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THE IMPACT OF MULTIPLE INVESTMENT OPPORTUNITIES ON THE INITIAL INVESTMENT

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The Impact of Multiple Investment Opportunities on the Initial Investment

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Abstract

Standard real options theory addresses a firm's investment problem from the perspective of a single investment, ignoring the impact of multiple investment options. This leads to opportunity loss, ultimately foregoing potential value enhancement for the firm. Intriguingly, our results suggest that by neglecting multiple investment options, firms may lose the potential value of size similar to the gains. Such loss in potential value gains is specifically more noticeable in low demand levels when uncertainty, investment cost differences, and discount factors are relatively low. Our findings indicate that considering multiple investment options influences the timing of investments. Specifically, each subsequent investment option reduces the value of waiting, leading to earlier investment undertakings.

Keywords: Investment analysis, Real options, Multiple Investments

1. Introduction

Most real options models explore the firm's investment timing from a single investment perspective where the firm needs to find the optimal time to undertake a lumpy investment project (McDonald and Siegel (1986), Pindyck (1988), Dixit and Pindyck (1994), Bertola (1988)). The main assumptions of the investment project include the irreversibility of investment costs and uncertainty in revenues. The major finding under these assumptions is that the firm has an incentive to delay its investment to gather more information. This *value of waiting* increases as the economic environment becomes less predictable. The complexity of the model increases with multiple investment opportunities due to the path-dependent nature of the investment times, as each decision is contingent upon the previous ones. We model the firm's investment problem using a discrete-time Markov decision process and employ value iteration to find the firm's optimal investment decision.

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Our main results show that the presence of multiple investment options diminishes the value of waiting and indicates a propensity for an earlier initial investment. Since investing earlier implies a greater value derived from undertaking the investment as the number of investments grows, it is crucial to measure the gain in value. To address this, we design a *k-myopic* analysis that considers only a limited set of k investment opportunities out of the total number of investments, providing a clear insight into the dynamics of investment decisions in environments with multiple investment opportunities. The $1 - \mathit{myopic}$ scenario where only one investment opportunity is considered is equivalent to the conventional real options models. The results show that by neglecting the subsequent investment options, as in the conventional models, firms may lose the potential value of size similar to gains. In particular, the loss in potential value gains is more noticeable when the demand level, uncertainty, cost differences, and discount factor are relatively low. Therefore, it is evident that the inclusion of additional investment options is imperative for firms aiming to maximize their value and make more informed investment decisions.

The variations in parameter values, namely investment cost and discount factor, confirm that considering multiple investment options prevents significant losses in potential value gains. In addition, the loss in potential value gain is amplified as the discount factor decreases. Under high uncertainty, firms should consider at least one additional investment option to avoid high losses in potential value gains. The additional investment options can offset the delayed investment thresholds caused by high uncertainty. On the other hand, the results suggest that the effect of considering the additional investment options is more noticeable in low demand levels when the uncertainty is low because transitions to other demand levels are less likely. We show that in all cases, our qualitative conclusions remain intact, i.e. considering additional investment options leads to earlier investment decisions, and prevents losses in potential value gains.

The seminal work by Dixit and Pindyck (1994) considers the standard problem of a firm that needs to find the optimal time to undertake a lumpy investment project. The idea of *lumpy* investment corresponds to scenarios in which firms must make significant, indivisible (or non-continuous) capital investments. Such investments are prevalent in sectors, among others, like automotive, brick, newspapers, and power plants. Wood (2005) state that investments in the British brick industry are lumpy because the major piece of equipment of a factory is a kiln and the size of the kiln depends on the technology acquired at the time of commissioning. Similarly, in the North American newspaper industry, the investment is lumpy because an investment project typically involves the installation of one paper machine at an existing or green-field site that has a minimum efficient scale of production of 220,000 tonnes per year (Booth et al. (1991), p.256). On the other hand, *incremental investments* are investments where the firm can invest in smaller, possibly continuous chunks, a scenario we do not explore in this paper.

The real options theory divides the investment projects into *lumpy* and *incremental* investments. While there is extensive literature on the application of real options theory on incremental and single lumpy investments, not much is said about one-time investments made with other future opportunities in mind. Our study fills this gap by extending the real options models from a single lumpy investment project to multiple lumpy investment projects. For instance, in the power plant industry, operators are often faced with several large investment decisions, often referred to as *lumpy* due to their substantial initial capital requirements and long-term implications. Consider a scenario where an operator is considering building a new power plant to meet the needs of rising urban areas. This would be a significant lumpy investment due to the high initial capital requirements, the long-term commitment, and the irreversible nature of the decision once the plant begins construction. Now, the power plant operator may have renewable energy investment options available such as solar and wind farms. Alternatively, the need to upgrade existing facilities for enhanced efficiency or to align with updated environmental regulations can arise. There may also be the prospect of expanding the operation to other regions where demand is high. Each of these represents a significant, discrete investment. Therefore, an attempt to value the investment project without considering other available investment options leads to opportunity loss and potential value gains.

There are a few studies that explore real options models in a discrete-time setting. Smit and Ankum (1993) explores how competition affects project value, combining investment timing, game theory, and industrial organization principles. Kulatilaka and Perotti (1998) explores strategic investment in growth options in uncertain, competitive markets, a theme further studied by Smit (2003) and Smit and Trigeorgis (2004). Garlappi (2004) examines value dynamics in competitive R&D, highlighting how firms' value responds differently to successes and failures, impacting risk premiums. Murto et al. (2004) looks into diverse investment opportunities for firms, comparing each to a set of real options. Savva and Scholtes (2005) studies bilateral agreements with downstream flexibility, emphasizing optionality's impact on partnership synergies. Brandao and Dyer (2005) presents a simplified method for project valuation under uncertainty, while Fontes and Fontes (2006) devises reliable and flexible investment strategies for capacity, endorsing Markov models.

The structure of the paper is outlined as follows. Section 2 introduces the model's framework. In Section 3, we present the solution method, and Section 4 demonstrates the impact of additional investment options. The robustness of the model is examined in Section 5, and the paper is concluded in Section 6.

2. Model Setup

This paper considers a single firm with multiple investment options. At each stage of the investment decision process, the firm considers follow-up investment options. For each project within the planning horizon, the firm faces a binary decision: *invest* or *not invest*. The firm can only invest once at a time by incurring an immediate irreversible investment cost, i.e., once the investment is made, recovering the cost of undertaking the project is not possible. The firm's revenue depends on the number of investments made.

The set of available investment options is denoted by $\mathcal{J} = \{1, 2, \ldots, m\}$. When considering an investment opportunity $i \in \mathcal{J}$, the firm weighs immediate revenues and discounted expected future revenues against the investment costs, I_i, incurred in making each investment. In the model, uncertainty is embedded in the level of demand, represented by the stochastic variable y. The firm earns immediate revenue of yD_ω, where D_ω represents the demand multiplier. This multiplier increases with the number of projects invested in, denoted by ω. In other words, the more projects a firm invests in, the greater the potential revenue, as indicated by an increase in the demand multiplier. The revenue is discounted by the factor $\gamma \in [0, 1)$. Let $Y = \{y_1, y_2, \ldots, y_n\}$ (where $n \in \mathbb{N}$) be the set of different demand levels, controlled by the transition matrix P_Y.

The order of investment plays a crucial role in determining the effectiveness of the investment strategy. The following lemma shows that the optimal order of investment is dictated by the order of investment cost.

Lemma 2.1. *Given a set of investment opportunities $\mathcal{J}$ and demand multipliers D_ω for the number of invested projects, $\omega \in \Omega$, investing in a non-decreasing order of investment cost I_i for all $i \in \mathcal{J}$ is optimal.*

Proof of lemma 2.1. For the sake of contradiction, assume there exists an optimal order of investment with a pair of investment opportunities $i, j \in \mathcal{J}$ such that i is invested before j and $I_i > I_j$. Now consider the new order of investment where i and j are swapped. The total cost of this new order up to the point of investing in j (which was i in the original order) is lower than in the original order due to $I_j < I_i$. Concurrently, the total cost up to the point of investing in i (which was j in the original) remains the same as in the original order since the costs of i and j are swapped. Given that the demand multiplier depends on the number of investments, the revenue generated at any point in the new order is equal to that in the original order. Consequently, the profit at each point in the new order is greater than or equal to that in the original order, which contradicts the assumption of the original order's optimality. □

This result suggests that the firm does not need to consider complex strategies that involve making investments in a specific sequence. Instead, the firm can simplify its strategy to prioritize investments in ascending order of their cost. We utilize this finding in the following to model our Markov decision process. While we utilize the lemma in formulating our model, a more general setup is given in Appendix A.

We model the described investment problem by employing finite Markov decision processes (MDP) (Bellman (1957), Howard (1960)). MDPs represent a subset of stochastic sequential mechanisms, with applications spanning diverse sectors as evidenced by Kydland and Prescott (1980), Berninghaus and Seifert-Vogt (1993), and White (1993). Fundamentally, within this model, the decision-making entity (in our case, the firm) operates within a dynamic environment where the state fluctuates stochastically given the firm's actions. The states embed information about the environment which affects the immediate reward obtained by the firm, and the probabilities of future transitions (Littman (2001)). Consequently, the firm's objective is to strategize actions optimizing a cumulative, long-term reward metric.

To represent our multiple investment options setting, we use an MDP with a finite set of states, $\mathcal{S} = \{(y, \omega) | y \in Y, \omega \in \Omega\}$. Let $\omega \in \Omega = \{0, 1, \ldots, m\}$ be the number of investments that have been made. We call $\omega \in \Omega$ the internal state of the firm which shows whether the firm has already invested, $\omega \neq 0$, or not invested, $\omega = 0$. In each state, we keep track of the demand level, $y \in Y$, and the internal state of the firm, $\omega \in \Omega$. Given that we are at state $s = (y, \omega) \in \mathcal{S}$, the firm takes action $a \in A_s$ depending on the value of the internal state $\omega \in \Omega$. If $\omega = m$, the firm has already invested in all investment opportunities, and the only option for the firm is to do nothing which is denoted by $a = 0$. If $\omega < m$ the firm has more investment options, and it needs to decide whether to *invest*, represented by "1", or *not invest*, represented by "0". The action space A_s, available to the firm at state $s = (y, \omega)$, is given by:

$$A_s := \begin{cases} \{0\} & \text{if } \omega = m \\ \\ \{0, 1\} & \text{otherwise} \end{cases}. \tag{1}$$

When an action is taken, the immediate reward function for the firm by taking action $a \in A_s$ at state $s = (y, \omega) \in \mathcal{S}$ can be formally expressed as:

$$R(y, \omega, a) := yD_{\omega+a} - aI_{\omega+a}, \forall y \in Y, \omega \in \Omega, a \in A_{(y,\omega)}, \tag{2}$$

and the internal state of the firm is updated as $\bar{\omega} = \omega + a$. The demand level is updated according to the dynamics, P_Y. The transition function π in our MDP is given by

$$\pi(\bar{y}, \bar{\omega}|y, \omega, a) = \begin{cases} P(\bar{y}|y) & \text{if } \bar{\omega} = \omega + a \\ \\ 0 & \text{otherwise} \end{cases}. \tag{3}$$

The equation implies that if $\bar{\omega}$ follows the update rule $\bar{\omega} = \omega + a$, the system transitions according to the demand dynamics $P(\bar{y}|y)$.

3. Solution Methodology

In this section, the value function is formulated based on the Markov decision process presented in Section 2. The value iteration algorithm is employed to solve the MDP, thereby determining the firm's optimal investment policy in the market.

3.1. Value iteration

Value Iteration is a numerical method for finding the optimal policy in an MDP that aids in decision-making by providing the maximum expected reward for all possible states, considering all possible actions (Bellman (1957) and Puterman (2014)). The primary objective of the MDP model, as described in Section 2, is to identify a policy - a sequence of actions - that maximizes the anticipated cumulative rewards. The algorithm iteratively improves the value function estimates until they converge to the optimal value. Once the optimal value function is obtained, the optimal policy can be directly determined. This method is crucial for finding an effective solution to an MDP where the objective is to maximize the cumulative reward (Sutton and Barto (2018)).

A policy is a set of rules that an agent employs to determine which action to take in a particular state to maximize its expected total reward over time. This is fundamentally a mapping from states to actions, serving as the entity's decision-making mechanism in response to its environments. A strategy can either be stochastic, averaging it assigns actions based on a probability distribution, or deterministic, which implies that it consistently selects a specific action for each state. As stated by Bertsekas (1987), every Markov decision process inherently has a deterministic stationary optimal policy. Therefore, a strategy, denoted as σ, is described as $\sigma : \mathcal{S} \to A$, making sure that $\sigma(s)$ maps to an action $a \in A_s$ for every state $s \in \mathcal{S}$. To evaluate the strategy σ, the value function is utilized. Let $V^{\sigma}(s)$ represent the discounted expected reward from state $s \in \mathcal{S}$ when adhering to the policy σ. It can be formally expressed through the following equation:

$$V^{\sigma}(s) = R(s, \sigma(s)) + \gamma \sum_{\bar{s} \in \mathcal{S}} \pi(\bar{s}|s, \sigma(s)) V^{\sigma}(\bar{s}), \quad \forall s \in \mathcal{S}. \tag{4}$$

Given an initial state $s \in \mathcal{S}$, the firm aims to find a policy σ that maximizes the total reward (i.e. $V^{\sigma}(s)$). Howard (1960) showed that there exists an optimal policy σ^* for any given initial state. The optimal value function, $V^*(s)$, can be found by

$$V^*(s) = \max_{a \in A_s} R(s, a) + \gamma \sum_{\bar{s} \in \mathcal{S}} \pi(\bar{s}|s, a) V^*(\bar{s}). \tag{5}$$

The Bellman equation provides a recursive computation for the value function of a policy in an MDP. It expresses that the value of a state under a certain policy is equal to the expected immediate reward plus the expected future value of the next state, considering that actions are chosen according to the policy. This recursive nature is fundamental for the iterative procedure of value iteration (Bellman (1957)).

Considering the investment problem in Section 2, if the firm at some state $s \in \mathcal{S}$ has already invested in all projects in the market (i.e. $\omega = m$) then the firm does not make a decision (i.e. $a = 0$). Therefore, from equation (5) the Bellman equation of the monopolistic firm is given by the linear system

$$V^*(y, \omega) = V^*(y, m) = y D_m + \gamma \mathbb{E}_P \left[V^*(\bar{y}, m) | y \right]. \tag{6}$$

If at state $s \in \mathcal{S}$, the firm has not invested in all projects (i.e. $\omega \neq m$), then for any given $y \in Y$, the Bellman equation of the monopolistic firm is given by the linear system

$$V^*(y,\omega) = \max \begin{cases} yD_\omega + \gamma \mathbb{E}_P \left[V^*(\bar{y},\omega) | y \right] & \text{if } a = 0, \\ yD_{\omega+1} - I_{\omega+1} + \gamma \mathbb{E}_P \left[V^*(\bar{y},\omega+1) | y \right] & \text{if } a = 1 \end{cases} \tag{7}$$

The value iteration algorithm iteratively updates the value of each state until the value function converges within a pre-defined threshold θ. The procedure starts by initializing the value function V, a policy σ, and a dictionary a^{dict} for storing the values of actions at each state. Then the algorithm iterates over the states and possible actions. The *Value Iteration* procedure returns the optimal value function V^* and the optimal policy σ^*. The optimal policy is determined by choosing the action that leads to the highest value in each state. By following this policy, the decision-maker can maximize the expected cumulative reward over time. The returned optimal value function V represents the maximum expected cumulative reward that can be achieved from each state, given that the decision-maker follows the optimal policy.

4. Impact of Additional Investment Options

The strategic decision-making process within firms often involves the evaluation of various investment options. In this context, understanding the impact of additional investment options is imperative, as these options can significantly influence the overall value of the investment portfolio. Starting with a single investment, the analysis is extended to explore markets characterized by up to 10 investments. This allows us to systematically observe the changes in both investment values and investment timing resulting from the sequential inclusion of additional investments following the initial one.

We use a *discrete-time discrete-state random walk* example in Chapter 3 of Dixit and Pindyck (1994) in this section to create the matrix of demand dynamics[1]. The dynamics of the demand are controlled by an $n \times n$ probability

[1]The stochastic processes considered in Dixit and Pindyck (1994) follow the fundamental assumption of the positive persistence of uncertainty. This fundamental assumption states that a higher current value of demand shifts the distribution of future values to the right. However, there are economic examples in the real world such as fidget spinners where the positive persistence of uncertainty assumption does not hold (see Faninam et al. (2023)). We analyze the investment timing of the first investment when the positive persistence of uncertainty does not hold in Section 5.4.

transition matrix where the random variable takes a jump either up or down with a probability half for each.

$$P_Y = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \end{array} \\ \begin{pmatrix} \mathbf{0.5} & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0.5} & \mathbf{0.5} \end{pmatrix} \end{array} . \tag{8}$$

The data presented in Table 1 illustrate the optimal actions and their corresponding values across various numbers of investments. We have assumed uniform investment costs across all investments to neutralize the impact of varying cost structures on our findings. This assumption is later revisited and modified in Section 5.1. Table 1 provides insights into how multiple investment options can affect investment timing by providing key metrics calculated by the value iteration algorithm. The values depicted in the table highlight the advantages of early investment when a wider range of options is available, emphasizing the significance of evaluating multiple investment options during the decision-making process. Each additional investment opportunity seems to facilitate earlier investment decisions, providing a higher perceived value from the investment (see also figure 1b).

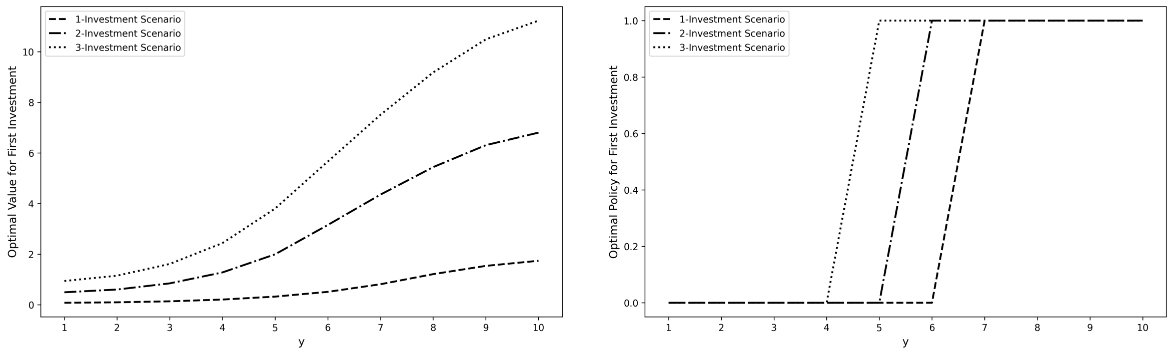
In scenarios involving a single investment, overlooking future investment opportunities results in a higher perceived value of waiting. This is evident when the demand level reaches y_7, at which point the value of investing (0.81176) surpasses that of waiting (0.77466). However, in a two-investment context, the firm factors in the potential of a subsequent project while deciding on the timing of the initial investment. This consideration lowers the critical demand level to y_6, where the value of investing (3.15304) exceeds the value of waiting (2.85647). This shift indicates that incorporating multiple investment options into the decision-making process reduces the value of waiting, particularly in our model where the demand multiplier D_ω increases linearly.

Once the firm commits to its first investment ($\omega = 1$), the threshold for investing in a second project drops to an even earlier demand level, y_5. This trend suggests that initial investments pave the way for subsequent ones, a natural outcome given the linear increase of the demand multiplier ($D_1 > D_0 > 0$). In a scenario with three investments, the firm weighs even more investment options. If the firm has yet to invest (D_0), it opts to invest at y_5, earlier than in both the two-investment and single-investment scenarios. This pattern underscores a key insight: the increase in available investment options correlates with a diminished value of waiting, leading to earlier investment decisions.

Table 1: The values are calculated using the value iteration algorithm. The parameter D_ω is defined such that $D_\omega = 0.5 + \omega, \omega \in \Omega$. The investment cost is $I_1 = I_2 = I_3 = 2.35$, $\gamma = 0.9$, $n = 10$, $Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \le l \le n, y = \frac{l}{n}\right\}$, and P_Y is given by equation (8).

Index	D_ω		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
			Single Investment									
First	D_0	action	0	0	0	0	0	0	$\mathbf{a_1}$	a_1	a_1	a_1
		optimal value	0.07988	0.09764	0.13709	0.20701	0.32293	0.51061	0.81176	1.21086	1.53707	1.73942
		waiting value	0.07988	0.09764	0.13709	0.20701	0.32293	0.51061	0.77466	1.05697	1.32763	1.47442
		investing value	-1.43943	-1.23708	-0.91087	-0.51176	-0.07453	0.37453	0.81176	1.21086	1.53707	1.73942
			Two Investments									
First	D_0	action	0	0	0	0	0	$\mathbf{a_1}$	a_1	a_1	a_1	a_1
		optimal value	0.49331	0.60294	0.84655	1.27829	1.99410	3.15304	4.35362	5.43982	6.30733	6.80327
		waiting value	0.49331	0.60294	0.84655	1.27829	1.99410	2.85647	3.86678	4.79743	5.50939	5.89977
		investing value	-0.89494	-0.57159	0.02350	0.89912	1.95494	3.15304	4.35362	5.43982	6.30733	6.80327
Second	D_1	action	0	0	0	0	$\mathbf{a_2}$	a_2	a_2	a_2	a_2	a_2
		optimal value	1.45506	1.77840	2.37350	3.24912	4.47640	5.82359	7.13528	8.33259	9.31122	9.91827
		waiting value	1.45506	1.77840	2.37350	3.24912	4.30494	5.50304	6.70362	7.78982	8.65733	9.15327
		investing value	0.38172	0.98877	1.96740	3.16471	4.47640	5.82359	7.13528	8.33259	9.31122	9.91827
			Three Investments									
First	D_0	action	0	0	0	0	$\mathbf{a_1}$	a_1	a_1	a_1	a_1	a_1
		optimal value	0.94061	1.14964	1.61414	2.43735	3.80218	5.66014	7.50697	9.18027	10.48910	11.22862
		waiting value	0.94061	1.14964	1.61414	2.43735	3.64387	5.08912	6.67818	8.09823	9.18400	9.77297
		investing value	-0.38303	0.05407	0.90196	2.20769	3.80218	5.66014	7.50697	9.18027	10.48910	11.22862
Second	D_1	action	0	0	0	$\mathbf{a_2}$	a_2	a_2	a_2	a_2	a_2	a_2
		optimal value	1.96697	2.40407	3.25196	4.57560	6.50588	8.60210	10.67714	12.56155	14.08149	14.98212
		waiting value	1.96697	2.40407	3.25196	4.55770	6.15218	8.01014	9.85697	11.53027	12.83910	13.57862
		investing value	0.92621	1.65425	2.90177	4.57560	6.50588	8.60210	10.67714	12.56155	14.08149	14.98212
Third	D_2	action	0	0	0	0	$\mathbf{a_3}$	a_3	a_3	a_3	a_3	a_3
		optimal value	3.27621	4.00425	5.25177	6.92560	9.02734	11.27265	13.45881	15.45433	17.08537	18.09712
		waiting value	3.27621	4.00425	5.25177	6.92560	8.85588	10.95210	13.02714	14.91155	16.43149	17.33212
		investing value	2.20287	3.21462	4.84567	6.84119	9.02734	11.27265	13.45881	15.45433	17.08537	18.09712

Figures 1a and 1b illustrate the effect of multiple investment options on the optimal value and the optimal policy for the initial investment. As demand, y, increases, Figure 1a shows a significant increase in the optimal value of the first investment. This is consistent with the idea that higher levels of demand represent more favorable market conditions. When comparing multiple investment scenarios, the optimal value increases significantly, highlighting the benefit of having more options to invest. Figure 1b shows that with more investment options available, firms are more likely to invest earlier.



(a) The optimal value

(b) The optimal policy

Figure 1: The optimal value of the optimal policy in different multiple investment scenarios. The parameter values are given by $D_\omega = 0.5 + \omega, I_1 = I_2 = I_3 = 2.35, \gamma = 0.9, n = 10, Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \le l \le n, y = \frac{l}{n}\right\}$, and P_Y is given by equation (8).

4.1. k-myopic policy

Quantifying the value of considering additional investment opportunities is essential, as many firms often face multiple investment opportunities when making an investment decision. We design a *k-myopic* analysis that considers only a limited set of k investments out of the total number of investments (m). The *k-myopic* is employed to quantify the loss in potential value gain by considering additional investment options for the initial investment. The loss is measured by comparing the optimal value of considering all investments and a limited (k) number of investments.

For instance, in a scenario with three investments, the firm's 1-myopic policy evaluates the first investment as if there is only one investment opportunity (the firm invests when the demand level is at y_7), yet optimally evaluates the subsequent investments (second and the third investments are made as soon as the demand level reaches y_4 and y_5, respectively). In essence, moving from a 1-myopic to a 2-myopic scenario or from a 2-myopic to a 3-myopic scenario demonstrates the impact of an additional investment option on the timing of the initial investment. For a k-myopic policy, the values are optimal when the total number of investments matches k. For instance, a 3-myopic policy indicates the optimal strategy with three total investments. The same logic holds across single and two-investment scenarios as well. Next, we evaluate these investment policies and measure the potential loss in value gains in different scenarios for the first investment.

We extend the previous example to $|\Omega| = 10$ investments with $n = 50$ and show the change in the investment threshold of the initial investment by considering additional investment options. Table 2 shows myopic values, optimal values, value loss, and the percentage loss by taking $k-$myopic policies for the first investment. The table is divided into parts for one, two, or three investments. In the case of a single investment, the $1-$myopic is optimal. However, for two or three investments, the $1-$myopic value is suboptimal, meaning there is some loss in value, especially when considering three investments. For instance, consider the two-investment scenario. The first row shows the optimal value by taking the optimal policy for the firm when it has not yet invested in any investments. The loss in value is calculated by comparing the 1-myopic value with the benchmark optimal values. At demand level y_1, the myopic value is 0.42845, while the optimal value is 0.49331, leading to a value loss of 0.06486, or a 13.15% loss. This pattern of loss in value continues for higher investment levels.

In a three-investment situation, at the same demand level (y_1), the myopic value decreases to 0.73879 compared to the optimal 0.94061, resulting in a more significant value loss of 0.20182, which translates to a 21.46% loss. The transition from a 1-myopic to a 2-myopic policy demonstrates a considerable improvement in investment decision-making by aligning closer to the optimal values and reducing value loss. For example, at the demand level y_6, under a 1-myopic policy, the myopic value is 4.72201 compared to the optimal value of 5.66014, resulting in a value loss of 0.93813, which translates to a percentage loss of around 16.57%. However, when the policy is adjusted to 2-myopic, the myopic value at the same demand level (y_6) becomes equal to the optimal value (5.66014), effectively eliminating the value loss and the associated percentage loss at this level. This elimination of loss at y_6 under the 2-myopic policy signifies that the policy is now capable of capturing the optimal investment strategy for that particular demand level, demonstrating the benefits of considering more future-oriented decision-making in investment scenarios.

Table 2: The values are calculated using the value iteration algorithm. The parameter D_ω is defined such that $D_\omega = 0.5 + \omega, \omega \in \Omega$. The investment cost is $I_1 = I_2 = I_3 = 2.35, \gamma = 0.9, n = 10, Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \le l \le n, y = \frac{l}{n}\right\}$, and P_Y is given by equation (8).

Index	D_ω	Value	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
		Single Investment										
First	D_0	optimal	0.07988	0.09764	0.13709	0.20701	0.32293	0.51061	0.81176	1.21086	1.53707	1.73942
		1-myopic	0.07988	0.09764	0.13709	0.20701	0.32293	0.51061	0.81176	1.21086	1.53707	1.73942
		loss	0	0	0	0	0	0	0	0	0	0
		% loss	0	0	0	0	0	0	0	0	0	0
		Two Investments										
First	D_0	optimal	0.49331	0.60294	0.84655	1.27829	1.99410	3.15304	4.35362	5.43982	6.30733	6.80327
		1-myopic	0.42845	0.52367	0.73525	1.11023	1.73193	2.73849	4.35362	5.43982	6.30733	6.80327
		loss	0.06486	0.07927	0.11130	0.16806	0.26217	0.41454	0	0	0	0
		% loss	13.14748	13.14745	13.14742	13.14741	13.14740	13.14739	0	0	0	0
		2-myopic	0.49331	0.60294	0.84655	1.27829	1.99410	3.15304	4.35362	5.43982	6.30733	6.80327
		loss	0	0	0	0	0	0	0	0	0	0
		% loss	0	0	0	0	0	0	0	0	0	0
		Three Investments										
First	D_0	optimal	0.94061	1.14964	1.61414	2.43735	3.80218	5.66014	7.50697	9.18027	10.48910	11.22862
		1-myopic	0.73879	0.90297	1.26781	1.91438	2.98637	4.72201	7.50697	9.18027	10.48910	11.22862
		loss	0.20182	0.24667	0.34634	0.52297	0.81581	0.93813	0	0	0	0
		% loss	21.45643	21.45641	21.45638	21.45637	21.45636	16.57433	0	0	0	0
		2-myopic	0.88557	1.08236	1.51969	2.29472	3.57968	5.66014	7.50697	9.18027	10.48910	11.22862
		loss	0.05504	0.06728	0.09446	0.14263	0.22250	0	0	0	0	0
		% loss	5.85196	5.85195	5.85195	5.85194	5.85194	0	0	0	0	0
		3-myopic	0.94061	1.14964	1.61414	2.43735	3.80218	5.66014	7.50697	9.18027	10.48910	11.22862
		loss	0	0	0	0	0	0	0	0	0	0
		% loss	0	0	0	0	0	0	0	0	0	0

Figure 2 illustrates the average loss across all states and its percentage deviation from the optimal value. In Figure 2, we observe a significant decrease in the average loss in potential value gains for the initial investment as the number of investment options increases. This decrease is particularly noticeable in a scenario where firms neglect to consider any subsequent investment options as in standard real options theory. With additional investment options, the average loss in potential value gains drops from 66.99% to 32.28%. The benefits of additional investments diminish as the number of investment options increases. For instance, the increase in value drops to nearly 16% when considering three investments. It further diminishes to a mere 0.02% with nine investments and ultimately results in no loss at all with ten investments.

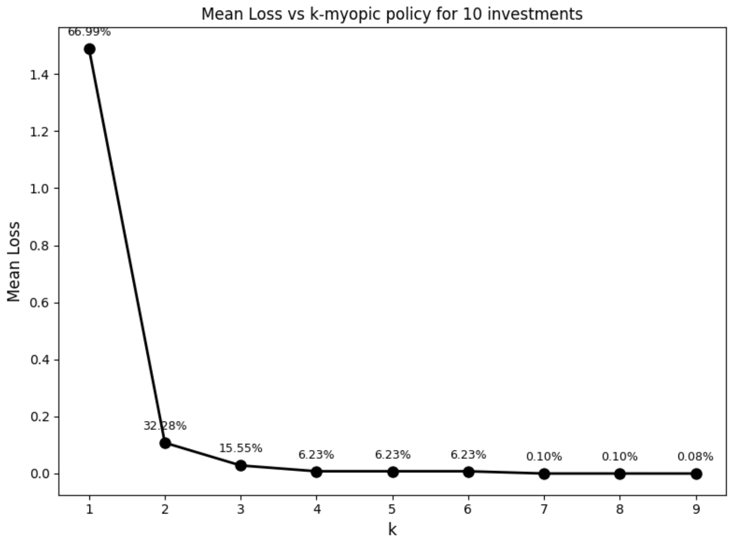


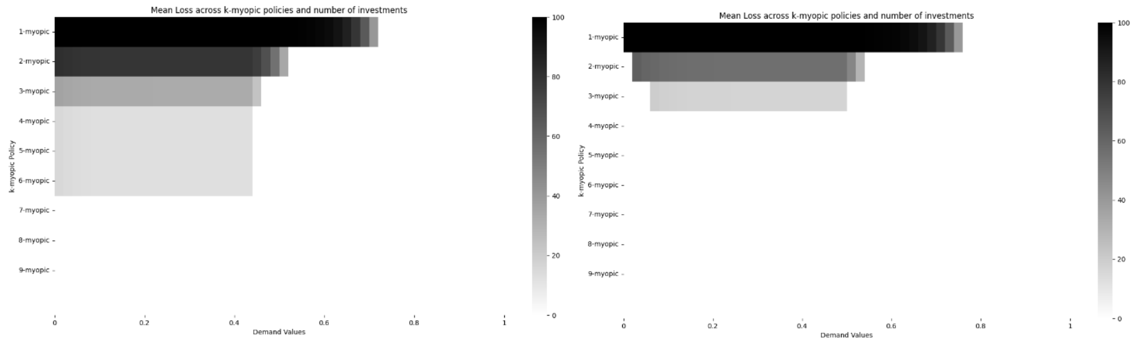
Figure 2: The average value loss by taking a k-myopic decision in the market. The parameter values are given by $D_\omega = 0.5 + \omega, I_\omega = 2.35, \gamma = 0.9, n = 50, Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \leq l \leq n, y = \frac{l}{n}\right\}$, and P_Y is created by equation (8).

5. Robustness of Results

In this section, two types of robustness checks are performed to verify the validity of our results. First, we consider the effect of changes in parameter values. In particular, we consider variations in investment costs and the discount factor. The results reveal that the propensity for an earlier initial investment remains the same for a wide range of values. Second, we consider changes in demand dynamics, in particular, uncertainty and the fundamental assumption of the positive persistence of uncertainty on the probability distribution of demand levels (see Dixit and Pindyck (1994) and Faninam et al. (2023)). The fundamental assumption of the positive persistence of uncertainty for demand implies that a higher current value of demand should shift the distribution of future values to the right. Dixit (1992) suggests that the positive persistence of uncertainty assumption may not be present in cases where large demand realizations lead to future decreases in demand. We show that changes in uncertainty and the absence of the positive persistence of uncertainty do not change our qualitative conclusions.

5.1. *Investment Cost*

The main results in Section 4 are derived under the assumption that investment cost is the same for investment opportunities. Here, we consider the case where the investment cost is linearly increasing. Figure 3 shows the sensitivity of the investment thresholds to different investment costs. With larger cost differences, the loss in value is higher in a 1-myopic scenario, but it decreases when more investment option is considered. In situations with large cost differences between options, considering only one additional investment option leads to significant value gains. The results show that as the cost differences between investment options grow, the firm can achieve near-optimal values by considering fewer investment options.



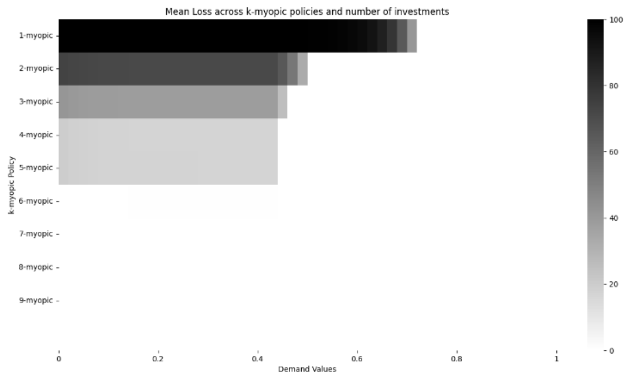
(a) Identical cost, $d = 0$.

(b) Non-identical cost, $d = 0.1$.

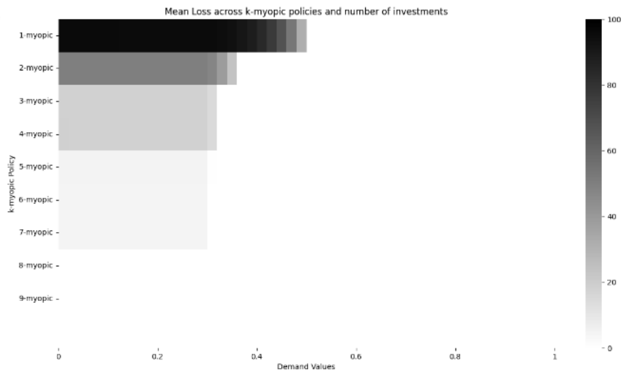
Figure 3: Comparing the average value loss by taking a k-myopic decision across different investment costs in the market. The parameter values are given by $D_\omega = 0.5 + \omega, I_\omega = 2.35 + d\omega, \gamma = 0.9, Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \leq l \leq n, y = \frac{l}{n}\right\}$, and P_Y is given by equation (8).

5.2. *Discount Factor*

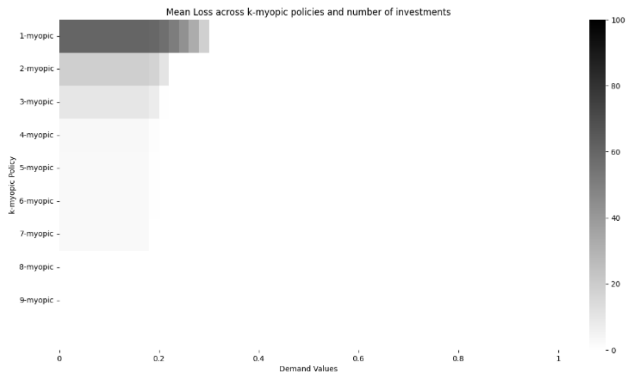
We examine the effect of different discount factors on the initial investment threshold, considering $|\Omega| = 10$ investments with $n = 50$. First, the results reveal that the accrued loss in potential value gain by taking a myopic decision is specifically large when the demand values are relatively lower, and this loss is further amplified as the discount factor decreases. The reason is that a lower discount factor might influence the timing of investments. With future returns devalued, the incentive to invest early (at medium demand levels) decreases. Firms might, therefore, delay investments until demand is high, leading to increased value loss at these demand levels due to suboptimal investment decisions.



(a) $\gamma = 0.85$



(b) $\gamma = 0.9$



(c) $\gamma = 0.95$

Figure 4: The effect of varying discount factors on the investment timing of the initial investment. The parameter values are given by $D_\omega = 0.5 + 0.75\omega, I_\omega = 2.35, \gamma = 0.9, n = 50, Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \leq l \leq n, y = \frac{l}{n}\right\}$, and P_Y is given by equation (8).

5.3. Uncertainty

The effect of different uncertainty levels is examined to study the threshold behavior of the initial investment and the loss in potential value gains by making myopic decisions. Entropy, in the context of information theory, quantifies the amount of uncertainty or randomness associated with a set of probabilities. Introduced into communication theory by Shannon (1948), entropy provides a measure of the information content of a random variable. Specifically, for a probability distribution, a higher entropy value indicates greater unpredictability or randomness, while a lower value indicates greater certainty.

In our context, entropy is used to classify the uncertainty in the distribution of transitions from one state to another.

Given a row vector $p = [p_1, p_2, \ldots, p_n]$ representing transition probabilities, the entropy $H(p)$ is defined as:

$$H(p) := -\sum_{i=1}^{n} p_i \log p_i, \tag{9}$$

where the logarithm has a base of 2. The entropy value is maximized when all p_i are equal, reflecting maximum unpredictability. It is minimized (equal to zero) when one of the p_i is 1 and the rest are 0, indicating certainty in the transition.

The proposed method first establishes entropy bounds. Given a matrix of dimension n, the minimum and maximum possible entropies are denoted as $E_{\min}(n)$ and $E_{\max}(n)$, respectively. Using these entropy extremes, two essential thresholds are defined. The first threshold, τ_1, is given by $\tau_1 = E_{\min}(n) + \frac{1}{3} \times (E_{\max}(n) - E_{\min}(n))$. The next threshold, τ_2, is defined as $\tau_2 = E_{\min}(n) + \frac{2}{3} \times (E_{\max}(n) - E_{\min}(n))$. Using these thresholds, the average entropy E_m is classified into three different categories. Specifically, for $E_m < \tau_1, \tau_1 \leq E_m < \tau_2$, and $E_m \geq \tau_2$, the system is described as having *low uncertainty*, *medium uncertainty*, and *high uncertainty*, respectively.

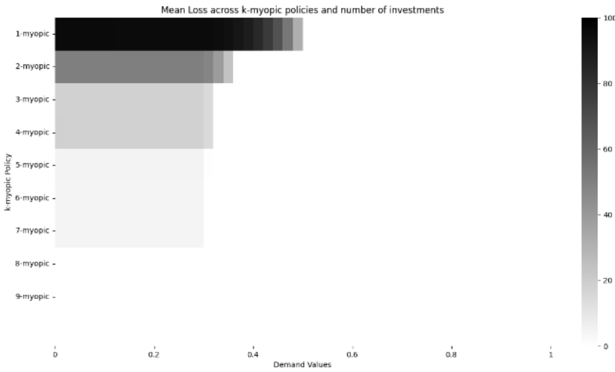
In equation (8), we formulated a transition matrix based on a simple random walk model with a 50% chance of moving up or down, and this structure represents low uncertainty based on the entropy measure. In transition matrices, the degree of uncertainty is reflected in the distribution of probabilities across demand levels. When future demand levels become less predictable, due to a greater spread of probabilities across multiple levels rather than just adjacent ones, the matrix represents higher uncertainty. To create such a matrix, as illustrated in equation (10), we make modifications to widen the range of potential transitions. Instead of only considering immediate neighboring demand levels (i.e., one step up or down), we incorporate more distant levels. Specifically, equation (10) is derived using the method outlined in the algorithm (2) found in Appendix B. In this method, the number of neighboring levels with positive transition probabilities, referred to as *spread*, is set to half of the total number of demand levels, and the initial value for the diagonal elements is set to $q = 0.2$. Similarly, by applying algorithm (2) with a spread of 5 demand levels and an initial diagonal value of $q = 0.5$, we derive the matrix with a medium uncertainty level based on the entropy measure as in equation (11).

$$
P_Y^{high\text{-}uncertainty} = \begin{array}{c|ccccccc}
 & y_1 & y_2 & y_3 & \cdots & y_{48} & y_{49} & y_{50} \\
\hline
y_1 & \mathbf{0.2308} & \mathbf{0.0308} & \mathbf{0.0308} & \cdots & 0 & 0 & 0 \\
y_2 & \mathbf{0.0296} & \mathbf{0.2296} & \mathbf{0.0296} & \cdots & 0 & 0 & 0 \\
y_3 & \mathbf{0.0286} & \mathbf{0.0286} & \mathbf{0.2286} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
y_{48} & 0 & 0 & 0 & \cdots & \mathbf{0.2286} & \mathbf{0.0286} & \mathbf{0.0286} \\
y_{49} & 0 & 0 & 0 & \cdots & \mathbf{0.0296} & \mathbf{0.2296} & \mathbf{0.0296} \\
y_{50} & 0 & 0 & 0 & \cdots & \mathbf{0.0308} & \mathbf{0.0308} & \mathbf{0.2308}
\end{array}. \tag{10}
$$

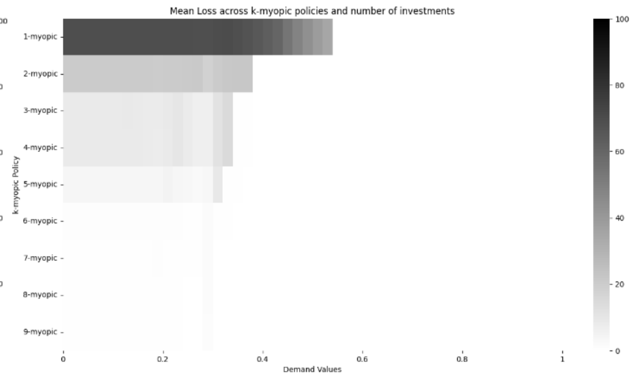
$$
P_Y^{medium\text{-}uncertainty} = \begin{array}{c|ccccccc}
 & y_1 & y_2 & y_3 & \cdots & y_{48} & y_{49} & y_{50} \\
\hline
y_1 & \mathbf{0.5833} & \mathbf{0.0833} & \mathbf{0.0833} & \cdots & 0 & 0 & 0 \\
y_2 & \mathbf{0.0714} & \mathbf{0.5714} & \mathbf{0.0714} & \cdots & 0 & 0 & 0 \\
y_3 & \mathbf{0.0625} & \mathbf{0.0625} & \mathbf{0.5625} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
y_{48} & 0 & 0 & 0 & \cdots & \mathbf{0.5625} & \mathbf{0.0625} & \mathbf{0.0625} \\
y_{49} & 0 & 0 & 0 & \cdots & \mathbf{0.0714} & \mathbf{0.5714} & \mathbf{0.0714} \\
y_{50} & 0 & 0 & 0 & \cdots & \mathbf{0.0833} & \mathbf{0.0833} & \mathbf{0.5833}
\end{array}. \tag{11}
$$

The findings indicate that the tendency for earlier investments due to additional options remains consistent across uncertainty levels. To further understand how uncertainty affects the threshold behavior of initial investment, we examine the distribution of loss in potential value gain resulting from myopic decisions (see Figure 5). Under high uncertainty, investments are delayed, which is the opposite effect of having more investment options. For 1-myopic decisions, this delay causes more loss in potential value gain, as the influence of uncertainty and additional options pull the initial investment threshold in opposite directions. However, when moving to a 2-myopic approach, the losses significantly decrease due to the positive impact of incorporating additional investment options. An important conclusion is that, under high uncertainty, firms should consider at least one additional investment option to avoid high losses in potential value gains. The additional investment options can offset the delayed investment thresholds caused by high uncertainty, effectively reducing losses from myopic investment decisions. Without this extra option, firms face greater loss in potential value gains due to further delays in investment thresholds, without the compensatory benefit of additional investment opportunities (see Figure 5).

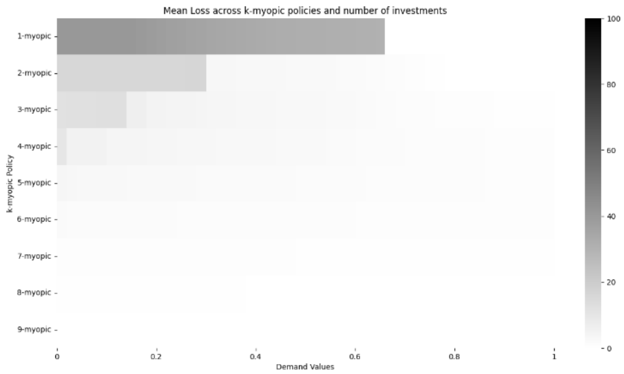
In scenarios with predictable outcomes (low uncertainty), deviations from the norm (e.g. transition to other demand levels) are less likely. Thus, in a medium demand level, when the uncertainty is low, the firm is likely to expect to remain at that medium demand level, leading to a higher loss in potential value gain if those decisions are not optimal.



(a) Low Uncertainty



(b) Medium uncertainty



(c) High uncertainty

Figure 5: Comparing the average value loss by taking a k-myopic decision across different uncertainty levels. The parameter values are given by $D_\omega = 0.5 + \omega, I_\omega = 2.35, \gamma = 0.9, n = 50$, $Y = \{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \leq l \leq n, y = \frac{l}{n}\}$, and P_Y is given by equation (8), (10), and (11).

5.4. Positive Persistence of Uncertainty & Demand Dynamics

The assumption of the positive persistence of uncertainty for demand implies that a higher current value of demand should shift the distribution of future values to the right. Dixit (1992) suggests that the positive persistence of uncertainty assumption may not be present in cases where large demand realizations lead to future decreases in demand (see Faninam et al. (2023)). This could occur in real-world scenarios where market dynamics and demand levels fluctuate unpredictably. In the context of our discrete-time setting, positive persistence of uncertainty (PPU) is formalized by the properties of a transition probability matrix for demand levels. Consider a finite set of demand levels $Y = \{y_1, y_2, \ldots, y_n\}$ where $y_1 < y_2 < \ldots < y_n$. Let P be an $n \times n$ transition probability matrix where each entry p_{ij} represents the probability of transitioning from demand level y_i to y_j in a single step. The system satisfies the Positive Persistence of Uncertainty (PPU) if the following condition is satisfied.

$$\sum_{l=j}^{n} p_{il} \geq \sum_{l=j}^{n} p_{kl}, \quad \text{for all } i, j, k \text{ such that } i < k. \tag{12}$$

This condition ensures that for any given demand level y_j, when starting from a lower state y_i, the cumulative probability of transitioning to a state less than or equal to y_j is at least as high as when starting from a higher state y_k. The algorithm (1) checks the existence of PPU for any given $n \times n$ matrix. The numerical results in Table 1 are given where the PPU holds.

Algorithm 1 Check PPU

Require: Matrix M, Tolerance η

Ensure: Boolean (True if M satisfies PPU, else False)

$n \leftarrow \text{rows}(M)$
CDF $\leftarrow$ cumulative sum along columns of M
for x_1 in 1 to n **do**
    for x_2 in $x_1 + 1$ to n **do**
        if not $\min(\text{CDF}[x_1, :] - \text{CDF}[x_2, :]) \geq -\eta$ **then**
            return False
return True

To investigate the effect of excluding positive persistence of uncertainty, the optimal investment decisions are derived from the value iteration algorithm with the same parameters as before, but with altered dynamics.

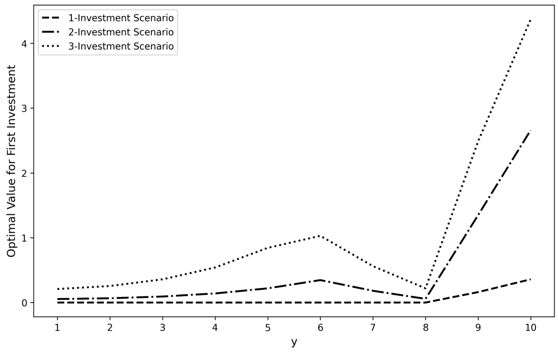
$$P_Y^{NPPU} = \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \\ y_9 \\ y_{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \end{array} \\ \begin{pmatrix} \mathbf{0.5} & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} & 0 & 0 \\ \mathbf{0.25} & \mathbf{0.75} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0.5} & 0 & \mathbf{0.5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0.5} & \mathbf{0.5} \end{pmatrix} \end{array}. \tag{13}$$

Table 3 illustrates the investment thresholds when the positive persistence of uncertainty assumption fails, showing the results of the value iteration algorithm. In a single investment scenario without PPU, the transition from not investing (0) to making the first investment (a_1) occurs only at the maximum demand level y_{10}, with a value of 0.2009. Considering two investments without PPU, the transition to a_1 occurs earlier at y_9, with a significant increase in value at y_{10}, reaching 2.96464. The effects of multiple investment opportunities on the initial investment threshold are similar to the case where the positive persistence of uncertainty assumption holds. Meaning that with additional investment options, investment actions tend to be initiated earlier. However, in this scenario the transitions to investment actions do not follow a consistent pattern, indicating disconnected investment regions (cite to our paper).

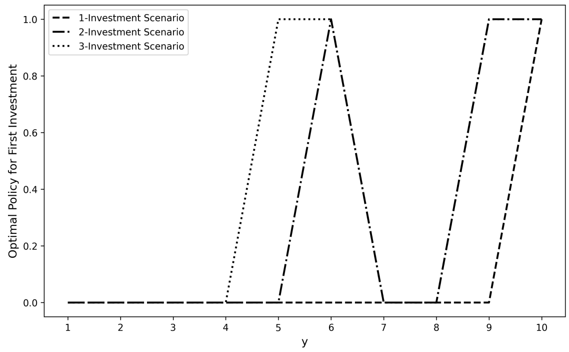
Table 3: The values are calculated using the value iteration algorithm. The parameter D_ω is defined such that $D_\omega = 0.5 + \omega$. The investment cost is $I_\omega = 2.35, \gamma = 0.9, n = 10, Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \le l \le n, y = \frac{l}{n}\right\}$, and P_Y is given by equation (13).

Index	D_ω	opt.	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
			Single Investment									
First	D_0	action	0	0	0	0	0	0	0	0	0	$\mathbf{a_1}$
		optimal value	0	0	0	0	0	0	0	0	0.16137	0.35860
		waiting value	0	0	0	0	0	0	0	0	0.16137	0.23398
		investing value	-1.58580	-1.41598	-1.16205	-0.89105	-0.66620	-0.56100	-0.67553	-1.158702	-0.15060	0.35860
			Two Investments									
First	D_0	action	0	0	0	0	0	$\mathbf{a_1}$	0	0	$\mathbf{a_1}$	a_1
		optimal value	0.05420	0.06624	0.09301	0.14045	0.21910	0.34644	0.18151	0.05691	1.34859	2.66080
		waiting value	-1.33406	-1.10829	-0.73004	-0.23872	0.17995	0.18027	-0.00398	-0.894370	1.22297	1.80422
		investing value	0.05420	0.06624	0.09301	0.14045	0.21910	0.34644	0.18151	0.05691	1.34859	2.66080
Second	D_1	action	0	0	0	0	a_2	a_2	a_2	0	a_2	a_2
		optimal value	1.01594	1.24171	1.61995	2.11128	2.70141	3.01700	2.67340	1.45563	4.24820	5.77580
		waiting value	1.01594	1.24171	1.61995	2.11128	2.52994	2.69644	2.34601	1.45563	3.69859	5.01080
		investing value	-0.05739	0.45207	1.21386	2.02687	2.70141	3.01700	2.67340	1.22390	4.24820	5.77580
			Three Investments									
First	D_0	action	0	0	0	0	$\mathbf{a_1}$	a_1	0	0	$\mathbf{a_1}$	a_1
		optimal value	0.20876	0.25515	0.35824	0.54094	0.84386	1.02941	0.56187	0.21920	2.48813	4.37142
		waiting value	0.20876	0.25515	0.35824	0.54094	0.70666	0.63258	0.56187	0.21920	2.06578	3.08680
		investing value	-1.11489	-0.84042	-0.35394	0.31129	0.84386	1.02941	0.50792	-0.66424	2.48813	4.37142
Second	D_1	action	0	0	0	a_2	a_2	a_2	a_2	0	a_2	a_2
		optimal value	1.23511	1.50958	1.99606	2.67919	3.54755	3.92444	2.85792	1.68576	5.74739	8.07800
		waiting value	1.23511	1.50958	1.99606	2.66129	3.19386	3.37941	2.85792	1.68576	4.83813	6.72142
		investing value	0.19435	0.75976	1.64586	2.67919	3.54755	3.92444	3.34495	1.48823	5.74739	8.07800
Third	D_2	action	0	0	0	0	a_3	a_3	a_3	0	a_3	a_3
		optimal value	2.54435	3.10976	3.99586	5.02919	6.06901	6.59500	6.02234	3.83823	8.64700	11.19300
		waiting value	2.54435	3.10976	3.99586	5.02919	5.89755	6.27444	5.69495	3.83823	8.09739	10.42800
		investing value	1.47101	2.32013	3.58976	4.94478	6.06901	6.59500	6.02234	3.60651	8.64700	11.19300

The difference in the investment timing emerges between two scenarios: (1) where the model satisfies the PPU assumption (*PPU*), and (2) where the PPU assumption is violated (*NPPU*). Under the PPU case, it is observed that the investments occur at an earlier stage in the sequence of demand levels as compared to the NPPU case. Furthermore, the values corresponding to each state are significantly higher in the PPU case than those in the NPPU case. Thus, the presence of PPU appears to increase the perceived value or outcome of the investment decision. In the absence of PPU, firms appear to initiate investments at higher demand levels, possibly waiting for clearer indicators of sustainable demand before taking action. These observed differences between PPU and NPPU scenarios provide valuable insights into the dynamics of investment decisions under different levels of uncertainty, reinforcing the robustness of the initial findings where PPU holds.



(a) The optimal value



(b) The optimal policy

Figure 6: The optimal value of a single-investment and the first investment of the two-investment scenario. The parameter values are given by $D_\omega = 0.5 + \omega, I_\omega = 2.35, \gamma = 0.9, n = 10, Y = \left\{y \in [0,1] \mid \exists l \in \mathbb{N} \text{ s.t. } 0 \le l \le n, y = \frac{l}{n}\right\}$, and P_Y is given by equation (13).

6. Conclusion

In standard real options theory, a value-maximizing firm evaluates a single investment project, ignoring the impact of multiple investment opportunities. This leads to opportunity loss, ultimately foregoing potential value enhancement for the firm. In this paper, we analyze the effect of considering multiple investment options when evaluating the initial investment opportunity. Our findings indicate that multiple investment options reduce the benefit of delaying investment and favor earlier investments to maximize value. In our analysis, we show that firms may lose the potential value of size similar to gains. This loss is especially pronounced with lower demand, uncertainty, cost variations, and discount rates. The variations in the parameter values, namely investment cost and discount factor, confirm that considering multiple investment options prevents significant losses in potential value gains. In addition, the loss of potential value gains increases as the discount factor decreases. Under high uncertainty, firms should consider at least one additional investment option to avoid large losses of potential value gains. The additional investment options can offset the delayed investment thresholds caused by high uncertainty, effectively reducing the losses from k-myopic investment decisions. On the other hand, when the uncertainty is low, the results suggest that the effect of additional investment options is more pronounced at low demand levels because transitions to other demand levels are less likely. In all cases, our qualitative conclusions remain intact, i.e. considering additional investment options leads to earlier investment decisions, and prevents the loss of potential value gains.

There are several promising avenues for extending the research presented in this paper. While our current model assumes that the order of investment options does not affect outcomes, future studies could explore more complex scenarios where the order of investments matters. This involves a more generalized model, as presented in Appendix A, that not only considers the order in which investments are made, but also examines how different combinations of investment options might yield different benefits. Another interesting extension would be to consider strategic investment in a multi-firm environment, where each firm has multiple investment options. This scenario would require a game-theoretic approach to identify equilibrium strategies among competing firms. In addition, incorporating more elements into the model, such as investment capacity, could provide more insight. However, this would significantly increase the complexity of the state space, underscoring the need for advanced computational techniques. In this regard, the use of reinforcement learning algorithms to approximate the value function represents a compelling direction for future research.

Appendix A. A More General Setup

In the model, uncertainty is embedded in the level of demand, represented by the stochastic variable y. The firm has access to the current level of demand. The set of available investment options is denoted by $\mathcal{J} = \{1, 2, \ldots, m\}$. When considering an investment in project $i \in \mathcal{J}$, the firm weighs immediate revenues and discounted expected future revenues against the investment costs, I_i, incurred in making the investment. Let D_ψ denote the demand multiplier where ψ is the number of projects invested. The paper defines $Y = \{y_1, y_2, \ldots, y_n\}$ (where $n \in \mathbb{N}$) as a bounded set of different demand levels, controlled by the transition function $P : Y \to [0, 1]$. The discount factor is denoted as $\gamma \in [0, 1)$.

In the multiple investment options setting, the MDP has a finite set of states, $\mathcal{S} = \{(y, \omega) | y \in Y, \omega \in \Omega =$

$\{0, 1, 2, \ldots, n\}\}$. In each state, we keep track of the demand level, $y \in Y$, and the internal state of the firm, $\omega \subseteq \Omega$. The action space A_s is defined such that the set of actions available to the firm at state $s = (y, \omega)$, denoted A_s, is given by:

$$A_s = \{0\} \cup (\mathcal{J} \setminus \Omega), \tag{A.1}$$

where the set subtraction $\mathcal{J} \setminus \Omega$ ensures that the firm only considers investing in projects where investments have not already been made. For example, consider a situation where there are five projects labeled sequentially from one to five, and investments have already been made in projects two and three. Given these pre-existing investments, with $\Omega = \{2, 3\}$, the action set A_s is defined as $A_s = \{0, 1, 4, 5\}$. Here, 0 represents the decision not to invest in any additional projects during this period, maintaining the current portfolio. averagewhile, 1, 4, and 5 represent the indices of the available projects for potential investment, ensuring no project receives investment more than once. For instance, choosing the action labeled "1" averages deciding to invest in project "1".

The immediate reward for the firm by taking action $a \in A_s$ at state $s = (y, \omega) \in \mathcal{S}$ is represented by the function $R : \mathcal{S} \times A \to \mathbb{R}$. If the firm decides to maintain its current investment portfolio (i.e. $a = 0$), the firm receives $R(y, \omega, a) = yD_{\psi + \mathbb{I}_{a\in(\mathcal{J}\setminus\Omega)}}$. On the contrary, if the firm decides to make an investment in project i (i.e. $a = i$ where $i \in (\mathcal{J} \setminus \Omega)$), the immediate reward is $R(y, \omega, a) = yD_{\psi + \mathbb{I}_{a\in(\mathcal{J}\setminus\Omega)}} - I_a$. Let $\psi = |\Omega| - 1$ denote the number of projects that have been already invested which is adjusted for the inclusion of "0" in the set Ω. Therefore, the reward function can be formally expressed as:

$$R(y, \omega, a) := yD_{\psi + \mathbb{I}_{a\in(\mathcal{J}\setminus\Omega)}} - \mathbb{I}_{a\in(\mathcal{J}\setminus\Omega)} I_a. \tag{A.2}$$

If the firm has already invested in all projects (i.e., $\psi = |\mathcal{J}|$), the only action it can take is to maintain its current portfolio (i.e., $a = 0$), and the firm receives a reward of $R(y, \psi, 0) = yD_\psi$. On the contrary, if there are still investment options (i.e., $\psi \neq |\mathcal{J}|$), the firm must make a decision from the set of possible actions $a \in A_s = \{0\} \cup (\mathcal{J} \setminus \Omega)$.

When an action is taken, the internal state of the firm is updated as

$$\bar{\omega} = \begin{cases} \omega + \mathbb{I}_{a\in(\mathcal{J}\setminus\Omega)} & \text{if } a \in \mathcal{J} \setminus \omega \\ \omega & \text{if } a = 0 \end{cases} \tag{A.3}$$

The demand level is updated according to the dynamics of the system, P_Y. The relation between the transition function

π and the demand dynamics P_Y is given by

$$\pi(\bar{y}, \bar{\omega}|y, \omega, a) = \begin{cases} P(\bar{y}|y) & \text{if } \bar{\omega} = \omega + \mathbb{I}_{a \in (\mathcal{J} \setminus \Omega)} \\ \\ 0 & \text{otherwise} \end{cases} . \tag{A.4}$$

This equation indicates that the probability of transitioning to the next state is dependent on both the current internal state $\omega \in \Omega$ and the action $a \in A_s$ taken at state $s \in \mathcal{S}$. The equation implies that if $\bar{\omega}$ follows the update rule $\bar{\omega} = \omega + \mathbb{I}_{a \in (\mathcal{J} \setminus \Omega)}$, the system transitions according to the demand dynamics $P(\bar{y}|y)$. If $\bar{\omega}$ does not adhere to this update rule (averageing no new investment is added to the portfolio), the probability of transitioning to the next state is zero.

Appendix B. Algorithms

Algorithm 2 generates a *high-uncertainty* transition matrix. For each state, it assigns a 20% likelihood to remain static and divides the remaining 80% among half of its neighboring states. This ensures elevated unpredictability in transitions. Finally, each row in the matrix is normalized to ensure probabilities sum up to 100%. Each diagonal state has a balanced 50% chance of remaining unchanged. The other 50% of the probability is then spread among states within a range of five neighboring states, introducing a slightly higher uncertainty. As with the previous matrix, each row is normalized to confirm that the total probabilities of transitions equal 100%. This method provides a mid-level unpredictability in state transitions, balancing between stability and variability.

Algorithm 2 Create Medium Uncertainty Matrix

Require: Number of states n_states, number of neighboring states with positive values $spread$, initial diagonal value q

Ensure: Transition probability matrix tpm

procedure CREATEUNCERTAINTYMATRIX($n\_states, spread, q$)

$tpm \leftarrow$ **zeros**(n_states, n_states)

for $i \leftarrow 0$ to $n\_states - 1$ **do**

$tpm[i][i] \leftarrow q$

$total\_adjacent\_prob \leftarrow 1.0 - tpm[i][i]$

$num\_adjacents \leftarrow 0$

for $offset \leftarrow -spread$ to $spread$ **do**

if $0 \leq i + offset < n\_states$ **then**

$num\_adjacents \leftarrow num\_adjacents + 1$

for $offset \leftarrow -spread$ to $spread$ **do**

if $0 \leq i + offset < n\_states$ **then**

$tpm[i][i + offset] \leftarrow tpm[i][i + offset] + \frac{total\_adjacent\_prob}{num\_adjacents}$

Normalize row i of tpm to sum up to 1.0

return tpm

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