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Abstract

We discuss three processes of price adjustment, respectively proposed by Smale (1976), van der Laan and Talman (1987), and Kamiya (1990). The latter two processes are guaranteed to converge to a competitive equilibrium for a generic set of exchange economies for any initial price system and the former process for a generic set of exchange economies for any initial price system such that one of the prices is zero. The simplest way to describe these processes is by characterizing the path of prices that they generate. Convergence proofs then rely on results from differential topology and establish that these paths have a manifold structure. The van der Laan and Talman (1987) process was shown by Herings (1997) to exhibit global and universal convergence. The required tools, involving regular constraint sets and manifolds with generalized boundary, are explained in detail and can be fruitfully applied in other domains as well. The paper concludes with an overview of globally and universally convergent processes in other environments like production economies, economies with price rigidities, and normal-form games.

KEYWORDS: General equilibrium, price adjustment, universal convergence, differential topology.

JEL CODES: C61, C62, C63, C68, D50.

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1 Introduction

The simplest price adjustment process studied in general equilibrium theory is the Walrasian tâtonnement process. The famous examples of Scarf (1960) made it apparent that Walrasian tâtonnement may not converge to a competitive equilibrium. The work of Sonnenschein (1972, 1973), Mantel (1974) and Debreu (1974), basically claiming that any continuous function satisfying homogeneity of degree zero and Walras' law is the excess demand function of some economy, makes clear that the examples of Scarf (1960) do not correspond to exceptional cases and that it is possible to construct many examples where Walrasian tâtonnement does not converge and displays highly irregular dynamic behavior. The work of Saari and Simon (1978) and Saari (1985) points out that simple adaptations of the Walrasian tâtonnement process will not have better convergence properties.

Still, at least three universally convergent price adjustment processes are known in the literature: Smale's global Newton method introduced in Smale (1976), the process of van der Laan and Talman (1987), and the process proposed in Kamiya (1990). Such processes can also be used to compute equilibria, thereby enabling comparative statics exercises as well as policy recommendations. See also Judd (1997) and Eaves and Schmedders (1999) for further elaborations on this point.

The global Newton method of Smale adjusts prices in such a way that the vector of excess demands remains proportional to the vector of excess demands at the initial price system. The method does converge to a competitive equilibrium for almost any economy, but it does not do so for any initial price system. Only when the initial price system is chosen such that the prices of some commodities are sufficiently close to zero, convergence to a competitive equilibrium can be guaranteed. The work by Keenan (1981) makes clear that there may exist an open set of starting price systems for which Smale's process does not converge to a competitive equilibrium.

Another universally convergent price adjustment process has been presented in Kamiya (1990). Kamiya's process can be interpreted as a weighted average of Smale's global Newton method and Walrasian tâtonnement. At the initial price system, the process corresponds to Walrasian tâtonnement, whereas it approaches Smale's global Newton method when prices are close to a Walrasian equilibrium. Under rather weak conditions on the excess demand function, convergence to a competitive equilibrium price system is guaranteed for almost every starting price system. Although the boundary conditions of Kamiya (1990) are weak, they are still more demanding than boundary conditions that are derived from assumptions on the primitives of the model.

Yet another price adjustment process has been proposed in van der Laan and Talman (1987). In this process, prices of commodities with positive excess demand are kept relatively maximal, that is relative to the initial price system, whereas prices of commodities with negative excess demand are kept relatively minimal. For this process universal convergence has been shown in Herings (1997). Under standard conditions on utility functions, consumption sets, and initial endowments, this price adjustment process converges to a Walrasian equilibrium price system for almost all economies and any starting price systems.

To show convergence of these processes, one first defines a set of price systems for which excess demands satisfy the properties of the price systems generated by the price adjustment process. For instance, for Smale's global Newton methods, this would be the set of price systems at which excess demand is proportional the excess demand at the initial price system. Next one shows that, for generic economies, this set consists of an arc connecting the initial price system and precisely one competitive equilibrium as boundary points, a finite number of arcs containing precisely two competitive equilibria, both being boundary points, and a finite number of loops that neither contain the initial price system nor any competitive equilibrium. The proofs for the van der Laan and Talman (1987) process have to deal with the complication that the path of prices generated by the process displays non-differentiabilities whenever one of the markets gets equilibrated during the process. To deal with such issues we need tools from differential topology like regular constraint sets and manifolds with generalized boundary. We explain these tools in detail as well as how they are applied in the convergence proof of the van der Laan and Talman (1987) process.

The price adjustment processes of Smale (1976), van der Laan and Talman (1987), and Kamiya (1990) are defined for exchange economies. The paper concludes with an overview of extensions of such processes to production economies and to processes where not only prices but also quantities may adjust. It also makes the connection to strategy adjustment processes in game theory. Although all processes discussed admit a natural economic or game-theoretic interpretation, they may also be considered as algorithms to compute an equilibrium. Finally, we briefly discuss the role of universally convergent price adjustment processes in experimental economics.

The organization of the paper is as follows. Section 2 provides the general set-up and Sections 3, 4, and 5 are devoted to the processes of Smale (1976), Kamiya (1990), and van der Laan and Talman (1987), respectively, which are defined by the properties of the orbits that they generate. Section 6 explains how these orbits can be obtained as the solutions to an appropriate system of differential equations. Section 7 presents an overview of the tools from differential topology that are needed for the convergence proofs and Section 8 explains how these tools are used in the convergence proof for the van der Laan and Talman (1987) process. Section 9 concludes the paper by discussing some further developments.

2 Price Adjustment Processes

We consider an economy $\mathcal{E}$ where L commodities are traded. Excess demand for the commodities in this economy as a function of prices is given by a function $z : P \to \mathbb{R}^L$, where P denotes the set of possible price vectors. Typical choices for P are $\mathbb{R}^L_+ \setminus \{0^L\} = \{p \in \mathbb{R}^L \mid p > 0^L\}$,[1] i.e., the non-negative orthant of $\mathbb{R}^L$ excluding the zero vector, or the set of strictly positive price vectors $\mathbb{R}^L_{++} = \{p \in \mathbb{R}^L \mid p \gg 0^L\}$. The excess demand function z satisfies *homogeneity* if, for every $p \in P$, for every $\lambda \in \mathbb{R}$ such that $\lambda p \in P$, $z(\lambda p) = z(p)$. Because of homogeneity, it is possible to normalize prices. In some of the price adjustment processes studied in this paper, it is convenient to require the sum of the prices to be equal to one and take P equal to the unit simplex $\Delta^L = \{p \in \mathbb{R}^L_+ \mid \sum_{\ell=1}^L p_\ell = 1\}$ or the relative interior of the unit simplex $\dot{\Delta}^L = \{p \in \mathbb{R}^L_{++} \mid \sum_{\ell=1}^L p_\ell = 1\}$, or to consider prices whose squares sum up to one and take P equal to $S^L = \{p \in \mathbb{R}^L_+ \mid \sum_{\ell=1}^L (p_\ell)^2 = 1\}$.

Apart from homogeneity, the excess demand function is assumed to satisfy *Walras' law*, the property that, for every $p \in P$, $p \cdot z(p) = 0$.

A Walrasian equilibrium is $p^* \in P$ such that $z(p^*) = 0^L$. We are interested in the question whether processes of price adjustment terminate in a Walrasian equilibrium, i.e., converge to a price vector at which excess demand of all commodities is equal to zero when starting from a price vector that does not correspond to a Walrasian equilibrium.

A standard way to define a price adjustment process is by means of a system of first-order differential equations

$$\frac{dp(t)}{dt} = f(p(t)),$$

where $p(t) \in P$ denotes the price vector reached at time $t \in \mathbb{R}_+$ and f is a function from P into $\mathbb{R}^L$. The initial price vector $p(0)$ is assumed to be given. The function f determines the way in which prices adjust. A common specification is that f is in proportion to excess demand at the price system p. This results in a price adjustment process that is known as Walrasian tâtonnement, described in Walras (1874), and formalized in Samuelson (1941).

Conditions for which the system of differential equations has a solution are well-known, see for instance Hirsch and Smale (1974) for details. The *orbit* $\gamma(p(0))$ is the set of price vectors that is generated by the system of first-order differential equations when the initial state is $p(0)$,

$$\gamma(p(0)) = \{p \in P \mid \exists t \geq 0,\ p = p(t)\}.$$

The closure of $\gamma(p(0))$ is denoted by $\overline{\gamma}(p(0))$ and is called an orbit as well.

Rather than describing the three price adjustment processes of interest in this paper by a system of differential equations, it will turn out to be more convenient to describe them

[1] We denote the L-dimensional vector of zeroes by 0^L and the L-dimensional vector of ones by 1^L.

by the orbit that they generate. The next three sections describe this orbit as the solution to a particular system of equations that has one degree of freedom, where each section is devoted to one particular price adjustment process.

3 Smale's Global Newton Method

To describe Smale's global Newton method, we take $P = \Delta^L$. The mean excess demand at $p \in \mathbb{R}^L_+ \setminus \{0^L\}$ is denoted by $\overline{z}(p) = \sum_{\ell=1}^L z_\ell(p)/L$.

We make the following assumption on the excess demand function z throughout this section.

Assumption 3.1: The excess demand function $z : \Delta^L \to \mathbb{R}^L$ is twice continuously differentiable, satisfies Walras' law, and has the following boundary behavior:

For every $p \in \Delta^L$ with at least one zero component, $z(p) - \overline{z}(p)1^L$ is not radially outward pointing, i.e., there is no $\mu > 0$ such that $z(p) - \overline{z}(p)1^L = \mu(p - (1/L)1^L)$.

Assumption 3.1 is weaker than the assumptions in Smale (1976), where a stronger condition on boundary behavior is stated. The assumption that for every $p \in \Delta^L$ with at least one zero component $z(p) - \overline{z}(p)1^L$ is not radially outward pointing is satisfied if, for every $p \in \Delta^L$ with at least one zero component, for some $\ell = 1, \ldots, L$ with $p_\ell = 0$, it holds that $z_\ell(p) > 0$, a rather natural requirement. To see this, observe that by Walras' law there is ℓ' with $p_{\ell'} > 0$ and $z_{\ell'}(p) \leq 0$, so $z_\ell(p) - \overline{z}(p) > z_{\ell'}(p) - \overline{z}(p)$, whereas $-1/L = p_\ell - 1/L < p_{\ell'} - 1/L$, which is easily seen to imply that $z(p) - \overline{z}(p)$ is not radially outward pointing at p.

Smale (1976) discusses a number of variations when defining the global Newton method. Here we follow Herings (2002) and combine the approaches as suggested in Smale (1976), p. 117, and Varian (1977). Let

$$D^L = \{p \in \mathbb{R}^L \mid \textstyle\sum_{\ell=1}^L (p_\ell - 1/L)^2 \leq 1 \text{ and } \sum_{l=1}^L p_l = 1\},$$

a disk containing Δ^L in its interior, and let $\widetilde{\pi} : D^L \to \Delta^L$ be the radial projection on Δ^L. For $p \in \Delta^L$ the radial projection of p on Δ^L is equal to p and for $p \in D^L \setminus \Delta^L$, the radial projection of p on Δ^L is given by the price system at which the line between p and $(1/L)1^L$ intersects the relative boundary of Δ^L, so

$$\widetilde{\pi}(p) = \begin{cases} p, & \text{if } p \in \Delta^L, \\ \frac{1/L}{1/L - \min_{\ell=1,\ldots,L} p_\ell} p + \frac{-\min_{\ell=1,\ldots,L} p_\ell}{1/L - \min_{\ell=1,\ldots,L} p_\ell}(1/L)1^L, & \text{otherwise.} \end{cases}$$

We apply Smale's global Newton method to the function $\widetilde{z} : D^L \to \mathbb{R}^L$ defined by

$$\widetilde{z}(p) = \widetilde{\pi}(p) - p + (1 - \|p - (1/L)1^L\|_2)\left(z(\widetilde{\pi}(p)) - \overline{z}(\widetilde{\pi}(p))1^L\right), \quad p \in D^L.$$

The function $\widetilde{z}$ extends a function equal to a positive multiple of $z - \overline{z}1^L$ from Δ^L to D^L. The vector $\widetilde{z}(p)$ is a weighted sum of the terms $\widetilde{\pi}(p) - p$ and $z(\widetilde{\pi}(p)) - \overline{z}(\widetilde{\pi}(p))1^L$, with weight 1 on the former term and non-negative weight $(1 - \|p - (1/L)1^L\|_2)$ on the latter term. Note that $\widetilde{\pi}(p) - p$ vanishes on Δ^L, while the contribution of the term $(1 - \|p - (1/L)1^L\|_2)z(\widetilde{\pi}(p)) - \overline{z}(\widetilde{\pi}(p))1^L$ is equal to zero on the relative boundary of D^L, where it holds that $\|p - (1/L)1^L\|_2 = 1$. The function $\widetilde{z}$ is therefore radially inward pointing on the relative boundary of D^L.

We argue next that the zero points of z and $\widetilde{z}$ coincide, so when a price adjustment process finds a zero point of $\widetilde{z}$, it has found a Walrasian equilibrium. Let p belong the relative boundary of D^L. The term $1 - \|p - (1/L)1^L\|_2$ vanishes and the remaining term is $\widetilde{\pi}(p) - p$, which is clearly not equal to zero. Let $p \in D^L \setminus \Delta^L$ be such that it does not belong to the relative boundary of D^L. Then $\widetilde{z}(p) = 0^L$ if and only if

$$\widetilde{\pi}(p) - p + (1 - \|p - (1/L)1^L\|_2)(z(\widetilde{\pi}(p)) - \overline{z}(\widetilde{\pi}(p))1^L) = 0^L,$$

so

$$\begin{aligned} z(\widetilde{\pi}(p)) - \overline{z}(\widetilde{\pi}(p))1^L &= \frac{1}{1 - \|p - (1/L)1^L\|_2}(p - \widetilde{\pi}(p)) \\ &= \frac{-L\min_{\ell=1,\ldots,L} p_\ell}{1 - \|p - (1/L)1^L\|_2}(\widetilde{\pi}(p) - (1/L)1^L), \end{aligned}$$

where the second equality follows from the definition of the radial projection at a price system in $D^L \setminus \Delta^L$. We find that z is radially outward pointing at $\widetilde{\pi}(p)$, a contradiction to Assumption 3.1. Finally, let $p \in \Delta^L$. Then $\widetilde{z}(p)$ is a positive multiple of $z(p) - \overline{z}(p)1^L$, so $\widetilde{z}(p) = 0^L$ if and only if $z(p) = 0^L$.

Take a starting price system p^0 in the relative boundary of D^L. Smale's process generates prices at which the excess demand is a non-negative multiple of the excess demand at p^0. The orbit generated by Smale's process therefore belongs to the set

$$\overline{P} = \{p \in D^L \mid \exists \theta \geq 0,\ \widetilde{z}(p) = \theta\widetilde{z}(p^0)\},$$

The choice of $\theta = 1$ shows that $p^0 \in \overline{P}$ and the choice of $\theta = 0$ guarantees that $p^* \in \overline{P}$ if p^* is a Walrasian equilibrium.

In general, the set $\overline{P}$ consists of multiple components. Suppose the component of $\overline{P}$ containing p^0 is homeomorphic to an interval, containing p^0 and a unique Walrasian equilibrium as its boundary points. The price adjustment process then generates the prices of that component and terminates in a uniquely specified Walrasian equilibrium. Smale (1976) shows that, for a generic economy, the component of $\overline{P}$ containing p^0 is indeed homeomorphic to an interval and connects p^0 to a Walrasian equilibrium.

Instead of defining Smale's global Newton method by the orbit that it generates, we can also describe it by a system of differential equations. To do so, drop the last component of $\widetilde{z}$ and denoted the resulting function by $\widehat{z}$. Consider the following system of differential equations:

$$\begin{aligned} \partial\widehat{z}(p)\frac{dp}{dt} &= -\lambda(p)\widehat{z}(p), \\ 1^{L\top}\frac{dp}{dt} &= 0, \end{aligned}$$

where λ is an arbitrary scalar function of p such that

$$\operatorname{sign}(\lambda(p)) = \operatorname{sign}\det\begin{pmatrix} -\partial\widehat{z}(p) \\ -1^{L\top} \end{pmatrix}.$$

Since

$$\begin{aligned} \textstyle\sum_{\ell=1}^{L}\widetilde{z}_\ell(p) &= \textstyle\sum_{\ell=1}^{L}(\widetilde{\pi}_\ell(p) - p_\ell) + (1 - \|p - (1/L)1^L\|_2)\sum_{\ell=1}^{L}(z_\ell(\widetilde{\pi}(p)) - \overline{z}(\widetilde{\pi}(p))) \\ &= 1 - 1 + (1 - \|p - (1/L)1^L\|_2)\cdot 0 = 0, \end{aligned}$$

it holds that $1^{L\top}\partial\widetilde{z}(p) = 0$. Then

$$\partial\widehat{z}(p)\tfrac{dp}{dt} = -\lambda(p)\widehat{z}(p)$$

implies

$$\partial\widetilde{z}_L(p)\tfrac{dp}{dt} = -\lambda(p)\widetilde{z}_L(p),$$

so the price of commodity L is adjusted on the basis of the same principles as the prices of the other commodities. The equation $1^{L\top}\frac{dp}{dt} = 0$ makes sure that the sum of the prices is kept equal to one, so prices remain in D^L. The fact that $\widetilde{z}$ is radially inward pointing on the relative boundary of D^L makes sure that prices move towards the relative interior of D^L. It is now immediate that the orbit of the system of differential equations belongs to the set $\overline{P}$.

The convergence of Smale's global Newton method to a Walrasian equilibrium depends crucially of the choice of p^0 in the relative boundary of D^L. Keenan (1981) shows that convergence may not hold for starting price systems in the relative interior of D^L. This is problematic from both a computational and an economic point of view. From a computational point of view, it makes sense to take the initial price system such that it is expected to be close to a Walrasian equilibrium price system. There is no reason to expect that such a price system is located near the relative boundary of D^L. From an economic point of view, it would be natural to take the equilibrium prices of the previous period as a starting point of the adjustment process to find the equilibrium prices of the current period. Again, there is no reason to expect the previous period's equilibrium prices to be close to the relative boundary of D^L.

4 The Price Adjustment Process of Kamiya

Kamiya (1990) introduces a price adjustment process that can be interpreted as a weighted average of Smale's global Newton method and Walrasian tâtonnement. This process addresses the problem that Smale's process may not converge when $p(0)$ belongs to the relative interior of D^L. For Kamiya's process it is convenient to normalize commodity prices by taking the sum of the squares of prices equal to one, so $P = S^L$.

We make the following assumption on the excess demand function throughout this section.

Assumption 4.1: The function $z : S^L \to \mathbb{R}^L$ is twice continuously differentiable, satisfies Walras' law, and has the following boundary behavior:

For every $p \in S^L$, for every $\ell = 1, \ldots, L$, $p_\ell = 0$ implies $z_\ell(p) > 0$.

We have argued in Section 3 that the boundary behavior on z imposed in Assumption 4.1 is more demanding than the boundary behavior required in Assumption 3.1.

Walras' law in combination with Assumption 4.1 implies that we may drop the last component of z in the search for an equilibrium. Also, it is possible to represent the price system by the first $L-1$ components only, and take the price system in the set

$$\dot{S}^{L-1} \;=\; \{p \in \mathbb{R}^{L-1}_+ \mid \textstyle\sum_{\ell=1}^{L-1}(p_\ell)^2 < 1\}.$$

We can then replace the function $z : S^L \to \mathbb{R}^L$ by the function $\widetilde{z} : \dot{S}^{L-1} \to \mathbb{R}^{L-1}$, where

$$\widetilde{z}_\ell(p) \;=\; z_\ell(p_1, \ldots, p_{L-1}, \sqrt{1 - \textstyle\sum_{k=1}^{L-1}(p_k)^2}), \quad \ell = 1, \ldots, L-1.$$

The function $\widetilde{z}$ is obtained by omitting the last component of z and making use of the price normalization.

Take a starting price system p^0 in the interior of $\dot{S}^{L-1}$. In Kamiya's process the prices are adjusted in such a way that $\widetilde{z}(p)$ is proportional to $p - p^0$. The orbit generated by Kamiya's process therefore belongs to the set

$$\overline{P} \;=\; \{p \in S^{L-1} \mid \exists \theta \in [0,1],\ (1-\theta)\widetilde{z}(p) = \theta(p - p^0)\}.$$

The choice of $\theta = 1$ yields p^0 as the unique solution, so $p^0 \in \overline{P}$. The choice of $\theta = 0$ guarantees that $(p^*_1, \ldots, p^*_{L-1}) \in \overline{P}$ if $(p^*_1, \ldots, p^*_{L-1}, p^*_L)$ is a Walrasian equilibrium with $\sum_{\ell=1}^{L}(p^*_\ell)^2 = 1$.

Again, suppose the component of $\overline{P}$ containing p^0 is homeomorphic to an interval, containing p^0 and a unique Walrasian equilibrium as its boundary points. The price adjustment process then generates the prices of that component and terminates in a uniquely specified Walrasian equilibrium. Kamiya (1990) shows that, for a generic economy, the component of P containing p^0 is indeed homeomorphic to an interval and connects p^0 to a Walrasian equilibrium. The considerable advantage of Kamiya's price adjustment process over the global Newton method, both form a computational and an economic point of view, is that it converges for any initial price system.

Instead of defining Kamiya's price adjustment process by the orbit that it generates, we can also describe it by a system of differential equations. If we denote the $(L-1) \times (L-1)$ identity matrix by I, then at prices p different from p^0 such that $\widetilde{z}(p) \neq 0^{L-1}$, Kamiya's process is defined by

$$\left(\frac{\partial \widetilde{z}(p)}{\|\widetilde{z}(p)\|_2} - \frac{I}{\|p - p^0\|_2} \right) \frac{dp}{dt} = -\lambda(p)\widetilde{z}(p),$$

where λ is an arbitrary scalar function of p such that

$$\text{sign}(\lambda(p)) = \text{sign det} \left(\frac{I}{\|p - p^0\|_2} - \frac{\partial \widetilde{z}(p)}{\|\widetilde{z}(p)\|_2} \right).$$

At $p = p^0$ and at $p \in \dot{S}^{L-1}$ such that $\widetilde{z}(p) = 0^{L-1}$, the price adjustment process is defined by taking a limit.

Kamiya's process can be seen to be a weighted average of Smale's global Newton method, $\partial \widetilde{z}(p) \frac{dp}{dt} = -\lambda(p)\widetilde{z}(p)$, and Walrasian tâtonnement, $\frac{dp}{dt} = \widetilde{z}(p)$. The weights depend on the norm of the excess demand and the distance between p and p^0. The process is approximately equal to Walrasian tâtonnement at prices close to p^0 and it approaches Smale's global Newton method when prices are close to a Walrasian equilibrium.

5 The Price Adjustment Process of van der Laan and Talman

Although Kamiya's process displays generic convergence to a Walrasian equilibrium for any initial price system, the required boundary behavior, though fairly natural, does not follow from standard assumptions on primitives like consumptions sets, utility functions, and initial endowments. This section presents a price adjustment process proposed by van der Laan and Talman (1987), which displays generic global and universal convergence, i.e., convergence takes place for any initial price system and for excess demand functions that can be derived from standard assumptions on the primitives.

To describe the price adjustment process of van der Laan and Talman (1987), we normalize prices of commodities to belong to $\dot{\Delta}^L$, i.e., the relative interior of the unit simplex.

We make the following assumption on the excess demand function throughout this section.

Assumption 5.1: The function $z : \dot{\Delta}^L \rightarrow \mathbb{R}^L$ is twice continuously differentiable, satisfies Walras' law, and has the following boundary behavior:

If $(p^n)_{n \in \mathbb{N}}$ is a sequence in $\dot{\Delta}^L$ converging to $\overline{p} \in \Delta^L \setminus \dot{\Delta}^L$, then $\lim_{n \to \infty} \|z(p^n)\|_\infty = +\infty$.

The boundary behavior postulated in Assumption 5.1 requires that excess demand explodes when one of the commodity prices converges to zero. Contrary to Assumption 4.1, it is possible to derive Assumption 5.1 from standard assumptions on consumption sets, utility functions, and initial endowments.

Take a starting price system p^0 in $\dot{\Delta}^L$. The dynamics of the price adjustment process can be described as follows. First, the sign of the excess demand is evaluated at the starting price system p^0. Typically it holds that for every commodity $\ell = 1, \ldots, L$, $z_\ell(p^0) \neq 0$. Initially, the price of a commodity ℓ with $z_\ell(p^0) < 0$ is decreased, while the price of a commodity ℓ with $z_\ell(p^0) > 0$ is increased. The ratio of prices of commodities for which there is a negative excess demand is kept constant among those commodities for which there is a negative excess demand, and similarly for the ratio of prices of commodities for which there is a positive excess demand. This generates a line segment of prices. The price system is adjusted in this way until a market, say the market for commodity ℓ^1, attains an equilibrium. Typically, there is exactly one market for which this happens.

The price adjustment continues by keeping the market for commodity ℓ^1 in equilibrium, while the price p_{ℓ^1} is relatively increased (decreased) if there was a negative (positive) excess demand in the market for commodity ℓ^1 before attaining equilibrium. Other prices are kept relatively minimal in case of a negative excess demand and relatively maximal in case of a positive excess demand. In general this generates a 1-dimensional curve of prices. Two situations can occur. Either another market, say the market of a commodity ℓ^2, attains an equilibrium. In this case the price system is adjusted in such a way that the markets of commodities ℓ^1 and ℓ^2 are kept in equilibrium, while the price of commodity ℓ^2 is relatively increased (decreased) when there was a negative (positive) excess demand on the market of commodity ℓ^2 before attaining equilibrium. Or the price on the market of commodity ℓ^1 becomes relatively minimal or maximal. In this case the market of commodity ℓ^1 is no longer kept in equilibrium but its excess demand is allowed to become negative or positive, respectively, while p_{ℓ^1} is kept relatively minimal or relatively maximal, respectively. The price adjustment process continues in this way until a Walrasian equilibrium is reached.

Formally, the orbit generated by van der Laan and Talman's process belongs to the set

$$\overline{P} = \{p \in \dot{\Delta}^L \mid \begin{array}{l} \text{for } \ell' = 1,\ldots,L, \quad z_{\ell'}(p) < 0 \Rightarrow \frac{p_{\ell'}}{p^0_{\ell'}} = \min_{\ell=1,\ldots,L} \frac{p_\ell}{p^0_\ell}, \\ \text{for } \ell' = 1,\ldots,L, \quad z_{\ell'}(p) > 0 \Rightarrow \frac{p_{\ell'}}{p^0_{\ell'}} = \max_{\ell=1,\ldots,L} \frac{p_\ell}{p^0_\ell}\}. \end{array} \tag{5.1}$$

For a price system to belong to $\overline{P}$ it should hold that the relative price of a commodity, i.e. the ratio of the price of a commodity and its initial price, is minimal if the commodity is in positive excess supply and it is maximal if the commodity is in positive excess demand. This is closely related to the ideas behind Walrasian tâtonnement, where prices of commodities in positive excess supply are decreased and those of commodities in positive excess demand are increased. The starting price system p^0 belongs to $\overline{P}$ as all relative prices are equal to one at p^0, so the relative prices of all commodities are both minimal and maximal at the same time. A Walrasian equilibrium p^* belongs to $\overline{P}$, since if $z(p^*) = 0^L$ then all implications in the definition of $\overline{P}$ are trivially met.

Herings (1997) shows, for a generic economy, that the component of P containing p^0 is indeed homeomorphic to an interval and connects p^0 to a Walrasian equilibrium. Like Kamiya's process, there is convergence to a Walrasian equilibrium for any starting price system. There are two differences with Kamiya's process. First, standard assumptions on the primitives suffice to guarantee convergence to a Walrasian equilibrium. Second, whereas the orbit of Kamiya's process is generically a differentiable manifold with boundary, the orbit of van der Laan and Talman's process is generically a piecewise differentiable manifold with boundary.

It is also possible to present the price adjustment process of van der Laan and Talman (1987) as the solution to a system of differential equations. In fact, it is possible to do so for any piecewise differentiable orbit. The next section explains how to do this in general.

6 From Orbits to Systems of Differential Equations

After making some minor modifications to the formulations in Sections 3 and 4, it is possible to formulate the orbits of the processes of Smale (1976) and Kamiya (1990) as the zero points of a function $f : [0,1] \times P \to \mathbb{R}^L$, where the set P is L-dimensional. The explicit construction for Smale's process is presented later in this section. The variable in the interval is denoted by λ and is equal to $1 - \theta$ for the processes of Smale and Kamiya. The variable λ is not explicit in the formulation of van der Laan and Talman's process, but as demonstrated in Herings (2002) it can be taken equal to $\max_{k,\ell \in L}(p_k/p^0_k - p_\ell/p^0_\ell)$ and belongs to the interval $[0, \max_{\ell=1,\ldots,L} 1/p^0_\ell]$. It is easy, though not essential, to renormalize prices to ensure that λ belongs to the interval $[0,1]$ as in the other two processes.

The orbits generated by the processes of Sections 3, 4, and 5 are well-behaved sets under suitable assumptions. We now ask the question whether it is possible to find a system of differential equations that generates a given orbit. Let $(\lambda(t), p(t))$ be the orbit corresponding to the zero points of f, parametrized by arc length t. The starting point of the price adjustment process equals $p(0)$ and corresponds to parameter $\lambda(0) = 0$. Assume that zero is a regular value of both f and of the restriction of f to $\{0, 1\} \times P$. Let $J(\lambda, p)$ denote the Jacobian of f evaluated at (λ, p), a matrix of dimension $L \times (L + 1)$. The regularity assumption implies that the Jacobian has rank L. It also implies that there is a unique vector $v(J(0, p(0))) \in \mathbb{R}^{L+1}$ such that $J(0, p(0)) \cdot v(J(0, p(0))) = 0^L$, $\|v(J(0, p(0)))\|_2 = 1$, and the first component of $v(J(0, p(0)))$ is positive. For $p \in P$, let $v(J(\lambda, p))$ denote the unique vector such that $J(\lambda, p) \cdot v(J(\lambda, p)) = 0^L$, $\|v(J(\lambda, p))\|_2 = 1$, and

$$\det\begin{pmatrix} J(\lambda, p) \\ v(J(\lambda, p))^\top \end{pmatrix} = \det\begin{pmatrix} J(0, p(0)) \\ v(J(0, p(0)))^\top \end{pmatrix}.$$

It can be shown that the orbit of zero points induced by f is generated by the system of differential equations

$$(\tfrac{d\lambda}{dt}, \tfrac{dp}{dt}) = v(J(\lambda, p))^\top,$$

see for instance Allgower and Georg (1983). This system of differential equations was first proposed by Davidenko (1953) and is also referred to as the system of Davidenko equations.

We now illustrate that using the Davidenko equations immediately leads to Smale's process. The function $f : [0, 1] \times P \to \mathbb{R}^L$ whose zero points correspond to the orbit of Smale's process is obtained by taking $P = \mathbb{R}^L_+ \setminus \{0^L\}$ and defining, for $(\lambda, p) \in [0, 1] \times P$,

$$f(\lambda, p) \quad = \quad \begin{cases} \widehat{z}(p) - (1 - \lambda)\widehat{z}(p^0), \\ 1 - \sum_{\ell=1}^{L} p_\ell. \end{cases}$$

It follows that

$$J(\lambda, p) = \begin{bmatrix} \widehat{z}(p^0) & \partial\widehat{z}(p) \\ 0 & 1^{L\top} \end{bmatrix}.$$

The Davidenko equations specify that

$$(\tfrac{d\lambda}{dt}, \tfrac{dp}{dt}) = v(J(\lambda, p)),$$

from which it follows that

$$\begin{aligned} \partial\widehat{z}(p)\frac{dp}{dt} \quad &= \quad -\frac{d\lambda}{dt}\widehat{z}(p^0), \\ 1^{L\top}\frac{dp}{dt} \quad &= \quad 0. \end{aligned}$$

Substituting $\widehat{z}(p) = (1 - \lambda)\widehat{z}(p^0)$ leads to the specification of Smale's process in Section 3.

7 Regular Constraint Sets and Manifolds with Generalized Boundary

In Sections 3, 4), and 5 it was asserted that the orbits generated by the various price adjustment processes are generically well-behaved sets. Section 6 required 0^L to be a regular value of f, the function whose zero points describe the orbit, and of the restriction of f to $\{0,1\} \times P$. This section presents the tools that can be used to give precise definitions of "well-behaved" and to demonstrate that certain sets are actually well-behaved. The exposition is based on Section 2.10 of Herings (1996). For further elaborations of this material, the reader is refererred to Milnor (1965), Golubitsky and Guillemin (1973), Jongen, Jonker, and Twilt (1983, 1986), and Mas-Colell (1985).

We use the notation $\mathbb{N} = \{0, 1, \ldots\}$, $\mathbb{N}_+ = \{1, 2, \ldots\}$, $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$, and $\mathbb{N}^*_+ = \mathbb{N}_+ \cup \{\infty\}$. Let X and Y be topological spaces. The set of continuous functions from X into Y is denoted by $C^0(X,Y)$. Let U be an open subset of $\mathbb{R}^m$ and V a non-empty subset of $\mathbb{R}^n$. For $r \in \mathbb{N}_+$, $C^r(U,V)$ is defined as the set of r times continuously differentiable functions from U into V. The set $C^\infty(U,V)$ is defined by $C^\infty(U,V) = \cap_{r \in \mathbb{N}} C^r(U,V)$.

A k-dimensional topological manifold, where $k \in \mathbb{N}$, is a set which is locally like $\mathbb{R}^k$. More formally, we have the following definition.

Definition 7.1 For $k \in \mathbb{N}$, a subset X of $\mathbb{R}^m$ is a *k-dimensional topological manifold* if for every element x of X there exists an open subset U of X containing x, an open subset V of $\mathbb{R}^k$, and an injective and surjective function $\varphi : U \to V$ such that $\varphi \in C^0(U,V)$ and $\varphi^{-1} \in C^0(V,U)$, i.e., $\varphi : U \to V$ is a homeomorphism.

Every neighborhood of a point in a k-dimensional topological manifold can be continuously deformed via a function φ into a set which looks like $\mathbb{R}^k$ and vice versa. Since only continuity is required, such a manifold can have kinks. For the special case where $k = 0$, the definition implies that a 0-dimensional topological manifold is a discrete set of points. The function φ is called a *coordinate system* for X around x. When the coordinate systems φ are required to be C^r diffeomorphisms instead of homeomorphisms, we obtain the definition of a C^r manifold.

Definition 7.2 For $k \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, a subset X of $\mathbb{R}^m$ is a *k-dimensional C^r manifold* if for every element x of X there exists an open subset U of X containing x, an open subset V of $\mathbb{R}^k$, and an injective and surjective function $\varphi : U \to V$ such that $\varphi \in C^r(U,V)$ and $\varphi^{-1} \in C^r(V,U)$, i.e., $\varphi : U \to V$ is a C^r diffeomorphism.

It follows from Definition 7.2 that the empty set is a k-dimensional C^∞ manifold for every $k \in \mathbb{N}$. When the coordinate systems are required to be differentiable as in Definition 7.2, it is not possible for the manifold to have kinks.

We next generalize the concept of a C^r manifold to a piecewise C^r manifold.

Definition 7.3 For $k \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, a subset X of $\mathbb{R}^m$ is a *k-dimensional piecewise C^r manifold* if X is a k-dimensional topological manifold being a finite union of C^r manifolds.

In Definition 7.3 it is allowed that the dimension of some of the C^r manifolds whose union is equal to X is less than k. A piecewise C^r manifolds can have kinks, but these kinks are restricted to occur at a finite number of lower-dimensional manifolds.

For $k \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, a characterization of a k-dimensional C^r manifold is given in the following theorem. Some authors use this characterization as a definition of a k-dimensional C^r manifold. A function $\varphi : U \to V$ is a C^r coordinate system of $\mathbb{R}^m$ around x if $x \in U$, U and V are open sets of $\mathbb{R}^m$, φ is injective and surjective, $\varphi \in C^r(U, V)$, and $\varphi^{-1} \in C^r(V, U)$.

Theorem 7.4 For $k \in \mathbb{N}_+$, for $r \in \mathbb{N}^*_+$, a subset X of $\mathbb{R}^m$ is a k-dimensional C^r manifold if and only if for every element x of X there exists a C^r coordinate system $\varphi : U \to V$ of $\mathbb{R}^m$ around x satisfying $\varphi(x) = 0^k$ and $\varphi(X \cap U) = \{y \in V \mid \forall i \in \{1, \ldots, m-k\},\ y_i = 0\}$.

To study globally and universally convergent price adjustment processes, the notion of a manifold is too restrictive. Typical choices for P like $\mathbb{R}^L_+ \setminus \{0^L\}$ or Δ^L are not even topological manifolds because they have a boundary. The same is true for the various sets $\overline{P}$ defined in Sections 3, 4, and 5. For the set Δ^L it holds that even the boundary has a boundary when $L \geq 3$. It turns out that the formulation of a manifold presented in Theorem 7.4 can be easily generalized to deal with manifolds that have boundaries, and where the boundaries may have boundaries themselves, and so on.

Definition 7.5 For $k \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, a subset X of $\mathbb{R}^m$ is a *k-dimensional C^r manifold with generalized boundary (MGB)* if for every element x of X there exists a C^r coordinate system $\varphi : U \to V$ of $\mathbb{R}^m$ around x and an integer $\ell(x)$, $0 \leq \ell(x) \leq k$, satisfying $\varphi(x) = 0^m$ and

$$\begin{aligned}\varphi(X \cap U) \quad = \quad \{y \in V | \quad & \forall i \in \{1, \ldots, m-k\},\ y_i = 0, \\ & \forall i \in \{m-k+1, \ldots, m-k+\ell(x)\},\ y_i \geq 0\}.\end{aligned}$$

The neighborhood of an element $\overline{x}$ in a k-dimensional MGB X looks like the set $\mathbb{R}^{\ell(\overline{x})}_+ \times$

$\mathbb{R}^{k-\ell(\overline{x})}$. The unit simplex Δ^L is an $(L-1)$-dimensional C^∞ MGB with, for every $x \in \Delta^L$, $\ell(x) = \#\{i \in \{1,\ldots,m\} \mid x_i = 0\}$.

For $k \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, let the set X be a k-dimensional C^r MGB. For $j \in \{0,\ldots,k\}$, define the set $B^j(X)$ by

$$B^j(X) = \{x \in X \mid \ell(x) = j\}.$$

For every $j \in \{0,\ldots,k\}$, a path-component of $B^j(X)$ is called a *stratum* of X. A stratum in $B^j(X)$ of a C^r MGB is a $(k-j)$-dimensional C^r manifold. The collection of strata of X forms a partition of X. The set $B^0(X)$ is called the *relative interior* of X and the set $X \setminus B^0(X)$ is called the *relative boundary* of X. If $X = B^0(X)$, then X is a k-dimensional C^r manifold. If $X = B^0(X) \cup B^1(X)$, then X is called a *manifold with boundary.* The following theorem reveals that compact 1-dimensional MGB's have a particularly nice structure.

THEOREM 7.6 *For $r \in \mathbb{N}^*_+$, let the set X be a compact 1-dimensional C^r MGB. Then X has a finite number of components, each being C^r diffeomorphic to either the unit circle or the unit interval.*

A set which is homeomorphic to the unit circle is called a *loop* and a set homeomorphic to the unit interval is called an *arc.* Theorem 7.6 therefore implies that a compact 1-dimensional C^r manifold with generalized boundary consists of a finite number of arcs and loops.

Many sets that play a role in economic theory are so-called regular constraint sets. Let U be an open set of $\mathbb{R}^m$ and let, for some $n^1 \in \mathbb{N}$, for every $i \in \{1,\ldots,n^1\}$, $g_i : U \to \mathbb{R}$, and, for some $n^2 \in \mathbb{N}$, for every $i \in \{1,\ldots,n^2\}$, $h_i : U \to \mathbb{R}$. Define the set $M[g,h]$ by

$$M[g,h] = \left\{ x \in U \;\middle|\; \forall i \in \{1,\ldots,n^1\},\ g_i(x) = 0,\ \forall i \in \{1,\ldots,n^2\},\ h_i(x) \geq 0 \right\}.$$

Notice that n^1 and n^2 are allowed to be zero in the definition above. For every element x of U, define the set $I^0(x) = \{i \in \{1,\ldots,n^2\} \mid h_i(x) = 0\}$.

DEFINITION 7.7 For $n^1, n^2 \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, let U be an open set of $\mathbb{R}^m$, for every $i \in \{1,\ldots,n^1\}$, $g_i \in C^r(U,\mathbb{R})$, and, for every $i \in \{1,\ldots,n^2\}$, $h_i \in C^r(U,\mathbb{R})$. The pair (g,h) is a C^r *regular constraint system* if for every $\overline{x} \in M[g,h]$ the vectors $(\partial_x g_i(\overline{x})_{i\in\{1,\ldots,n^1\}}, \partial_x h_i(\overline{x})_{i\in I^0(\overline{x})})$ are linearly independent. The set X is a C^r *regular constraint set (RCS)* if there exists a C^r regular constraint system (g,h) such that $X = M[g,h]$.

The next result shows the relation between an RCS and an MGB.

THEOREM 7.8 *For $r \in \mathbb{N}^*_+$, let the subset X of $\mathbb{R}^m$ be a C^r RCS and let (g,h) be a C^r*

regular constraint system such that $M[g,h] = X$. *If* g *has* n^1 *components, then* X *is an* $(m - n^1)$*-dimensional* C^r *MGB and, for every* $x \in X$, $\ell(x) = \# I^0(x)$.

Theorem 7.8 is very convenient to show that a certain set is an MGB. Let us illustrate this for the unit simplex Δ^L. Define the functions $g : \mathbb{R}^L \to \mathbb{R}$ and, for every $i \in \{1, \ldots, L\}$, $h_i : \mathbb{R}^L \to \mathbb{R}$ by

$$\begin{array}{rcll} g(x) & = & \sum_{j \in \{1,\ldots,L\}} x_j - 1, & x \in \mathbb{R}^L, \\ h_i(x) & = & x_i, & x \in \mathbb{R}^L. \end{array}$$

It is easy to verify that (g, h) is a C^∞ regular constraint system. Since $M[g,h] = \Delta^L$, it holds that the unit simplex Δ^L is a C^∞ RCS and by Theorem 7.8 a C^∞ MGB.

Next, the tangent space and the tangent cone of a manifold with generalized boundary are defined.

Definition 7.9 For $k \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, let X be a k-dimensional C^r MGB. Let $\overline{x}$ be an element of X and let φ be a C^r coordinate system for $\mathbb{R}^m$ around $\overline{x}$. The *tangent space* of X at $\overline{x}$, denoted by $T_{\overline{x}}X$, is the set $\partial\varphi^{-1}(0^m)(\{0^{m-k}\} \times \mathbb{R}^k)$ and the *tangent cone* of X at $\overline{x}$, denoted by $C_{\overline{x}}X$, is the set $\partial\varphi^{-1}(0^m)(\{0^{m-k}\} \times \mathbb{R}_+^{k-\ell(\overline{x})} \times \mathbb{R}^{\ell(\overline{x})})$.

It can be shown that both the tangent space $T_{\overline{x}}X$ and the tangent cone $C_{\overline{x}}X$ as defined in Definition 7.9 do not depend on the choice of the coordinate system. It is easily verified that a tangent cone is indeed a cone. Since φ^{-1} is a C^r diffeomorphism, it holds that $\partial\varphi^{-1}(0^m)$ is an invertible matrix and therefore $T_{\overline{x}}X$ is a k-dimensional vector space. The tangent cone $C_{\overline{x}}X$ is a k-dimensional MGB.

In case a set X is an RCS, the following theorem gives an easy way to determine the tangent space of X at an element $\overline{x}$ of X.

Theorem 7.10 *For* $k \in \mathbb{N}$, *for* $r \in \mathbb{N}^*_+$, *let the subset* X *of* $\mathbb{R}^m$ *be a* k*-dimensional* C^r *RCS, let* $\overline{x}$ *be an element of* X, *and let the pair of functions* (g,h) *be a regular constraint system such that* $X = M[g,h]$. *Then*

$$T_{\overline{x}}X = \left\{ x \in \mathbb{R}^m \,\middle|\, \partial g(\overline{x})(x) = 0^{m-k} \right\}.$$

For $r \in \mathbb{N}^*_+$, let the subsets X of $\mathbb{R}^m$ and Y of $\mathbb{R}^n$ be C^r manifolds, let $\overline{x}$ be an element of X, and let f be a function of $C^r(X, Y)$. Let U be an open set of $\mathbb{R}^m$ such that $X \subset U$ and let the function $g \in C^r(U, \mathbb{R}^n)$ be such that, for every $x \in X$, $g(x) = f(x)$. It can be shown that $\partial g(\overline{x})_{|T_{\overline{x}}X}$ is a function from $T_{\overline{x}}X$ into $T_{f(\overline{x})}Y$. Moreover, the function $\partial g(\overline{x})_{|T_{\overline{x}}X}$ does not depend on the choice of the function g. The *derivative* of f at $\overline{x}$, denoted by

$\partial f(\overline{x})$, is defined by $\partial f(\overline{x}) = \partial g(\overline{x})_{|T_{\overline{x}}X}$. The element $\overline{x}$ is called a *regular point* of f if $\partial f(\overline{x})(T_{\overline{x}}X) = T_{f(\overline{x})}Y$. Otherwise $\overline{x}$ is called a *critical point* of f. Let an element $\overline{y}$ of Y be given. The element $\overline{y}$ is called a *critical value* of f if it is the image of a critical point of f. Otherwise $\overline{y}$ is called a *regular value* of f. Notice that every element y of $Y \setminus f(X)$ is a regular value of f.

For $r \in \mathbb{N}_+^* \setminus \{1\}$, for $m \in \mathbb{N}_+ \setminus \{1\}$, let the subset X of $\mathbb{R}^m$ be an m-dimensional C^r manifold with boundary. Let $\overline{x}$ be an element of $B^1(X)$. To each element $\overline{x}$ we associate the vector $\widehat{g}(\overline{x})$ of $\mathbb{R}^m$ satisfying $\forall x \in T_{\overline{x}}B^1(X)$, $\widehat{g}(\overline{x}) \cdot x = 0$, $\forall x \in C_{\overline{x}}X$, $\widehat{g}(\overline{x}) \cdot x \leq 0$, and $\|\widehat{g}(\overline{x})\|_2 = 1$. It is easily verified that the vector $\widehat{g}(\overline{x})$ is uniquely determined. Obviously, it belongs to the set $\widetilde{B}^{m-1}(0^m, 1) = \{x \in \mathbb{R}^m \mid \sum_{j=1}^m (x_j)^2 = 1\}$. The function $\widehat{g} : B^1(X) \to \widetilde{B}^{m-1}(0^m, 1)$, obtained by associating with every element x of $B^1(X)$ the vector $\widehat{g}(x)$, is called the *Gauss map* of $B^1(X)$. The function $\widehat{g}$ is continuously differentiable and, for every $x \in B^1(X)$, $\partial\widehat{g}(x)$ is a function from $T_xB^1(X)$ into $T_xB^1(X)$. For every $x \in B^1(X)$, the determinant of the linear function $\partial\widehat{g}(x)$, is called the *Gaussian curvature* of $B^1(X)$ at x.

Let C^1 manifolds X, Y, and Z, Z being a subset of Y, an element $\overline{x}$ of X, and a function $f \in C^1(X, Y)$ be given. The function f is said to intersect Z *transversally* at $\overline{x} \in X$, denoted by $f \pitchfork Z$ at $\overline{x}$, if

$$f(\overline{x}) \notin Z, \text{ or } f(\overline{x}) \in Z \text{ and } T_{f(\overline{x})}Z + \partial f(\overline{x})(T_{\overline{x}}X) = T_{f(\overline{x})}Y.$$

The function f is said to intersect Z *transversally* if $f \pitchfork Z$ at every $x \in X$. The following theorem follows almost immediately from the definition of transversality.

Theorem 7.11 *For $k^1, k^2, k^3 \in \mathbb{N}$, let X be a k^1-dimensional C^1 manifold, Y a k^2-dimensional C^1 manifold, $Z \subset Y$ a k^3-dimensional C^1 manifold, and $f \in C^1(X, Y)$ be such that $f \pitchfork Z$. If $k^1 - k^2 + k^3 < 0$, then $f^{-1}(Z) = \emptyset$.*

The following result is complementary to Theorem 7.11 and applies to the case where $k^1 - k^2 + k^3 \geq 0$.

Theorem 7.12 *For $k^1, k^2, k^3 \in \mathbb{N}$, for $r \in \mathbb{N}_+^*$, let X be a k^1-dimensional C^r manifold, Y a k^2-dimensional C^r manifold, $Z \subset Y$ a k^3-dimensional C^r manifold, and $f \in C^r(X, Y)$ be such that $f \pitchfork Z$. If $k^1 - k^2 + k^3 \geq 0$, then $f^{-1}(Z)$ is a $(k^1 - k^2 + k^3)$-dimensional C^r manifold.*

The following result is an easy corollary to Theorems 7.11 and 7.12.

Theorem 7.13 *For $k^1, k^2 \in \mathbb{N}$, for $r \in \mathbb{N}_+^*$, let X be a k^1-dimensional C^r manifold, Y a k^2-dimensional C^r manifold, and $f \in C^r(X, Y)$. Let the element $\overline{y}$ of Y be a regular* *value of f. If $k^1 - k^2 < 0$, then $f^{-1}(\{\overline{y}\}) = \emptyset$, and if $k^1 - k^2 \geq 0$, then $f^{-1}(\{\overline{y}\})$ is a $(k^1 - k^2)$-dimensional C^r manifold.*

To state the last result in this section, we need the notion of Lebesgue measure zero of a set in a manifold X. For $k \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, let the subset X of $\mathbb{R}^m$ be a k-dimensional C^r manifold and let S be a subset of X. Then the set S is said to have *Lebesgue measure zero* in X if there exists a countable cover $\{U^n \mid n \in \mathbb{N}\}$ of S and, for all $n \in \mathbb{N}$, a coordinate system φ^n with domain U^n such that $\varphi^n(U^n \cap S)$ has Lebesgue measure zero. In case $X \subset \mathbb{R}^m$ is an m-dimensional C^r manifold, then the notions of Lebesgue measure zero and Lebesgue measure zero in X coincide.

Theorem 7.14 *For $k^1, k^2, k^3 \in \mathbb{N}$, for $r \in \mathbb{N}^*_+$, let X^1 be a k^1-dimensional C^r manifold, X^2 a C^r manifold, Y a k^2-dimensional C^r manifold, $Z \subset Y$ a k^3-dimensional C^r manifold, and $f \in C^r(X^1 \times X^2, Y)$ with $r > \max(\{0, k^1 - k^2 + k^3\})$. For every $x^2 \in X^2$, define the function $f^{x^2} \in C^r(X^1, Y)$ by*

$$f^{x^2}(x^1) = f(x^1, x^2), \quad x^1 \in X^1.$$

Then $f \pitchfork Z$ implies $f^{x^2} \pitchfork Z$, except for x^2 in a subset of X^2 having Lebesgue measure zero in X^2.

8 Convergence Proof

So far, an economy $\mathcal{E}$ was characterized by an excess demand function $z : P \to \mathbb{R}^L$. We now go back to the primitives and derive the excess demand function from the consumption choices of a finite set M of consumers that face a budget constraint.

For every consumer $i \in M$, the consumption set X^i and the preference relation $\preceq^i$ are assumed to be given in this section as is the starting price system $p^0 \in \dot{\Delta}^L$. After choosing, for every $i \in M$, initial endowments $\omega^i \in X^i$ we obtain an economy $\mathcal{E} = ((X^i, \preceq^i, \omega^i)_{i \in M})$. To show that, generically in initial endowments, a price adjustment process is globally and universally stable, a number of standard assumptions on consumption sets and preference relations are made.

A1. For every consumer $i \in M$, the consumption set X^i is equal to $\mathbb{R}^L_{++}$.

Consider some consumer $i \in M$. The preference relation $\preceq^i$ satisfies the *boundary condition* if, for every $\overline{x}^i \in X^i$, the closure of the set $\{x^i \in X^i \mid \overline{x}^i \preceq^i x^i\}$ of consumption bundles weakly preferred to $\overline{x}^i$ is contained in $\mathbb{R}^L_{++}$. The preference relation $\preceq^i$ is *of the class C^r* for some $r \in \mathbb{N}^*_+$ if the set $\{(x^i, \overline{x}^i) \in X^i \times X^i \mid x^i \sim^i \overline{x}^i\}$ of pairs of consumption bundles among which i is indifferent is a $(2L - 1)$-dimensional C^r manifold.

Let $\preceq^i$ be complete, transitive, continuous, monotonic, i.e., a consumption bundle containing more of every commodity than another consumption bundle is preferred to this other consumption bundle by consumer i, and of the class C^r. For every $\overline{x}^i \in X^i$, we define the sets $I^i(\overline{x}^i) = \{x^i \in X^i \mid x^i \sim^i \overline{x}^i\}$ and $P^i(\overline{x}^i) = \{x^i \in X^i \mid \overline{x}^i \preceq^i x^i\}$ as the sets of consumption bundles indifferent to $\overline{x}^i$ and weakly preferred to $\overline{x}^i$, respectively, by consumer i. It holds that $P^i(\overline{x}^i)$ is an L-dimensional C^r manifold with boundary, with boundary equal to the $(L-1)$-dimensional C^r manifold $I^i(\overline{x}^i)$. The real number $c^i(\overline{x}^i)$ is defined as the Gaussian curvature of $P^i(\overline{x}^i)$ at $\overline{x}^i$. The preference relation $\preceq^i$ is said to have *non-zero Gaussian curvature* if, for every $x^i \in X^i$, $c^i(x^i) \neq 0$. Intuitively, $\preceq^i$ has non-zero Gaussian curvature if indifference curves are nowhere flat.

A2. For every consumer $i \in M$, the preference relation $\preceq^i$ is complete, transitive, continuous, strongly monotonic, strongly convex, of the class C^3, satisfies the boundary condition, and has non-zero Gaussian curvature.

If the economy $\mathcal{E} = ((X^i, \preceq^i, \omega^i)_{i \in M})$ satisfies Assumptions A1 and A2 and for every consumer $i \in M$ it holds that $\omega^i \in X^i$, then it can be shown that the total excess demand function $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ of the economy $\mathcal{E}$ belongs to $C^2(\mathbb{R}^L_{++}, \mathbb{R}^L)$.

Let $(X^i, \preceq^i)_{i \in M}$ satisfy Assumptions A1-A2 and let $p^0 \in \dot{\Delta}^L$ be the starting price system. The set Ω of initial endowments is defined by $\Omega = \prod_{i \in M} \mathbb{R}^L_{++}$. The next definition introduces the notion of regularity for initial endowments $\omega \in \Omega$, which means that the set $\overline{P}_\omega$ as defined in equation (5.1) of Section 5 has a nice structure.

Definition 8.1 Let $(X^i, \preceq^i)_{i \in M}$ satisfy Assumptions A1 and A2 and let $p^0 \in \dot{\Delta}^L$ be the starting price system. The *set of regular initial endowments* Ω^* is the set of initial endowments ω of Ω for which the components of the set $\overline{P}_\omega$ as defined in (5.1) for the economy $\mathcal{E} = ((X^i, \preceq^i, \omega^i)_{i \in M})$ with starting price system p^0 are given by

1. an arc containing p^0 and precisely one Walrasian equilibrium price system as boundary points,
2. a finite number of arcs containing precisely two Walrasian equilibrium price systems both being boundary points,
3. a finite number of loops containing neither p^0 nor any Walrasian equilibrium price system.

If initial endowments ω are regular, then the price adjustment process converges to a uniquely determined Walrasian equilibrium. Moreover, since one Walrasian equilibrium is connected to the starting price system, the other Walrasian equilibria come in pairs, and all Walrasian equilibria belong to $\overline{P}_\omega$, the number of Walrasian equilibria is odd.

The result that we are after is that the set of non-regular initial endowments is small, both in a topological and in a measure theoretic sense. This is the content of Theorem 8.2, stated as Theorem 2.3 in Herings (1997).

THEOREM 8.2 *Let* $(X^i, \preceq^i)_{i \in M}$ *satisfy Assumptions A1 and A2 and let* $p^0 \in \dot{\Delta}^L$ *be the starting price system. Then the set of non-regular initial endowments* $\Omega \setminus \Omega^*$ *has a closure in* Ω *with Lebesgue measure zero.*

The proof of Theorem 8.2 conveys that the set of initial endowments $\omega \in \Omega$ for which the set $\overline{P}_\omega$ is a compact 1-dimensional piecewise C^2 manifold, i.e., a 1-dimensional topological manifold that is a finite union of C^2 manifolds, some possibly of lower dimension, is such that its complement in Ω has a closure with Lebesgue measure zero. All these initial endowments are regular, as the conditions required for $\overline{P}_\omega$ in Definition 8.1 are clearly satisfied.

Theorem 8.2 confirms the well-known result of Dierker (1972) that, generically, an economy has an odd number of Walrasian equilibria.

To prove Theorem 8.2, it is useful to decompose the set $\overline{P}_\omega$ on the basis of the sign of excess demands at prices in $\overline{P}_\omega$. To do so, we define the set of sign vectors

$$S = \{s \in \{-1, 0, +1\}^L \mid \exists \ell^1, \ell^2 \in \{1, \ldots, L\} \text{ such that } s_{\ell^1} = -1 \text{ and } s_{\ell^2} = +1\}.$$

For every sign vector $s \in S$, we define the sets $L^-(s)$, $L^0(s)$, and $L^+(s)$ by $L^-(s) = \{\ell \in \{1, \ldots, L\} \mid s_\ell = -1\}$, $L^0(s) = \{\ell \in \{1, \ldots, L\} \mid s_\ell = 0\}$, and $L^+(s) = \{\ell \in \{1, \ldots, L\} \mid s_\ell = +1\}$. The number of elements in the sets $L^-(s)$, $L^0(s)$, and $L^+(s)$ are denoted by $\ell^-(s)$, $\ell^0(s)$, and $\ell^+(s)$, respectively.

For every $\omega \in \Omega$, for every $s \in S$, we define the set $\overline{P}_\omega(s)$ of price systems by

$$\begin{aligned} \overline{P}_\omega(s) \; = \; \Big\{ p \in \dot{\Delta}^L \Big| \; & \forall \ell \in L^-(s), \; z_\ell(p, \omega) \leq 0 \text{ and } \tfrac{p_\ell}{p^0_\ell} = \min\left(\left\{\tfrac{p_k}{p^0_k} | k \in \{1, \ldots, L\}\right\}\right) \\ & \forall \ell \in L^0(s), \; z_\ell(p, \omega) = 0, \\ & \forall \ell \in L^+(s), \; z_\ell(p, \omega) \geq 0 \text{ and } \tfrac{p_\ell}{p^0_\ell} = \max\left(\left\{\tfrac{p_k}{p^0_k} \Big| k \in \{1, \ldots, L\}\right\}\right)\Big\}, \end{aligned}$$

To make the dependence on ω explicit, the domain of the total excess demand function z is now equal to $\mathbb{R}^L_{++} \times \Omega$. It is easily seen that the set $\overline{P}_\omega$ is equal to $\cup_{s \in S} \overline{P}_\omega(s)$.

Let $s \in S$, $\ell^- \in L^-(s)$, and $\ell^+ \in L^+(s)$. Without loss of generality, it can be assumed that $L^0(s) = \{1, \ldots, \ell^0(s)\}$, $L^-(s) = \{\ell^0(s) + 1, \ldots, \ell^0(s) + \ell^-(s)\}$, and $L^+(s) = \{\ell^0(s) +$

$\ell^-(s)+1,\ldots,L\}$. Let $\omega \in \Omega$. Then $p \in \overline{P}_\omega(s)$ if and only if $p \in \mathbb{R}^L_{++}$ and

$$
\begin{aligned}
z_\ell(p,\omega) &= 0, \qquad \ell \in L^0(s), && (8.1)\\
p_\ell p^0_{\ell+1} - p_{\ell+1}p^0_\ell &= 0, \qquad \ell \in \{\ell^0(s)+1,\ldots,\ell^0(s)+\ell^-(s)-1\}, && (8.2)\\
p_\ell p^0_{\ell+1} - p_{\ell+1}p^0_\ell &= 0, \qquad \ell \in \{\ell^0(s)+\ell^-(s)+1,\ldots,L-1\}, && (8.3)\\
\textstyle\sum_{\ell\in\{1,\ldots,L\}} p_\ell - 1 &= 0, && (8.4)\\
-z_\ell(p,\omega) &\geq 0, \qquad \ell \in L^-(s), && (8.5)\\
z_\ell(p,\omega) &\geq 0, \qquad \ell \in L^+(s), \text{ if } \ell^0(s) \leq L-3, && (8.6)\\
p_\ell p^0_{\ell^-} - p_{\ell^-}p^0_\ell &\geq 0, \qquad \ell \in L^0(s), && (8.7)\\
p_{\ell^+}p^0_\ell - p_\ell p^0_{\ell^+} &\geq 0, \qquad \ell \in L^0(s), && (8.8)\\
p_{\ell^+}p^0_{\ell^-} - p_{\ell^-}p^0_{\ell^+} &\geq 0. && (8.9)
\end{aligned}
$$

Notice that if $\ell^-(s) = 1$, then (8.2) is vacuous. The same holds for (8.3) if $\ell^+(s) = 1$. Since $\ell^-(s)$ and $\ell^+(s)$ are both greater than or equal to one, there are all together $L-1$ equations in (8.1)-(8.4). If $\ell^0(s) > L-3$, so $\ell^0(s) = L-2$, then $\ell^-(s) = \ell^+(s) = 1$. In this case the inequality in (8.6) follows by Walras' law from equality (8.1) and inequality (8.5), so inequality (8.6) is redundant.

Let a sign vector $s \in S$ be given. It is shown in the following that for almost every $\omega \in \Omega$ the set of price systems satisfying (8.1)–(8.9) is a 1-dimensional C^2 manifold with boundary. This is achieved by showing that, for almost every $\omega \in \Omega$, (8.1)–(8.9) yields a regular constraint system as defined in Definition 7.7.

To show Theorem 8.2 it is convenient to define, for every $s \in S$, for every $\omega \in \Omega$, the set $Q_\omega(s)$ by

$$
Q_\omega(s) = \left\{ p \in \dot{\Delta}^{L-1} \,\middle|\, \begin{array}{ll} \frac{p_{\ell'}}{p^0_{\ell'}} = \frac{p_{\ell''}}{p^0_{\ell''}}, & \forall \ell',\ell'' \in L^-(s), \\ z_\ell(p,\omega) = 0, & \forall \ell \in L^0(s), \\ \frac{p_{\ell'}}{p^0_{\ell'}} = \frac{p_{\ell''}}{p^0_{\ell''}}, & \forall \ell',\ell'' \in L^+(s) \end{array} \right\}.
$$

Let $s \in S$. Clearly, for every $\omega \in \Omega$, $\overline{P}_\omega(s) \subset Q_\omega(s)$, the difference between these two sets being that no inequality constraints are taken into account in the definition of $Q_\omega(s)$. In Lemma 8.3 it is shown that there exists a subset $\overline{\Omega}$ of Ω such that $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero and for every $\omega \in \overline{\Omega}$ the set $Q_\omega(s)$ is a 1-dimensional C^2 manifold. Hence, it can be shown to consist of a number of disjoint sets that are diffeomorphic to either the unit circle or the open unit interval.

For every $s \in S$, the function $\psi^s : \mathbb{R}^L_{++} \times \Omega \to \mathbb{R}^{L-1}$ is defined such that, for every $(p,\omega) \in \mathbb{R}^L_{++} \times \Omega$, $\psi^s(p,\omega)$ is the left-hand side of (8.1)-(8.4). For every $s \in S$, for every $\omega \in \Omega$, the function $\psi^{s,\omega} : \mathbb{R}^L_{++} \to \mathbb{R}^{L-1}$ is defined by, for every $p \in \mathbb{R}^L_{++}$, $\psi^{s,\omega}(p) = \psi^s(p,\omega)$. Notice that, for every $s \in S$, for every $\omega \in \Omega$, $Q_\omega(s) = \psi^{s,\omega^{-1}}(\{0^{L-1}\})$.

Lemma 8.3 *Let $(X^i, \preceq^i)_{i \in M}$ satisfy Assumptions A1 and A2 and let $p^0 \in \dot{\Delta}^{L-1}$ be the starting price system. Let $s \in S$. Then there exists a subset $\overline{\Omega}$ of Ω such that $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero and, for every $\omega \in \overline{\Omega}$, $\psi^{s,\omega} \pitchfork \{0^{L-1}\}$ and $Q_\omega(s)$ is a 1-dimensional C^2 manifold.*

Proof: The matrix of partial derivatives of ψ^s evaluated at $(\overline{p}, \overline{\omega}) \in \mathbb{R}^L_{++} \times \Omega$ satisfying $\psi^s(\overline{p}, \overline{\omega}) = 0^{L-1}$ is denoted by $\overline{M}$ and is given in Table 1. Moreover, in Table 1 two submatrices $\overline{M}^1$ and $\overline{M}^2$ of $\overline{M}$ are defined. It is shown that the matrix $\overline{M}$ has rank $L-1$. First it is proved that, for every $i \in M$, $\partial_{\omega^i} z(\overline{p}, \overline{\omega})$ has rank $L-1$. Notice that, for every $i \in M$, $\overline{p}^\top \partial_{\omega^i} z(\overline{p}, \overline{\omega}) = 0^{L^\top}$ and $\partial_{\omega^i} z(\overline{p}, \overline{\omega}) = \partial_{w^i} d^i(\overline{p}, \overline{p} \cdot \overline{\omega}^i)\overline{p}^\top - I^L$, where $d^i(\overline{p}, \overline{p} \cdot \overline{\omega}^i)$ denotes the demand of consumer i at prices $\overline{p}$ and income $\overline{p} \cdot \overline{\omega}^i$. Then, for every $i \in M$,

$$\partial_{\omega^i} z(\overline{p}, \overline{\omega}) \left(\overline{p}_{\ell''} e^L(\ell') - \overline{p}_{\ell'} e^L(\ell'')\right) = \overline{p}_{\ell'} e^L(\ell'') - \overline{p}_{\ell''} e^L(\ell'), \ \forall \ell', \ell'' \in \{1, \ldots, L\},$$

where, for $\ell \in \{1, \ldots, L\}$, $e^L(\ell)$ denotes the ℓ-th L-dimensional unit vector, so the rank of $\partial_{\omega^i} z(\overline{p}, \overline{\omega})$ is equal to $L-1$.

Let some $i \in M$ be given. Consider the first $\ell^0(s)$ rows of $\partial_{\omega^i} z(\overline{p}, \overline{\omega})$. These rows are independent. Suppose not, then $\ell^0(s) \leq L-2$ implies the existence of $y \in \mathbb{R}^L \setminus \{0^L\}$ such that $y_{L-1} = y_L = 0$ and $y^\top \partial_{\omega^i} z(\overline{p}, \overline{\omega}) = 0^{L^\top}$. Since $\overline{p}^\top \partial_{\omega^i} z(\overline{p}, \overline{\omega}) = 0^{L^\top}$, this implies that the rank of $\partial_{\omega^i} z(\overline{p}, \overline{\omega})$ is less than or equal to $L-2$, a contradiction.

Now let $y \in \mathbb{R}^{L-1}$ be such that $y^\top \overline{M} = 0^{LM+L^\top}$. From the previous paragraph it follows that $y^\top \partial_{\omega^1} \psi^s(\overline{p}, \overline{\omega}) = 0^{L^\top}$ implies, for every $\ell \in \{1, \ldots, \ell^0(s)\}$, $y_\ell = 0$.

Suppose $y_{L-1} \neq 0$. Without loss of generality, it can be assumed that $y_{L-1} < 0$. If $\ell^0(s) \geq 1$ or $\ell^-(s) = 1$, then $0 > y_{L-1} = y^\top \partial_{p_1} \psi^s(\overline{p}, \overline{\omega}) = 0$, yielding a contradiction. If $\ell^0(s) = 0$ and $\ell^-(s) \geq 2$, then $y_{L-1} < 0$ and $y^\top \partial_{p_1} \psi^s(\overline{p}, \overline{\omega}) = y_1 p^0_{\ell^0(s)+2} + y_{L-1} = 0$ implies $y_1 > 0$. It is easily seen that, for every $\ell \in \{1, \ldots, \ell^-(s)-2\}$, $y_\ell > 0$ and $y^\top \partial_{p_{\ell+1}} \psi^s(\overline{p}, \overline{\omega}) = 0$ implies $y_{\ell+1} > 0$, so $y_{\ell^-(s)-1} > 0$. It follows that $y^\top \partial_{p_{\ell^-(s)}} \psi^s(\overline{p}, \overline{\omega}) < 0$, leading to a contradiction. Consequently, $y_{L-1} = 0$.

The independence of the rows of $\overline{M}^1$ and of the rows of $\overline{M}^2$ yields $y_{\ell^0(s)+1} = \cdots = y_{L-2} = 0$. So, $y = 0^{L-1}$ and $\overline{M}$ has rank $L-1$.

Since $\overline{M}$ has rank $L-1$, it follows that ψ^s intersects $\{0^{L-1}\}$ transversally, $\psi^s \pitchfork \{0^{L-1}\}$. From our assumptions on the utility functions, it follows that $\psi^s \in C^2(\mathbb{R}^L_{++} \times \Omega, \mathbb{R}^{L-1})$. Moreover, $\mathbb{R}^L_{++}$ is an L-dimensional C^∞ manifold, Ω is an LM-dimensional C^∞ manifold, $\mathbb{R}^{L-1}$ is an $(L-1)$-dimensional C^∞ manifold, and $\{0^{L-1}\}$ is a 0-dimensional C^∞ manifold. Let the set $\overline{\Omega}$ be defined by $\overline{\Omega} = \{\omega \in \Omega \mid \psi^{s,\omega} \pitchfork \{0^{L-1}\}\}$. It follows from Theorem 7.14 that the set $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero in Ω. Since Ω is an LM-dimensional C^∞ manifold that is a subset of $\mathbb{R}^{LM}$, it follows that the set $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero. For every $\omega \in \overline{\Omega}$, $\psi^{s,\omega}$ is a function from an L-dimensional C^∞ manifold into an $(L-1)$-dimensional C^∞ manifold, $\psi^{s,\omega} \in C^2(\mathbb{R}^L_{++}, \mathbb{R}^{L-1})$, and $\psi^{s,\omega} \pitchfork \{0^{L-1}\}$, so ${\psi^{s,\omega}}^{-1}(\{0^{L-1}\})$ and hence $Q_\omega(s)$ is a 1-dimensional C^2 manifold by Theorem 7.13. $\square$

$$
\overline{M} = \begin{array}{|c|c|c|c|c}
\hline
\multicolumn{3}{|c|}{\begin{array}{c}\partial_p z_1(\overline{p},\overline{\omega})\\ \vdots \\ \partial_p z_{\ell^0(s)}(\overline{p},\overline{\omega})\end{array}} & \begin{array}{c}\partial_\omega z_1(\overline{p},\overline{\omega})\\ \vdots \\ \partial_\omega z_{\ell^0(s)}(\overline{p},\overline{\omega})\end{array} & \ell^0(s) \\
\hline
0^{(\ell^-(s)-1)\times \ell^0(s)} & \overline{M}^1 & 0^{(\ell^-(s)-1)\times \ell^+(s)} & 0^{(\ell^-(s)-1)\times LM} & \ell^-(s)-1 \\
\hline
\multicolumn{2}{|c|}{0^{(\ell^+(s)-1)\times(\ell^0(s)+\ell^-(s))}} & \overline{M}^2 & 0^{(\ell^+(s)-1)\times LM} & \ell^+(s)-1 \\
\hline
\multicolumn{3}{|c|}{1^{L^\top}} & 0^{LM^\top} & 1 \\
\hline
\multicolumn{3}{c}{L} & LM &
\end{array}
$$

$$
\overline{M}^1 = \begin{array}{|cccccc|}
\hline
p^0_{\ell^0(s)+2} & -p^0_{\ell^0(s)+1} & & & 0^{\ell^-(s)-2^\top} & \\
0 & p^0_{\ell^0(s)+3} & -p^0_{\ell^0(s)+2} & & 0^{\ell^-(s)-3^\top} & \\
& & \ddots & \ddots & & \\
& 0^{\ell^-(s)-3^\top} & & p^0_{\ell^0(s)+\ell^-(s)-1} & -p^0_{\ell^0(s)+\ell^-(s)-2} & 0 \\
& 0^{\ell^-(s)-2^\top} & & & p^0_{\ell^0(s)+\ell^-(s)} & -p^0_{\ell^0(s)+\ell^-(s)-1} \\
\hline
\end{array}
$$

$$\ell^-(s)$$

$$
\overline{M}^2 = \begin{array}{|cccccc|}
\hline
p^0 p^0_{\ell^0(s)+\ell^-(s)+2} & -p^0_{\ell^0(s)+\ell^-(s)+1} & & & 0^{\ell^+(s)-2^\top} & \\
0 & p^0_{\ell^0(s)+\ell^-(s)+3} & -p^0_{\ell^0(s)+\ell^-(s)+2} & & 0^{\ell^+(s)-3^\top} & \\
& & \ddots & \ddots & & \\
& 0^{\ell^+(s)-3^\top} & & p^0_{L-1} & -p^0_{L-2} & 0 \\
& 0^{\ell^+(s)-2^\top} & & & p^0_L & -p^0_{L-1} \\
\hline
\end{array}
$$

$$\ell^+(s)$$

Table 1: The matrix $\overline{M}$.

For every $s \in S$, for every $\omega \in \Omega$, for every $\ell^1 \in L^-(s) \cup L^+(s)$, the set $Q_\omega(s, \ell^1)$ is defined by

$$Q_\omega(s,\ell^1) = \left\{ p \in \dot{\Delta}^{L-1} \;\middle|\; \begin{array}{ll} \frac{p_{\ell'}}{p^0_{\ell'}} = \frac{p_{\ell''}}{p^0_{\ell''}}, & \forall \ell', \ell'' \in L^-(s), \\ z_\ell(p,\omega) = 0, & \forall \ell \in L^0(s) \cup \{\ell^1\}, \\ \frac{p_{\ell'}}{p^0_{\ell'}} = \frac{p_{\ell''}}{p^0_{\ell''}}, & \forall \ell', \ell'' \in L^+(s) \end{array} \right\}.$$

It is easily verified that $p \in Q_\omega(s, \ell^1)$ if and only if $p \in \mathbb{R}^L_{++}$, p satisfies equations (8.1)-(8.4), and

$$z_{\ell^1}(p,\omega) = 0. \tag{8.10}$$

For every $s \in S$, for every $\ell^1 \in L^-(s) \cup L^+(s)$, the function $\psi^s_{\ell^1} : \mathbb{R}^L_{++} \times \Omega \to \mathbb{R}^L$ is defined such that, for every $(p,\omega) \in \mathbb{R}^L_{++} \times \Omega$, $\psi^s_{\ell^1}(p,\omega)$ is the left-hand side of (8.1)-(8.4) and (8.10). For every $s \in S$, for every $\ell^1 \in L^-(s) \cup L^+(s)$, for every $\omega \in \Omega$, the function $\psi^{s,\omega}_{\ell^1} : \mathbb{R}^L_{++} \to \mathbb{R}^L$ is defined by, for every $p \in \mathbb{R}^L_{++}$, $\psi^{s,\omega}_{\ell^1}(p) = \psi^s_{\ell^1}(p,\omega)$.

Lemma 8.4 *Let $(X^i, \preceq^i)_{i \in M}$ satisfy Assumptions A1 and A2 and let $p^0 \in \dot{\Delta}^{L-1}$ be the starting price system. Let $s \in S$ and $\ell^1 \in L^-(s) \cup L^+(s)$. Then there exists a subset $\overline{\Omega}$ of Ω such that $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero and, for every $\omega \in \overline{\Omega}$, $\psi^{s,\omega}_{\ell^1} \pitchfork \{0^L\}$ and $Q_\omega(s,\ell^1)$ is a 0-dimensional manifold.*

The proof of Lemma 8.4 involves similar techniques as the proof of Lemma 8.3.

For every $s \in S$ with $\ell^0(s) \leq L - 3$, for every $\omega \in \Omega$, for every $\ell^1, \ell^2 \in L^-(s) \cup L^+(s)$ with $\ell^1 \neq \ell^2$, the set $Q_\omega(s, \ell^1, \ell^2)$ is defined by

$$Q_\omega(s,\ell^1,\ell^2) = \left\{ p \in \dot{\Delta}^{L-1} \;\middle|\; \begin{array}{ll} \frac{p_{\ell'}}{p^0_{\ell'}} = \frac{p_{\ell''}}{p^0_{\ell''}}, & \forall \ell', \ell'' \in L^-(s), \\ z_\ell(p,\omega) = 0, & \forall \ell \in L^0(s) \cup \{\ell^1, \ell^2\}, \\ \frac{p_{\ell'}}{p^0_{\ell'}} = \frac{p_{\ell''}}{p^0_{\ell''}}, & \forall \ell', \ell'' \in L^+(s) \end{array} \right\}.$$

Let a sign vector $s \in S$ with $\ell^0(s) \leq L-3$ and some $\ell^1, \ell^2 \in L^-(s) \cup L^+(s)$ with $\ell^1 \neq \ell^2$ be given. Let some $\omega \in \Omega$ be given. It is easily verified that $p \in Q_\omega(s, \ell^1, \ell^2)$ if and only if $p \in \mathbb{R}^L_{++}$, p satisfies the equations (8.1)-(8.4), and

$$z_{\ell^1}(p,\omega) = 0, \tag{8.11}$$

$$z_{\ell^2}(p,\omega) = 0. \tag{8.12}$$

For every $s \in S$ with $\ell^0(s) \leq L - 3$, for every $\ell^1, \ell^2 \in L^-(s) \cup L^+(s)$ with $\ell^1 \neq \ell^2$, the function $\psi^s_{\ell^1,\ell^2} : \mathbb{R}^L_{++} \times \Omega \to \mathbb{R}^{L+1}$ is defined such that, for every $(p,\omega) \in \mathbb{R}^L_{++} \times \Omega$, $\psi^s_{\ell^1,\ell^2}(p,\omega)$ is the left-hand side of (8.1)-(8.4), (8.11), and (8.12). For every $s \in S$ with

$\ell^0(s) \leq L-3$, for every $\ell^1, \ell^2 \in L^-(s) \cup L^+(s)$ with $\ell^1 \neq \ell^2$, for every $\omega \in \Omega$, the function $\psi^{s,\omega}_{\ell^1,\ell^2} : \mathbb{R}^L_{++} \to \mathbb{R}^{L+1}$ is defined by, for every $p \in \mathbb{R}^L_{++}$, $\psi^{s,\omega}_{\ell^1,\ell^2}(p) = \psi^s_{\ell^1,\ell^2}(p, \omega)$.

Lemma 8.5 *Let $(X^i, \preceq^i)_{i \in M}$ satisfy Assumptions A1-A2 and let $p^0 \in \dot{\Delta}^{L-1}$ be the starting price system. Let $s \in S$ be such that $\ell^0(s) \leq L-3$ and let $\ell^1, \ell^2 \in L^-(s) \cup L^+(s)$ be such that $\ell^1 \neq \ell^2$. Then there exists a subset $\overline{\Omega}$ of Ω such that $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero and, for every $\omega \in \overline{\Omega}$, $\psi^{s,\omega}_{\ell^1,\ell^2} \pitchfork \{0^{L+1}\}$ and $Q_\omega(s, \ell^1, \ell^2)$ is an empty set.*

Notice that the condition $\ell^0(s) \leq L-3$ is crucial since for an admissible sign vector s with $\ell^0(s) = L-2$ a corresponding set $Q_\omega(s, \ell^1, \ell^2)$ is equal to the set of Walrasian equilibrium price systems in $\overline{P}_\omega(s)$ of the economy $\mathcal{E} = ((X^i, \preceq^i, \omega^i)_{i \in M})$. Clearly, it cannot be shown that this set is empty for almost every $\omega \in \Omega$.

Let $\omega \in \Omega$. If, for every $\ell \in \{1, \ldots, L\}$, $z_\ell(p^0, \omega) \neq 0$, then it holds that $p^0 \in \overline{P}_\omega(s)$ for a uniquely determined admissible sign vector $s \in S$. Therefore, it is shown in Lemma 8.6 that there exists a subset $\overline{\Omega}$ of Ω such that $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero and, for every $\omega \in \overline{\Omega}$, for every $\ell \in \{1, \ldots, L\}$, $z_\ell(p^0, \omega) \neq 0$. For every $\ell \in \{1, \ldots, L\}$, the function $\psi_\ell : \{p^0\} \times \Omega \to \mathbb{R}$ is defined by, for every $\omega \in \Omega$, $\psi_\ell(p^0, \omega) = z_\ell(p^0, \omega)$. For every $\ell \in \{1, \ldots, L\}$, for every $\omega \in \Omega$, the function $\psi^\omega_\ell : \{p^0\} \to \mathbb{R}$ is defined by $\psi^\omega_\ell(p^0) = \psi_\ell(p^0, \omega)$.

Lemma 8.6 *Let $(X^i, \preceq^i)_{i \in M}$ satisfy Assumptions A1 and A2 and let $p^0 \in \dot{\Delta}^{L-1}$ be the starting price system. Then there exists a subset $\overline{\Omega}$ of Ω such that $\Omega \setminus \overline{\Omega}$ has Lebesgue measure zero and, for every $\omega \in \overline{\Omega}$, for every $\ell \in \{1, \ldots, L\}$, $\psi^\omega_\ell \pitchfork \{0\}$ and $z_\ell(p^0, \omega) \neq 0$.*

The proof of Lemma 8.6 uses similar techniques as the proof of Lemma 8.3.

With this preliminary work out of the way, the proof of Theorem 8.2 consists of three parts. In the first part, which combines Lemmas 8.3, 8.4, 8.5, and 8.6, it is shown that, for almost every $\omega \in \Omega$, for every $s \in S$, the set $\overline{P}_\omega(s)$ is a compact 1-dimensional C^2 manifold with boundary. In this part of the proof, Lemma 8.5 is needed to show that, generically, at most one of the inequalities in (8.5)–(8.9) can be binding at the same time. In the second part, such a generic $\omega \in \Omega$ is fixed, and the sets $\overline{P}_\omega(s)$ corresponding to the various sign vectors $s \in S$ are linked together. Here Lemma 8.5 is crucial again, since it ensures that a boundary point of $\overline{P}_\omega(s)$ is also a boundary point of $\overline{P}_\omega(s')$ for a uniquely determined sign vector $s' \in S \setminus \{s\}$, unless the boundary point is equal to the starting price system p^0 or a Walrasian equilibrium price system p^*. It then follows that for almost every $\omega \in \Omega$ the set $\overline{P}_\omega$ consists of a finite number of arcs and loops. The argument corresponds to a non-linear version of the door-in door-out principle. There is a unique arc having the starting price system p^0 and a unique Walrasian equilibrium price system as boundary points. The other arcs have two Walrasian equilibrium price systems as boundary points, whereas the loops contain no Walrasian equilibrium price systems. Therefore, the set $\Omega \setminus \Omega^*$ has Lebesgue measure zero. In the third part of the proof it is shown that the closure in Ω of the set $\Omega \setminus \Omega^*$ has Lebesgue measure zero. For details of these three parts, the reader is referred to Herings (1997).

9 Further Developments

The price adjustment processes of Smale (1976), van der Laan and Talman (1987), and Kamiya (1990) have an appealing economic interpretation where the adjustment of prices is related to the values of excess demands. Another price adjustment process which admits such an interpretation was introduced in Joosten and Talman (1998), where prices of commodities with the highest excess demand are kept above their initial values, the prices of commodities with the lowest excess demand are allowed to be lower than their initial values, and the prices of other commodities are equal to their initial values. The literature has also considered various price adjustment processes for production economies. Economies with linear or constant returns to scale production were studied by van den Elzen (1993, 1997) and van den Elzen, van der Laan, and Talman (1994) and economies with non-convex production technologies by van den Elzen and Kremers (2006).

Bénassy (1975), Drèze (1975), and Younès (1975) studied exchange economies where price adjustment is subject to rigidities. At a fix-price equilibrium, equality of supply and demand is achieved by adjusting quantities rather than prices. Chapter 11 of Herings (1996) presents a globally and universally converging quantity adjustment process that ends up at a fix-price equilibrium. Herings, van der Laan, Talman, and Venniker (1997), Herings, van der Laan, and Venniker (1998), and Herings, van der Laan, and Talman (1999) study processes where both prices and quantities adjust and eventually a Walrasian equilibrium is reached.

In this paper, the differentiable approach is emphasized. Herings (2002) presents an alternative approach towards the understanding of the universal convergence of price adjustment processes that does not rely on differentiability and points out that convergence can be understood from fixed-point theory, more precisely the fixed point theorem of Browder (1960). Methods from differential topology can be used to show that for a generic economy, where the primitives satisfy suitable differentiability assumptions, the initial price system is connected by exactly one path to a Walrasian equilibrium. Browder's theorem can be used to show that for every economy, where the primitives satisfy suitable continuity assumptions, the initial price system is always connected to a Walrasian equilibrium, though not necessarily by a uniquely specified path. Herings (2002) also extends these insights to strategy adjustment processes used in non-cooperative game theory like the tracing procedure of Harsanyi (1975) and the equilibrium selection procedure of McKelvey and Palfrey (1995).

In the tracing procedure, the players initially choose a best response against a given common prior distribution. Next, they lower the weight on the prior and give some weight to the initial best responses to form a new prior against which a best response is played. This process of updating the initial prior on the basis of best responses continues until the weight on the prior is zero and a Nash equilibrium is reached. Herings and Peeters (2001) show that the tracing procedure converges to a Nash equilibrium for a generic finite normal-form game. Herings and Peeters (2004) extend the tracing procedure to the class of stochastic games and show convergence to subgame perfect equilibrium in stationary strategies for a generic stochastic game.

Jackson and Wolinsky (1996) studies network formation games and introduces the widely used concept of pairwise stability. A network is pairwise stable if it is robust against unilateral link deletion and bilateral link creation. Most of the literature considers unweighted networks, but recently, Bich and Morhaim (2020) study a weighted version, where the strength of each link is measured by a continuous variable. Agents can unilaterally decide to decrease the link strength, whereas it requires the consent of both agents to increase it. Bich and Morhaim (2020) prove that pairwise stable networks exist if all agents have quasi-concave and continuous utility functions. Herings and Zhan (2022) reformulate the network formation problem as a non-cooperative game played by the links and adapt the linear tracing procedure of Harsanyi (1975) to this problem. They show that for a generic network formation problem, a uniquely defined pairwise stable network is selected by the corresponding strategy adjustment process.

McKelvey and Palfrey (1995) study finite normal-form games and assume that each player's payoff is subject to random error. Their concept of quantal response equilibrium is consistent in the sense that all players maximize their utility given the choices made by the other players, and the utility maximizing behavior of a player, together with the error structure, leads to the mixed strategy against which the other players optimize. Quantal response equilibria are quite successful in describing the behavior of participants in experiments. McKelvey and Palfrey (1995) also consider a procedure similar to the tracing procedure to select a Nash equilibrium. Start with the quantal response equilibrium where choices are completely determined by the error terms and follow the path of quantal response equilibria that results when the error terms vanish. McKelvey and Palfrey (1995) show that for almost all games, a unique Nash equilibrium is selected in this way.

Among other frameworks, Tyson (2021) considers finite normal-form games where players are assumed to be exponential satisficers. For a satisficing player, there is a positive probability that the perceived payoffs of different strategies are equal, even when actual payoffs are different. The exponential satisficing model depends on a preference resolution parameter $\gamma \geq 0$, where $\gamma = 0$ implies that all pure strategies are perceived indifferent and higher values of γ correspond to a higher probability that perceived preferences are in line with actual preferences. Tyson also studies the limit, when γ tends to infinity, of exponential satisficing equilibria, as a way to select Nash equilibria.

We have considered price adjustment processes with a natural economic interpretation and briefly discussed strategy adjustment processes with an appealing game-theoretic description. These processes can also be seen as algorithms to compute an economic or game-theoretic equilibrium and thereby contribute to the vast literature on equilibrium computation as pioneered by Scarf (1967), Kuhn (1968), and Eaves (1972). To numerically follow the price adjustment processes presented in this paper, one can either resort to piecewise linear approximations of the problem of interest and then follow the exact price adjustment process for the piecewise linear approximation, or use predictor-corrector methods to approximately follow the price adjustment process for the exact problem. For numerical details, we refer the reader to Garcia and Zangwill (1981) and Allgower and Georg (2012).

There is some experimental support for price adjustment processes like the global Newton method. Asparouhova and Bossaerts (2009) performed experiments where a large pool of subjects trade in securities via a double auction mechanism. It turns out that security prices respond to excess demand of all securities, so not only to their own excess demand as in the Walrasian tâtonnement process. In fact, the authors provide evidence that the resulting price adjustment process coincides with the global Newton method.

Still, in other experimental designs, price adjustment seems to be described fairly well by Walrasian tâtonnement. For instance, in an economic environment based on Scarf's example, Anderson, Plott, Shimomura, and Granat (2004) find strong support for the hypothesis that price dynamics are mainly driven by a market's excess demand, and therefore prices follow the path predicted by Walrasian tâtonnement. This finding is corroborated in the experimental work by Gillen, Hirota, Hsu, Plott, and Rogers (2021), who find that price changes for a good are largely determined by excess demand in its own market with only second-order influences of excess demands in other markets. These findings raise the question whether globally and universally convergent price adjustment processes can be used to stabilize markets. The experimental work by Goeree and Lindsay (2016) suggests that this is the case. They also construct an economic environment based on Scarf's example, use the global Newton method to adjust prices, and observe convergence to a competitive equilibrium.

References

Asparouhova, E., and P. Bossaerts (2009), "Modelling Price Pressure in Financial Markets," *Journal of Economic Behavior and Organization, 72,* 119–130.

Allgower, E.L., and K. Georg (2012), *Numerical Continuation Methods: An Introduction,* Springer Verlag, New York.

Allgower, E.L., and K. Georg (1983), "Predictor-Corrector and Simplicial Methods for Approximating Fixed Points and Zero Points of Nonlinear Mappings," in A. Bachem, M. Grötchel, and B. Korte (eds.), *Mathematical Programming; The State of the Art; Bonn 1982,* Springer-Verlag, Berlin, 15-56.

Anderson, C., C.R. Plott, K.I. Shimomura, and S. Granat (2004), "Global Instability in Experimental General Equilibrium: The Scarf Example," *Journal of Economic Theory, 115,* 209–249.

Bénassy, J.-P. (1975), "Neo-Keynesian Disequilibrium Theory in a Monetary Economy," *Review of Economic Studies, 42,* 503–523.

Bich, P., and L. Morhaim (2020), "On the Existence of Pairwise Stable Weighted Networks," forthcoming in *Mathematics of Operations Research.*

Browder, F.E. (1960), "On Continuity of Fixed Points under Deformations of Continuous Mappings," *Summa Brasiliensis Mathematicae, 4,* 183-191.

Davidenko, D. (1953), "On a New Method of Numerical Solution of Systems of Nonlinear Equations," *Doklady Akad. Nauk USSR, 88,* 601-602. (In Russian)

Debreu, G. (1974), "Excess Demand Functions," *Journal of Mathematical Economics, 1,* 15-21.

Dierker, E. (1972), "Two Remarks on the Number of Equilibria of an Economy," *Econometrica, 40,* 951–953.

Drèze, J.H. (1975), "Existence of an Exchange Equilibrium under Price Rigidities," *International Economic Review, 16,* 301–320.

Eaves, B.C. (1972), "Homotopies for Computation of Fixed Points," *Mathematical Programming, 3,* 1–22.

Eaves, B.C., and K. Schmedders (1999), "General Equilibrium Models and Homotopy Methods," *Journal of Economic Dynamics and Control, 23,* 1249-1279.

Elzen, A.H. van den (1993), *Adjustment Processes for Exchange Economies and Noncooperative Games,* Lecture Notes in Economics and Mathematical Systems, 402, Springer-Verlag, Berlin.

Elzen, A.H. van den (1997), "An Adjustment Process for the Standard Arrow/Debreu Model with Production," *Journal of Mathematical Economics, 27,* 315-324.

Elzen, A.H. van den, and H. Kremers (2006), "An Adjustment Process for Nonconvex Production Economies," *Journal of Mathematical Economics, 42,* 1–13.

Elzen, A.H. van den, G. van der Laan, and A.J.J. Talman (1994), "An Adjustment Process for an Economy with Linear Production Technologies," *Mathematics of Operations Research, 19,* 341-351.

Garcia, C.B., and W.I. Zangwill (1981), *Pathways to Solutions, Fixed Points, and Equilibria,* Prentice-Hall Series in Computational Mathematics, Prentice-Hall, Englewood Cliffs.

Gillen, B.J., M. Hirota, M. Hsu, M., C.R. Plott, and B.W. Rogers (2021), "Divergence and Convergence in Scarf Cycle Environments: Experiments and Predictability in the Dynamics of General Equilibrium Systems," *Economic Theory, 71,* 1033–1084.

Goeree, J., and L. Lindsay (2016), "Market Design and the Stability of General Equilibrium," *Journal of Economic Theory, 165,* 37–68.

Golubitsky, M., and V. Guillemin (1973), *Stable Mappings and Their Singularities,* Springer-Verlag, New York.

Govindan, S., and R. Wilson (2003), "A Global Newton Method to Compute Nash Equilibria," *Journal of Economic Theory, 110,* 65–86.

Harsanyi, J.C. (1975), "The Tracing Procedure: A Bayesian Approach to Defining a Solution for n-Person Noncooperative Games," *International Journal of Game Theory, 4,* 61-94.

Herings, P.J.J. (1996), *Static and Dynamic Aspects of General Disequilibrium Theory,* Theory and Decision Library, Series C: Game Theory, Mathematical Programming and Operations Research, Kluwer Academic Publishers, Norwell, Massachusetts.

Herings, P.J.J. (1997), "A Globally and Universally Stable Price Adjustment Process," *Journal of Mathematical Economics, 27,* 163-193.

Herings, P.J.J. (2002), "Universally Converging Adjustment Processes - a Unifying Approach," *Journal of Mathematical Economics, 38,* 341–370.

Herings, P.J.J., G. van der Laan, and A.J.J. Talman (1999), "Price-Quantity Adjustment in a Keynesian Economy," in P.J.J. Herings, G. van der Laan, and A.J.J. Talman (eds.), *The Theory of Markets,* North-Holland, Amsterdam, 27-57.

Herings, P.J.J., G. van der Laan, A.J.J. Talman, and R. Venniker (1997), "Equilibrium Adjustment of Disequilibrium Prices," *Journal of Mathematical Economics, 27,* 53-77.

Herings, P.J.J., G. van der Laan, and R.J.G. Venniker (1998), "The Transition from a Drèze Equilibrium to a Walrasian Equilibrium," *Journal of Mathematical Economics, 29,* 303-330.

Herings, P.J.J., and R.J.A.P. Peeters (2001), "A Differentiable Homotopy to Compute Nash Equilibria of n-Person Games," *Economic Theory, 18,* 159-186.

Herings, P.J.J., and R.J.A.P. Peeters (2004), "Stationary Equilibria in Stochastic Games: Structure, Selection, and Computation," *Journal of Economic Theory, 118,* 32–60.

Herings, P.J.J., and Y. Zhan (2022), "The Computation of Pairwise Stable Networks," forthcoming in *Mathematical Programming.*

Hirsch, M.W., and S. Smale (1974), *Differential Equations, Dynamical Systems, and Linear Algebra,* Academic Press, New York.

Jackson, M.O., and A. Wolinsky (1996), "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory, 71,* 44–74.

Jongen, H.Th., P. Jonker, and F. Twilt (1983), *Nonlinear Optimization in $\mathbb{R}^n$, I. Morse Theory, Chebyshev Approximation,* Methoden und Verfahren der Mathematischen Physik, 29, Peter Lang, Frankfurt.

Jongen, H.Th., P. Jonker, and F. Twilt (1986), *Nonlinear Optimization in $\mathbb{R}^n$, II. Transversality, Flows, Parametric Aspects,* Methoden und Verfahren der Mathematischen Physik, 32, Peter Lang, Frankfurt.

Joosten, R.A.M.G., and A.J.J. Talman (1997), "A Globally Convergent Price Adjustment Process for Exchange Economies," *Journal of Mathematical Economics, 29,* 15-26.

Judd, K.L. (1997), "Computational Economics and Economic Theory: Subsitutes or Complements ?" *Journal of Economic Dynamics and Control, 21,* 907-942.

Kamiya, K. (1990), "A Globally Stable Price Adjustment Process," *Econometrica, 58,* 1481-1485.

Keenan, D.C. (1981), "Further Remarks on the Global Newton Method," *Journal of Mathematical Economics, 8,* 159-165.

Kuhn, H.W. (1968), "Simplicial Approximation of Fixed Points," *Proceedings of the National Academy of Sciences of the United States of America, 61,* 1238–1242.

Laan, G. van der, and A.J.J. Talman (1987), "A Convergent Price Adjustment Process," *Economics Letters, 23,* 119-123.

Mantel, R.R. (1974), "On the Characterization of Aggregate Excess Demand," *Journal of Economic Theory, 7,* 348-353.

Mas-Colell, A. (1985), *The Theory of General Economic Equilibrium, A Differentiable Approach,* Cambridge University Press, Cambridge.

McKelvey, R.D., and T.R. Palfrey (1995), "Quantal Response Equilibria for Normal Form Games," *Games and Economic Behavior, 10,* 6-38.

Milnor, J.W. (1965), *Topology from the Differentiable Viewpoint,* The University Press of Virginia, Charlottesville.

Saari, D.G. (1985), "Iterative Price Mechanisms," *Econometrica, 53,* 1117-1131.

Saari, D.G., and C.P. Simon (1978), "Effective Price Mechanisms," *Econometrica, 46,* 1097-1125.

Samuelson, P.A. (1941), "The Stability of Equilibrium: Comparative Statics and Dynamics," *Econometrica, 9,* 97–120.

Scarf, H. (1960), "Some Examples of Global Instability of the Competitive Equilibrium," *International Economic Review, 1,* 157-172.

Scarf, H. (1967), "The Approximation of Fixed Points of a Continuous Mapping," *SIAM Journal on Applied Mathematics, 15,* 1328–1343.

Scarf, H. (1973), *The Computation of Economic Equilibria,* Yale University Press, New Haven.

Smale, S. (1976), "A Convergent Process of Price Adjustment and Global Newton Methods," *Journal of Mathematical Economics, 3,* 107-120.

Sonnenschein, H. (1972), "Market Excess Demand Functions," *Econometrica, 40,* 549-563.

Sonnenschein, H. (1973), "Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Functions?," *Journal of Economic Theory, 6,* 345-354.

Tyson, C.J. (2021), "Exponential Satisficing," *American Economic Journal: Microeconomics, 13,* 439–467.

Younès, Y. (1975), "On the Role of Money in the Process of Exchange and the Existence of a Non-Walrasian Equilibrium," *Review of Economic Studies, 42,* 489–501.

Varian, H.R. (1977), "A Remark on Boundary Restrictions in the Global Newton Method," *Journal of Mathematical Economics, 4,* 127-130.