \mathbb{Z}_{\geq 0}$ and let $\mathcal{L}_{k,m}$ denote the *line bundle of Siegel-Jacobi forms* on $U_{g,N}$ of *weight* k and *index* m. It is endowed with a canonical smooth invariant hermitian metric $h_{k,m}$. The first Chern form $c_1(\mathcal{L}_{k,m}, h_{k,m})$ is a semipositive $(1,1)$-form on $U_{g,N}$.

The next result is a special case of Theorem 6.3.

Theorem D. *The smooth hermitian line bundle $(\mathcal{L}_{k,m}, h_{k,m})$ has a canonical extension $(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m})$ as a $\mathbb{Q}$-line bundle with a plurisubharmonic metric with toroidal, and hence almost asymptotically algebraic, singularities over $\overline{U}_{g,N}$.*

Let $n = \dim U_{g,N} = g + g(g+1)/2$. Let s be any non-zero rational section of $\mathcal{L}_{k,m}$, and let $D(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m}, s)$ be the b-divisor associated to s and the metric $\overline{h}_{k,m}$. As is verified by Proposition 5.8, the b-divisor $D(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m}, s)$ is independent of the choice of the compactification $\overline{U}_{g,N}$.

Theorem D together with Theorem A and Theorem C imply the following.

Theorem E. *(Theorem 6.8) Let notations be as above. Then the equalities*

$$\int_{U_{g,N}} c_1(\mathcal{L}_{k,m}, h_{k,m})^n = D(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m}, s)^n = \operatorname{vol}_{\mathcal{J}}(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m})$$

hold in $\mathbb{R}_{\geq 0}$.

Note that this generalizes (1.5) and (1.6) to higher degrees. In our follow-up paper [7] we shall prove that the b-divisors associated to $(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m})$ are toroidal b-divisors on the smooth toroidal variety $\overline{U}_{g,N}$ in the sense of [6]. Along the way we show that these b-divisors are *not* given by a divisor on any single model of $\overline{U}_{g,N}$. This shows that it is really necessary to consider these limits of divisors on all models. Moreover, we will use the techniques developed in the present paper to compute the asymptotic growth of the dimensions of spaces of Siegel–Jacobi forms.

1.3 Structure of the paper

The purpose of Section 2 is to review several analytic notions such as singular hermitian metrics, (quasi-)plurisubharmonic functions and metrics, Lelong numbers, multiplier ideals and the multiplier ideal volume. We discuss various notions of singularity type for quasi-plurisubharmonic functions and review Demailly's regularization theorem as well as the notion of non-pluripolar products. We state an important monotonicity result for non-pluripolar products due to Boucksom, Eyssidieux, Guedj and Zeriahi.

In Section 3 we introduce the notion of almost asymptotically algebraic singularities and discuss various examples. We will see that good plurisubharmonic and toroidal plurisubharmonic metrics have almost asymptotically algebraic singularities. We prove that in the case of almost asymptotically algebraic singularities, the multiplier ideal volume and the non-pluripolar volume coincide.

In Section 4 we recall the notion of b-divisors and review part of the work of Dang and Favre on approximable nef b-divisors. In Section 5 we show how to associate a b-divisor to a line bundle with a plurisubharmonic metric with Zariski unbounded locus. We show that such b-divisors are approximable nef, and prove our Chern–Weil type result for almost asymptotically algebraic singularities.

Finally in Section 6 we discuss the biextension line bundle, and more generally the line bundle of Siegel–Jacobi forms, on the universal abelian variety as an example of a line bundle with a plurisubharmonic metric with almost asymptotically algebraic singularities.

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2 Analytic preliminaries

In this section X denotes a complex manifold of pure dimension n.

2.1 Singular hermitian metrics

We refer to [20], [21] and [36] (see also [8], [12] and [29]) for definitions and proofs of the analytic properties given in this section. We take our notion of plurisubharmonic (psh) functions from [36, Chapter 3].

Definition 2.1. Let U be an open coordinate subset of X that we identify with an open subset of $\mathbb{C}^n$. A function $\varphi\colon U \to \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic (psh)* if it satisfies the following two conditions:

1. φ is upper semicontinuous and is not identically $-\infty$ on any connected component of U;
2. for every $z \in U$ and $a \in \mathbb{C}^n$ the function in one complex variable

$$\zeta \longmapsto \varphi(z + a\zeta) \in \mathbb{R} \cup \{-\infty\}$$

is either identically $-\infty$ or subharmonic in each connected component of the open set $\{\zeta \in \mathbb{C} \mid z + a\zeta \in U\}$.

A function $\varphi\colon U \to \mathbb{R} \cup \{-\infty\}$ on an arbitrary open subset U of X is called *psh* if U can be covered by open coordinate subsets U_i and each $\varphi|_{U_i}$ is psh.

We write dd^c for the operator $\frac{i}{\pi}\partial\bar{\partial}$. The following characterization of psh functions, that does not refer to coordinate charts, is used frequently.

Proposition 2.2. *Let $U \subset X$ be an open set and let $\varphi\colon U \to \mathbb{R} \cup \{-\infty\}$ be a measurable function. Then φ is psh if and only if the following two conditions are satisfied:*

1. *The function φ is* strongly upper semicontinuous. *That is, for all $V \subset U$ of total Lebesgue measure, and all $x \in U$, the condition*

$$\varphi(x) = \limsup_{\substack{y \to x \\ y \in V}} \varphi(y)$$

holds.

2. *The function φ is locally integrable and the $(1,1)$-current $dd^c\varphi$ is positive.*

Examples of psh functions are given by the functions $\frac{1}{2}\log(|f_1|^2 + \cdots + |f_N|^2)$ where $f_1, \ldots, f_N \in \mathcal{O}_X(U)$ are non-zero holomorphic functions.

Remark 2.3. If T is a closed positive $(1,1)$-current, then T is locally exact and so is locally of the form $dd^c\varphi$ for some psh function φ; we call such φ *local potentials* of T. We say T has *bounded local potentials* if the φ can be chosen to be bounded, and similarly with 'continuous' or 'smooth' instead of 'bounded'.

When X is compact, any global psh function on X is constant. To have a rich global theory we need to allow some flexibility.

Definition 2.4. 1. An upper semicontinuous function $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ is called *quasi-plurisubharmonic (quasi-psh)* if φ is locally of the form $u + f$, where u is psh and f is smooth.

2. Let θ be a smooth closed $(1,1)$-form on X. A measurable function $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ is called *θ-psh* if φ is locally integrable, strongly upper semicontinuous and $dd^c\varphi + \theta$ is a positive current.

When T is a current we write $T \geq 0$ to express that T is positive.

Lemma 2.5. *Let $\varphi\colon X \to \mathbb{R} \cup \{-\infty\}$ be a function.*

1. *If φ is θ-psh for some smooth closed $(1,1)$-form θ then φ is quasi-psh.*
2. *If X is compact Kähler with Kähler form ω and φ is quasi-psh then there is a real number $a > 0$ such that φ is $(a\omega)$-psh.*

Proof. For (1), assume that φ is θ-psh for some smooth closed $(1,1)$-form θ. Then there is a covering of X by open subsets U_i and on each U_i there is a smooth function f_i with $dd^c f_i = \theta$. Then, on U_i we write $\varphi = (\varphi + f_i) - f_i$. The function $-f_i$ is smooth and the function $\varphi + f_i$ is locally integrable, strongly upper semi-continuous and satisfies

$$dd^c(\varphi + f_i) = dd^c\varphi + \theta \geq 0\,.$$

Therefore $\varphi + f_i$ is psh and φ is quasi-psh.
For (2), assume that φ is quasi-psh and X is compact Kähler with Kähler form ω. There is a finite open cover $\{U_i\}_i$ of X, and for each i a decomposition $\varphi = f_i + \gamma_i$ with f_i psh and γ_i smooth. After shrinking the U_i if necessary we can assume that, for each i, the form $dd^c\gamma_i$ can be extended to a smooth form on an open neighbourhood of the compact set $\overline{U_i}$. Then there is a real number $a > 0$ such that $dd^c\gamma_i + a\omega|_{U_i} \geq 0$ for all i. It follows that

$$(dd^c\varphi + a\omega)\,|_{U_i} = dd^c f_i + dd^c\gamma_i + a\omega|_{U_i} \geq 0$$

for all i. Therefore, φ is $(a\omega)$-psh. ☐

In algebraic geometry it is often convenient to work with the related concept of *psh metrics* on a line bundle. Let $\mathcal{L}$ be a line bundle on X and fix a trivialization $\{(U_i, s_i)\}$ of $\mathcal{L}$ with transition functions $\{g_{ij}\}$. A *hermitian metric* on $\mathcal{L}$ is a collection of measurable functions

$$h = \{\varphi_i\colon U_i \to \mathbb{R} \cup \{\pm\infty\}\}\ ,$$

such that

$$e^{-\varphi_i} = |g_{ij}|e^{-\varphi_j} \tag{2.1}$$

on $U_i \cap U_j$. The function φ_i determines the *norm* $h(s_i)$ of the trivializing sections s_i by the formula

$$\varphi_i(z) = -\log h(s_i(z)), \quad z \in U_i .$$

The condition (2.1) is equivalent to

$$\log h(s_i) - \log h(s_j) = \log |s_i/s_j| .$$

Then, the norm $h(s)$ of any section s at a point z is given by

$$\log h(s(z)) = \log |s(z)/s_i(z)| + \log h(s_i(z)) = \log |s(z)/s_i(z)| - \varphi_i(z) ,$$

if $z \in U_i$, and this is easily verified to be independent of the choice of i. The functions $\varphi_i = -\log h(s_i)$ are called *local potentials* of the metric h.

DEFINITION 2.6. The metric h is called *singular* (resp. *psh*, *continuous*, *smooth*) if the local potentials φ_i are locally integrable (resp. psh, continuous, smooth).

The notion of singular (resp. psh, continuous, smooth) metric readily generalizes to the context of $\mathbb{Q}$-line bundles on X (see [4, Definition 2.10] for a discussion of this terminology).

Remark 2.7. The global relation between psh metrics on line bundles on X and θ-psh functions on X is given as follows. Choose a smooth reference metric h_0 on $\mathcal{L}$ and write

$$\theta = c_1(\mathcal{L}, h_0).$$

Then θ is a smooth closed $(1,1)$-form on X. Note that for every trivializing open subset U_i and trivializing section s_i of $\mathcal{L}$ over U_i one has

$$\theta|_{U_i} = -dd^c \log(h_0(s_i)).$$

Now choose a non-zero rational section s of $\mathcal{L}$. Then the maps

$$h \longmapsto \varphi = -\log(h(s)/h_0(s)) \text{ and } \varphi \longmapsto \theta + dd^c\varphi$$

do not depend on the choice of section. The first map is a bijection between the set of psh metrics on $\mathcal{L}$ and the set of θ-psh functions on X. If X is compact connected, the second map induces a bijection between the set of θ-psh functions on X up to constants and the set of positive $(1,1)$-currents in the cohomology class $c_1(\mathcal{L})$ of $\mathcal{L}$. If h is a psh metric on $\mathcal{L}$ and $\varphi = -\log(h(\cdot)/h_0(\cdot))$ the corresponding θ-psh function, we will use the notation $c_1(\mathcal{L}, h) = \theta + dd^c\varphi$ for the first Chern current associated to h.

2.2 LELONG NUMBERS, EQUIVALENCE OF SINGULARITIES, MULTIPLIER IDEALS

A first measure of the singularities of a psh function is given by its Lelong numbers. Let T be a closed positive $(1,1)$-current on the complex manifold X. The *Lelong number* $\nu(T, x)$ of T at a point $x \in X$ is given by

$$\nu(T, x) = \lim_{r \to 0^+} \nu(T, x, r),$$

where $\nu(T,x,r)$ is computed in an open coordinate neighborhood of x as the integral

$$\nu(T,x,r) = \frac{1}{(2\pi r^2)^{n-1}} \int_{B(x,r)} T(z) \wedge \left(i\partial\bar{\partial}|z|^2\right)^{n-1}.$$

Here, $B(x,r)$ denotes the ball with center x and radius r. Several important properties of Lelong numbers are stated in [22, Section 2.B]. In particular, they are non-negative real numbers which are invariant under holomorphic changes of local coordinates. Further, the Lelong number $\nu(T,x)$ is additive in T. If φ is a psh local potential of T as in Remark 2.3 then we write

$$\nu(\varphi,x) = \nu(T,x).$$

The Lelong numbers of psh functions have the following characterization.

Proposition 2.8. *Let $U \subset X$ be an open coordinate subset and let φ be a psh function on U. Let $x \in U$. Then the equality*

$$\nu(\varphi,x) = \sup\left\{\gamma \geq 0 \;\middle|\; \varphi(z) \leq \gamma \log|z-x| + O(1) \text{ near } x\right\}$$

holds. In particular, if $\varphi = \log|f|$ with $f \in \mathcal{O}_X(U)$ holomorphic, then $\nu(\varphi,x) = \mathrm{ord}_x(f)$, where $\mathrm{ord}_x(f)$ is the largest power of the maximal ideal of x which contains f.

Definition 2.9. A morphism $\pi\colon X' \to X$ of complex manifolds is called a *modification* if it is proper and there exists a nowhere-dense analytic subset $Z \subseteq X$ such that the map $\pi^{-1}(X \setminus Z) \to X \setminus Z$ given by restricting π is an isomorphism, and such that $X' \setminus \pi^{-1}Z$ is nowhere dense in X'.

Given a modification $\pi\colon X' \to X$ and a psh function φ on an analytic open subset U of X, the composition $\varphi \circ \pi$ is a psh function on $\pi^{-1}(U)$. Hence for any positive $(1,1)$-current T on X the Lelong number $\nu(T,x)$ at a point $x \in X'$ is well defined. Furthermore, one can define the Lelong number $\nu(T,P)$ at any prime divisor P on X' by

$$\nu(T,P) \coloneqq \nu(T,\eta),$$

where η is a very general point of P. If T is of the form $c_1(\mathcal{L},h)$ for a psh metric h on a line bundle $\mathcal{L}$, we write

$$\nu(h,P) \coloneqq \nu(T,P).$$

Remark 2.10. Let T be a closed positive $(1,1)$-current on X. We briefly recall the Siu decomposition of T on X (and refer to [8, Section 2.2.1] for details). This decomposes T uniquely as a sum

$$T = R + \sum_k \nu\left(T, Y_k\right) \delta_{Y_k},$$

where the sum is over an (at most countably infinite) family of 1-codimensional subvarieties Y_k of X. Here, δ_{Y_k} denotes the integration current determined by Y_k, and R is a closed positive $(1,1)$-current whose Lelong number on any prime divisor on X is zero. In Section 5 we will relate the Siu decomposition of T to the so-called b-divisor associated to a psh metric (see Remark 5.6).

The Lelong numbers allow us to classify singularities of psh functions in the sense that more singular psh functions have bigger Lelong numbers. But in some situations, the classification of singularities by Lelong numbers is too crude, and a more refined classification is needed. Such a more refined classification is given by the concept of *type of singularity* of a psh function.

Definition 2.11. Let $U \subset X$ be an open subset and let φ, ψ be two psh functions on U. We say that φ is *more singular* than ψ at a point $u \in U$ if there exists an open neighbourhood $u \in V \subseteq U$ and a constant $C \in \mathbb{R}$ such that $\varphi \leq \psi + C$ on V. We say that φ is more singular than ψ, denoted $\varphi \prec \psi$, if it is so at every $u \in U$.
We write $\varphi \sim \psi$ if $\varphi \prec \psi$ and $\psi \prec \varphi$. In this case we say that φ and ψ have *equivalent singularities*. This notion can be easily extended to the set of quasi-psh functions on X and defines an equivalence relation on this set. Given a quasi-psh function φ on X, we denote by $[\varphi]$ its equivalence class, and call $[\varphi]$ the *type of singularity* of φ.

The above classification of singularities of quasi-psh functions carries over to closed positive $(1,1)$-currents and to psh metrics on $\mathcal{L}$ by using local potentials. If T and T' are two closed positive $(1,1)$-currents, we write $T \prec T'$ if there is an open covering $\{U_i\}$ of X and local psh potentials φ_i and φ'_i of T and T' on U_i such that $\varphi_i \prec \varphi'_i$. Similarly, given two psh metrics h and h' on the line bundle $\mathcal{L}$, we write $h \prec h'$ if $c_1(\mathcal{L}, h) \prec c_1(\mathcal{L}, h')$.

Remark 2.12. Let $T \prec T'$ be two closed positive $(1,1)$-currents on X. Then their Lelong numbers satisfy

$$\nu(T, x) \geq \nu(T', x)$$

for every point $x \in X'$ in any modification $\pi \colon X' \to X$ of X. In particular if T and T' have equivalent singularities, then they have the same Lelong numbers at every prime divisor on every modification of X.
The converse does not hold as the following example shows. The example is local for ease of writing but can easily be made into a global one.

Example 2.13. Consider the function $f(z) = -\log(-\log(z\bar{z}))$ on the disk $z\bar{z} < 1$. Then $\nu(f, q) = 0$ for all q in the disk, but f is not bounded below. So f has the same Lelong numbers as a constant function in any point but $f \not\sim 1$. This is a counterexample to the converse of Remark 2.12 because any modification of the disk is the disk itself.

Two psh functions which have the same Lelong numbers on X need not have the same Lelong numbers on a modification. In fact, more is true: it can happen that $\nu(f,x) \leq \nu(g,x)$ for all $x \in X$ and even $\nu(f,x_0) < \nu(g,x_0)$ for a point $x_0 \in X$, but $\nu(f,y) > \nu(g,y)$ for some point y above x_0 in some modification X' of X.

Example 2.14. Consider the functions f and g in the polydisk

$$E = \{(x,y) \in \mathbb{C}^2 \mid x\bar{x} < 1,\, y\bar{y} < 1\}$$

given by

$$f(x,y) = \frac{.9}{2}\log(x\bar{x} + (y\bar{y})^2)\,, \qquad g(x,y) = \frac{1}{2}\log((x\bar{x})^2 + y\bar{y})\,,$$

Then $\nu(f,0) = .9 < 1.0 = \nu(g,0)$. Nevertheless

$$g(0,y) - f(0,y) = -.4\log(y\bar{y})$$

is not bounded above. So $\nu(f,q) \leq \nu(g,q)$ for any point q in the polydisk but $g \not\precsim f$.

However, if we consider the chart of the blow-up of the polydisk at $(0,0)$ with coordinates (s,t) with $x = st$, $y = s$ and if we let p be the point $s = t = 0$ on the blow-up, then $\nu(f,p) = 1.8$ while $\nu(g,p) = 1$.

A convenient way to encode Lelong numbers is by means of multiplier ideal sheaves.

Definition 2.15. Let φ be a quasi-psh function on X. Then the *multiplier ideal sheaf* $\mathcal{J}(\varphi)$ of φ is the coherent ideal sheaf of $\mathcal{O}_X$-modules given locally by those holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally integrable. If $\mathcal{L}$ is a line bundle with a psh metric h then the multiplier ideal sheaf $\mathcal{J}(h)$ of h is defined to be the multiplier ideal sheaf of the quasi-psh function $\varphi = -\log(h(s)/h_0(s))$ for any smooth reference metric h_0 and any non-zero rational section s of $\mathcal{L}$.

It turns out that the multiplier ideal sheaves of all multiples of a given quasi-psh function on X give the same information as the Lelong numbers on all points of all modifications of X. See Proposition 2.28 below for a precise statement. Hence it follows from Example 2.13 that for a general quasi-psh function also its multiplier ideal is not enough to recover the singularity type.

2.3 Algebraic singularities

From now on we will assume that X is a projective complex manifold. An important class of quasi-psh functions on X consists of those having algebraic singularities.

Definition 2.16. A quasi-psh function φ on X is said to have *algebraic singularities* if there is a constant $c \in \mathbb{Q}_{\geq 0}$ and φ can be written locally as

$$\varphi = \frac{c}{2}\log\left(|f_1|^2 + \cdots + |f_N|^2\right) + \lambda, \tag{2.2}$$

where λ is a bounded function and the f_j are non-zero algebraic functions.

Remark 2.17. There is also the related notion of *analytic* singularities. In the projective context the only difference with the notion of algebraic singularities is to allow the constant c to be a real number.

Following [22, 1.10], if the quasi-psh function φ on X has algebraic singularities with constant c, we can associate to it a coherent sheaf of ideals $\mathcal{I}(\varphi/c)$, in the following way. Since X is compact, we can assume that there is a finite covering of X and φ has the form (2.2) on each open of the covering. Then $\mathcal{I}(\varphi/c)$ is defined to be the ideal sheaf of holomorphic functions h satisfying

$$|h| \leq C(|f_1| + \cdots + |f_N|)$$

for some constant C. This is a globally defined ideal sheaf of $\mathcal{O}_X$ locally equal to the integral closure of the ideal generated by $(f_1, \dots, f_N)$. Since X is assumed to be projective, the coherent ideal sheaf $\mathcal{I}(\varphi/c)$ is the analytification of an algebraic coherent ideal sheaf on X.

In contrast with what happens for arbitrary quasi-psh functions, for quasi-psh functions with algebraic singularities, we can recover the singularity type from the multiplier ideal sheaf. Following [22, Remark 5.9], the multiplier ideal sheaf $\mathcal{J}(\varphi)$ of a quasi-psh function φ with algebraic singularities is easy to describe. Assume first that there is a constant $c \in \mathbb{Q}_{\geq 0}$ and an effective Cartier divisor with simple normal crossings $D = \sum_i \alpha_i D_i$ on X where the D_i are irreducible such that locally φ can be written as

$$\frac{c}{2} \log |g|^2 + \lambda, \tag{2.3}$$

where λ is bounded and g is a local equation for D. Then

$$\mathcal{J}(\varphi) = \mathcal{O}_X\Big(- \sum_i \lfloor c\alpha_i \rfloor D_i \Big). \tag{2.4}$$

In particular $\mathcal{J}((k/c)\varphi) = \mathcal{I}(k\varphi/c)$ for every integer $k \geq 0$.

Assume now that φ has algebraic singularities and, as before, let $c \in \mathbb{Q}_{\geq 0}$ be the constant appearing in Definition 2.16. Then there exists a modification $\pi \colon X_\pi \to X$ such that $\pi^{-1}\mathcal{I}(\varphi/c) \cdot \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(-D)$ for a simple normal crossings divisor $D = \sum_i \alpha_i D_i$ on X_π. Let $R_\pi = \sum_i \rho_i D_i$ be the zero divisor of the Jacobian function of π. Then combining (2.4) with the direct image formula [22, Proposition 5.8] we obtain

$$\mathcal{J}(\varphi) = \pi_* \mathcal{O}_{X_\pi}(R_\pi - \sum_i \lfloor c\alpha_i \rfloor D_i) = \pi_* \mathcal{O}_{X_\pi}(\sum_i (\rho_i - \lfloor c\alpha_i \rfloor) D_i). \tag{2.5}$$

Therefore, for every integer $k > 0$

$$\mathcal{J}((k/c)\varphi) = \pi_* \big(\mathcal{O}_{X_\pi}(R_\pi) \otimes (\pi^{-1}\mathcal{I}(k\varphi/c) \cdot \mathcal{O}_{X_\pi})\big). \tag{2.6}$$

This description shows that, if φ has algebraic singularities with constant c, then the asymptotic properties of the family of ideals $\mathcal{J}((k/c)\varphi)$, $k > 0$ and that of the family $\mathcal{I}((k\varphi)/c)$, $k > 0$ are similar. The following is an example of this property.

Lemma 2.18. *Let $\mathcal{L}$ be a line bundle on the projective complex manifold X provided with a smooth reference metric h_0 and a psh metric h. Let $\theta = c_1(\mathcal{L}, h_0)$ and let $\varphi = -\log(h(s)/h_0(s))$ be the resulting θ-psh function as in Remark 2.7, where s is any non-zero rational section of $\mathcal{L}$. If φ has algebraic singularities with constant c, then*

$$\lim_{k\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{J}(k\varphi))}{k^n/n!} = \lim_{\substack{k\to\infty\\ kc\in\mathbb{Z}}} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{I}((kc\varphi)/c))}{k^n/n!}.$$

Proof. Let $\pi\colon X_\pi \to X$ be a modification such that $\pi^{-1}\mathcal{I}(\varphi/c)\cdot\mathcal{O}_{X_\pi} = \mathcal{O}(-D)$, with $D = \sum_i \alpha_i D_i$ a simple normal crossings divisor on X_π. As before let $R_\pi = \sum_i \rho_i D_i$ be the zero divisor of the Jacobian function of π. Then by the descriptions (2.5) and (2.6) of the multiplier ideal in the case of algebraic singularities we deduce

$$\begin{aligned}
\lim_{k\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{J}(k\varphi))}{k^n/n!} &= \lim_{k\to\infty} \frac{h^0\big(X_\pi, \pi^*\mathcal{L}^{\otimes k} \otimes \mathcal{O}(\sum_i(\rho_i - \lfloor kc\alpha_i\rfloor)D_i)\big)}{k^n/n!} \\
&= \lim_{\substack{k\to\infty\\ kc\in\mathbb{Z}}} \frac{h^0\big(X_\pi, \pi^*\mathcal{L}^{\otimes k} \otimes \mathcal{O}(\sum_i(\rho_i - kc\alpha_i)D_i)\big)}{k^n/n!} \\
&= \lim_{\substack{k\to\infty\\ kc\in\mathbb{Z}}} \frac{h^0\big(X_\pi, \pi^*\mathcal{L}^{\otimes k} \otimes \mathcal{O}(-\sum_i kc\alpha_i D_i)\big) + O\left(k^{n-1}\right)}{k^n/n!} \\
&= \lim_{\substack{k\to\infty\\ kc\in\mathbb{Z}}} \frac{h^0\big(X_\pi, \pi^*\mathcal{L}^{\otimes k} \otimes \mathcal{O}(-\sum_i kc\alpha_i D_i)\big)}{k^n/n!} \\
&= \lim_{\substack{k\to\infty\\ kc\in\mathbb{Z}}} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{I}((kc\varphi)/c))}{k^n/n!}.
\end{aligned}$$

□

The next lemma shows that, after a modification, a quasi-psh function with algebraic singularities is always of the form (2.3).

Lemma 2.19. *Let φ be a quasi-psh function on X with algebraic singularities and let c be the constant in equation (2.2). Let $\pi\colon X_\pi \to X$ be a modification such that $\pi^{-1}\mathcal{I}(\varphi/c)\cdot\mathcal{O}_{X_\pi}$ is locally principal. Let $U \subset X_\pi$ be an open subset such that $\pi^{-1}\mathcal{I}(\varphi/c)\cdot\mathcal{O}_{X_\pi}$ is generated by a holomorphic function g on U. Then*

$$\pi^*\varphi - \frac{c}{2}\log|g|^2$$

is locally bounded on U.

Proof. After shrinking U if necessary, we can assume that there is an open set $V \subset X$ where φ has the shape (2.2) and $U \subset \pi^{-1}(V)$. To simplify the notation we will not distinguish between functions on X and on X_π as these spaces agree on a dense open subset.

Since g is a generator of $\mathcal{I}(\varphi/c)$ and the functions f_i belong to this ideal, there are holomorphic functions b_i such that $f_i = b_i g$. Then

$$\varphi = \frac{c}{2}\log\left(|f_1|^2+\cdots+|f_N|^2\right)+\lambda = \frac{c}{2}\log|g|^2+\frac{c}{2}\log\left(|b_1|^2+\cdots+|b_N|^2\right)+\lambda.$$

Since λ is locally bounded, it suffices to prove that $\log\left(|b_1|^2+\cdots+|b_N|^2\right)$ is locally bounded. Assume that this is not the case. Since $|b_1|^2+\cdots+|b_N|^2$ is continuous, the only possibility for the logarithm not to be locally bounded is that the functions b_i have a common zero. Assume that x is such that $b_i(x)=0$ for all i. Since $\mathcal{I}(\varphi/c)_x$ is the integral closure of the ideal $I=(f_1,\dots,f_N)$, there exist an integer $r\geq 1$ and elements $\alpha_j\in I^j$ for $j=1,\dots,r$ such that

$$g^r+\alpha_1 g^{r-1}+\cdots+\alpha_r=0. \tag{2.7}$$

Assume that $\operatorname{ord}_x(g)=k$. Since $b_i(x)=0$ for all i, we have that $\operatorname{ord}_x f_i>k$ for all i. Hence any element $\alpha_j\in I^j$ has $\operatorname{ord}_x \alpha_j>jk$. Then condition (2.7) implies that $\operatorname{ord}_x g^r>kr$, which contradicts the fact that $\operatorname{ord}_x g^r=kr$. We conclude that the functions b_i do not have a common zero and $\log\left(|b_1|^2+\cdots+|b_N|^2\right)$ is locally bounded. □

2.4 Demailly's regularization theorem

Next we discuss Demailly's regularization theorem for closed positive $(1,1)$-currents. Roughly speaking it states that, for X a projective complex manifold, any θ-psh function on X can be approximated by functions with algebraic singularities. There are several versions in the literature, depending on the properties we want for the approximating functions. The version we use here can be obtained combining the local version in [23, Theorem 4.2] with the global version in [24, Theorem 3.2] (see also [22, Theorem 13.2] and [20, Theorem 1.1] for the original statement and the heart of the proof). We will use Demailly's regularization theorem in Section 3.2 to give criteria that ensure a psh metric has almost asymptotically algebraic singularities.

Theorem 2.20. *(Demailly's regularization theorem) Let X be a projective complex manifold of dimension n with Kähler form ω and let θ be a smooth $(1,1)$-form on X. Let T be a closed positive $(1,1)$-current in the same cohomology class as θ. Write $T=\theta+dd^c\varphi$ with φ a θ-psh function. Then there exists a sequence $(\varphi_m)_{m\geq 1}$ of quasi-psh functions on X satisfying the following properties.*

1. *Each φ_m has algebraic singularities. Moreover, for each m there is a modification $\pi_m\colon X_{\pi_m}\to X$, a simple normal crossings divisor D_m on X_{π_m}, and a rational number $c_m>0$ such that, locally on X_{π_m},*

$$\varphi_m\circ\pi_m = c_m\log|g|+f$$

where g is a local equation for D_m and f is smooth (and not just locally bounded as in Lemma 2.19).

2. *The sequence $(\varphi_m)_{m\geq 0}$ is non-increasing and there is a sequence of positive real numbers $(a_m)_{m\geq 0}$ converging monotonically to zero, such that φ_m is $(\theta + a_m\omega)$-psh.*

3. *For every coordinate open set U and relatively compact open subset $V \subset\subset U$ there are constants $C_1, C_2 > 0$ such that for all $m \in \mathbb{Z}_+$, $z \in V$ and $r \in \mathbb{R}_+$ with $r < d(z, \partial V)$,*

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{\|x-z\|<r} \varphi(x) + \frac{1}{m} \log \frac{C_2}{r^n}. \tag{2.8}$$

4. *For all $x \in X$, the Lelong numbers of φ and φ_m satisfy the condition*

$$\nu(\varphi, x) - \frac{n}{m} \leq \nu(\varphi_m, x) \leq \nu(\varphi, x). \tag{2.9}$$

   In particular the Lelong numbers of the functions φ_m on the points of X converge monotonically and uniformly to the Lelong numbers of the function φ.

We refer to any sequence of approximations $(\varphi_m)_{m\geq 1}$ with properties (1)–(4) from the theorem as a *Demailly approximating sequence* of the function φ. Note that in particular $\varphi \prec \varphi_m$ for a Demailly approximating sequence.

2.5 Non-pluripolar products

Generalizing a construction due to Bedford and Taylor [3], it has been shown in [10] that given closed positive $(1,1)$-currents $T_1, \ldots, T_p$ on the projective complex manifold X one has a non-pluripolar product

$$\langle T_1 \wedge \cdots \wedge T_p \rangle$$

of these currents with good properties. The result is a *closed* positive (p,p)-current which does not charge pluripolar sets. As before let $n = \dim X$.

Definition 2.21. Let $T_1, \ldots, T_p$ be closed positive $(1,1)$-currents on X. The *non-pluripolar product*

$$\langle T_1 \wedge \cdots \wedge T_p \rangle$$

is the (p,p)-current on X determined as follows. For $i = 1, \ldots, p$ let θ_i be a smooth $(1,1)$-form in the same cohomology class as T_i and let φ_i be a θ_i-psh function with $\theta_i + dd^c\varphi_i = T_i$. For every $k \geq 0$ write U_k for the set

$$U_k = \{x \in X \,|\, \varphi_i(x) \geq -k\,,\ i = 1, \ldots, p\}\,.$$

Then for every smooth $(n-p, n-p)$-form η one sets

$$\langle T_1 \wedge \cdots \wedge T_p \rangle (\eta) = \lim_{k\to\infty} \int_{U_k} T_1 \wedge \cdots \wedge T_p \wedge \eta.$$

The current $\langle T_1 \wedge \cdots \wedge T_p \rangle$ is independent of the choices of the θ_i and φ_i.

Remark 2.22. For the existence of the non-pluripolar product a hypothesis like X being Kähler is needed (see [10, Proposition 1.6], and the examples before and the remark after it).

Remark 2.23. It is clear from the definition that when $T_1, \dots, T_n$ restrict to smooth differential forms on the dense open $U \subset X$, we have

$$\int_X \langle T_1 \wedge \dots \wedge T_n \rangle = \int_U T_1 \wedge \dots \wedge T_n\,.$$

In particular we see that in this case the improper integral $\int_U T_1 \wedge \dots \wedge T_n$ is well-defined and finite.

In the case of currents with locally bounded potentials, the non-pluripolar product agrees with the cohomology product, as is shown by the next lemma.

Lemma 2.24. *Let $T_1, \dots, T_k$ be closed positive $(1,1)$-currents on X with locally bounded potentials and let $\theta_{k+1}, \dots, \theta_n$ be smooth closed $(1,1)$-forms on X. Choose smooth closed $(1,1)$-forms θ_i in the cohomology class of T_i for $i = 1, \dots, k$. Let φ_i for $i = 1, \dots, k$ be locally bounded quasi-psh functions satisfying $\theta_i + dd^c\varphi_i = T_i$. Then*

$$\int_X \theta_1 \wedge \dots \wedge \theta_n = \int_X T_1 \wedge \dots \wedge T_k \wedge \theta_{k+1} \wedge \dots \wedge \theta_n = \int_X \langle T_1 \wedge \dots \wedge T_k \wedge \theta_{k+1} \wedge \dots \wedge \theta_n \rangle,$$

where the middle product is defined by Bedford-Taylor theory [2]. In particular, the cohomology classes $\mathrm{cl}(\theta_i) = \mathrm{cl}(T_i)$ are nef for $i = 1, \dots, k$.

Proof. Since the currents T_i are positive, the functions φ_i are θ_i-psh. Since they are locally bounded and X is compact, they are bounded. Then, Definition 2.21 of the non-pluripolar product immediately implies the equality of the second and third integrals.

We prove the first equality by induction on k. If $k = 0$ there is nothing to prove. Assume that $k > 0$ and that the result is true for $k-1$. Bedford-Taylor theory [2] provides us with positive currents

$$T_1 \wedge \dots \wedge T_{k-1}, \qquad T_1 \wedge \dots \wedge T_k,$$

and a current $\varphi_k T_1 \wedge \dots \wedge T_{k-1}$ satisfying

$$dd^c(\varphi_k T_1 \wedge \dots \wedge T_{k-1}) = T_1 \wedge \dots \wedge T_k - T_1 \wedge \dots \wedge T_{k-1} \wedge \theta_k.$$

Therefore, using the induction hypothesis and the fact that the integral over X of an exact current is zero we obtain

$$\int_X T_1 \wedge \dots \wedge T_k \wedge \theta_{k+1} \wedge \dots \wedge \theta_n = \int_X T_1 \wedge \dots \wedge T_{k-1} \wedge \theta_k \wedge \dots \wedge \theta_n = \int_X \theta_1 \wedge \dots \wedge \theta_n.$$

This proves the first equality. Finally, let C be a closed curve on X. Then for each $i = 1, \dots, k$ we have

$$\mathrm{cl}(\theta_i) \cdot C = \int_C \theta_i = \int_C \langle T_i \rangle \geq 0\,.$$

This shows that the cohomology class $\mathrm{cl}(\theta_i) = \mathrm{cl}(T_i)$ is nef. $\square$

The non-pluripolar product is clearly symmetric and it is multilinear in the following sense.

Proposition 2.25 ([10, Proposition 1.4]). *Let T_1', T_1, $T_2, \dots, T_p$ be closed positive $(1,1)$-currents. Then for every pair of positive real numbers α, β the relation*

$$\langle(\alpha T_1 + \beta T_1') \wedge T_2 \wedge \cdots \wedge T_p\rangle = \alpha\langle T_1 \wedge T_2 \wedge \cdots \wedge T_p\rangle + \beta\langle T_1' \wedge T_2 \wedge \cdots \wedge T_p\rangle$$

is satisfied.

The following is a monotonicity property of non-pluripolar products with respect to singularity type. Following [10, Definition 1.2] we say that a quasi-psh function φ on X has *small unbounded locus* if there exists a (locally) complete pluripolar closed subset A of X such that φ is locally bounded on $X \setminus A$.

Theorem 2.26 ([10, Theorem 1.16]). *Let θ be a smooth closed $(1,1)$-form. For $i = 1, \dots, n$, let $\{\varphi_i\}$ and $\{\psi_i\}$ be two collections of θ-psh functions with small unbounded locus such that $\varphi_i \prec \psi_i$ for all i. Then the non-pluripolar products satisfy*

$$\int_X \langle(\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n)\rangle \leq \int_X \langle(\theta + dd^c\psi_1) \wedge \cdots \wedge (\theta + dd^c\psi_n)\rangle.$$

The main technical difficulty we face now is that in Demailly's regularization theorem (Theorem 2.20) we have very good control on the Lelong numbers of a quasi-psh function by means of approximating sequences, but not on the type of singularity. By contrast, in order to apply Theorem 2.26 on the monotonicity of non-pluripolar products we need control on the type of singularity of our quasi-psh functions. In the long run this will force us to restrict the space of quasi-psh functions we can consider.

2.6 Algebraic singularity type

Multiplier ideals allow us to define the algebraic type of singularity of a quasi-psh function. The notion of algebraic type has been introduced in [30]. We will follow [18] though as the main source for our discussion. We continue to assume that X is a projective complex manifold.

Definition 2.27. Let φ and ψ be two quasi-psh functions on X. Then φ is said to be *algebraically more singular* than ψ (denoted $\varphi \prec_{\mathcal{J}} \psi$) if for all real numbers $a > 0$ the inclusion $\mathcal{J}(a\varphi) \subset \mathcal{J}(a\psi)$ holds. We say that φ and ψ have the same *algebraic singularity type*, denoted $\varphi \simeq_{\mathcal{J}} \psi$, if $\varphi \prec_{\mathcal{J}} \psi$ and $\psi \prec_{\mathcal{J}} \varphi$.

The algebraic singularity type is governed by the Lelong numbers not just on X but on all modifications of X, as the following result shows.

Proposition 2.28 ([18, Corollary 2.16]). *Let φ and ψ be two quasi-psh functions on X. The following assertions are equivalent:*

1. *$\varphi \prec_{\mathcal{J}} \psi$;*
2. *for every modification $Y \to X$ and for every $y \in Y$ the inequality $\nu(\varphi, y) \geq \nu(\psi, y)$ holds.*

The type of singularity and the algebraic type of singularity allow us to attach two envelopes to a quasi-psh function. The first one was introduced in [39] and the second in [18].

Definition 2.29. Let θ be a smooth closed $(1,1)$-form and φ a θ-psh function on X. Then the *envelope of the singularity type* of φ is the function

$$P[\varphi] = \sup\{\psi \ \theta\text{-psh } \mid \psi \prec \varphi, \psi \leq 0\}^*$$

on X, where f^* denotes the upper semicontinuous regularization of f. The *envelope of the algebraic singularity type* of φ is

$$P[\varphi]_{\mathcal{J}} = \sup\{\psi \ \theta\text{-psh} \mid \psi \prec_{\mathcal{J}} \varphi, \psi \leq 0\}^*.$$

The following are basic properties of the envelopes.

Proposition 2.30 ([18, Proposition 2.19]). *Let φ be a θ-psh function on X. Then*

1. *$P[\varphi]$ and $P[\varphi]_{\mathcal{J}}$ are θ-psh functions on X.*
2. *$P[\varphi]_{\mathcal{J}} \simeq_{\mathcal{J}} \varphi$.*
3. *$\varphi \prec P[\varphi] \prec P[\varphi]_{\mathcal{J}}$. Moreover $P\left[P[\varphi]_{\mathcal{J}}\right] = P[\varphi]_{\mathcal{J}}$.*

The following result due to Darvas and Xia indicates that the difference between the envelope of the singularity type and the envelope of the algebraic singularity type governs when the non-pluripolar product is well behaved.
Let $\mathcal{L}$ be a line bundle on X provided with a smooth reference metric h_0 and a psh metric h. Let $\theta = c_1(\mathcal{L}, h_0)$ be the first Chern form and $\varphi = -\log(h(s)/h_0(s))$ the resulting θ-psh function.

Theorem 2.31 ([18, Theorem 5.5]). *Assume that $\mathcal{L}$ is ample and θ is a Kähler form on X. Let $n = \dim X$. Then the limit*

$$\operatorname{vol}_{\mathcal{J}}(\mathcal{L}, h) = \lim_{k\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{J}(k\varphi))}{k^n/n!} \tag{2.10}$$

exists, and we have

$$\operatorname{vol}_{\mathcal{J}}(\mathcal{L}, h) = \int_X \langle(\theta + dd^c P[\varphi]_{\mathcal{J}})^{\wedge n}\rangle \geq \int_X \langle(\theta + dd^c \varphi)^{\wedge n}\rangle\,.$$

If $\int_X \langle(\theta + dd^c\varphi)^{\wedge n}\rangle > 0$ then the following assertions are equivalent:

1. *$\operatorname{vol}_{\mathcal{J}}(\mathcal{L}, h) = \int_X \langle(\theta + dd^c\varphi)^{\wedge n}\rangle$;*

2. $P[\varphi] = P[\varphi]_{\mathcal{J}}$.

The limit in (2.10) is called the *multiplier ideal volume* of the pair $(\mathcal{L}, h)$, whereas the integral $\int_X \langle(\theta + dd^c\varphi)^{\wedge n}\rangle$ is called the *non-pluripolar volume* of $(\mathcal{L}, h)$. In [18, Theorem 5.5] one may find other statements equivalent to conditions 1. and 2. above. The precise statements will not be needed here but roughly speaking the equality $P[\varphi] = P[\varphi]_{\mathcal{J}}$ holds if and only if φ can be "very well approximated" (in what is called the $d_{\mathcal{S}}$*-distance*) by algebraic singularities.

We take the above result as an indication that for a reasonable Chern–Weil and Hilbert-Samuel theory to hold, one should deal with quasi-psh functions that are well approximated by algebraic singularities in some sense.

Remark 2.32. As was mentioned in the introduction, we note that the above result was very recently extended and generalized to the case where $\mathcal{L}$ is only assumed to be pseudoeffective, see [19, Theorem 1.1].

2.7 Good psh metrics

Based on this idea, in Section 3 we will introduce a large class of singularities where the equality of non-pluripolar volume and multiplier ideal volume can be seen to be satisfied. For this class (the class of "almost asymptotically algebraic singularities") we will be able to prove a reasonable Chern–Weil type statement. As a warm-up, we examine here already the case of "good" metrics in the sense of Mumford [35] and in the next section the case of algebraic singularities.

Example 2.33. Consider $X = \mathbb{P}^1$ with homogeneous coordinates $(x : y)$ and absolute coordinate $t = x/y$. Let $\mathcal{L} = \mathcal{O}_X(1)$. The space of global sections of $\mathcal{L}$ can be identified with the space of linear forms in the variables x, y. We consider the psh metric h on $\mathcal{L}$ given by

$$-\log h(y) = \begin{cases} 1 + \log|t|, & \text{if } |t| \geq 1/e, \\ -\log(-\log(|t|)) & \text{if } |t| \leq 1/e. \end{cases}$$

This metric is singular at the point $t = 0$. The interest of this singularity is that (up to multiplying by a normalization factor) it is equivalent to the singularity of the invariant (i.e., Hodge) metric on the line bundle of modular forms on a modular curve at a cusp (see Section 6.2 for further discussion).

We choose now a smooth metric h_0 on $\mathbb{P}^1$. A canonical choice is the Fubini-Study metric, given by

$$-\log h_0(y) = \frac{1}{2}\log(1 + |t|^2).$$

Let ω be the first Chern form of $(\mathcal{L}, h_0)$. Then the function

$$\varphi(t) = \begin{cases} 1 + \log|t| - \log(1 + |t|^2)/2 & \text{if } |t| \geq 1/e, \\ -\log(-\log(|t|)) - \log(1 + |t|^2)/2 & \text{if } |t| \leq 1/e, \end{cases}$$

is ω-psh. All the Lelong numbers of the function φ are zero. For $t \neq 0$ this is clear because the function φ is locally bounded in $\mathbb{P}^1 - \{(0:1)\}$. And at $t = 0$ this follows from the fact that φ grows at most as the logarithm of the logarithm. It follows that $P[\varphi]_{\mathcal{J}} = 0$. Also the growth of the function φ at $t = 0$ shows that the current $dd^c\varphi$ does not charge any pluripolar set. Therefore

$$\int_{\mathbb{P}^1} \langle \omega + dd^c\varphi \rangle = \int_{\mathbb{P}^1} \omega + dd^c\varphi = \int_{\mathbb{P}^1} \omega = \int_{\mathbb{P}^1} \langle \omega + dd^c P[\varphi]_{\mathcal{J}} \rangle = 1.$$

From Theorem 2.31 we may deduce that $P[\varphi] = 0$.

Example 2.34. The previous example can be generalized to the setting of good metrics in the sense of Mumford [35]. Let X be a smooth complex variety with $\dim X = n$ and $D \subset X$ a normal crossings divisor. Let $\mathcal{L}$ be a line bundle on X and h a singular metric on $\mathcal{L}$. The metric h is said to be *good* if h is smooth on $X \setminus D$, and for every holomorphic chart U of X with coordinates $z_1, \dots, z_n$ in which D has the equation $z_1 \cdots z_k = 0$, each generating section s of $\mathcal{L}$ on U and each vector field v on U there is a neighborhood V of $(0, \dots, 0)$ in which the estimates

1. $h(s), h^{-1}(s) \leq C \left(\sum_{i=1}^k - \log |z_i| \right)^{2m}$ for some $C \in \mathbb{R}_{>0}$ and $m \in \mathbb{N}$,

2. $\|h(s)^{-1} \partial h(s)(v)\|^2 \leq C \sum_{i=1}^k \frac{1}{|z_i|^2 (\log |z_i|)^2}$ for some $C \in \mathbb{R}_{>0}$

hold. Good metrics appear naturally when considering toroidal compactifications of locally symmetric spaces, see [35].
Assume that X is projective and that h is, at the same time, good and psh. Choose a smooth reference metric h_0 on $\mathcal{L}$ and write $\omega = c_1(\mathcal{L}, h_0)$ and $\varphi = -\log(h(s)/h_0(s))$, so that φ is ω-psh.
By [35, Proposition 1.2] we have that the Lelong numbers of φ are zero on all points of all modifications of X and that

$$\int_X \langle (\omega + dd^c\varphi)^n \rangle = \int_{X \setminus D} (\omega + dd^c\varphi)^n = \int_X \omega^n = \deg(\mathcal{L}).$$

We see in particular that $P[\varphi]_{\mathcal{J}} = 0$.
Now assume that $\mathcal{L}$ is moreover ample. Then $\deg(\mathcal{L}) > 0$ and hence we have $\int_X \langle (\omega + dd^c\varphi)^n \rangle > 0$. Also, since $\mathcal{J}(k\varphi)$ is an ideal sheaf

$$\lim_{k \to \infty} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{J}(k\varphi))}{k^n/n!} \leq \lim_{k \to \infty} \frac{h^0(X, \mathcal{L}^{\otimes k})}{k^n/n!} = \deg(\mathcal{L}).$$

We may deduce from Theorem 2.31 that equality holds in the latter and that $P[\varphi] = P[\varphi]_{\mathcal{J}} = 0$.

2.8 Multiplier ideal volume equals non-pluripolar volume in the case of algebraic singularities

The purpose of this section is to show that multiplier ideal volume equals non-pluripolar volume in the case of algebraic singularities.
Let θ be a closed smooth $(1,1)$-form on the pure-dimensional projective complex manifold X and let φ be a θ-psh function on X. Let $n = \dim X$.

Lemma 2.35. *Assume that φ has algebraic singularities as in Definition 2.16. Let $c \in \mathbb{Q}_{\geq 0}$ be the constant associated to φ. Take a modification $\pi\colon X_\pi \to X$ such that $\pi^{-1}\mathcal{I}(\varphi/c)\cdot\mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(-D)$ for an effective simple normal crossings divisor D. Let $\mathrm{cl}(\pi^*\theta)$ and $[D]$ denote the cohomology classes of the closed smooth $(1,1)$-form $\pi^*\theta$ and the divisor D on X_π. Then $\mathrm{cl}(\pi^*\theta) - c[D]$ is a nef class on X_π and the equality*

$$(\mathrm{cl}(\pi^*\theta) - c[D])^n = \int_X \left\langle (\theta + dd^c\varphi)^{\wedge n} \right\rangle \tag{2.11}$$

holds in $\mathbb{Q}_{\geq 0}$.

Proof. The Siu decomposition (see Remark 2.10) of $\pi^*\left(\theta + dd^c\varphi\right)$ on X_π is

$$\pi^*(\theta + dd^c\varphi) = T' + c\,\delta_D\,, \tag{2.12}$$

with T' a closed positive $(1,1)$-current with locally bounded potentials representing the class $\mathrm{cl}(\pi^*\theta) - c[D]$. It follows from Lemma 2.24 that $\mathrm{cl}(\pi^*\theta) - c[D]$ is a nef class. Next we note that

$$\int_X \langle\theta + dd^c\varphi\rangle^n = \int_{X_\pi} \langle T'\rangle^n$$

as T' agrees with $\theta + dd^c\varphi$ on $X_\pi \setminus D$. As T' represents the cohomology class $\mathrm{cl}(\pi^*\theta) - c[D]$ on X_π and moreover is positive with locally bounded potentials we have by Lemma 2.24 that

$$\int_{X_\pi} \langle T'\rangle^n = (\mathrm{cl}(\pi^*\theta) - c[D])^n\ .$$

This proves the required equality. It is clear that the degree $(\mathrm{cl}(\pi^*\theta) - c[D])^n$ is a rational number. ☐

The next result is proved in [18, Theorem 2.26], under the hypothesis that $\mathcal{L}$ is ample with a Kähler metric. We give an alternative proof, and remove the hypothesis that $\mathcal{L}$ is ample.

Theorem 2.36. *Let $\mathcal{L}$ be a line bundle on X provided with a smooth reference metric h_0 and a psh metric h. Let $\theta = c_1(\mathcal{L}, h_0)$ be the first Chern form and $\varphi = -\log(h(s)/h_0(s))$ the resulting θ-psh function as in Remark 2.7. If φ has algebraic singularities then the equality*

$$\lim_{k\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{J}(k\varphi))}{k^n/n!} = \int_X \langle(\theta + dd^c\varphi)^{\wedge n}\rangle$$

holds in $\mathbb{Q}_{\geq 0}$.

Proof. Let $c \in \mathbb{Q}_{\geq 0}$ be the constant associated to φ as in Definition 2.16. Let $\pi\colon X_\pi \to X$ be a modification such that $\pi^{-1}\mathcal{I}(\varphi/c)\cdot\mathcal{O}_{X_\pi} = \mathcal{O}(-D)$ for $D = \sum_i \alpha_i D_i$ an effective simple normal crossings divisor on X_π. By Lemma 2.35 we have that $\pi^*\mathcal{L}\otimes\mathcal{O}(-cD)$ is nef, and the equality

$$\deg(\pi^*\mathcal{L}\otimes\mathcal{O}(-cD)) = \int_X \langle(\theta + dd^c\varphi)^{\wedge n}\rangle \tag{2.13}$$

holds in $\mathbb{Q}_{\geq 0}$. Let ℓ be a positive integer such that $\ell c \in \mathbb{Z}$. Denote by $\operatorname{vol}(\mathcal{M})$ the volume of a line bundle $\mathcal{M}$. By Lemma 2.18

$$\begin{aligned}
\lim_{k\to\infty}\frac{h^0(X,\mathcal{L}^{\otimes k}\otimes\mathcal{J}(k\varphi))}{k^n/n!} &= \lim_{k\to\infty}\frac{h^0(X,\mathcal{L}^{\otimes k\ell}\otimes\mathcal{I}((k\ell c\varphi)/c))}{(k\ell)^n/n!}\\
&= \lim_{k\to\infty}\frac{h^0\big(X_\pi,\pi^*\mathcal{L}^{\otimes k\ell}\otimes\mathcal{O}(-\ell cD)^{\otimes k}\big)}{(k\ell)^n/n!}\\
&= \frac{\operatorname{vol}(\pi^*\mathcal{L}^{\otimes\ell}\otimes\mathcal{O}(-\ell cD))}{\ell^n}.
\end{aligned}$$

As $\pi^*\mathcal{L}^{\otimes\ell}\otimes\mathcal{O}(-\ell cD)$ is nef, we have the Hilbert–Samuel formula

$$\operatorname{vol}(\pi^*\mathcal{L}^{\otimes\ell}\otimes\mathcal{O}(-\ell cD)) = \deg(\pi^*\mathcal{L}^{\otimes\ell}\otimes\mathcal{O}(-\ell cD)),$$

and the result follows by applying (2.13). □

3 Almost asymptotically algebraic singularities

We continue to assume that X is a pure-dimensional projective complex manifold.

3.1 Definition and first examples

Let θ be a closed smooth $(1,1)$-form and ω a Kähler form on X. The following terminology has been introduced by Rashkovskii [38] in the local case of isolated singularities, but can be adapted to the global case of θ-psh functions.

Definition 3.1. A θ-psh function φ on X is said to have *asymptotically algebraic singularities with respect to* θ if there is a sequence of quasi-psh functions $(\psi_m)_{m\geq 1}$ with algebraic singularities and a sequence of positive real numbers $(a_m)_{m\geq 1}$ converging monotonically to zero such that for each $m \geq 1$ the function ψ_m is $(\theta + a_m\omega)$-psh and the inequalities

$$(1+\frac{1}{m})\psi_m \prec \varphi \prec (1-\frac{1}{m})\psi_m$$

hold.

We will omit the addition "with respect to θ" from the terminology if the form θ is clear from the context.

For instance, if φ has isolated singularities, and is tame in the sense of [11] or is exponentially Hölder, then it has asymptotically algebraic singularities, see [38, Examples 3.6 and 3.7].

We introduce the following slightly weaker notion.

Definition 3.2. A θ-psh function φ on X is said to have *almost asymptotically algebraic singularities with respect to* θ if there exists a quasi-psh function f with algebraic singularities, a sequence of quasi-psh functions $(\psi_m)_{m\geq 1}$ with algebraic singularities and a sequence of positive real numbers $(a_m)_{m\geq 1}$ converging monotonically to zero such that for all $m \geq 1$ the function ψ_m is $(\theta + a_m\omega)$-psh and the inequalities

$$\psi_m + \frac{1}{m} f \prec \varphi \prec \psi_m$$

hold. A psh metric h on a line bundle $\mathcal{L}$ on X such that θ represents the cohomology class $c_1(\mathcal{L})$ is said to have almost asymptotically algebraic singularities if the corresponding θ-psh function has almost asymptotically algebraic singularities with respect to θ.

Again, we will omit the addition "with respect to θ" from the terminology if the form θ is clear from the context.

Lemma 3.3. *If h is a psh good metric (in the sense of Example 2.34) on a line bundle $\mathcal{L}$, and h_0 is a smooth metric on $\mathcal{L}$, then h has almost asymptotically algebraic singularities with respect to $\theta \coloneqq c_1(\mathcal{L}, h_0)$*

Proof. Let D be a divisor as in Example 2.34, s a non-zero rational section of $\mathcal{L}$, and set

$$\varphi = -\log(h(s)/h_0(s)).$$

This is a θ-psh function, with singularities contained in D. Choose an effective normal crossings divisor A such that $|D| \subset |A|$ and such that $\mathcal{O}(A)$ admits a smooth psh metric (for example, we could take A ample); choose one such metric. Let 1 be the canonical section of $\mathcal{O}(A)$ (so $\operatorname{div}(1) = A$), and write $f = \log \|1\|$. We claim that for every $m > 0$ the inequalities

$$\frac{1}{m} f \prec \varphi \prec 0$$

hold. That $\varphi \prec 0$ follows from the assumption that h be psh; we will use goodness to establish the other inequality. We work locally around a point $x \in X$, where we can assume that our rational section s is in fact a generating section of $\mathcal{L}$, so that $\varphi \sim -\log h(s)$. We write $z_1, \dots, z_a$ for local defining equations for the branches of A through x, and we order them so that D is cut out by $\prod_{i=1}^{b} z_i$ for some $b \leq a$. Then

$$f \sim \sum_{i=1}^{a} a_i \log|z_i| \tag{3.1}$$

where the $a_i \in \mathbb{Z}_{>0}$ are the multiplicities of the branches cut out by z_i in the divisor A. Now from the definition of a good metric we see that

$$h(s) \leq C\left(\sum_{i=1}^{b} -\log|z_i|\right)^{2M} \tag{3.2}$$

for some positive integer M and positive real number C. Hence

$$-2M\log\left(\sum_{i=1}^{b} -\log|z_i|\right) \prec \varphi, \tag{3.3}$$

from which we see that

$$\frac{1}{m}f \prec \varphi \tag{3.4}$$

for all $m \geq 1$. □

Remark 3.4.

1. The quasi-psh functions with almost asymptotically algebraic singularities form a convex cone. More precisely, if φ_1 is a θ_1-psh function with almost asymptotically algebraic singularities with respect to θ_1, and φ_2 is a θ_2-psh function with almost asymptotically algebraic singularities with respect to θ_2, then $\varphi_1 + \varphi_2$ is a $(\theta_1 + \theta_2)$-psh function with almost asymptotically algebraic singularities with respect to $\theta_1 + \theta_2$.
2. The notion of (almost) asymptotically algebraic singularities is birationally invariant. Hence all of the results concerning almost asymptotically algebraic singularities continue to hold if we pass to a modification of X.
3. Any psh metric with almost asymptotically algebraic singularities has small unbounded locus. Indeed, if $\psi + f \prec \varphi \prec \psi$ with ψ, f quasi-psh with algebraic singularities, then φ is locally bounded away from the singular loci of ψ and f, which are proper Zariski closed, and in particular pluripolar, sets.

Lemma 3.5. *The notion of almost asymptotically algebraic singularities does not depend on the choice of Kähler metric ω. Moreover we can choose the function f in Definition 3.2 to be ω-psh and the sequence (a_m) to be $a_m = \frac{1}{m}$.*

Proof. Assume that φ has almost asymptotically algebraic singularities with respect to θ. Let ω' be another Kähler metric on X. Then by the compactness of X there is a real number $a > 0$ such that $\omega \leq a\omega'$. Therefore, if ψ_m is $(\theta + a_m\omega)$-psh, then it is also $(\theta + a_m a\omega')$-psh. So after setting $(a_m a)$ instead of (a_m) we see that φ satisfies Definition 3.2 for ω'.
Since f is quasi-psh, by Lemma 2.5 there is a real number $b > 0$ such that f is $(b\omega)$-psh and hence f/b is ω-psh. Choose an increasing sequence m_k of integers satisfying

$$a_{m_k} \leq \frac{1}{k}, \qquad \text{and} \qquad m_k > kb. \tag{3.5}$$

This can easily be achieved as (a_m) converges to zero. Writing $\psi'_k = \psi_{m_k}$ and $f' = f/b$ we have that ψ'_k is $(\theta + a_{m_k}\omega)$-psh. By the first condition in (3.5), the function ψ'_k is $(\theta + (1/k)\omega)$-psh. Using the second condition in (3.5) we find

$$\psi'_k + \frac{1}{k} f' = \psi_{m_k} + \frac{1}{bk} f \prec \psi_{m_k} + \frac{1}{m_k} f \prec \varphi.$$

It follows that the functions ψ'_k and f' satisfy the conditions of Definition 3.2 where f' is ω-psh and where $a_k = 1/k$. □

Lemma 3.6. *If φ has asymptotic algebraic singularities with respect to θ, then it has almost asymptotically algebraic singularities with respect to θ.*

Proof. Let ψ_m be a sequence of functions satisfying Definition 3.1. For $m \geq 2$ we have the chain of inequalities

$$\frac{3}{2}\psi_2 \prec \varphi \prec \left(1 - \frac{1}{m}\right)\psi_m \prec \frac{1}{2}\psi_m,$$

which implies $\psi_m \succ 3\psi_2$. Therefore we have the chain of inequalities

$$\left(1 - \frac{1}{m}\right)\psi_m + \frac{1}{m} 6\psi_2 \prec \left(1 - \frac{1}{m}\right)\psi_m + \frac{2}{m}\psi_m \prec \varphi \prec \left(1 - \frac{1}{m}\right)\psi_m\,,$$

showing that φ has almost asymptotically algebraic singularities. □

The converse is not true. We will illustrate this with the function φ of Example 2.33. We use the notations in that example. Note that the function

$$f(t) = \log|t| - \log(1 + |t|^2)/2$$

is ω-psh. For all $m \in \mathbb{Z}_{>0}$ we have

$$\frac{1}{m} f \prec \varphi \prec 0.$$

Indeed the only point where f and φ are not locally bounded is the point $t = 0$. Close to this point φ has a singularity of the shape $-\log(-\log|t|)$, while f/m has a singularity of the shape $\log(|t|)/m$ which is more singular for all values of m. So, taking $\psi_m = 0$ in Definition 3.2, we see that φ has almost asymptotically algebraic singularities. Nevertheless φ does not have asymptotically algebraic singularities. Assume that it satisfies Definition 3.1 for a family of functions ψ_m with algebraic singularities. Then for $m = 2$ we have

$$\frac{3}{2}\psi_2 \prec \varphi \prec \frac{1}{2}\psi_2$$

where ψ_2 has algebraic singularities. Since the Lelong numbers of φ are zero, the right inequality implies that $0 \prec \psi_2$. But this contradicts the left inequality as $0 \not\prec \varphi$.

3.2 Some criteria for almost asymptotically algebraic singularities

The purpose of this section is to exhibit some useful criteria that allow to verify that a quasi-psh function has almost asymptotically algebraic singularities. Our results are based on Demailly's regularization theorem (Theorem 2.20). We continue with our assumption that X is a projective complex manifold. We fix a background Kähler form ω on X.

Definition 3.7. Let U be a Euclidean open subset of X with norm $\|\cdot\|$. An upper semi-continuous function $\varphi\colon U \to \mathbb{R}\cup\{-\infty\}$ is said to be *meromorphically Lipschitz* if there exists a finite open coordinate covering $\{U_i\}$ of U and for each open U_i there is a regular algebraic function f_i on U_i such that

$$\varphi(x) - \varphi(y) \leq \frac{\|x-y\|}{|f_i(y)|}, \quad x, y \in U_i\,.$$

Lemma 3.8. *Assume we are given a finite open covering $\{U_i\}$ of X and for each i a regular algebraic function f_i on U_i. For each i let $V_i \subset\subset U_i$ be a relatively compact open subset such that the collection $\{V_i\}$ is still an open cover of X. Then there exists a quasi-psh function φ on X with algebraic singularities such that, for all i, the inequality*

$$\varphi|_{V_i} \leq \log|f_i| \tag{3.6}$$

holds.

Proof. For each i let D_i be the divisor of f_i on U_i. Let E be an effective divisor on X with

$$E|_{U_i} \geq D_i \quad \text{for all}\, i\,. \tag{3.7}$$

Choose a smooth hermitian metric h_0 on $\mathcal{O}(E)$ and let s be a global section of $\mathcal{O}(E)$ with $\operatorname{div}(s) = E$. By condition (3.7), the function s/f_i is regular on U_i. Therefore $h_0(s)/|f_i|$ is continuous on U_i, hence $\log(h_0(s)) - \log|f_i|$ is bounded above on the compact subset $\overline{V_i}$. Let M be a real number such that for all i we have the bound $\log(h_0(s)) - \log|f_i| \leq M$ on $\overline{V_i}$. Let $\theta = c_1(\mathcal{O}(E), h_0)$ be the first Chern form of $\mathcal{O}(E)$ with smooth metric h_0. Then the function $\varphi = \log(h_0(s)) - M$ is θ-psh, has algebraic singularities and satisfies the inequalities (3.6). □

Let θ be a smooth $(1,1)$-form on X.

Proposition 3.9. *Let φ be a θ-psh function on X that can be written locally as a sum $\varphi = \phi + \gamma$ with ϕ meromorphically Lipschitz and γ bounded. Then φ has almost asymptotically algebraic singularities with respect to θ.*

Proof. Let $n = \dim X$, and choose finite open coordinate coverings $\{U_i\}$ and $\{V_i\}$ of X with $V_i \subset\subset U_i$. By Demailly's regularization theorem (Theorem 2.20) there exist constants C_1 and C_2, a sequence of functions φ_m with algebraic singularities on X satisfying, in each V_i,

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{\|x-z\|<r} \varphi(x) + \frac{1}{m} \log \frac{C_2}{r^n} \tag{3.8}$$

and a sequence of positive real numbers $(a_m)_{m>0}$ converging monotonically to zero such that each function φ_m is $(\theta + a_m\omega)$-psh. Note that $\varphi \prec \varphi_m$.
By our assumption on φ, after taking a finite refinement of the open cover we can assume that there exist functions f_i that are regular on U_i and such that on each U_i the estimate

$$\varphi(x) - \varphi(z) \leq \frac{\|x-z\|}{|f_i(z)|} + C_3$$

holds for a constant C_3. By Lemma 3.8 there is a quasi-psh function ψ on X with algebraic singularities and such that $\psi|_{V_i} \leq \log |f_i|$ for all i.
Taking now $r = |f_i(z)|$, we deduce from (3.8) that

$$\begin{aligned}\varphi_m(z) &\leq \varphi(z) + \frac{r}{|f_i(z)|} + C_3 + \frac{1}{m} \log \frac{C_2}{r^n} \\ &\leq \varphi(z) + 1 + C_3 + \frac{1}{m} \log C_2 - \frac{n}{m} \log |f_i(z)| \\ &\leq \varphi(z) + C_4 - \frac{n}{m} \psi(z).\end{aligned}$$

The function $f(z) = n\psi(z)$ is quasi-psh with algebraic singularities and we have found that for every m the estimate

$$\varphi_m + \frac{1}{m} f \prec \varphi \prec \varphi_m$$

holds. This concludes the proof of the proposition. □

We can use Proposition 3.9 to see that toroidal singularities are almost asymptotically algebraic. The following definition is inspired by the fact that if $g\colon \mathbb{R}^k_{>0} \to \mathbb{R}$ is a bounded-above convex function, then the function

$$g(-\log |z_1|, \dots, -\log |z_k|)$$

is a psh function on $D(1)^k$. Here $D(1)$ is the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

Definition 3.10. Let $U \subset X$ be Zariski open with $D = X \setminus U$ a normal crossings divisor. A quasi-psh function φ on X is said to have *toroidal singularities* (with respect to D) if φ is locally bounded on U and there exists an open coordinate covering $\{V_i\}$ of X such that on each V_i the divisor D has equation $z_1 \cdots z_{k_i} = 0$ and the restriction $\varphi|_{V_i \cap U}$ can be written as

$$\varphi|_{V_i \cap U} = \gamma + g(-\log |z_1|, \dots, -\log |z_{k_i}|)\,, \tag{3.9}$$

where γ is locally bounded on V_i and g is a bounded above convex Lipschitz continuous function defined on a quadrant $\{u_j \geq M_j | j = 1, \dots, k_i\} \subseteq \mathbb{R}^{k_i}$. We say that a psh metric on a line bundle $\mathcal{L}$ on X has toroidal singularities if any corresponding θ-psh function has toroidal singularities. Here θ is any smooth closed differential form representing the cohomology class $c_1(\mathcal{L})$. Note that this is equivalent to any local psh potentials being of the form (3.9).

Proposition 3.11. *A toroidal θ-psh function on X has almost asymptotically algebraic singularities with respect to θ.*

Proof. By Proposition 3.9 it suffices to check that, if g is a bounded above convex Lipschitz continuous function defined on a quadrant $K = \{u_j \geq M_j | j = 1, \dots, k\}$, then $g(-\log|z_1|, \dots, -\log|z_k|)$ is meromorphically Lipschitz in the polydisk $|z_j| < e^{-M_j}$. We first observe that, if $s, t > 0$ are real numbers, then

$$\log(s) - \log(t) \leq \frac{|s-t|}{t}. \tag{3.10}$$

Indeed, if $s \leq t$ then the right hand side is greater or equal to zero while the left hand side is smaller or equal than zero. And if $t < s$ then

$$\log(s) - \log(t) = \int_t^s \frac{d\xi}{\xi} \leq \frac{s-t}{t}.$$

Next we see that since g is bounded above convex Lipschitz continuous on the quadrant K the function g is non-increasing on each semi-line of the form $u + \lambda w$, $\lambda \geq 0$, for $u \in K$ and w a vector with non-negative entries. Therefore, if $u = (u_1, \dots, u_k)$ and $v = (v_1, \dots, v_k)$ are points in K then

$$g(u) - g(v) \leq \sum_{j\colon v_j > u_j} C(v_j - u_j), \tag{3.11}$$

where C is the Lipschitz constant of g. Now using equations (3.10) and (3.11) we obtain for $z, x \in \mathbb{C}^k$ with $|z_j|, |x_j| < e^{-M_j}$ for $j = 1, \dots, k$ that

$$\begin{aligned}
&g(-\log|z_1|, \dots, -\log|z_k|) - g(-\log|x_1|, \dots, -\log|x_k|) \\
&\leq \sum_{j\colon |z_j| > |x_j|} C(\log|z_j| - \log|x_j|) \\
&\leq \sum_{j\colon |z_j| > |x_j|} C\frac{|x_j - z_j|}{|x_j|} \\
&\leq C' \frac{\|x - z\|}{\prod |x_j|}
\end{aligned}$$

for some constant C', proving the claim. $\square$

3.3 Multiplier ideal volume equals non-pluripolar volume in the case of almost asymptotically algebraic singularities

We continue to assume that X is a projective pure-dimensional complex manifold. Let $n = \dim X$.
The main result of this section is that for a quasi-psh function with almost asymptotically algebraic singularities the multiplier ideal volume and the non-pluripolar volume are equal. See Theorem 3.13.

Lemma 3.12. *Let $D \subset X$ be a smooth divisor, A an ample divisor and $\ell \in \mathbb{Z}_{>0}$. We denote by $\mathcal{O}_{\ell D}$ the coherent sheaf on X defined by the exact sequence*

$$0 \longrightarrow \mathcal{O}_X(-\ell D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\ell D} \to 0\,.$$

Then for all $k \in \mathbb{Z}_{>0}$ we have

$$h^0(X, \mathcal{O}_{\ell D}(kA)) \leq \ell k^{n-1} D \cdot A^{n-1}.$$

Proof. For every $0 \leq j \leq \ell - 1$ there is an exact sequence

$$0 \longrightarrow \frac{\mathcal{O}(-(j+1)D)}{\mathcal{O}(-\ell D)} \longrightarrow \frac{\mathcal{O}(-jD)}{\mathcal{O}(-\ell D)} \longrightarrow \mathcal{O}_D(-jD) \longrightarrow 0.$$

Adding up, these exact sequences imply that

$$\begin{aligned} h^0(X, \mathcal{O}_{\ell D}(kA)) &\leq \sum_{j=0}^{\ell-1} h^0(X, \mathcal{O}_D(kA - jD)) \\ &\leq \sum_{j=0}^{\ell-1} h^0(X, \mathcal{O}_D(kA))) \leq \ell k^{n-1} D \cdot A^{n-1}. \end{aligned}$$

□

Let $\mathcal{L}$ be a line bundle on X provided with a smooth reference metric h_0 and a psh metric h. Let $\theta = c_1(\mathcal{L}, h_0)$ be the first Chern form and $\varphi = -\log(h(s)/h_0(s))$ the associated θ-psh function. As before we let

$$\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h) = \lim_{k\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes k} \otimes \mathcal{J}(k\varphi))}{k^n/n!}$$

be the multiplier ideal volume of $(\mathcal{L}, h)$. The following can be seen as a Hilbert–Samuel type formula.

Theorem 3.13. *If φ has almost asymptotically algebraic singularities with respect to θ then the equality*

$$\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h) = \int_X \langle (\theta + dd^c\varphi)^{\wedge n} \rangle \tag{3.12}$$

holds in $\mathbb{R}_{\geq 0}$.

Proof. By Lemma 3.5 we can find an ample line bundle $\mathcal{O}(D)$ on X with a smooth metric such that its first Chern form is a Kähler form ω, an ω-psh function f with algebraic singularities, and a sequence of quasi-psh functions φ_m with algebraic singularities such that φ_m is $(\theta + (1/m)\omega)$-psh and

$$\varphi_m + \frac{1}{m} f \prec \varphi \prec \varphi_m \,. \tag{3.13}$$

Also, for each $m > 0$ each of the three functions separated by the two inequalities in (3.13) is $(\theta + (2/m)\omega)$-psh by Lemma 2.5. We recall from Remark 3.4 that almost asymptotically algebraic singularities have small unbounded locus. Thus we can apply the monotonicity property of the non-pluripolar product (Theorem 2.26) and conclude that for each fixed $m > 0$ that

$$\begin{aligned}\int_X \langle (\theta + \frac{2\omega}{m} + dd^c\varphi_m + \frac{1}{m} dd^c f)^n \rangle \leq \int_X \langle (\theta + \frac{2\omega}{m} + dd^c\varphi)^n \rangle \\ \leq \int_X \langle (\theta + \frac{2\omega}{m} + dd^c\varphi_m)^n \rangle.\end{aligned} \tag{3.14}$$

Now set

$$A_m \coloneqq \int_X \langle (\theta + \frac{2\omega}{m} + dd^c\varphi_m + \frac{1}{m} dd^c f)^n \rangle - \int_X \langle (\theta + \frac{\omega}{m} + dd^c\varphi_m)^n \rangle \,.$$

By the multi-additivity of the non-pluripolar product (Proposition 2.25) we have

$$\begin{aligned}A_m &= \sum_{k=1}^{n} \binom{n}{k} \int_X \langle (\theta + \frac{\omega}{m} + dd^c\varphi_m)^{n-k} (\frac{\omega}{m} + \frac{1}{m} dd^c f)^k \rangle \\ &= \frac{1}{m} \sum_{k=1}^{n} \frac{1}{m^{k-1}} \binom{n}{k} \int_X \langle (\theta + \frac{\omega}{m} + dd^c\varphi_m)^{n-k} (\omega + dd^c f)^k \rangle \,.\end{aligned}$$

Note that $A_m \geq 0$ since all the summands on the right of the above expression are positive. We see that there is a constant C such that $0 \leq A_m \leq C/m$ for all m so

$$\lim_{m\to\infty} A_m = 0.$$

Similarly,

$$\begin{aligned}\lim_{m\to\infty} \int_X \langle (\theta + \frac{2\omega}{m} + dd^c\varphi_m)^n \rangle - \int_X \langle (\theta + \frac{\omega}{m} + dd^c\varphi_m)^n \rangle = 0, \\ \lim_{m\to\infty} \int_X \langle (\theta + \frac{2\omega}{m} + dd^c\varphi)^n \rangle - \int_X \langle (\theta + dd^c\varphi)^n \rangle = 0.\end{aligned}$$

We conclude that when $m \to \infty$ each of the three terms in the inequality (3.14) converges to

$$\int_X \langle (\theta + dd^c\varphi)^n \rangle.$$

Next, for each $m \geq 0$ and each $k > 0$ with $m|k$, by the monotonicity of multiplier ideals with the type of singularity, we obtain the inequalities

$$\begin{aligned} h^0\big(X, \mathcal{L}^k \otimes \mathcal{O}(2(k/m)D) & \otimes \mathcal{J}(k(\varphi_m + f/m))\big) \\ & \leq h^0\left(X, \mathcal{L}^k \otimes \mathcal{O}(2(k/m)D) \otimes \mathcal{J}(k\varphi)\right) \\ & \leq h^0\left(X, \mathcal{L}^k \otimes \mathcal{O}(2(k/m)D) \otimes \mathcal{J}(k(\varphi_m)\right). \end{aligned} \tag{3.15}$$

Since φ_m and f have algebraic singularities, by Theorem 2.36 we have the statement of the theorem, that is, the equality in (3.12), for the terms at the left and at the right of the chain of inequalities (3.15). Using the convergence to $\int_X \langle(\theta + dd^c\varphi)^n\rangle$ of all three terms in the chain (3.14) we deduce that

$$\lim_{m\to\infty} \lim_{\substack{k\to\infty \\ m|k}} \frac{h^0\left(X, \mathcal{L}^k \otimes \mathcal{O}(2(k/m)D) \otimes \mathcal{J}(k\varphi)\right)}{k^n/n!} = \int_X \langle(\theta + dd^c\varphi)^n\rangle.$$

It remains to show that

$$\lim_{m\to\infty} \lim_{\substack{k\to\infty \\ m|k}} \frac{h^0\left(X, \mathcal{L}^k \otimes \mathcal{O}(2(k/m)D) \otimes \mathcal{J}(k\varphi)\right)}{k^n/n!} = \lim_{k\to\infty} \frac{h^0\left(X, \mathcal{L}^k \otimes \mathcal{J}(k\varphi)\right)}{k^n/n!}. \tag{3.16}$$

Fix $k = \ell m$. Replacing D by a positive multiple we may assume that D is effective and that $\mathcal{L}(D)$ is ample. Then the exact sequence

$$0 \to \mathcal{O}(-2\ell D) \to \mathcal{O} \to \mathcal{O}_{2\ell D} \to 0$$

implies that

$$\begin{aligned} h^0\big(X, \mathcal{L}^{\ell m} \otimes \mathcal{O}(2\ell D) & \otimes \mathcal{J}(\ell m\varphi)\big) - h^0\big(X, \mathcal{L}^{\ell m} \otimes \mathcal{J}(\ell m\varphi)\big) \\ & \leq h^0\left(X, \mathcal{L}^{\ell m} \otimes \mathcal{O}_{2\ell D}(2\ell D) \otimes \mathcal{J}(\ell m\varphi)\right) \\ & \leq h^0\left(X, \mathcal{L}^{\ell m} \otimes \mathcal{O}_{2\ell D}(2\ell D)\right), \end{aligned}$$

where the last inequality follows from the fact that $\mathcal{J}(\ell m\varphi)$ is an ideal sheaf. By Bertini we can further assume that D is smooth. Then writing $\mathcal{L} = \mathcal{O}_X(E)$ for a divisor E we calculate

$$\begin{aligned} h^0\left(X, \mathcal{O}_{2\ell D}(\ell m E + 2\ell D)\right) & \leq h^0\left(X, \mathcal{O}_{2\ell D}(\ell m(E + D))\right) \\ & \leq 2\ell(\ell m)^{n-1} D \cdot (E + D)^{n-1}, \end{aligned} \tag{3.17}$$

where the last inequality is an application of Lemma 3.12 with $A = E + D$. In particular,

$$\frac{h^0\left(X, \mathcal{O}_{2\ell D}(\ell m E + 2\ell D)\right)}{(\ell m)^n/n!} \to 0 \tag{3.18}$$

as $k = \ell m \to \infty$. This implies equation (3.16). ☐

Combining Theorem 2.31 and Theorem 3.13 we obtain the following corollary.

Corollary 3.14. *Let the situation be as in Theorem 3.13. Suppose that moreover we have $\int_X \langle(\theta + dd^c\varphi)^n\rangle > 0$, and that $\mathcal{L}$ is ample. Then $P[\varphi] = P[\varphi]_{\mathcal{J}}$, where $P[\varphi]$ and $P[\varphi]_{\mathcal{J}}$ are the envelopes of singularity type from Definition 2.29.*

We do not expect that conversely, the equality $P[\varphi] = P[\varphi]_{\mathcal{J}}$ for a quasi-psh function φ implies that φ has almost asymptotically algebraic singularities.

Remark 3.15. Let h be a psh metric on $\mathcal{L}$ with small unbounded locus. We do not assume that h has almost asymptotically algebraic singularities. Then we still have the inequality

$$\mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h) \geq \int_X \langle c_1(\mathcal{L}, h)^n\rangle.$$

Indeed, consider a Demailly approximation sequence $\{\varphi_m\}_{m\in\mathbb{N}}$ for φ. Then $\varphi \prec \varphi_m$ and in the same way as in the proof of Theorem 3.13, the inequality follows. This recovers the inequality in the result of Darvas and Xia in Theorem 2.31, without the assumption that $\mathcal{L}$ is ample.

Remark 3.16. It follows from the monotonicity properties of the non-pluripolar product and Theorem 3.13 that the multiplier ideal volume has the following continuity property on the space of psh metrics with almost asymptotically algebraic singularities: let φ be a quasi-psh function with almost asymptotically algebraic singularities with respect to θ. Let φ_m an approximating sequence of quasi-psh functions satisfying the conditions of Definition 3.2. Then

$$\lim_{m\to\infty} \mathrm{vol}_{\mathcal{J}}(\mathcal{L}, \varphi_m) = \mathrm{vol}_{\mathcal{J}}(\mathcal{L}, \varphi).$$

4 b-divisors

In this section we discuss Weil and Cartier $\mathbb{R}$-b-divisors on compact algebraic complex manifolds. This is essentially Shokurov's notion of birational divisors, or b-divisors, see [41]. For more background concerning b-divisors we refer to [9] and [11]; see also [6] and [5] for a discussion of the toroidal and the toric cases, respectively.

4.1 Basic definitions

Throughout this section X is a compact algebraic complex manifold (this section is purely algebraic, so if preferred the reader can work with finite-type algebraic varieties over any field of characteristic zero). We write $\mathrm{Div}_{\mathbb{R}}(X)$ for the set of Weil divisors on X with real coefficients, viewed as a real vector space (generally of infinite dimension). We endow it with the direct limit topology with respect to its finite dimensional subspaces. Explicitly, a sequence of divisors $(D_i)_{i\geq 0}$ converges to a divisor D in $\mathrm{Div}_{\mathbb{R}}(X)$ if there is a divisor A, such that $\mathrm{supp}(D_i) \subset \mathrm{supp}(A)$ for all $i \geq 0$ and $(D_i)_{i\geq 0}$ converges to D in the finite dimensional vector space of real divisors with support contained in $\mathrm{supp}(A)$.

In Definition 2.9 we defined a modification of complex manifolds. We note that if $\pi\colon X' \to X$ is a modification then X' is also a compact algebraic complex manifold.

Definition 4.1. The set of *models of* X is

$$R(X) \coloneqq \left\{\pi\colon X_\pi \to X \,\middle|\, \pi \text{ is a modification}\right\}.$$

We view $R(X)$ as a full subcategory of the category of complex manifolds over X, in particular morphisms are over X. Maps of models are unique if they exist, and are necessarily proper and bimeromorphic.

Hironaka's resolution of singularities implies that $R(X)$ is a directed set, where we set $\pi' \geq \pi$ if there exists a morphism $\mu\colon X_{\pi'} \to X_\pi$.
Consider a pair $\pi' \geq \pi$ in $R(X)$, and let $\mu\colon X_{\pi'} \to X_\pi$ be the corresponding modification. We have a pullback map

$$\mu^*\colon \operatorname{Div}_{\mathbb{R}}(X_\pi) \longrightarrow \operatorname{Div}_{\mathbb{R}}(X_{\pi'})$$

and a pushforward map

$$\mu_*\colon \operatorname{Div}_{\mathbb{R}}(X_{\pi'}) \longrightarrow \operatorname{Div}_{\mathbb{R}}(X_\pi)$$

of divisors. Both maps are continuous.

Definition 4.2. The group of *Cartier* $\mathbb{R}$*-b-divisors on* X is the direct limit

$$\text{C-b-Div}_{\mathbb{R}}(X) \coloneqq \varinjlim_{\pi\in R(X)} \operatorname{Div}_{\mathbb{R}}(X_\pi),$$

taken in the category of topological vector spaces, with maps given by the pullback maps. The resulting topology is called the *strong* topology. The group of *Weil* $\mathbb{R}$*-b-divisors on* X is the inverse limit

$$\text{W-b-Div}_{\mathbb{R}}(X) \coloneqq \varprojlim_{\pi\in R(X)} \operatorname{Div}_{\mathbb{R}}(X_\pi),$$

taken in the category of topological vector spaces, with maps given by the pushforward maps. The resulting topology is called the *weak* topology.

Remark 4.3. As a set, $\text{C-b-Div}_{\mathbb{R}}(X)$ can be seen as the disjoint union of the sets $\operatorname{Div}_{\mathbb{R}}(X_\pi)$ modulo the equivalence relation which sets two divisors equal if they coincide after pullback to a common modification. The set $\text{W-b-Div}_{\mathbb{R}}(X)$ can be seen as the subset of $\prod_{\pi\in R(X)} \operatorname{Div}_{\mathbb{R}}(X_\pi)$ given by the elements $\mathbb{D} = (D_\pi)_{\pi\in R(X)}$ satisfying the compatibility condition that for each $\pi' \geq \pi$ we have $\mu_* D_{\pi'} = D_\pi$, where μ is the corresponding modification.

Definition 4.4. Let $\mathbb{D}$ be a Cartier $\mathbb{R}$-b-divisor. A *determination* of $\mathbb{D}$ is a representative D of the equivalence class given by $\mathbb{D}$ as described in Remark 4.3. If X_π is the modification where D lives, we say that $\mathbb{D}$ is *determined* in X_π. If $\mathbb{D} = (D_\pi)_{\pi\in R(X)}$ is a Weil $\mathbb{R}$-b-divisor, then for $\pi \in R(X)$, the divisor D_π is called the *incarnation* of $\mathbb{D}$ on X_π.

Remark 4.5. 1. For every modification $\mu\colon X_{\pi'} \to X_{\pi}$, the identity $\mu_*\mu^* = \mathrm{Id}$ holds. Therefore, the natural map $\text{C-b-Div}_{\mathbb{R}}(X) \to \text{W-b-Div}_{\mathbb{R}}(X)$ is injective. This map can be described as follows. Let $D \in \mathrm{Div}_{\mathbb{R}}(X_{\pi_0})$ be any determination of a Cartier b-divisor. Then for each $\pi \in R(X)$ we choose any element $\pi' \in R(X)$ such that $\pi' \geq \pi_0$ and $\pi' \geq \pi$. Let $\mu_0\colon X_{\pi'} \to X_{\pi_0}$ and $\mu\colon X_{\pi'} \to X_{\pi}$ be the corresponding modifications. Then $D_\pi \coloneqq \mu_*\mu_0^*D$ does not depend on the choice of π' and the image of the Cartier $\mathbb{R}$-b-divisor given by D is the Weil $\mathbb{R}$-b-divisor $(D_\pi)_\pi$. From now on we will identify, as a set, $\text{C-b-Div}_{\mathbb{R}}(X)$ with its image in $\text{W-b-Div}_{\mathbb{R}}(X)$, and by $\mathbb{R}$-*b-divisor* we will mean a Weil $\mathbb{R}$-b-divisor.

2. The injection $\text{C-b-Div}_{\mathbb{R}}(X) \hookrightarrow \text{W-b-Div}_{\mathbb{R}}(X)$ as described above is continuous, but is *not* a homeomorphism onto its image. In fact, a net of Cartier $\mathbb{R}$-b-divisors $\{\mathbb{D}_i\}_{i\in I}$ converges in $\text{C-b-Div}_{\mathbb{R}}(X)$ to a Cartier $\mathbb{R}$-b-divisor $\mathbb{D}$ if and only if the following is satisfied. There exists a model π such that $\mathbb{D}$ and all the $\mathbb{D}_i$ are determined in π, and if $D, D_i \in \mathrm{Div}_{\mathbb{R}}(X_\pi)$ are determinations of $\mathbb{D}$ and $\mathbb{D}_i$, respectively, then
$$D = \lim_{i\in I} D_i$$
in $\mathrm{Div}_{\mathbb{R}}(X_\pi)$. On the other hand, a net of $\mathbb{R}$-b-divisors $\{\mathbb{D}_i\}_{i\in I}$ converges in $\text{W-b-Div}_{\mathbb{R}}(X)$ to an $\mathbb{R}$-b-divisor $\mathbb{D}$ if and only if for each model $\pi \in R(X)$, we have that
$$D_\pi = \lim_{i\in I} D_{i,\pi}$$
in $\mathrm{Div}_{\mathbb{R}}(X_\pi)$.

3. Any $\mathbb{R}$-b-divisor $\mathbb{D} = (D_\pi)_{\pi\in R(X)}$ is the limit of its incarnations D_π. It follows that $\text{C-b-Div}_{\mathbb{R}}(X)$ is dense in $\text{W-b-Div}_{\mathbb{R}}(X)$.

4. It is natural in many situations to consider also integral or rational coefficients. The definitions are then easily adapted.

4.2 Nef and approximable nef b-divisors

Definition 4.6 ([9]). A Cartier $\mathbb{R}$-b-divisor $\mathbb{D} \in \text{C-b-Div}_{\mathbb{R}}(X)$ is *nef* if $D_\pi \in \mathrm{Div}_{\mathbb{R}}(X_\pi)$ is nef for one (and hence for every) determination $\pi \in R(X)$ of $\mathbb{D}$. A Weil $\mathbb{R}$-b-divisor $\mathbb{D} \in \text{W-b-Div}_{\mathbb{R}}(X)$ is *nef* if it is a limit (in the weak topology) of a net of nef Cartier $\mathbb{R}$-b-divisors.

Remark 4.7. It is a priori not clear that if a nef Weil $\mathbb{R}$-b-divisor is Cartier, then it is nef as a Cartier $\mathbb{R}$-b-divisor. This is known to be true in the toroidal setting [6, Lemma 4.24] and if we work with algebraic varieties over a countable field (instead of complex manifolds), see [16, Corollary 4].

Definition 4.8 ([16, Section 2]). Let $\mathbb{D}$ be a nef $\mathbb{R}$-b-divisor on X. Then $\mathbb{D}$ is called *approximable nef* if $\mathbb{D}$ can be written as a limit

$$\mathbb{D} = \lim_{i\in\mathbb{N}} \mathbb{E}_i$$

of a sequence of nef Cartier $\mathbb{R}$-b-divisors satisfying the monotonicity property $\mathbb{E}_i \geq \mathbb{E}_j$ whenever $i \leq j$. We call such a sequence an *approximating sequence.*

Remark 4.9. In [16, Theorem 5] Dang and Favre show that in the case of algebraic varieties over a countable field, *any* nef Weil $\mathbb{R}$-b-divisor is approximable nef. We will show in Section 5.2 that any b-divisor that comes from a psh metric in a suitable sense is approximable nef.

4.3 Intersection products of approximable nef b-divisors

Let $n = \dim X$, let $\mathbb{E}_1, \ldots, \mathbb{E}_{n-1}$ be Cartier $\mathbb{R}$-b-divisors on X, and let $\mathbb{D}$ be a Weil $\mathbb{R}$-b-divisor on X. If $\pi \in R(X)$ is such that all $\mathbb{E}_i$ are determined in π, then the real-valued intersection number

$$E_{1,\pi} \cdots E_{n-1,\pi} \cdot D_\pi$$

is independent of the choice of π, by the projection formula. This yields a well-defined intersection product

$$\text{C-b-Div}_{\mathbb{R}}(X) \times \ldots \times \text{C-b-Div}_{\mathbb{R}}(X) \times \text{W-b-Div}_{\mathbb{R}}(X) \longrightarrow \mathbb{R}. \qquad (4.1)$$

Extending this to an intersection product on all Weil $\mathbb{R}$-b-divisors seems too much to ask for, in general. However, if $\mathbb{D}_1, \ldots, \mathbb{D}_n$ are *approximable nef* $\mathbb{R}$-b-divisors and $(D_{i,r})_r$ are approximating sequences for the $\mathbb{D}_i$, then one can show that the limit

$$\lim_{r\to\infty} D_{1,r} \cdots D_{n,r} \qquad (4.2)$$

exists in $\mathbb{R}_{\geq 0}$ and is independent of the choice of approximating sequences. This yields a top intersection product on the space of approximable nef b-divisors, which is continuous with respect to approximating sequences. The details can be found in [16, Section 3]. We mention that in [16] the authors work over a countable ground field, but one can either check that this part of their argument does not use the countability assumption, or apply the following lemma:

Lemma 4.10. *Let $\mathbb{D} \in$ W-b-Div$_{\mathbb{R}}(X)$ be a limit of a sequence of Cartier divisors. Then there exists a countable subfield $L \subseteq \mathbb{C}$ over which both X and $\mathbb{D}$ are defined.*

5 The b-divisor associated to a psh metric

5.1 The definition of the b-divisor

Let X be a projective complex manifold of dimension n. Let $\mathcal{L}$ be a line bundle on X, and let h be a psh metric on $\mathcal{L}$ (see Definition 2.6). If $\pi\colon X_\pi \to X$ is a model in the category $R(X)$ (see Definition 4.1), we define the anti-effective $\mathbb{R}$-divisor

$$Z(\mathcal{L},h)_\pi \coloneqq \sum_P -\nu(h,P)P \tag{5.1}$$

on X_π, where the sum is over all prime divisors P on X_π. In general this sum need not be finite, so we make the following definition.

Definition 5.1. The psh metric h on $\mathcal{L}$ has *Zariski unbounded locus* if there exists a non-empty Zariski open subset $U \subseteq X$ such that the local potentials of h are locally bounded on U.

Example 5.2. By Remark 3.4, any psh metric with almost asymptotically algebraic singularities in the sense of Definition 3.2 has Zariski unbounded locus.

Lemma 5.3. *Assume that the psh metric h on $\mathcal{L}$ has Zariski unbounded locus. Then, given any model $\pi\colon X_\pi \to X$ in $R(X)$, for only finitely many prime divisors P on X_π we have that $\nu(h,P)$ is non-zero. Moreover, if $\mu\colon X_{\pi'} \to X_\pi$ is a map of models then we have an equality of Weil $\mathbb{R}$-divisors*

$$\mu_* Z(\mathcal{L},h)_{\pi'} = Z(\mathcal{L},h)_\pi. \tag{5.2}$$

on $X_{\pi'}$.

Proof. This is an easy consequence of the definitions. □

Definition 5.4. Assume that the psh metric h has Zariski unbounded locus. The Weil $\mathbb{R}$-b-divisor $Z(\mathcal{L},h) \in \text{W-b-Div}_{\mathbb{R}}(X)$ is defined by

$$Z(\mathcal{L},h) \coloneqq (Z(\mathcal{L},h)_\pi)_{\pi\in R(X)}\,.$$

It follows from Lemma 5.3 that this is indeed a Weil $\mathbb{R}$-b-divisor.
Let s be a non-zero rational section of $\mathcal{L}$ and write $D = \operatorname{div}(s)$, seen as a Cartier $\mathbb{R}$-b-divisor on X. Then we define the Weil $\mathbb{R}$-b-divisor

$$D(\mathcal{L},h,s) \coloneqq D + Z(\mathcal{L},h).$$

We observe that the formation of $D(\mathcal{L},h,s)$ is multiplicative in the sense that $D(\mathcal{L}_1\otimes\mathcal{L}_2, h_1\otimes h_2, s_1\otimes s_2) = D(\mathcal{L}_1,h_1,s_1) + D(\mathcal{L}_2,h_2,s_2)$ whenever $(\mathcal{L}_1,h_1)$, $(\mathcal{L}_2,h_2)$ are line bundles with psh metrics with Zariski unbounded locus on X and s_1, s_2 are non-zero rational sections of $\mathcal{L}_1$ resp. $\mathcal{L}_2$.
We write

$$D(\mathcal{L},h,s) = (D(\mathcal{L},h,s)_\pi)_{\pi\in R(X)}\,,$$

where

$$D(\mathcal{L},h,s)_\pi = \pi^* D + Z(\mathcal{L},h)_\pi = \operatorname{div}_{X_\pi}(s) + Z(\mathcal{L},h)_\pi\,.$$

Example 5.5. Let notations be as above and assume that h has algebraic singularities. Then $D(\mathcal{L},h,s)$ belongs to C-b-Div$_{\mathbb{R}}(X)$. Indeed, it can be computed as follows. Fix a reference smooth metric h_0 on $\mathcal{L}$ and write $\varphi = -\log(h(s)/h_0(s))$ and $\theta = c_1(\mathcal{L},h_0)$. As in Remark 2.7 we have that φ is a θ-psh function on X and it has algebraic singularities by assumption. Let c be the rational constant from Definition 2.16 for φ and write $\mathcal{I} = \mathcal{I}(\varphi/c)$. It follows Lemma 2.19 from there is a model $\pi\colon X_\pi \to X$ in $R(X)$ such that $\pi^{-1}\mathcal{I}\cdot\mathcal{O}_{X_\pi} = \mathcal{O}(-D)$ for an effective simple normal crossings divisor D, and such that

$$D(\mathcal{L},h,s) = \operatorname{div}(s) - cD$$

as Cartier b-divisors. Note that $D(\mathcal{L},h,s)$ is actually a $\mathbb{Q}$-b-divisor in this case.

Note that also a psh metric that is good in the sense of Mumford gives rise to a Cartier b-divisor (all Lelong numbers are zero on all modifications). This shows that the converse to the statement in the example does not hold as good metrics do not necessarily have algebraic singularities.

Remark 5.6. Let T be the closed positive $(1,1)$-current on X given as $T = c_1(\mathcal{L},h)$ and let s be a non-zero rational section of $\mathcal{L}$. We can relate the b-divisor $D(\mathcal{L},h,s)$ to the Siu decomposition of T on X (see Remark 2.10). Recall that this decomposes T uniquely as a sum

$$T = R + \sum_k \nu\left(T,Y_k\right)\delta_{Y_k},$$

where the sum is over an at most countable family of 1-codimensional subvarieties Y_k of X and R is a closed positive $(1,1)$-current whose Lelong number on any divisor is zero. The sum $\sum_k \nu\left(T,Y_k\right)\delta_{Y_k}$ is called the *divisor part of* T. If h has Zariski unbounded locus, then the family $\{Y_k\}$ is finite and the divisor part agrees with $-Z(\mathcal{L},h)_{\mathrm{id}}$.

Also, for each $\pi \in R(X)$ we may consider the closed positive current $T_\pi = c_1\left(\pi^*\mathcal{L},\pi^*h\right)$ on X_π, where π^*h denotes the pullback metric whose local potentials are defined by pulling back the local psh potentials of h. Then the divisor part of T_π agrees with $-Z(\mathcal{L},h)_\pi$.

We see that the b-divisor $D(\mathcal{L},h,s)$ encodes the divisor parts of the Siu decomposition of the pullbacks of T on all models $\pi \in R(X)$, and hence it also encodes in some sense the parts of higher codimension of the singular locus of the metric.

Example 5.7. Let h and h' be two psh metrics on $\mathcal{L}$ with Zariski unbounded locus. Let s be a non-zero rational section of $\mathcal{L}$ and denote by $D(\mathcal{L},h,s)$ and $D(\mathcal{L},h',s)$ the associated $\mathbb{R}$-b-divisors on $R(X)$. Fix a smooth reference metric h_0 on $\mathcal{L}$ and write $\varphi = -\log\left(h(s)/h_0(s)\right)$ and $\varphi' = -\log\left(h'(s)/h_0(s)\right)$. Let $\theta = c_1(\mathcal{L},h_0)$. Then φ and φ' are θ-psh functions on X. Recall the notion of algebraic singularity type and the equivalence relation $\prec_{\mathcal{J}}$ from Definition 2.27. We have

$$D(\mathcal{L},h,s) \geq D(\mathcal{L},h',s) \quad \text{iff} \quad \varphi' \prec_{\mathcal{J}} \varphi. \tag{5.3}$$

This is just a rephrasing of Proposition 2.28.

If h is a psh metric on $\mathcal{L}$ with Zariski unbounded locus and $U \subset X$ is a dense Zariski open subset such that the local potentials of h are locally bounded on U, then, for any rational section s of $\mathcal{L}$, the b-divisor $D(\mathcal{L}, s, h)$ only depends on the restriction of $(\mathcal{L}, s, h)$ to U. More concretely,

Proposition 5.8. *Let $U \subset X$ be a dense Zariski open subset and $\mathcal{L}$ a line bundle on U with a psh metric h. Let X_1 and X_2 be two compactifications of U and $\mathcal{L}_1$ and $\mathcal{L}_2$ line bundles on X_1 and X_2 together with isomorphisms $\mathcal{L}_1|_U \xrightarrow{\sim} \mathcal{L}^{\otimes e_1}$ and $\mathcal{L}_2|_U \xrightarrow{\sim} \mathcal{L}^{\otimes e_2}$ for some integers $e_1, e_2 > 0$. Assume moreover that the metric h extends to singular psh metrics h_1 and h_2 on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Let s be a non-zero rational section of $\mathcal{L}$, so $s^{\otimes e_i}$ is a non-zero rational section of $\mathcal{L}_i$. Then*

$$\frac{1}{e_1} D(\mathcal{L}_1, s^{\otimes e_1}, h_1) = \frac{1}{e_2} D(\mathcal{L}_2, s^{\otimes e_2}, h_2).$$

Proof. By considering a high enough model dominating both X_1 and X_2, and high enough powers of the $\mathcal{L}_i$, we may reduce to the case $X = X_1 = X_2$, $e_1 = e_2 = 1$. Then by symmetry we may further reduce to the case where $\mathcal{L}_1 = \mathcal{L}_2 \otimes \mathcal{O}(D)$ with D an effective divisor such that $\operatorname{supp}(D) \subset X \setminus U$. Let P be a prime divisor on any modification X' of X; we need to show that

$$\operatorname{ord}_P(s, \mathcal{L}_1) - \nu(P, h_1, \mathcal{L}_1) = \operatorname{ord}_P(s, \mathcal{L}_2) - \nu(P, h_2, \mathcal{L}_2). \tag{5.4}$$

Let $r = \operatorname{coeff}_P(D)$. Then

$$\operatorname{ord}_P(s, \mathcal{L}_1) - \operatorname{ord}_P(s, \mathcal{L}_2) = r. \tag{5.5}$$

Let g be a local equation of P and for $i = 1, 2$, let s_i be local invertible sections of the $\mathcal{L}_i$'s at P. We can write $s_2 = s_1 \cdot g^r$. Hence

$$\log \|s_2\| = \log \|s_1\| + r \log \|s_1\|$$

which implies that

$$r = \nu(P, h_1, \mathcal{L}_1) - \nu(P, h_2, \mathcal{L}_2). \tag{5.6}$$

Putting (5.5) and (5.6) together yields (5.4). □

5.2 b-divisors coming from psh metrics are approximable nef

In this section we show that the $\mathbb{R}$-b-divisors associated to psh metrics with Zariski unbounded locus are approximable nef (see Definition 4.8).

As before let X be a projective complex manifold of dimension n, let $\mathcal{L}$ on X be a line bundle, and let h be a psh metric on $\mathcal{L}$. We choose a canonical divisor K_X on X, a very ample line bundle B on X, and a smooth hermitian metric g on B. Let $\omega \coloneqq c_1(B, g)$ and assume it is a Kähler form on X. We essentially globalize some of the arguments found in [11, Sections 5.1 and 5.2].

We start with some preparatory results (the first of which only needs X to be a compact Kähler manifold).

Theorem 5.9 (Analytic Nadel Vanishing, cf. [32, Theorem 9.4.21]). *Assume that* $c_1(\mathcal{L}, h) \geq \epsilon\omega$ *for some* $\epsilon > 0$. *Then*

$$H^i(X, \mathcal{O}_X(K_X) \otimes \mathcal{L} \otimes \mathcal{J}(h)) = 0 \text{ for } i > 0.$$

Theorem 5.10 ([34, Lecture 14]). *Let* $\mathcal{F}$ *be a coherent sheaf on* X *such that*

$$H^i(X, \mathcal{F} \otimes B^{\otimes(k-i)}) = 0 \text{ for all } i > 0 \text{ and } k \geq 0.$$

Then $\mathcal{F}$ *is globally generated.*

Corollary 5.11. *Assume that* $c_1(\mathcal{L}, h) \geq \epsilon\omega$ *for some* $\epsilon > 0$. *Then*

$$\mathcal{O}(K_X) \otimes \mathcal{L} \otimes \mathcal{J}(h) \otimes B^{\otimes n}$$

is globally generated.

Proof. Let $\mathcal{F} = \mathcal{O}(K_X) \otimes \mathcal{L} \otimes \mathcal{J}(h) \otimes B^{\otimes n}$. Then $\mathcal{F}$ is coherent. By Theorem 5.10 it suffices to show that

$$H^i(X, \mathcal{O}(K_X) \otimes \mathcal{L} \otimes \mathcal{J}(h) \otimes B^{\otimes n} \otimes B^{\otimes(k-i)}) = 0 \text{ for } i > 0 \text{ and } k \geq 0.$$

For $i > n$ this follows from the fact that $\dim X = n$, and if $i \leq n$ the vanishing follows from Theorem 5.9 applied to $\mathcal{L} \otimes B^{\otimes(n+k-i)}$. □

Definition 5.12. Let $\mathcal{J}$ be a non-zero coherent sheaf of ideals on X, with its canonical rational section 1. If X_π is a model on which the inverse image ideal sheaf $\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{X_\pi}$ of $\mathcal{J}$ is invertible, then the pullback of 1 in $\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{X_\pi}$ defines a divisor on X_π. This determines an anti-effective Cartier b-divisor on X, independent of the choice of X_π, which we denote by $Z(\mathcal{J})$.

Lemma 5.13. *In the notation of Definition 5.12, suppose that* $\mathcal{L} \otimes \mathcal{J}$ *is globally generated. Then the line bundle* $\pi^*\mathcal{L} \otimes \left(\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{X_\pi}\right)$ *on* X_π *is globally generated, in particular, it is nef.*

Proof. The sheaf $\pi^*(\mathcal{L} \otimes \mathcal{J})$ is globally generated, and the canonical epimorphism $\pi^*(\mathcal{L} \otimes \mathcal{J}) \to \pi^*\mathcal{L} \otimes \left(\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{X_\pi}\right)$ shows that the latter is globally generated. It is therefore the pullback of $\mathcal{O}(1)$ under a morphism to projective space, and so it is nef. □

Definition 5.14. If $\pi\colon X_\pi \to X$ is a model in $R(X)$, then the relative dualising sheaf is *canonically* trivial over the locus where the map π is an isomorphism, and so comes with a canonical rational section, which is in fact a regular section, and defines an effective relative canonical *divisor* K_π. If $\mu\colon X_\pi \to X_{\pi'}$ is a map of models then $K_\pi = \mu^*K_{\pi'} + K_\mu$ (see also [11, Section 3.1]), so the K_π assemble to an effective $\mathbb{R}$-b-divisor on X.

We mention that K_π can equivalently be computed by taking the jacobian ideal sheaf $\mathrm{Jac}(\pi)$.

Lemma 5.15. *Assume that the psh metric h has Zariski unbounded locus, and let D be a reduced effective divisor on X outside of which $(\mathcal{L}, h)$ has bounded local potentials. For $\pi \in R(X)$, we let $\pi^* D_{\text{red}}$ denote the reduced divisor with same support as $\pi^* D$. Let $\mathbb{A} = (A_\pi)_\pi$ be the $\mathbb{R}$-b-divisor given on each model $\pi \in R(X)$ by*

$$A_\pi = \pi^* D_{\text{red}} + K_\pi.$$

Then we have the following inequalities of $\mathbb{R}$-b-divisors

$$Z(\mathcal{J}(h)) - \mathbb{A} \le Z(\mathcal{L}, h) \le Z(\mathcal{J}(h)). \tag{5.7}$$

We refer to Definition 2.15 for the definition of the multiplier ideal sheaf $\mathcal{J}(h)$ associated to the metric h.

Proof. We follow the idea of the proof of [11, Theorem 5.1]. Let $\pi \in R(X)$ and let U be a small ball in X_π, which we identify with a small ball centered at the origin in $\mathbb{C}^n$. We assume that $\pi^*\mathcal{L}$ admits a generating section ξ over U, and we let $\varphi = -\log h(\xi)$, which is then a psh function on U. We need to check the inequalities of divisors

$$Z(\mathcal{J}(\varphi)) - A_\pi|_U \le Z(\varphi) \le Z(\mathcal{J}(\varphi))$$

on $U \subset X_\pi$.
The inequality $Z(\varphi) \le Z(\mathcal{J}(\varphi))$ is standard and follows from the Ohsawa-Takegoshi extension theorem (e.g. see the proof of the first inequality in part a) of [22, Theorem 13.2]).
It thus remains to show the inequality $Z(\mathcal{J}(\varphi)) - A_\pi|_U \le Z(\varphi)$. We first check this result 'before blowing up'; in other words, we let P be a prime divisor of X. If P is not contained in the support of D then both sides of the inequality vanish at P and we are done. So assume that P is contained in the support of D, so $\operatorname{ord}_P \mathbb{A} = 1$. Then $\mathcal{J}(\varphi)$ is principal at the generic point of P, say generated by f. Then

$$\operatorname{ord}_P Z(\mathcal{J}(\varphi)) = -\operatorname{ord}_P f = -\nu(\log|f|, p)\,,$$

where p is a generic point of P. To prove the claim it suffices then to show that

$$\nu(\log|f|, p) \ge \nu(\varphi, p) - 1.$$

This comes down to showing that $\varphi(z) \le c\log|z| + O(1)$ implies that we have $c \le 1 + \operatorname{ord}_p(f)$. Hence assume that $\varphi(z) \le c\log|z| + O(1)$. Then $e^{2\varphi} \ll |z|^{2c}$ and thus $|f|^2 e^{-2\varphi} \gg |z|^{2\operatorname{ord}_p f}|z|^{-2c}$. Since $|f|^2 e^{-2\varphi}$ is assumed to be integrable it follows that $|z|^{2\operatorname{ord}_p(f)-2c}$ is integrable. Hence $2\operatorname{ord}_p(f) - 2c \ge -2$ as required.
It remains to treat the case where P is an exceptional prime divisor on some X_π. We denote by $\operatorname{Jac}(\pi)$ the jacobian ideal sheaf of π, and recall that the relative canonical divisor K_π is the divisor cut out by $\operatorname{Jac}(\pi)$. On a small disc in X_π centered at a generic point p of P, write j for a generator of $\operatorname{Jac}(\pi)$. Then if f

on X is a function such that $|f|^2 e^{-2\phi}$ is locally integrable on X we have that $|f|^2 e^{-2\phi} |j|^2$ is locally integrable on X_π. Hence we see that

$$\operatorname{ord}_P Z(\mathcal{J}(\phi \circ \pi)) \leq \operatorname{ord}_P Z(\mathcal{J}(\phi)) + \operatorname{ord}_P K_\pi.$$

Using this inequality, the claim follows as in the case where P was a prime divisor on X. $\square$

Note that the $\mathbb{R}$-b-divisor $\mathbb{A}$ only depends on the divisor D on which h has singularities; in particular, if we replace $(\mathcal{L}, h)$ by some positive power, then $\mathbb{A}$ need not change. Hence, from the previous lemma we obtain the following corollary.

Corollary 5.16.

$$Z(\mathcal{L}, h) = \lim_{k \to \infty} \frac{1}{k} Z\left(\mathcal{J}(h^{\otimes k})\right).$$

Remark 5.17. Stated in a different language Corollary 5.16 can also be found in [18, Proposition 2.14], with essentially the same proof.

We are now ready to state and prove the main result of this section.

Theorem 5.18. *Let X be a projective complex manifold of dimension n, let $\mathcal{L}$ on X be a line bundle, and let h be a psh metric on $\mathcal{L}$ with Zariski unbounded locus. Let s be a non-zero rational section of $\mathcal{L}$. Then the associated $\mathbb{R}$-b-divisor $D(\mathcal{L}, h, s) = \text{b-div}(s) + Z(\mathcal{L}, h)$ is approximable nef.*

In particular, by what was said in Section 4.3 the $\mathbb{R}$-b-divisor $D(\mathcal{L}, h, s)$ has a well-defined degree $D(\mathcal{L}, h, s)^n \in \mathbb{R}_{\geq 0}$.

Proof. We choose a canonical divisor K_X on X, a very ample line bundle B on X, and a smooth hermitian metric g on B. Let $\omega := c_1(B, g)$ which we assume to be a Kähler form on X. Let $k \in \mathbb{Z}_{>0}$. By Corollary 5.11 applied to $\mathcal{L}^{\otimes k} \otimes B$, using that $c_1(\mathcal{L}^{\otimes k} \otimes B) \geq \omega$ and that $\mathcal{J}(h^k) = \mathcal{J}(h^k g)$, we see that the sheaf on X given by

$$\mathcal{O}(K_X) \otimes \mathcal{L}^{\otimes k} \otimes \mathcal{J}(h^k) \otimes B^{\otimes n+1} \tag{5.8}$$

is globally generated. Choose a non-zero rational section b of B, determining a Cartier $\mathbb{R}$-b-divisor $\operatorname{div}(b)$ on X. We also write K_X for the Cartier $\mathbb{R}$-b-divisor determined by K_X. Recall that every $Z(\mathcal{J}(h^{\otimes k}))$ is Cartier. By Lemma 5.13 and the fact that (5.8) is globally generated, the Cartier $\mathbb{R}$-b-divisor given by

$$K_X + k \operatorname{div}(s) + Z(\mathcal{J}(h^k)) + (n+1) \operatorname{div}(b)$$

is nef. Dividing by k we find

$$\frac{1}{k} K_X + \operatorname{div}(s) + \frac{1}{k} Z(\mathcal{J}(h^k)) + \frac{n+1}{k} \operatorname{div}(b) \tag{5.9}$$

is again a nef Cartier $\mathbb{R}$-b-divisor. Let $C := K_X + (n+1)\operatorname{div}(b)$, a divisor on X. Write $C = E - P$ where E is effective and P is nef. Then for each k we have

$$\frac{1}{k}K_X + \operatorname{div}(s) + \frac{1}{k}Z(\mathcal{J}(h^k)) + \frac{n+1}{k}\operatorname{div}(b) + \frac{P}{k} = \operatorname{div}(s) + \frac{1}{k}Z(\mathcal{J}(h^k)) + \frac{E}{k}.$$

Hence we see that

$$\operatorname{div}(s) + \frac{1}{k}Z(\mathcal{J}(h^k)) + \frac{E}{k}$$

is a nef $\mathbb{R}$-b-Cartier divisor, as it is a sum of such. Since E is effective, the sequence E/k is decreasing as $k \to \infty$. Next, we have that a subsequence of $Z(\mathcal{J}(h^k))/k$ is decreasing. Indeed, this follows from the fact that for all k, we have

$$\frac{1}{k}Z(\mathcal{J}(h^k)) \geq \frac{1}{2k}Z(\mathcal{J}(h^{2k})),$$

which follows from the additivity property stated in Lemma 5.19 below. The proof is then concluded by Corollary 5.16. □

Lemma 5.19 ([22, Theorem 14.2]). *Let φ_1 and φ_2 be psh functions. Then $\mathcal{J}(\varphi_1 + \varphi_2) \subseteq \mathcal{J}(\varphi_1)\mathcal{J}(\varphi_2)$.*

5.3 A Chern–Weil type result for psh metrics with almost asymptotically algebraic singularities

Let X be a projective complex manifold of dimension n and let $\mathcal{L}$ be a line bundle on X. Let h be a psh metric on $\mathcal{L}$, let s be a non-zero rational section of $\mathcal{L}$ and let $D(\mathcal{L}, h, s)$ be the associated $\mathbb{R}$-b-divisor on X as in Definition 5.4. We fix a smooth reference metric h_0 on $\mathcal{L}$ and write $\theta = c_1(\mathcal{L}, h_0)$.

Theorem 5.20. *Assume that the psh metric h has almost asymptotically algebraic singularities with respect to θ. Then the equality*

$$D(\mathcal{L}, h, s)^n = \int_X \langle c_1(\mathcal{L}, h)^n \rangle$$

holds in $\mathbb{R}_{\geq 0}$. Here $\langle\cdot\rangle$ denotes the non-pluripolar product of closed positive $(1,1)$-currents, and $D(\mathcal{L}, h, s)^n$ is the degree of the approximable nef $\mathbb{R}$-b-divisor $D(\mathcal{L}, h, s)$.

Proof. Write $\mathbb{D} = D(\mathcal{L}, h, s)$ and $T = c_1(\mathcal{L}, h)$. If h has algebraic singularities then the result follows from combining Lemma 2.35 and Example 5.5.
Assume now that h has almost asymptotically algebraic singularities. Let B be a very ample line bundle on X, g a Kähler metric on B, and set $\omega = c_1(B, g)$ and $\varphi = -\log(h(s)/h_0(s))$. By Lemma 3.5 we may choose an ω-psh function f with algebraic singularities on X, and for every integer $m > 0$ an $(\frac{1}{m}\omega + \theta)$-psh function φ_m with algebraic singularities on X such that

$$\varphi_m + \frac{1}{m}f \prec \varphi \prec \varphi_m\,.$$

Each φ_m defines a psh metric h_m with algebraic singularities on $\mathcal{L} \otimes B^{\otimes \frac{1}{m}}$, and each $\varphi_m + \frac{1}{m} f$ defines a psh metric h'_m with algebraic singularities on $\mathcal{L} \otimes B^{\otimes \frac{2}{m}}$. Choose a non-zero rational section t of B. Since the Lelong numbers of φ_m converge to the Lelong numbers of φ, the $D(\mathcal{L} \otimes B^{\otimes \frac{1}{m}}, h_m, st^{\frac{1}{m}})$ converge to $\mathbb{D}$, hence the $D(\mathcal{L} \otimes B^{\otimes \frac{1}{m}}, h_m, st^{\frac{1}{m}})^n$ converge to $\mathbb{D}^n$. Similarly the $D(\mathcal{L} \otimes B^{\otimes \frac{2}{m}}, h'_m, st^{\frac{2}{m}})^n$ converge to $\mathbb{D}^n$. The case of algebraic singularities yields that

$$\int_X \langle (\theta + \frac{1}{m}\omega + dd^c \varphi_m)^n \rangle = D(\mathcal{L} \otimes B^{\otimes \frac{1}{m}}, h_m, st^{\frac{1}{m}})^n$$

and

$$\int_X \langle (\theta + \frac{2}{m}\omega + dd^c (\varphi_m + \frac{1}{m} f))^n \rangle = D(\mathcal{L} \otimes B^{\otimes \frac{2}{m}}, h'_m, st^{\frac{2}{m}})^n$$

for each m. Since the θ-psh function φ is also $\frac{1}{m}\omega + \theta$-psh, we can apply monotonicity of non-pluripolar products (Theorem 2.26) to deduce

$$\int_X \langle (T + \frac{1}{m}\omega)^n \rangle = \int_X \langle (\theta + \frac{1}{m}\omega + dd^c \varphi)^n \rangle \leq \int_X \langle (\theta + \frac{1}{m}\omega + dd^c \varphi_m)^n \rangle$$

for each m, and similarly

$$\int_X \langle (\theta + \frac{2}{m}\omega + dd^c (\varphi_m + \frac{1}{m} f))^n \rangle \leq \int_X \langle (\theta + \frac{2}{m}\omega + dd^c \varphi)^n \rangle = \int_X \langle (T + \frac{2}{m}\omega)^n \rangle.$$

Now by the multi-linearity of the non-pluripolar product as expressed in Proposition 2.25 we see

$$\int_X \langle (T + \frac{1}{m}\omega) \rangle^n = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{m^i} \int_X \langle T^{n-i} \omega^i \rangle, \tag{5.10}$$

so that

$$\begin{aligned} \int_X \langle T^n \rangle &= \lim_{m\to\infty} \int_X \langle (T + \frac{1}{m}\omega) \rangle^n \leq \lim_{m\to\infty} \int_X \langle (\theta + \frac{1}{m}\omega + dd^c \varphi_m)^n \rangle \\ &= \lim_{m\to\infty} D(\mathcal{L} \otimes B^{\otimes \frac{1}{m}}, h_m, st^{\frac{1}{m}})^n = \mathbb{D}^n \,. \end{aligned} \tag{5.11}$$

Similarly

$$\begin{aligned} \mathbb{D}^n = \lim_{m\to\infty} \mathbb{D}(\mathcal{L} \otimes B^{\otimes \frac{2}{m}}, h'_m, st^{\frac{2}{m}})^n &= \lim_{m\to\infty} \int_X \langle (\theta + \frac{2}{m}\omega + dd^c (\varphi_m + \frac{1}{m} f))^n \rangle \\ &\leq \lim_{m\to\infty} \int_X \langle (T + \frac{2}{m}\omega)^n \rangle = \int_X \langle T^n \rangle. \end{aligned}$$

□

Example 5.21. (Chern–Weil for good and psh metrics) Suppose that h is, at the same time, good and psh as in Example 2.34. Then h has almost asymptotically algebraic singularities by Lemma 3.3. Since all the Lelong numbers at all points on all modifications of X are zero, we have that $D(\mathcal{L}, h, s) = \mathrm{div}(s)$, seen as a Cartier b-divisor. Following the computations in Example 2.34, we have that

$$D(\mathcal{L}, h, s)^n = \mathrm{div}_X(s)^n = \deg(\mathcal{L}) = \int_X \langle c_1(\mathcal{L}, h)^n \rangle \,,$$

and this verifies the Theorem in the case of good and psh metric.

Remark 5.22. Let h be a psh metric on $\mathcal{L}$ with Zariski unbounded locus. We do not assume that h has almost asymptotically algebraic singularities. Then we still have the inequality

$$D(\mathcal{L}, h, s)^n \geq \int_X \langle c_1(\mathcal{L}, h)^n \rangle.$$

Indeed, consider a Demailly approximation sequence $\{\varphi_m\}_{m\in\mathbb{N}}$ for φ. Then $\varphi \prec \varphi_m$ and in the same way as in the proof of Theorem 5.20, see in particular equation (5.11), the inequality follows.

Combining Theorem 3.13 and Theorem 5.20 we obtain the following b-divisorial analogue of the classical Hilbert–Samuel type statement for nef line bundles.

Corollary 5.23. *Let assumptions be as in Theorem 5.20. Then the equality*

$$D(\mathcal{L}, h, s)^n = \mathrm{vol}_{\mathcal{J}}(\mathcal{L}, h)$$

holds in $\mathbb{R}_{\geq 0}$.

6 The line bundle of Siegel–Jacobi forms

The purpose of this section is to exhibit an application of our results in the context of the line bundle of Siegel–Jacobi forms on the universal abelian variety over the fine moduli space $A_{g,N}$ of principally polarized complex abelian varieties of dimension g with level N structure. The results in this section form a generalization of the main results of [15].

6.1 The biextension metric on the Poincaré bundle

We start by showing that the Poincaré bundle on an abelian scheme has a natural psh extension over any smooth toroidal compactification of the abelian scheme.

Let S be a smooth complex algebraic variety, and let $\pi\colon U \to S$ be an abelian scheme with zero section $e\colon S \to U$. That is, the morphism π is proper and smooth, and the fibers of π are abelian varieties with origin determined by the section e. We have a tautological Poincaré line bundle $\mathcal{P}$ on the fiber product $U \times_S U^\vee$, where $\pi^\vee\colon U^\vee \to S$ denotes the dual abelian scheme. The Poincaré bundle $\mathcal{P}$ comes equipped with a rigidification along the zero section and with a canonical smooth hermitian metric h_0 as explained in [33, Section 3]. When $\lambda\colon U \to U^\vee$ is a polarization of abelian schemes over S, we define $\mathcal{B}_\lambda$ to be

the line bundle $(\mathrm{Id}, \lambda)^*\mathcal{P}$ on U, equipped with the pullback metric, which we denote by h_λ.
The line bundle $\mathcal{B}_\lambda$ is an example of a *biextension line bundle* associated to a polarized variation of pure Hodge structures of weight -1, as discussed in [28, Sections 6 and 7]. In our case, the underlying variation of pure Hodge structures of weight -1 is given by the local system $\mathcal{H} = R^1\pi_*\mathbb{Z}_U(1)$, and the polarization of the variation $\mathcal{H}$ is induced by the polarization λ.
As a special case of general results such as [27, Theorem 13.1] or [37, Theorem 8.2], or alternatively by a computation in local coordinates using the explicit formulas for h_0 in [14, Section 2], we have that the metric h_λ on the biextension line bundle $\mathcal{B}_\lambda$ is *semipositive*.
Let $X \supset U$ be a smooth projective compactification of U, and assume that $D = X \setminus U$ is a normal crossings divisor on X. We will assume throughout that the pullback of the variation of pure Hodge structure $\mathcal{H}$ from S to U has unipotent monodromy around each local branch of D.
Let D^{sing} denote the singular locus of D. As a special case of [13, Theorems 24 and 27] we have the following result.

Theorem 6.1. *There exists a positive integer e such that:*

(i) The semipositive line bundle $\mathcal{B}_\lambda^{\otimes e}$ on U has a unique extension $\widetilde{\mathcal{B}_\lambda^{\otimes e}}$ as a continuously metrized line bundle over the locus $X \setminus D^{\mathrm{sing}}$.

(ii) The continuously metrized line bundle $\widetilde{\mathcal{B}_\lambda^{\otimes e}}$ on $X \setminus D^{\mathrm{sing}}$ has a unique extension as a line bundle with a psh metric $\overline{\mathcal{B}_\lambda^{\otimes e}}$ over X.

Remark 6.2. To avoid carrying the exponent e around in what follows, we can think of $\left(\overline{\mathcal{B}_\lambda^{\otimes e}}\right)^{\otimes \frac{1}{e}}$ as an extension of $\mathcal{B}_\lambda$ as a metrised $\mathbb{Q}$-line bundle over X, and we simply denote it $\overline{\mathcal{B}}_\lambda$ (which is independent of the choice of e). We say a metric on a $\mathbb{Q}$-line bundle $\mathcal{L}$ on a projective complex manifold is psh (or toroidal, or has almost asymptotically algebraic singularities, ...) if some positive tensor power of $\mathcal{L}$ which is a line bundle has this property.

Following terminology introduced in [28] we call the $\mathbb{Q}$-line bundle $\overline{\mathcal{B}}_\lambda$ the *Lear extension* of $\mathcal{B}_\lambda$ over X. We will denote by $\overline{h}_\lambda$ the natural psh metric on $\overline{\mathcal{B}}_\lambda$ which is given by Theorem 6.1. Note that the singularities of the psh metric $\overline{h}_\lambda$ are contained in the codimension two locus D^{sing} of X.
The main result of this section is as follows.

Theorem 6.3. *The $\mathbb{Q}$-line bundle $\overline{\mathcal{B}}_\lambda$ with psh metric $\overline{h}_\lambda$ on X has toroidal singularities in the sense of Definition 3.10. In particular, the psh metric $\overline{h}_\lambda$ has almost asymptotically algebraic singularities.*

Note that the second part of the theorem follows from the first by Proposition 3.11.
We start with two lemmas.

Lemma 6.4. *Let $V \subseteq \mathbb{R}^k$ be any subset, and let $g\colon V \to \mathbb{R}$ be a Lipschitz continuous function with Lipschitz constant c. Then g can be extended to a Lipschitz continuous function $\mathbb{R}^k \to \mathbb{R}$ with constant c.*

Proof. One extension is given by $y \mapsto \inf_{x\in V}(g(x) + c|y - x|)$. □

Lemma 6.5. *Let $r \in \mathbb{Z}_{>0}$. Let $A_1, \dots, A_k$ be positive semi-definite $r \times r$-matrices such that for all $x_1, \dots, x_k \in \mathbb{R}_{>0}$ we have $\sum_{i=1}^k x_iA_i$ of full rank. Let $c_1, \dots, c_k \in \mathbb{R}^r$ be column vectors. Then the smooth function*

$$g = \left(\sum_{i=1}^k x_iA_ic_i\right)^t \cdot \left(\sum_{i=1}^k x_iA_i\right)^{-1} \cdot \left(\sum_{i=1}^k x_iA_ic_i\right) \tag{6.1}$$

on $\mathbb{R}^k_{>0}$ has a unique continuous extension to the whole of $\mathbb{R}^k_{\geq 0}$, and the extension is Lipschitz continuous.

Proof. Define $P = \{(x_1, \dots, x_k) \in \mathbb{R}^k_{>0} | x_1 < x_2 < \cdots < x_k\}$. Then the union of the translates of P by the action of the symmetric group $\mathfrak{S}_k$ is dense in $\mathbb{R}^k_{>0}$; by symmetry and Lemma 6.4 it suffices to show that g is Lipschitz continuous on P. Writing $y_1 = x_1$, $y_i = x_i - x_{i-1}$ for $i = 2, \dots, k$ we find that $x_i = \sum_{j=1}^i y_j$ and that P is parametrized by $y_1 > 0, \dots, y_k > 0$. Note that

$$\sum_{i=1}^k x_iA_i = \sum_{i=1}^k y_i \sum_{j=i}^k A_j \quad \text{and} \quad \sum_{i=1}^k x_iA_ic_i = \sum_{i=1}^k y_i \sum_{j=i}^k A_jc_j.$$

By [14, Lemma 3.5] we have, writing $\tilde{A}_i = \sum_{j=i}^k A_j$, that the *flag condition* $\operatorname{Ker} \tilde{A}_i \subseteq \operatorname{Ker} \tilde{A}_{i+1}$ holds for $i = 1, \dots, k-1$. Moreover we have $\operatorname{Im}(\tilde{A}_i) = \sum_{j=i}^k \operatorname{Im}(A_j)$. It follows that there exist vectors $\tilde{c}_i \in \mathbb{R}^r$ such that

$$\sum_{j=i}^k A_jc_j = \tilde{A}_i\tilde{c}_i\,.$$

Replacing A_i by $\tilde{A}_i$, x_i by y_i and c_i by $\tilde{c}_i$ we are reduced to proving the Lipschitz continuity of g on $\mathbb{R}^k_{>0}$ under the extra hypothesis that the matrices $A_1, \dots, A_k$ satisfy the above introduced flag condition. We do this by showing that the partial derivatives of the smooth function g are bounded on $\mathbb{R}^k_{>0}$. To shorten the notation we set $z = \sum_{i=1}^k x_iA_ic_i$ and $\Omega = \sum_{i=1}^k x_iA_i$. Then $g = z^t\Omega^{-1}z$ and a small computation shows

$$\frac{\partial g}{\partial x_i} = 2(A_ic_i)^t\Omega^{-1}z - z^t\Omega^{-1}A_i\Omega^{-1}z\,. \tag{6.2}$$

We see that it suffices to show that $\Omega^{-1}z$ is bounded on $\mathbb{R}^k_{>0}$. Write $r_j = \operatorname{rk}(A_j)$ for $j = 1, \dots, k$. We have $r = r_1 \geq \cdots \geq r_k \geq 1$ by the flag condition. Now

[14, Lemma 3.6] states the existence of a constant c_1 such that for all integers $1 \leq \alpha, \beta \leq r$ we have

$$\left(\Omega^{-1}\right)_{\alpha,\beta} \leq \frac{c_1}{\sum_{j:r_j \geq \min(\alpha,\beta)} x_j} \leq \frac{c_1}{x_1} . \tag{6.3}$$

Hence there exists a constant c_2 such that

$$\left(\Omega^{-1}\right)_{\alpha,\beta} z_\beta \leq \frac{c_2 \sum_{j:r_j \geq \beta} x_j}{\sum_{j:r_j \geq \min(\alpha,\beta)} x_j} \leq c_2 , \tag{6.4}$$

showing that $\Omega^{-1} z$ is bounded on $\mathbb{R}^k_{>0}$. □

Proof of Theorem 6.3. Let $D(\epsilon)$ denote the open disk in $\mathbb{C}$ with radius ϵ. Write $n = \dim X$. Choose a point $p \in D$. By [13, Theorem 81] there exist a small positive number ϵ, an open neighborhood V of p in X, a coordinate chart $V \xrightarrow{\sim} D(\epsilon)^n$ with $p \mapsto 0$, and a generating section s of $\mathcal{B}_\lambda$ over $U \cap V$. Assume that $D \cap V$ is given by the equation $z_1 \cdots z_k = 0$. Our task is to show that the psh function $-\log h_\lambda(s)$ can be written on $U \cap V$ as a sum

$$-\log h_\lambda(s) = \gamma + g(-\log|z_1|, \ldots, -\log|z_k|) , \tag{6.5}$$

where γ is locally bounded on V and g is a convex Lipschitz continuous function defined on some quadrant $\{x_j \geq M_j \mid j = 1, \ldots, k\} \subseteq \mathbb{R}^k$.
For this we invoke [14, Theorem 1.1] and its proof in [14, Section 4]; these give us the expansion (6.5) for $-\log h_\lambda(s)$ with γ locally bounded on V and with g convex on $\mathbb{R}^k_{>0}$ and given as

$$g = \left(\sum_{i=1}^k x_i A_i c_i\right)^t \cdot \left(\sum_{i=1}^k x_i A_i\right)^{-1} \cdot \left(\sum_{i=1}^k x_i A_i c_i\right) \tag{6.6}$$

up to a linear form in the x_i. Here $A_1, \ldots, A_k$ are positive semi-definite $r \times r$-matrices for some $r \in \mathbb{Z}_{>0}$ such that for all $x_1, \ldots, x_k \in \mathbb{R}_{>0}$ we have $\sum_{i=1}^k A_i x_i$ of full rank and where $c_1, \ldots, c_k \in \mathbb{R}^r$. Then by Lemma 6.5 the function g is Lipschitz continuous on any closed quadrant contained in $\mathbb{R}^k_{>0}$.

□

Note that the psh metric $\overline{h}_\lambda$ has its unbounded locus supported on the boundary divisor $D = X \setminus U$. Let s be a non-zero rational section of the line bundle $\mathcal{B}_\lambda$. We have a natural associated $\mathbb{R}$-b-divisor $D(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda, s)$ on the category $R(X)$. We note that $D(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda, s)$ is actually a $\mathbb{Q}$-b-divisor, since on each model in $R(X)$ its incarnation is given by the Lear extension of $\mathcal{B}_\lambda$, which is a $\mathbb{Q}$-line bundle. By Proposition 5.8, the b-divisor $D(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda, s)$ is independent of the choice of the chosen compactification X of U.
Let $\mathrm{vol}_{\mathcal{J}}(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda)$ denote the multiplier ideal volume of the psh line bundle $(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda)$ on X. By combining Theorem 6.3 with the Hilbert–Samuel formula in Theorem 3.13 and the Chern-Weil formula in Theorem 5.20 we find the following result.

Theorem 6.6. *Let $n = \dim U$. The equalities*

$$\mathrm{vol}_{\mathcal{J}}(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda) = \big(D(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda, s)\big)^n = \int_X \langle c_1(\overline{\mathcal{B}}_\lambda, \overline{h}_\lambda)^n \rangle = \int_U c_1(\mathcal{B}_\lambda, h_\lambda)^n$$

hold in $\mathbb{R}_{\geq 0}$.

6.2 The line bundle of Siegel–Jacobi forms and its invariant metric

The aim of this final section is to discuss a variant of Theorem 6.6 in the context of Siegel–Jacobi forms. Let $g \in \mathbb{Z}_{\geq 1}$ and $N \in \mathbb{Z}_{\geq 3}$ and let $A_{g,N}$ denote the fine moduli space of principally polarized complex abelian varieties of dimension g and level N. This is a smooth quasi-projective complex algebraic variety of dimension $g(g+1)/2$. Let $\pi\colon U_{g,N} \to A_{g,N}$ be the universal family of abelian varieties. Following the constructions in Section 6.1, the tautological polarization on $U_{g,N}$ gives rise to a canonical biextension line bundle $\mathcal{B}$ on $U_{g,N}$ equipped with a smooth semipositive hermitian metric h.

We write $\mathcal{H}$ for the tautological polarized variation of pure Hodge structure $R^1\pi_*\mathbb{Z}_{U_{g,N}}(1)$ of weight -1 on $A_{g,N}$. We denote by $\mathcal{F} = \mathcal{F}^0(\mathcal{H}_{\mathbb{C}} \otimes \mathcal{O}_S)$ the associated Hodge bundle on $A_{g,N}$, and write $\mathcal{M} = \bigwedge^g \mathcal{F}$ for its determinant. The tautological polarization of $\mathcal{H}$ induces a smooth hermitian metric h^{inv} on $\mathcal{M}$ called the *invariant metric* or *Hodge metric*. By [40, Lemmas 7.18 and 7.19], the metric h^{inv} is semi-positive.

Definition 6.7. Let $k, m \in \mathbb{Z}_{\geq 0}$. The line bundle

$$\mathcal{L}_{k,m} = \pi^*\mathcal{M}^{\otimes k} \otimes \mathcal{B}^{\otimes m}$$

on $U_{g,N}$ is called the *line bundle of Siegel–Jacobi forms of weight k and index m.*

Note that each line bundle $\mathcal{L}_{k,m}$ has a natural smooth hermitian metric $h_{k,m}$ obtained by taking appropriate tensor product combinations of the metrics h^{inv} on $\mathcal{M}$ and h on $\mathcal{B}$. As the metrics h^{inv} and h are semipositive, it is clear that for each $k, m \in \mathbb{Z}_{\geq 0}$ the smooth hermitian metric $h_{k,m}$ is a semipositive metric on $\mathcal{L}_{k,m}$.

The work done in [26, Chapter VI] and [35, Section 3] allows to choose:

- a projective smooth toroidal compactification $\overline{A}_{g,N}$ of $A_{g,N}$,
- an extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ over $\overline{A}_{g,N}$ as a line bundle,
- a projective smooth toroidal compactification $\overline{U}_{g,N}$ of $U_{g,N}$, and
- a map $\overline{\pi}\colon \overline{U}_{g,N} \to \overline{A}_{g,N}$ extending the projection map $\pi\colon U_{g,N} \to A_{g,N}$,

such that

- the metric h^{inv} extends as good psh metric $\overline{h}^{\mathrm{inv}}$ over $\overline{\mathcal{M}}$, and

- the pullback of the variation $\mathcal{H}$ from $A_{g,N}$ to $U_{g,N}$ has unipotent monodromy around each local branch of the normal crossings boundary divisor $D = \overline{U}_{g,N} \setminus U_{g,N}$.

As the psh metric $\overline{h}^{\mathrm{inv}}$ is good, it has in particular almost asymptotically algebraic singularities, see Lemma 3.3. Further it follows from Theorem 6.1 that the biextension line bundle $\mathcal{B}$ on $U_{g,N}$ has a Lear extension $\overline{\mathcal{B}}$ over $\overline{U}_{g,N}$, and from Theorem 6.3 that the metric h has a natural extension $\overline{h}$ over the $\mathbb{Q}$-line bundle $\overline{\mathcal{B}}$ as a psh metric with toroidal, and in particular almost asymptotically algebraic, singularities.

We define

$$\overline{\mathcal{L}}_{k,m} = \overline{\pi}^{*}\overline{\mathcal{M}}^{\otimes k} \otimes \overline{\mathcal{B}}^{\otimes m},$$

a $\mathbb{Q}$-line bundle on $\overline{U}_{g,N}$ extending the line bundle $\mathcal{L}_{k,m}$, and denote by $\overline{h}_{k,m}$ the metric on $\overline{\mathcal{L}}_{k,m}$ induced from $\overline{h}^{\mathrm{inv}}$ on $\overline{\mathcal{M}}$ and $\overline{h}$ on $\overline{\mathcal{B}}$ by taking the appropriate tensor product combinations.

As both $\overline{h}^{\mathrm{inv}}$ and $\overline{h}$ have almost asymptotically algebraic singularities, we conclude by Remark 3.4 that the natural metric $\overline{h}_{k,m}$ on the $\mathbb{Q}$-line bundle $\overline{\mathcal{L}}_{k,m}$ is a psh metric with almost asymptotically algebraic singularities. Further we note that the unbounded locus of the psh metric $\overline{h}_{k,m}$ has support contained in the boundary divisor $D = \overline{U}_{g,N} \setminus U_{g,N}$.

Let $g \in \mathbb{Z}_{\geq 1}$, $N \in \mathbb{Z}_{\geq 3}$, $k, m \in \mathbb{Z}_{\geq 0}$. Let s be a non-zero rational section of the line bundle $\mathcal{L}_{k,m}$ of Siegel–Jacobi forms on $U_{g,N}$ of weight k and index m. Let $D(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m}, s)$ be the $\mathbb{R}$-b-divisor associated to the Lear extension $\overline{\mathcal{L}}_{k,m}$ over $\overline{U}_{g,N}$, its canonical psh metric $\overline{h}_{k,m}$ and the rational section s. As follows from Proposition 5.8, the b-divisor $D(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m}, s)$ is independent of the choice of compactification $\overline{U}_{g,N}$.

Applying Theorem 3.13 and Theorem 5.20 we obtain the following result.

Theorem 6.8. *Let $n = \dim U_{g,N} = g + g(g+1)/2$. Let $\mathrm{vol}_{\mathcal{J}}(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m})$ denote the multiplier ideal volume of the psh line bundle $(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m})$. Then the equalities*

$$\begin{aligned}
\mathrm{vol}_{\mathcal{J}}(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m}) &= \big(D(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m}, s)\big)^{n} \\
&= \int_{\overline{U}_{g,N}} \langle c_1(\overline{\mathcal{L}}_{k,m}, \overline{h}_{k,m})^{n} \rangle \\
&= \int_{U_{g,N}} c_1(\mathcal{L}_{k,m}, h_{k,m})^{n}
\end{aligned}$$

hold in $\mathbb{R}_{\geq 0}$.

In the case of the modular curve $Y(N) = A_{1,N}$ and for $k = m = 4$ this reproduces the main result of [15]. We observe that in [15], volume, degree and integral were each calculated independently, and the outcomes were seen to be equal by inspection. Here we see a more intrinsic approach.

Remark 6.9. The above Chern–Weil type result can be extended, with proofs carrying over mutatis mutandis, to the more general setting of a finite self-product of the universal abelian scheme over any PEL type Shimura variety. The necessary smooth projective toroidal compactifications are provided in this setting by the work of Lan [31].

A On the non-continuity of the volume function

In Theorem 2.31 the multiplier ideal volume of a line bundle with a psh metric is defined. This volume function has nice properties for metrics with almost asymptotically algebraic singularities. For instance, it agrees with the non-pluripolar volume (Theorem 3.13) and it is continuous with respect to good algebraic approximations (see Remark 3.16 for the precise statement).

Another approach to define the volume of a line bundle with a psh metric, assuming it has Zariski unbounded locus, is to measure the abundance of global sections for multiples of the associated b-divisor. In fact, if the b-divisor is Cartier, this is the classical definition of the volume of a divisor. In this appendix we give the definition of the volume of a b-divisor that is the analogue of the volume function of a classical divisor. We can then ask whether this volume function is continuous with respect to the weak topology. The results of [6, §5] show that this question has an affirmative answer in the toroidal case. Nevertheless, we give an example showing that, in general, this volume function is not continuous for approximable nef b-divisors, even in the big case. We don't know if the constructed example can be realized as the b-divisor associated to a psh metric.

We start with some definitions. Let X be a smooth projective complex variety with function field F. Let $\mathbb{D} = (D_\pi)_{\pi \in R(X)}$ be a Weil $\mathbb{R}$-b-divisor on X as in Section 4. We define

$$\mathcal{L}(\mathbb{D}) = \{0 \neq f \in F \mid \forall \pi \in R(X) :\ D_\pi + \operatorname{div}(f) \geq 0\} \cup \{0\}\,.$$

The *volume* $\operatorname{vol}(\mathbb{D})$ of the b-divisor $\mathbb{D}$ is defined via the formula

$$\operatorname{vol}(\mathbb{D}) \coloneqq \limsup_{\ell} \frac{\dim \mathcal{L}(\ell\mathbb{D})}{\ell^d/d!}. \tag{A.1}$$

We say $\mathbb{D}$ is *big* if $\operatorname{vol}(\mathbb{D}) > 0$.

The example is constructed in two steps. The first one is a preparatory step.

Step 1:

Let X be a smooth projective surface, and A, B divisors on X meeting transversely at a point p and let $b \geq 1$ be an integer. Setting $X_0 = X$ and $p_0 = p$ we make the following recursive definition, which is illustrated in Figure 1:

1. For $0 \leq i < b$, let X_{i+1} be the blow up of X_i at p_i, and E_{i+1} the exceptional locus of the blowup;

2. For any $i \in \{1, \dots, b\}$ and any divisor F on X_j for $0 \leq j \leq i$, we write $\widehat{F}$ for the strict transform of F on X_i (note that $F = \widehat{F}$ if $i = j$).

3. Let p_{i+1} be the unique point of intersection of $\widehat{B}$ with E_{i+1} on X_{i+1}.

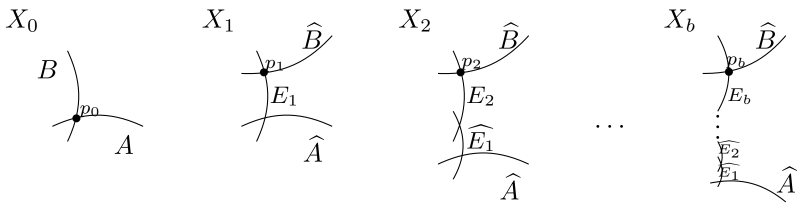

Figure 1: The surfaces X_b

We easily compute some intersection numbers on the surface X_b for any $b > 0$:

- $\widehat{E_i} \cdot \widehat{E_j} = 1$ if $|i - j| = 1$ and 0 if $|i - j| \geq 2$;
- $(E_b)^2 = -1$;
- For all $i < b$ we have $\left(\widehat{E_i}\right)^2 = -2$;
- $\left(\widehat{A}\right)^2 = A^2 - 1$ and $\left(\widehat{B}\right)^2 = B^2 - b$.

Now let D be a nef divisor on X whose support does not contain p.
On X_b we define the $\mathbb{Q}$-divisor

$$D_b = \pi^* D - \sum_{i=1}^{b} \frac{i}{b} \widehat{E_i},$$

where $\pi \colon X_b \to X$ denotes the corresponding sequence of blow-ups.

Lemma A.1. *For each integer $b \geq 1$, the following equalities hold true:*

1. $(D_b)^2 = D^2 - \frac{1}{b}$,
2. $D_b \cdot \widehat{A} = D \cdot A - \frac{1}{b}$,
3. $D_b \cdot \widehat{E_i} = 0$ *for* $i < b$,
4. $D_b \cdot E_b = \frac{1}{b}$,
5. $D_b \cdot \widehat{B} = D \cdot B - 1$.

Proof. For 1. first observe that $\pi^* D \cdot \widehat{E_i} = 0$ for all i since D does not pass through p. We then calculate

$$\begin{aligned}(D_b)^2 &= D^2 - 2\sum_{i=1}^{b-1} \frac{i^2}{b^2} - 1 + 2\sum_{i=1}^{b-1} \frac{i(i+1)}{b^2} \\ &= D^2 - 1 + 2\sum_{i=1}^{b-1} \frac{i}{b^2} \\ &= D^2 - \frac{1}{b}.\end{aligned}$$

Similarly, $\widehat{A} \cdot \widehat{E_i} = 0$ for $1 < i \leq b$ and $\widehat{B} \cdot \widehat{E_i} = 0$ for $1 \leq i < b$, so we compute

$$D_b \cdot \widehat{A} = D \cdot A - \frac{1}{b}\widehat{E_1} \cdot \widehat{A} = D \cdot A - \frac{1}{b},$$

$$D_b \cdot \widehat{B} = D \cdot B - E_b \cdot B = D \cdot B - 1,$$

which gives 2. and 5. On the other hand, we have

$$D_b \cdot \widehat{E_1} = \frac{-1}{b}\left(\widehat{E_1}\right)^2 - \frac{2}{b}\widehat{E_2} \cdot \widehat{E_1} = \frac{2}{b} - \frac{2}{b} = 0,$$

and for $2 \leq i \leq b-1$ we get

$$D_b \cdot \widehat{E_i} = -\frac{i-1}{b}\widehat{E_{i-1}} \cdot \widehat{E_i} - \frac{i}{b}\left(\widehat{E_i}\right)^2 - \frac{i+1}{b}\widehat{E_{i+1}} \cdot \widehat{E_i} = -\frac{i-1}{b} + \frac{2i}{b} - \frac{i+1}{b} = 0,$$

which gives 3. Finally, we compute

$$D_b \cdot E_b = -\frac{b-1}{b}\widehat{E_{b-1}} \cdot E_b - (E_b)^2 = -\frac{b-1}{b} + 1 = \frac{1}{b},$$

giving 4. □

The second step is the main construction.

Step 2:

Now we let $X = \mathbb{P}^2$, and we choose a line L on X and $(p_k)_{k \geq 0}$ an ordered countable set of distinct points on L. Let H be a line not passing through any p_k, and let $D = 2H$, a divisor on X. For each k we choose B_k a line through p_k distinct from L.

Set $X'_0 = X$ and $D'_0 = D$. For any k we let X'_{k+1} and D'_{k+1} be the result of applying Construction 1 to X'_k, $A = \widehat{L}$, $B = B_k$, $p = p_k$ and D'_k with $b = 2^k$. By Lemma A.1, for any k, we find that

$$(D'_k)^2 = D^2 - \frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2^k} = 3 + \frac{1}{2^k}.$$

Lemma A.2. *D'_k is nef.*

Proof. First we compute (using Lemma A.1) that

$$D'_k \cdot \widehat{L} = D \cdot A - \sum_{i=1}^{k} \frac{1}{2^i} = 1 + \frac{1}{2^k} \geq 0,$$

$$D'_k \cdot \widehat{B_i} = D \cdot B_i - 1 = 1 \geq 0,$$

and $D'_k \cdot E \geq 0$ for any irreducible exceptional divisor E. Now let $C \subset X'_k$ be any irreducible curve distinct from $\widehat{L}$, $\widehat{B_i}$, and exceptional curves. Denote by $E_{i,j}$ the i'th exceptional divisor on X'_j. Then

$$\sum_{j=1}^{k} \sum_{i=1}^{2^j} \widehat{E_{ij}} \cdot C \leq L \cdot \pi_* C,$$

where $\pi \colon X'_k \to X$ denotes the corresponding sequence of blow-ups. Indeed,

$$\begin{aligned} L \cdot \pi_* C &= \pi^* L \cdot C \\ &= (\widehat{L} + \sum_{i,j} \widehat{E_{i,j}}) \cdot C \\ &= \widehat{L} \cdot C + \sum_{i,j} \widehat{E_{ij}} \cdot C, \end{aligned}$$

each summand of which is non-negative. We then compute

$$\begin{aligned} D'_k \cdot C &= \pi^* D \cdot C - \sum_{i,j} \frac{i}{2^j} \widehat{E_{i,j}} \cdot C \\ &\geq \pi^* D \cdot C - \sum_{i,j} \widehat{E_{i,j}} \cdot C \\ &\geq \pi^* D \cdot C - \pi^* L \cdot C \\ &= L \cdot \pi_* C \geq 0, \end{aligned}$$

where the first equality on the last line holds because $D \sim 2L$. □

Now, the sequence $\{D'_k\}_{k \in \mathbb{N}}$ converges in the weak topology and defines an approximable nef b-divisor on $R(X)$ which we denote by $\mathbb{D}$.
Given $k \geq 0$, let f be a rational function on X such that

$$k\mathbb{D} + \operatorname{div} f \geq 0.$$

Since f must cancel infinitely many exceptional divisors over L with multiplicity k we see that $\operatorname{ord}_L f \geq k$, and in fact this condition is equivalent to canceling all the negative multiples of exceptional divisors. Since the sum of these exceptional parts is by definition given by $2H - \mathbb{D}$ we see that

$$\operatorname{vol}(\mathbb{D}) = \operatorname{vol}(2H - L) = 1/2.$$

This implies in particular that $\mathbb{D}$ is big. On the other hand, since the D'_k's are nef, we have

$$2\operatorname{vol}(D'_k) = (D'_k)^2 = 3 + \frac{1}{2^k}.$$

Hence $\lim_k \operatorname{vol}(D'_k) = 3/2$. In conclusion, we see that the volume function is not continuous in the space of approximable big and nef b-divisors. In consequence, for an approximable nef b-divisor, the volume and the degree do not agree necessarily.

Remark A.3.

1. In [6, Theorem 5.13] it is shown that, for toroidal nef b-divisors, the degree and the volume agree. In that paper it was also announced that the volume was not necessarily continuous in the non-toroidal case. This is the promised example.
2. To a psh metric with almost asymptotically algebraic singularities, we can associate an approximable nef b-divisor (Theorem 5.18) and the multiplier ideal volume of the metric agrees with the degree of this b-divisor (Corollary 5.23). Then, [6, Theorem 5.13] and Corollary 5.23 imply that, for toroidal psh metrics, the multiplier ideal volume of the psh metric agrees with the volume of the associated b-divisor.
3. It would be interesting to know whether the equality between the multiplier ideal volume of a psh metric and the volume of the associated b-divisor continues to be true in the general case of almost asymptotically algebraic psh metrics. Note that if one can show that the above example can be realized as the b-divisor of an almost asymptotically algebraic psh metric, we obtain a negative answer to this question.

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