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Matching extension and distance spectral radius

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ABSTRACT

A graph is called k-extendable if each k-matching can be extended to a perfect matching. We give spectral conditions for the k-extendability of graphs and bipartite graphs using Tutte-type and Hall-type structural characterizations. Concretely, we give a sufficient condition in terms of the spectral radius of the distance matrix for the k-extendability of a graph and completely characterize the corresponding extremal graphs. A similar result is obtained for bipartite graphs.

1. Introduction

In this paper, we give conditions for the extendability of matchings in a graph in terms of the spectral radius of the distance matrix. We say that a graph is *k-extendable* if each matching consisting of k edges can be extended to a perfect matching. Historically, matching extension was born out of the canonical decomposition theory for graphs with

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perfect matchings [19]. The study of the concept of k-extendability gradually evolved from the concept of so-called elementary (that is, 1-extendable) bipartite graphs. Hetyei [12] provided four useful characterizations of elementary bipartite graphs. Lovász [18] showed that the class of bipartite elementary graphs plays an important role in the structure of graphs with a perfect matching. The first results on k-extendable graphs (for arbitrary k) were obtained by Plummer [22]. In 1980, he studied the properties of k-extendable graphs and showed that nearly all k-extendable graphs ($k \geq 2$) are $(k-1)$-extendable and $(k+1)$-connected. Motivated by this work, many researchers further looked at the relationship between k-extendability and other graph parameters, e.g., degree [1,29], connectivity [17], genus [6,24] and toughness [25]. We refer the interested reader also to three surveys [26–28] and to the list of references therein.

With the development of spectral graph theory, also the relation between matchings and graph eigenvalues was studied, e.g., [9,15,21] for adjacency eigenvalues, [3,11] for Laplacian eigenvalues and [16,31] for distance eigenvalues. Recently, Fan and Lin [7] investigated the k-extendability of graphs from an adjacency spectral perspective. In this paper, we study the relationship between k-extendability and the distance spectral radius.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *distance* between v_i and v_j, denoted by $d(v_i, v_j)$, is the length of a shortest path from v_i to v_j. The *distance matrix* of G, denoted by $D(G)$, is a real symmetric matrix whose (i, j)-entry is $d(v_i, v_j)$. The distance matrix of a graph was introduced by Graham and Pollak [10] to study the routing of messages or data between computers, and this motivated much additional work on the distance matrix. For example, Merris [20] provided an estimation of the spectrum of the distance matrix of a tree. Since then, there has been a lot of research on distance matrices and their spectra; see the three surveys [2,13,14].

By the Perron-Frobenius Theorem, the spectral radius of the distance matrix of G, which we denote by $\partial(G)$, equals its largest eigenvalue, and is called the *distance spectral radius* of G. As is usual when studying the distance spectrum, we will assume that the graphs under consideration are connected.

Denote by $\vee$ and $\cup$ the *join* and *union* of two graphs, respectively. Furthermore, we denote by $K_{a,b} \diamond K_{c,d}$ the bipartite graph obtained from the union of $K_{a,b}$ and $K_{c,d}$ by adding all edges between the parts of the sizes b and c. A bipartite graph is called *balanced* if both parts of the bipartition have equal size. Clearly, every bipartite graph with a perfect matching (and hence a k-extendable bipartite graph) must be balanced.

Zhang and Lin [31, Thm. 1.1] showed that the graph of order $2n$ with the smallest distance spectral radius that does not have a perfect matching is $K_{n-1} \vee (n+1)K_1$ for $n \leq 4$ and $K_1 \vee (K_{2n-3} \cup 2K_1)$ for $n \geq 5$. They [31, Thm. 1.2] also showed that the balanced bipartite graph of order $2n$ with the smallest distance spectral radius that does not have a perfect matching is $K_{n-1,n-2} \diamond K_{1,2}$.

Note that the existence of a perfect matching can be considered as 0-extendability. Instead, we will focus on k-extendability for $k \geq 1$ and prove that the graph of order $2n$ with smallest distance spectral radius that is not k-extendable is $K_{2k} \vee (K_{2n-2k-1} \cup K_1)$.

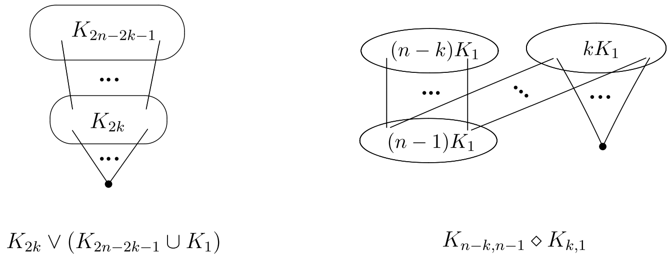

Fig. 1. The extremal graphs that are not k-extendable.

The balanced bipartite graph of order $2n$ with the smallest distance spectral radius that is not k-extendable is $K_{n-k,n-1} \diamond K_{k,1}$.

See Fig. 1 for a picture of the extremal graphs that are not k-extendable. It is clear that these graphs are quite different from the extremal graphs that do not have a perfect matching (mentioned above; see Zhang and Lin [31]), and that they do have perfect matchings.

Our paper is further organized as follows: In Section 2, we introduce some lemmas on the structure of non-extendable graphs (Section 2.1) and distance spectral radius (Section 2.2). In Section 3, we determine the graph with the smallest distance spectral radius among all non-extendable graphs of given order (Theorem 3.1). In Section 4, we give a sufficient condition in terms of the distance spectral radius for the k-extendability of a bipartite graph (Theorem 4.1). In Section 5, we finish the paper with an analogous result on k-factor-criticality in graphs (Theorem 5.2).

2. Preliminaries

2.1. The structure of non-extendable graphs

We start with a Tutte-type characterization for k-extendable graphs obtained by Chen [5]. For any $S \subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S and let $G - S$ be the subgraph induced by $V(G) \setminus S$. Denote the number of odd components in G by $o(G)$.

Lemma 2.1. *[5, Lemma 1] Let $k \geq 1$. A graph G is k-extendable if and only if*

$$o(G - S) \leq |S| - 2k$$

for any $S \subseteq V(G)$ that contains a k-matching.

A Hall-type condition for bipartite graphs to be k-extendable was obtained by Plummer [23]. For any $S \subseteq V(G)$, let $N(S)$ be the set of all neighbors of the vertices in S.

Lemma 2.2. *[23, Thm. 2.2] Let $k \geq 1$ and let G be a connected bipartite graph with parts U and W. Then the following are equivalent:*

(i) *G is k-extendable;*
(ii) *$|U| = |W|$ and for all nonempty subsets X of U, if $|X| \le |U| - k$, then $|N(X)| \ge |X| + k$;*
(iii) *For all $u_1, u_2, \dots, u_k \in U$ and $w_1, w_2, \dots, w_k \in W$, the graph $G' = G - \{u_1, \dots, u_k, w_1, \dots, w_k\}$ has a perfect matching.*

2.2. The distance spectral radius

An elementary, but fundamental result to compare the distance spectral radii of a graph and a spanning subgraph can be obtained by the Rayleigh quotient and the Perron-Frobenius Theorem.

Lemma 2.3. *Let G be a connected graph with $u, v \in V(G)$ and $uv \notin E(G)$, then*

$$\partial(G) > \partial(G + uv).$$

Proof. Let $\mathbf{x}$ be a Perron eigenvector for $\partial(G + uv)$, so that $\mathbf{x}$ is positive in all entries. Note that $D(G) = D(G + uv) + M$, where M is a nonzero nonnegative matrix. Then

$$\begin{aligned}\partial(G) &\ge \frac{\mathbf{x}^\top (D(G+uv)+M)\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \frac{\mathbf{x}^\top D(G+uv)\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} + \frac{\mathbf{x}^\top M\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} \\ &> \frac{\mathbf{x}^\top D(G+uv)\mathbf{x}}{\mathbf{x}^\top \mathbf{x}} = \partial(G+uv). \quad \square\end{aligned}$$

We also need a result mentioned as Claim 1 of the proof of Theorem 1.1 in [31].

Lemma 2.4. *[31, pp. 317–319] Let $p \ge 2$ and $n_i \ge 1$ for $i = 1, \dots, p$. If $\sum_{i=1}^{p} n_i = n - s$ where $s \ge 1$, then*

$$\partial(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_p})) \ge \partial(K_s \vee (K_{n-s-p+1} \cup (p-1)K_1))$$

with equality if and only if $n_i = 1$ for $i = 2, \dots, p$.

3. Extendability and the distance spectral radius of graphs

Using the Tutte-type characterization in Lemma 2.1 and the lemmas in Section 2.2 on the distance spectral radius, we will now prove our main result.

Theorem 3.1. *Let $k \ge 1$ and $n \ge k + 1$. Let G be a connected graph of order $2n$. If*

$$\partial(G) \le \partial(K_{2k} \vee (K_{2n-2k-1} \cup K_1)),$$

then G is k-extendable unless $G = K_{2k} \vee (K_{2n-2k-1} \cup K_1)$.

Proof. Note that $K_{2k} \vee (K_{2n-2k-1} \cup K_1)$ is not k-extendable. We will prove that if G is not k-extendable, then $\partial(G) \geq \partial(K_{2k} \vee (K_{2n-2k-1} \cup K_1))$ with equality only if $G = K_{2k} \vee (K_{2n-2k-1} \cup K_1)$.

Suppose that G is not k-extendable with $2n$ vertices where $n \geq k+1$, then by Lemma 2.1, there exists some nonempty subset S of $V(G)$, say of size s, such that $s \geq 2k$ and $o(G-S) > s-2k$. Because G has an even order, $o(G-S)$ and s have the same parity, so we have $o(G-S) \geq s-2k+2$. We may assume that all components of $G-S$ are odd, otherwise, we can move one vertex from each even component to the set S, and consequently, the number of odd components and the size of S increase by the same amount, so that all assumption remains valid. We may also assume that the number of odd components equals $s-2k+2$, for additional odd components (of which there are an even number) may be added to one of the other odd components (as we will not use that a component is connected). Let the odd number n_i be the cardinality of the i-th odd component of $G-S$. It is clear then that G is a spanning subgraph of

$$K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{s-2k+2}})$$

for some odd integers $n_1, n_2, \ldots, n_{s-2k+2}$ and $\sum_{i=1}^{s-2k+2} n_i = 2n-s$. By Lemma 2.3, we have

$$\partial(G) \geq \partial(K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{s-2k+2}}))$$

where equality holds if and only if $G = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{s-2k+2}})$. Write

$$G^{(s)} = K_s \vee (K_{2n-2s+2k-1} \cup (s-2k+1)K_1).$$

By Lemma 2.4, it is clear that

$$\partial(G) \geq \partial(G^{(s)})$$

where equality holds if and only if $G = G^{(s)}$. Note that $2n = s + \sum_{i=1}^{s-2k+2} n_i \geq 2s-2k+2$ and $G^{(2k)} = K_{2k} \vee (K_{2n-2k-1} \cup K_1)$. As the latter is our claimed extremal graph, let us denote its distance spectral radius $\partial(G^{(2k)})$ by ∂^*. The main idea of the following is to show that $\partial(G^{(s)}) > \partial^*$ when $n \geq s-k+1$ and $s \geq 2k+1$. Let $\mathbf{x}$ be the unit Perron eigenvector of $D(G^{(2k)})$, hence $D(G^{(2k)})\mathbf{x} = \partial^*\mathbf{x}$. It is well-known that $\mathbf{x}$ is constant on each part corresponding to an equitable partition [4, §2.5]. Thus we may set

$$\mathbf{x} = (\underbrace{a, \ldots, a}_{2k}, \underbrace{b, \ldots, b}_{2n-2k-1}, c)^\top$$

where $a, b, c \in \mathbb{R}^+$. In order to compare the appropriate spectral radii, we refine the partition, and write $\mathbf{x}$ as follows:

$$\mathbf{x} = (\underbrace{a, \ldots, a}_{2k}, \underbrace{b, \ldots, b}_{s-2k}, \underbrace{b, \ldots, b}_{2n-2s+2k-1}, \underbrace{b, \ldots, b}_{s-2k}, c)^\top.$$

Accordingly, $D(G^{(s)}) - D(G^{(2k)})$ is partitioned as

$$\begin{array}{r} \\ 2k \\ s-2k \\ 2n-2s+2k-1 \\ s-2k \\ 1 \end{array} \begin{array}{c} \begin{array}{ccccc} 2k & s-2k & 2n-2s+2k-1 & s-2k & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -J \\ 0 & 0 & 0 & J & 0 \\ 0 & 0 & J & J-I & 0 \\ 0 & -J & 0 & 0 & 0 \end{bmatrix} \end{array}$$

where each J is an all-ones matrix of appropriate size and I is an identity matrix. Then we have

$$\begin{aligned} \partial(G^{(s)}) - \partial^* &\geq \mathbf{x}^\top (D(G^{(s)}) - D(G^{(2k)}))\mathbf{x} \\ &= (s-2k)b \cdot [(4n-3s+2k-3)b - 2c]. \end{aligned}$$

Note that $s \geq 2k+1$ and $b > 0$, hence it suffices to show that

$$(4n - 3s + 2k - 3)b - 2c > 0. \tag{3.1}$$

From the equation $D(G^{(2k)})\mathbf{x} = \partial^*\mathbf{x}$, we obtain that

$$\begin{cases} \partial^* \cdot a = (2k-1)a + (2n-2k-1)b + c \\ \partial^* \cdot b = 2ka + (2n-2k-2)b + 2c \\ \partial^* \cdot c = 2ka + 2(2n-2k-1)b \end{cases}$$

which implies

$$c = \left(1 + \frac{2n-2k-2}{\partial^* + 2}\right) b.$$

By substituting this and $s \leq n + k - 1$ into (3.1), it follows that instead of the latter, it suffices to prove that

$$\partial^* > 2 + \frac{4}{n-k-2},$$

unless $n = k+2$. But this easily follows from the bound $\partial^* > \min_i r_i(D(G^{(2k)})) = 2n-1$, where $r_i(A)$ denotes the i-th row sum of a matrix A (note that $n \geq 4$ if $n \geq k+3$). Note that there is a strict inequality in this bound because the row sums are not constant; we need this strict inequality below.

Thus, except for the case $n = k+2$, which only occurs in conjunction with $n = s-k+1$, the proof is finished. For the remaining case, the above approach does not work, and the distance spectral radius is quite close to the claimed optimal value. If $n = k + 2$ and $n = s - k + 1$, then $G^{(s)} = K_{2k+1} \vee 3K_1$, whereas $G^{(2k)} = K_{2k} \vee (K_3 \cup K_1)$. Thus, for $G^{(s)}$, there is a clear equitable partition with two parts, and $\partial(G^{(s)})$ is the largest eigenvalue of the corresponding quotient matrix (of $D(G^{(s)})$)

$$\begin{bmatrix} 2k & 3 \\ 2k+1 & 4 \end{bmatrix}$$

which has characteristic polynomial

$$\phi_s(x) = x^2 - (2k+4)x + 2k - 3.$$

Similarly, ∂^* is the largest eigenvalue of the quotient matrix

$$\begin{bmatrix} 2k-1 & 3 & 1 \\ 2k & 2 & 2 \\ 2k & 6 & 0 \end{bmatrix}$$

of $D(G^{(2k)})$, and hence ∂^* is the largest root of the characteristic polynomial

$$\phi(x) = x^3 - (2k+1)x^2 - (4k+14)x + 4k - 12.$$

Next, we let $\varphi_s(x) = (x+3)\phi_s(x)$. In this way, we make sure that $\varphi_s(x) - \phi(x) = -x + 2k + 3$. As noted before, we have that $\partial^* > 2n - 1 = 2k + 3$ and hence

$$\phi_s(\partial^*) = \tfrac{1}{\partial^*+3}\varphi_s(\partial^*) = \tfrac{1}{\partial^*+3}(\varphi_s(\partial^*) - \phi(\partial^*)) = \tfrac{1}{\partial^*+3}(-\partial^* + 2k + 3) < 0,$$

which implies that $\partial(G^{(s)}) > \partial^*$. □

4. Bipartite graphs

We will next restrict our attention to bipartite graphs. Instead of the Tutte-type characterization, we will now use the Hall-type characterization in Lemma 2.2.

Theorem 4.1. *Let $k \geq 1$ and $n \geq k+1$. Let G be a connected balanced bipartite graph of order $2n$. If*

$$\partial(G) \leq \partial(K_{n-k,n-1} \diamond K_{k,1}),$$

then G is k-extendable unless $G = K_{n-k,n-1} \diamond K_{k,1}$.

Proof. Note that the bipartite graph $K_{n-k,n-1} \diamond K_{k,1}$ is not k-extendable. We will prove that if G is bipartite but not k-extendable, then $\partial(G) \geq \partial(K_{n-k,n-1} \diamond K_{k,1})$ with equality only if $G \cong K_{n-k,n-1} \diamond K_{k,1}$.

Let G be a balanced connected bipartite graph with parts U and W (each of size n). Suppose that G is not k-extendable, then by Lemma 2.2, there exists some nonempty subset X, say of size s, of U such that $|N(X)| \leq s+k-1$ and $s \leq n-k$. We now proceed in a similar way as in Section 3. Here we have that G is a spanning subgraph of

$$B^{(s)} = K_{s,s+k-1} \diamond K_{n-s,n-s-k+1},$$

and, by Lemma 2.3, we have that

$$\partial(G) \geq \partial(B^{(s)})$$

where equality holds if and only if $G \cong B^{(s)}$.

It is clear that $B^{(1)} \cong B^{(n-k)} = K_{n-k,n-1} \diamond K_{k,1}$, which is our claimed extremal graph, and let us denote its distance spectral radius $\partial(B^{(1)})$ by ∂^*. Note that more generally, $B^{(s)} \cong B^{(n-k+1-s)}$, so it suffices to show that $\partial(B^{(s)}) > \partial^*$ for $\frac{1}{2}(n-k+1) \leq s \leq n-k-1$. Note that such s only occur when $n \geq k+3$.

Let $\mathbf{z}$ be the unit Perron eigenvector of $D(B^{(1)})$, hence $D(B^{(1)})\mathbf{z} = \partial^*\mathbf{z}$. As before, $\mathbf{z}$ is constant on each part corresponding to an equitable partition [4, §2.5], hence we may set

$$\mathbf{z} = (\underbrace{a_1, \ldots, a_1}_{n-k}, \underbrace{a_2, \ldots, a_2}_{k}, \underbrace{b_1, \ldots, b_1}_{n-1}, b_2)^\top$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}^+$. Again, we refine the partition, and write

$$\mathbf{z} = (\underbrace{a_1, \ldots, a_1}_{s}, \underbrace{a_1, \ldots, a_1}_{n-k-s}, \underbrace{a_2, \ldots, a_2}_{k}, \underbrace{b_1, \ldots, b_1}_{s+k-1}, \underbrace{b_1, \ldots, b_1}_{n-s-k}, b_2)^\top.$$

Accordingly, we can partition $D(B^{(s)}) - D(B^{(1)})$ as

$$\begin{array}{r} \\ s \\ n-k-s \\ k \\ s+k-1 \\ n-s-k \\ 1 \end{array} \begin{array}{c} \begin{array}{cccccc} s & n-k-s & k & s+k-1 & n-s-k & 1 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 2J & 0 \\ 0 & 0 & 0 & 0 & 0 & -2J \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2J & 0 & 0 & 0 & 0 & 0 \\ 0 & -2J & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

and obtain that

$$\begin{aligned}\partial(B^{(s)}) - \partial^* &\geq \mathbf{z}^\top (D(B^{(s)}) - D(B^{(1)})\mathbf{z} \\ &= 4(n-s-k)a_1(sb_1 - b_2). \end{aligned} \tag{4.1}$$

Since $D(B^{(1)})\mathbf{z} = \partial^*\mathbf{z}$, we find that

$$\begin{cases} \partial^* \cdot b_1 = (n-k)a_1 + ka_2 + 2(n-2)b_1 + 2b_2, \\ \partial^* \cdot b_2 = 3(n-k)a_1 + ka_2 + 2(n-1)b_1. \end{cases}$$

Because $s \geq \frac{1}{2}(n-k+1)$, we may assume that $s \geq 3$, except for the case that both $n = k+3$ and $s = 2$ (which case we will discuss below). If indeed $s \geq 3$, then

$$\partial^* \cdot (sb_1 - b_2) \geq \partial^* \cdot (3b_1 - b_2) = 2ka_2 + (4n-10)b_1 + 6b_2 > 0,$$

and hence (4.1) shows that $\partial(B^{(s)}) - \partial^* > 0$. Therefore, except for the case $s = 2$ and $n = k+3$, the proof is finished.

In the remaining case, we will again consider the quotient matrices and their characteristic polynomials. We now have $B^{(2)} = K_{2,k+1} \diamond K_{k+1,2}$, which has an equitable partition with four parts, and $\partial(B^{(2)})$ is the largest eigenvalue of the corresponding quotient matrix

$$\begin{bmatrix} 2 & 2k+2 & k+1 & 6 \\ 4 & 2k & k+1 & 2 \\ 2 & k+1 & 2k & 4 \\ 6 & k+1 & 2k+2 & 2 \end{bmatrix},$$

which has characteristic polynomial

$$\begin{aligned}\phi_2(x) =& x^4 - (4k+4)x^3 + (3k^2 - 6k - 53)x^2 \\ &+ (12k^2 + 88k - 68)x - 36k^2 + 72k - 20.\end{aligned}$$

Similarly, $B^{(1)} = K_{3,k+2} \diamond K_{k,1}$ has quotient matrix (of $D(B^{(1)})$)

$$\begin{bmatrix} 0 & 2k+4 & k & 9 \\ 2 & 2k+2 & k & 3 \\ 1 & k+2 & 2k-2 & 6 \\ 3 & k+2 & 2k+2 & 4 \end{bmatrix},$$

with characteristic polynomial

$$\begin{aligned}\phi_1(x) =& x^4 - (4k+4)x^3 + (3k^2 - 6k - 45)x^2 \\ &+ (12k^2 + 72k - 52)x - 24k^2 + 64k + 28.\end{aligned}$$

Note that ∂^* is the largest root of $\phi_1(x)$ and $\partial^* > \min_i r_i(D(B^{(1)}) = 3k + 7$. Then

$$\begin{aligned}\phi_2(\partial^*) = \phi_2(\partial^*) - \phi_1(\partial^*) &= -8{\partial^*}^2 + (16k-16)\partial^* - 12k^2 + 8k - 48 \\ &\leq -8(3k+7)^2 + (16k-16)(3k+7) - 12k^2 + 8k - 48 \\ &< 0\end{aligned}$$

and hence $\partial(B^{(2)}) > \partial^*$. □

5. Concluding remarks

We have studied the relationship between extendability of matchings and the distance spectral radius of a graph. Related to extendability is the concept of k-factor-criticality, which was introduced by Favaron [8] and Yu [30], independently. Based on results of k-extendability, Yu [30] generalized the idea of k-extendability to $k\frac{1}{2}$-extendability for graphs of odd order. Besides, Favaron [8] extended some results on factor-critical and bicritical graphs.

A graph G is said to be *k-factor-critical*, if $G - S$ has a perfect matching for every subset $S \subseteq V(G)$ with $|S| = k$. It is clear that if a graph G is $2k$-factor-critical then it must be k-extendable. Note that for bipartite graphs, one needs another definition that includes balancedness.

A Tutte-type characterization of k-factor-criticality due to Yu [30] and independently Favaron [8], is as follows.

Lemma 5.1. *[30, Thm. 2.11][8, Thm. 3.5] Let $k \geq 1$. A graph G of order n is k-factor-critical if and only if $n \equiv k \pmod 2$ and*

$$o(G - S) \leq |S| - k$$

for any subset $S \subseteq V(G)$ with $|S| \geq k$.

By using this characterization and a similar analysis as for Theorem 3.1, we can obtain a sufficient condition in terms of the distance spectral radius to determine whether a graph is k-factor-critical.

Theorem 5.2. *Let $k \geq 1$ and $n \equiv k \pmod 2$ with $n \geq k+2$. Let G be a connected graph of order n. If*

$$\partial(G) \leq \partial(K_k \vee (K_{n-k-1} \cup K_1)),$$

then G is k-factor-critical unless $G = K_k \vee (K_{n-k-1} \cup K_1)$.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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