Lie derivatives of the shape operator of real hypersurfaces in the complex and complex hyperbolic quadrics

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Abstract

On a real hypersurface in the complex quadric or the complex hyperbolic quadric we can consider the Levi-Civita connection and, for any nonnull real number k, the k-th generalized Tanaka-Webster connection. We also have a differential operator of first order of Lie type associated to the k-th generalized Tanaka-Webster connection. We classify real hypersurfaces in the complex quadric and the complex hyperbolic quadric for which the Lie derivative and the Lie type differential operator coincide when they act on the shape operator of the real hypersurface either in the direction of the structure vector field or in any direction of the maximal holomorphic distribution.

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1 Introduction.

Let $(\tilde{M}, J, g)$ be a Kähler manifold and M a real hypersurface of $\tilde{M}$, that is, a submanifold of codimension 1 with local normal unit vector field N. Then M inherits an almost constant metric structure (ϕ, η, g, ξ). Let ∇ denote the Levi-Civita connection on M and S its shape operator associated to N.

As M has an almost contact metric structure, for any nonnull real number k we can define the so called k-th generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ on M by

$$\hat{\nabla}_X^{(k)}Y = \nabla_X Y + g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$$

for any X, Y tangent to M (see [3]). Let us call $F_X^{(k)}Y = g(\phi SX, Y)\xi - \eta(Y)\phi SX - k\eta(X)\phi Y$, for any X, Y tangent to M . $F_X^{(k)}$ is called the k-th Cho operator on M associated to X. Notice that if $X \in \mathcal{C}$, the maximal holomorphic distribution on M given by all the vector fields orthogonal to ξ, the associated Cho operator does not depend on k and we will denote it simply by F_X. We also have $\hat{\nabla}^{(k)}\phi = 0$, $\hat{\nabla}^{(k)}\eta = 0$, $\hat{\nabla}^{(k)}g = 0$ and $\hat{\nabla}^{(k)}\xi = 0$.

The torsion of the connection $\hat{\nabla}^{(k)}$ is given by $T^{(k)}(X, Y) = F_X^{(k)}Y - F_Y^{(k)}X$ for any X, Y tangent to M. From $T^{(k)}$ we can define the torsion operator associated to the vector field X tangent to M by $T_X^{(k)}Y = T^{(k)}(X, Y)$ for any Y tangent to M.

Let $\mathfrak{L}$ denote the Lie derivative of the real hypersurface M. Then $\mathfrak{L}_X Y = \nabla_X Y - \nabla_Y X$ for any X, Y tangent to M. Associated to the k-th generalized Tanaka-Webster connection $\hat{\nabla}^{(k)}$ we can consider a differential operator of first order by $\mathfrak{L}_X^{(k)}Y = \hat{\nabla}_X^{(k)}Y - \hat{\nabla}_Y^{(k)}X = \mathfrak{L}_X Y + T_X^{(k)}Y$, for any X, Y tangent to M.

In this paper we will consider real hypersurfaces in $\bar{M}^m(\epsilon)$, $\epsilon = \pm 1$, where $\bar{M}^m(1)$ will denote the complex quadric $Q^m = SO_{m+2}/SO_m SO_2$ and $\bar{M}^m(-1)$ will denote the complex hyperbolic quadric $Q^{m*} = SO^o_{2,m}/SO_2 SO_m$. Both are Hermitian symmetric spaces of rank 2 and are equipped with two geometric structures: a Kähler structure J and a parallel rank 2 subbundle $\mathfrak{A}$ of the endomorphism bunddle $End(T\bar{M}^m(\epsilon))$, which consists of all the real structures on the tangent space of $\bar{M}^m(\epsilon)$. For any $A \in \mathfrak{A}$ the following relations hold: $A^2 = I$ and $AJ = -JA$. A nonzero tangent vector W at a point of $\bar{M}^m(\epsilon)$ is called *singular* if it is tangent to more than one maximal flat in $\bar{M}^m(\epsilon)$. There are two types of singular tangent vectors for $\bar{M}^m(\epsilon)$:

- If there exists a conugation $A \in \mathfrak{A}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
- If there exists a conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that $W/\|W\| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{A}$-isotropic,

where $V(A) = \{X \in T_{[z]}\bar{M}^m(\epsilon) | AX = X\}$ and $JV(A) = \{X \in T_{[z]}\bar{M}^m(\epsilon) | AX - X\}$, $[z] \in \bar{M}^m(\epsilon)$, are the (+1)-eigenspace and the (-1)-eigenspace for the involution A.

The tangent bundle TM of the real hypersurface M splits orthogonally into

$$TM = \mathcal{C} \oplus \mathcal{F},$$

where $\mathcal{C} = ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J.

We sill say that M is Hopf if its Reeb vector field is principal. That is $S\xi = \alpha\xi$ for a certain function α.

The study of real hypersurfaces M in Q^m was initiated by Berndt and Suh in [1]. In this paper the geometric properties of real hypersurfaces M in complex quadric Q^m , which are tubes of radius r, $0 < r < \pi/2$, around the totally geodesic $\mathbb{C}P^n$ in Q^m , when $m = 2n$ are presented. The condition of isometric Reeb flow is equivalent to the commuting condition of the shape operator S with the structure tensor ϕ of M. The classification of such real hypersurfaces in Q^m is obtained in [2]:

Proposition 1.1 *The following statements hold for a tube M of radius r, $0 < r < \pi/2$ around the totally geodesic $\mathbb{C}P^n$ in Q^m , $m = 2n$:*

1. *M is a Hopf hypersurface.*
2. *The normal bundle of M consists of $\mathfrak{A}$-isotropic singular tangent vectors of Q^m .*
3. *M has four distinct principal curvatures, unless $m = 2$ in which case M has two distinct principal curvatures, which are given in the following matrix*

Principal curvature	*Eigenspace*	*Multiplicity*
$2\cot(2r)$	$\mathcal{F}$	1
$\cot(r)$	$\nu_z\mathbb{C}P^n \ominus [\xi]$	$2n-2$
$-\tan(r)$	$T_z\mathbb{C}P^n \ominus [A\xi]$	$2n-2$
0	$[A\xi]$	2

4. *The shape operator commutes with the structure tensor field ϕ, i.e. $S\phi = \phi S$.*

5. *M is a homogeneous hypersurface.*

Moreover, such tubes are the unique real hypersurfaces in Q^m satisfying $S\phi = \phi S$.

In the case of the complex hyperbolic quadric Q^{m*} we have a similar result, [7]

Proposition 1.2 *Let M be a tube around the totally geodesic $\mathbb{C}H^n$ in Q^{m*}, $m = 2n$, or the horosphere in Q^{m*} whose center at the infinity is in the equivalent class of an $\mathfrak{A}$-isotropic singular geodesic in Q^{m*}. Then the following statement holds:*

1. *M is a Hopf hypersurface.*
2. *The tangent bundle TM and the normal bundle νM of M consists of $\mathfrak{A}$-isotropic singular tangent vectors of Q^{m*}.*
3. *M has four (or three) distinct principal curvatures which are given in the following table*

Principal curvature	*Eigenspace*	*Multiplicity*
$2\coth(2r)$,2	$\mathcal{F}$	1
$\coth(r)$,1	$\nu\mathbb{C}H^n \ominus \mathbb{C}\nu M$	$2n-2$
$\tanh(r)$,1	$T\mathbb{C}H^n \ominus (\mathcal{C} \ominus \mathcal{Q})$	$2n-2$
0	$\mathcal{C} \ominus \mathcal{Q}$	2

4. *The shape operator commutes with the structure tensor field ϕ, i.e. $S\phi = \phi S$.*
5. *The Reeb flow on M is an isometric flow.*

Real hypersurfaces appearing in Proposition 1.2 are the unique ones in Q^{m*} satisfying condition 4.

Recently, in [5] we have proved the following

Theorem 1.1 *There does not exist any real hypersurface in Q^m, $m \geq 3$, such that the Lie derivative and the differential operator $\mathfrak{L}^{(k)}$ coincide on S, that is, $\mathfrak{L}S = \mathfrak{L}^{(k)}S$, for any nonnull k.*

In this paper we will study real hypersurfaces in $\bar{M}^m(\epsilon)$ such that either $\mathfrak{L}_\xi S = \mathfrak{L}_\xi^{(k)} S$ or $\mathfrak{L}_X S = \mathfrak{L}_X^{(k)} S$ for any $X \in \mathcal{C}$. We will obtain the following results:

Theorem 1.2 *Let M be a real hypersurface in $\bar{M}^m(\epsilon)$, $m \geq 3$, and any nonnull real number k. Then $\mathfrak{L}_\xi S = \mathfrak{L}_\xi^{(k)} S$ if and only if either i) $\epsilon = 1$ and M is an open part of a tube of radius r, $0 < r < \frac{\pi}{2}$, around the totally geodesic $\mathbb{C}P^n$ in Q^m, $m = 2n$, or ii) $\epsilon = -1$ and M is an open part of a tube around a totally geodesic $\mathbb{C}H^n$ in Q^{m*}, $m = 2n$, or a horosphere whose center at infinity is $\mathfrak{A}$-isotropic singular.*

Theorem 1.3 1. *There does not exist any real hypersurface in Q^m, $m \geq 3$, such that $\mathfrak{L}_X S = \mathfrak{L}_X^{(k)} S$, for any $X \in \mathcal{C}$ and any nonnull real number k.*

2. *Let M be a real hypersurface in Q^{m*}, $m \geq 3$, and k a nonnull real number. Then $\mathfrak{L}_X S = \mathfrak{L}_X^{(k)} S$ for any $X \in \mathcal{C}$ if and only if $k^2 = 1$, M is Hopf and N is either $\mathfrak{A}$-isotropic or $\mathfrak{A}$-principal. In both cases M has two distinct constant principal curvatures, k and $-k$, and in the first case the eigenspace corresponding to k is $T_k = Span\{\xi, A\xi, AN\} \oplus \mathcal{C}_k$, with $\phi\mathcal{C}_k = T_{-k}$; in the second case $T_k = Span\{\xi\} \oplus \mathcal{C}_k$, with $\phi\mathcal{C}_k = T_{-k}$.*

As a conclusion of both Theorems we obtain

Corollary 1.1 *There does not exist any real hypersurface in Q^{m*}, $m \geq 3$, such that $\mathfrak{L}S = \mathfrak{L}^{(k)}S$, for any nonnull real number k.*

2 Preliminaries.

For the study and notations of the geometry of the complex quadric Q^m and its real hypersurfaces see [6] and [4]. In the case of the complex hyperbolic quadric Q^{m*} see [7].

Let M be a real hypersurface in $\bar{M}^n(\epsilon)$ and N a unit normal vector field of M. Any vector field X tangent to M satisfies the relation

$$JX = \phi X + \eta(X)N. \tag{2.1}$$

The tangential component of the above relation defines on M a skew-symmetric tensor field of type (1,1) ϕ, named the structure tensor. The structure vector field ξ is defined by $\xi = -JN$ and is called the Reeb vector field. The 1-form η is given by $\eta(X) = g(X, \xi)$ for any vector field X tangent to M . So, on M an almost contact metric structure (ϕ, ξ, η, g) is defined. The elements of the almost contact structure satisfy the following relations

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for all tangent vectors X, Y to M. Relation (2.1) implies $\phi\xi = 0$.

At each point $[z] \in M$ we define

$$\mathcal{Q}_{[z]} = \{X \in T_{[z]}M | AX \in T_{[z]}M \text{ for all } A \in \mathfrak{A}_{[z]}\}$$

which is the maximal $\mathfrak{A}_{[z]}$-invariant subspace of $T_{[z]}M$. Then for real hypersurfaces in $\bar{M}^n(\epsilon)$ we have the following, [2], [7],

Lemma 2.1 *Let M be a real hypersurface in $\bar{M}^m(\epsilon)$. Then the following statements are equivalent:*

1. *The normal vector $N_{[z]}$ of M is $\mathfrak{A}$-principal.*
2. $\mathcal{Q}_{[z]} = \mathcal{C}_{[z]}$.
3. *There exists a real structure $A \in \mathfrak{A}_{[z]}$ such that $AN_{[z]} \in \mathbb{C}\nu_{[z]}M$.*

Assume now that the normal vector $N_{[z]}$ of M is not $\mathfrak{A}$-principal. Then there exists a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$\begin{aligned} N_{[z]} &= \cos(t)Z_1 + \sin(t)JZ_2, \\ AN_{[z]} &= \cos(t)Z_1 - \sin(t)JZ_2, \end{aligned} \tag{2.3}$$

where Z_1, Z_2 are orthonormal vector in $V(A)$ and $0 < t \leq \frac{\pi}{4}$. Moreover, the above relations due to $\xi = -JN$ imply

$$\begin{aligned}\xi_{[z]} &= -\cos(t)JZ_1 + \sin(t)Z_2, \\ A\xi_{[z]} &= \cos(t)JZ_1 + \sin(t)Z_2.\end{aligned} \tag{2.4}$$

So we have $g(AN_{[z]}, \xi_{[z]}) = 0$, $g(N_{[z]}, AN_{[z]}) = cos(2t) = -g(\xi_{[z]}, A\xi_{[z]})$.

The Codazzi equation of M is given by

$$\begin{gathered}g((\nabla_X S)Y - (\nabla_Y S)X, Z) = \epsilon\{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ +g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z)\}\end{gathered} \tag{2.5}$$

for any X, Y, Z tangent to M.

The shape operator of a real hypersurface M in Q^m is denoted by S. The real hypersurface is called *Hopf hypersurface* if the Reeb vector field is an eigenvector of the shape operator, i.e.

$$S\xi = \alpha\xi, \tag{2.6}$$

where $\alpha = g(S\xi, \xi)$ is the Reeb function. In this case, taking $Z = \xi$ in (2.5) we get

$$Y(\alpha) = \xi(\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi) \tag{2.7}$$

Finally, [2], [7], we have the following

Lemma 2.2 *Let M be a Hopf real hypersurface in $\bar{M}^m(\epsilon)$, $m \geq 3$. Then the tensor field $2S\phi S - \alpha(\phi S + S\phi)$ leaves $\mathcal{Q}$ and $\mathcal{C} \ominus \mathcal{Q}$ invariant and we have $2S\phi S - \alpha(\phi S + S\phi) = -2\phi$ on $\mathcal{Q}$ and $2S\phi S - \alpha(\phi S + S\phi) = -2\beta^2\phi$ on $\mathcal{C} \ominus \mathcal{Q}$, where $\beta = g(A\xi, \xi) = -2cos(2t)$.*

3 Proof of Theorem 1.2

Let M be a real hypersurface in Q^m such that $(\mathfrak{L}_\xi S)Y = (\mathfrak{L}_\xi^{(k)} S)Y$ for any Y tangent to M. This yields

$$-\phi S^2 Y + S\phi SY = g(\phi S\xi, SY)\xi - \eta(SY)\phi S\xi - k\phi SY - g(\phi S\xi, Y)S\xi + \eta(Y)S\phi S\xi + kS\phi Y \tag{3.1}$$

for any $Y \in TM$.

Let us suppose that M is non Hopf. Then at a point $p \in M$ we can write $S\xi = \alpha\xi + \beta U$, for a nonnull real number β and a unit vector $U \in \mathcal{C}_p$. Therefore on an open neighborhood of p we have a similar expression where α and $\beta \neq 0$ are functions on such a neighborhood and U a unit vector field in $\mathcal{C}$. Then (3.1) gives

$$\begin{aligned}\beta g(\phi U,SY)\xi-\beta\eta(SY)\phi U-k\phi SY-\beta g(\phi U,Y)S\xi\\+\beta\eta(Y)S\phi U+kS\phi Y+\phi S^2Y-S\phi SY=0\end{aligned}\tag{3.2}$$

for any Y tangent to M. If we take the scalar product of (3.2) and ξ we get $\beta g(S\phi U,Y)-\alpha\beta g(\phi U,Y)+k\beta g(\phi Y,U)-\beta g(\phi SY,U)=0$. As $\beta\neq 0$ this yields $2g(S\phi U,Y)=(\alpha+k)g(\phi U,Y)$ for any Y tangent to M. Therefore

$$S\phi U=\frac{\alpha+k}{2}\phi U.\tag{3.3}$$

If in (3.2) we take $Y=\xi$ we obtain $-k\beta\phi U+\beta\phi SU=0$. Once again, $\beta\neq 0$ yields $\phi SU=k\phi U$. Applying ϕ we obtain

$$SU=\beta\xi+kU.\tag{3.4}$$

Taking now $Y=\phi U$ in (3.2) and bearing in mind (3.4) we have $\beta(\frac{k-\alpha}{2})\xi+(k(\frac{\alpha+k}{2})-\beta^2-\frac{(\alpha+k)^2}{4})U=(\frac{k-\alpha}{2})SU$. If we suppose $k=\alpha$ we have $k^2-\beta^2-k^2=-\beta^2=0$, which is impossible. Thus $k-\alpha\neq 0$ and

$$SU=\beta\xi+(\frac{2}{k-\alpha})(\frac{k^2-\alpha^2}{4}-\beta^2)U.\tag{3.5}$$

From (3.4) and (3.5) it follows $k=(\frac{2}{k-\alpha})(\frac{k^2-\alpha^2}{4}-\beta^2)$. This yields $2k^2-2k\alpha-k^2+\alpha^2=-2\beta^2$, that is, $(k-\alpha)^2=-2\beta^2$, which is impossible.

Therefore M must be Hopf. We write $S\xi=\alpha\xi$ and (3.1) gives $-\phi S^2Y+S\phi SY=-k\phi SY+kS\phi Y$, for any Y tangent to M, or equivalently $(S\phi-\phi S)SY=k(S\phi-\phi S)Y$, for any Y tangent to M.

Now $S\phi-\phi S$ is a symmetric operator on M. Moreover, for any X,Y tangent to M we have $g((S\phi-\phi S)SX,Y)=kg((S\phi-\phi S)X,Y)=kg(X,(S\phi-\phi S)Y)=g(X,(S\phi-\phi S)SY)=g(S(S\phi-\phi S)X,Y)$. That is, $(S\phi-\phi S)S=S(S\phi-\phi S)$. Then we can find an orthonormal basis of TM that at the same time diagonalizes $S\phi-\phi S$ and S. Let Y be a vector field of such a base satisfying $(S\phi-\phi S)Y=\lambda Y$ and $SY=\mu Y$. Thus $S\phi Y-\mu\phi Y=\lambda Y$. Its scalar product with Y gives $0=\lambda$. Therefore, any eigenvalue of $S\phi-\phi S$ is null. This yields $S\phi=\phi S$ and the result follows from Propositions 1.1 and 1.2.

4 Proof of Theorem 1.3

Let M be a real hypersurface in Q^m such that $(\mathfrak{L}_XS)Y=(\mathfrak{L}^{(k)}_XS)Y$ for any $X\in\mathcal{C}$ and any Y tangent to M. Then we have

$$\begin{aligned}g(\phi SX,SY)\xi-\eta(SY)\phi SX+g(\phi S^2Y,X)\xi+k\eta(SY)\phi X\\-g(\phi SX,Y)S\xi+\eta(Y)S\phi SX+g(\phi SY,X)S\xi-k\eta(Y)S\phi X=0\end{aligned}\tag{4.1}$$

for any $X\in\mathcal{C}$, Y tangent to M.

First suppose that M is Hopf with $S\xi = \alpha\xi$. Taking $Y = \xi$ in (4.1) we obtain

$$-\alpha\phi SX + k\alpha\phi X + S\phi SX - kS\phi X = 0 \tag{4.2}$$

for any $X \in \mathcal{C}$. Bearing in mind (4.2), (4.1) becomes

$$g(\phi SX, SY)\xi - g(\phi S^2Y, X)\xi - \alpha g(\phi SX, Y)\xi + \alpha g(\phi SY, X)\xi = 0 \tag{4.3}$$

for any $X \in \mathcal{C}$, Y tangent to M. Its scalar product with ξ yields

$$S\phi SX + S^2\phi X - \alpha\phi SX - \alpha S\phi X = 0 \tag{4.4}$$

for any $X \in \mathcal{C}$.

The first term in (4.2) has no component in ξ. Taking the scalar product of (4.2) and any $Y \in \mathcal{C}$ we also obtain

$$\alpha S\phi X - k\alpha\phi X - S\phi SX + k\phi SX = 0 \tag{4.5}$$

for any $X \in \mathcal{C}$. Adding (4.2) and (4.5) we obtain $(k-\alpha)\phi SX + (\alpha - k)S\phi X = 0$ for any $X \in \mathcal{C}$. Thus we have two possibilities:

The first one is $\alpha \neq k$. As M is Hopf, we obtain $\phi S = S\phi$. Take a unit $X \in \mathcal{C}$ such that $SX = \lambda X$ and $Y = \phi X$ in (4.1). As $\phi S = S\phi$ we obtain $2g(SX, SX)\xi - 2g(SX, X)S\xi = 0$. Therefore $\lambda(\lambda - \alpha) = 0$. Thus either $\lambda = 0$ or $\lambda = \alpha$. In the case of $\epsilon = 1$, from Proposition 1.1, in $\mathcal{C}$ there are nonnull principal curvatures. Thus there exists a principal curvature $\lambda = \alpha \neq 0$. Therefore we must have either $2cot(2r) = cot(r)$ or $2cot(2r) = -tan(r)$. As $2cot(2r) = cot(r) - tan(r)$ in both cases we arrive to a contradiction. A similar reasoning from Proposition 1.2 in the case of $\epsilon = -1$ proves that this possibility does not occur.

We have obtained then that $\alpha = k$. In this case (4.2) becomes

$$-k\phi SX + k^2\phi X + S\phi SX - kS\phi X = 0 \tag{4.6}$$

for any $X \in \mathcal{C}$ and (4.4) would be

$$S\phi SX + S^2\phi X - k\phi SX - kS\phi X = 0 \tag{4.7}$$

for any $X \in \mathcal{C}$. The scalar product of (4.7) and $Z \in \mathcal{C}$ yields

$$-S\phi SX - \phi S^2 X + kS\phi X + k\phi SX = 0 \tag{4.8}$$

for any $X \in \mathcal{C}$. Take a unit $X \in \mathcal{C}$ such that $SX = \lambda X$. From (4.6) we get $-k\lambda\phi X + k^2\phi X + \lambda S\phi X - kS\phi X = 0$. This implies

$$(\lambda - k)S\phi X = k(\lambda - k)\phi X. \tag{4.9}$$

From (4.9) we have two possibilities: Either $\lambda = k$ or $\lambda \neq k$. In this case $S\phi X = k\phi X$. Introducing this ϕX in (4.7) and (4.8) we obtain $S^2X = k^2X$. That is, $\lambda^2 = k^2$ and as $\lambda \neq k$, we must have $\lambda = -k$.

We have proved that in $\mathcal{C}$ the unique possible eigenvalues are k and $-k$. If any eigenvalue in $\mathcal{C}$ is equal to k we should have $S\phi = \phi S$, but as $\alpha = k$, we should have, for $\epsilon = 1$, $2cot(2r) = k = cot(r) = -tan(r)$ and $k = 2k$, which is impossible. We obtain a similar contradiction in the case $\epsilon = -1$.

Therefore there exists a unit $X \in \mathcal{C}$ such that $SX = -kX$ and $S\phi X = k\phi X$. Introducing this X in the Codazzi equation we have $g((\nabla_X S)\phi X - (\nabla_{\phi X} S)X, \xi) = \epsilon\{-2 + g(X, AN)g(A\phi X, \xi) - g(\phi X, AN)$ $g(AX, \xi) + g(X, A\xi)g(JA\phi X, \xi) - g(\phi X, A\xi)g(JAX, \xi)\} = \epsilon\{-2 + 2g(X, AN)^2 + 2g(X, A\xi)^2\}$.

On the other hand $(\nabla_X S)\phi X - (\nabla_{\phi X} S)X = k\nabla_X \phi X - S\nabla_X \phi X + k\nabla_{\phi X} X + S\nabla_{\phi X} X$. Therefore $g((\nabla_X S)\phi X - (\nabla_{\phi X} S)X, \xi) = -kg(X, SX) + kg(X, SX) - 2kg(X, \phi S\phi X) = 2kg(S\phi X, \phi X) = 2k^2$.

We have obtained $2k^2 = \epsilon\{-2 + 2g(X, AN)^2 + 2g(X, A\xi)^2\}$. If $\epsilon = 1$,we get $1 < k^2 + 1 = g(X, AN)^2 + g(X, A\xi)^2 \leq 1$, giving a contradiction.

Suppose $\epsilon = -1$. From (2.7), the fact of k being constant and $g(\xi, AN) = 0$ we get $g(Y, AN)g(\xi, A\xi) = 0$ for any Y tangent to M. Therefore, if $g(\xi, A\xi) = 0$, $cos(2t) = 0$, $t = \frac{\pi}{4}$ and N is $\mathfrak{A}$-isotropic. If not, $g(AN, Y) = 0$ for any Y tangent to M and N is $\mathfrak{A}$-principal.

If N is $\mathfrak{A}$-isotropic, for any $X \in \mathcal{Q}$ we have $g(X, AN) = g(X, A\xi) = 0$ and $2k^2 = 2$. That is, $k^2 = 1$. If N is $\mathfrak{A}$-principal, from Lemma 2.2, $2S\phi S - k(\phi S + S\phi) = -2\phi$ on $\mathcal{C}$. As there exists a unit $X \in \mathcal{C}$ such that $SX = -kX$ and $S\phi X = k\phi X$, $(\phi S + S\phi)X = -k\phi X + k\phi X = 0$. Therefore, $S\phi SX = -\phi X = -kS\phi X = -k^2\phi X$ and again $k^2 = 1$. This finishes the proof for Hopf real hypersurfaces.

Let us suppose now that M is non Hopf. As in Theorem 1.2, we write $S\xi = \alpha\xi + \beta U$ on a certain open subset of M, for a unit $U \in \mathcal{C}$ and a nonvanishing fuction β on such a subset.

Taking $Y = \xi$ in (4.1) we obtain

$$\begin{gathered} -\beta g(S\phi U, X)\xi - \alpha\phi SX - \alpha\beta g(\phi U, X)\xi - \beta g(\phi SU, X)\xi \\ +k\alpha\phi X + S\phi SX + \beta g(\phi U, X)S\xi - kS\phi X) = 0. \end{gathered} \tag{4.10}$$

for any $X \in \mathcal{C}$. Its scalar product with ξ yields $-2\beta g(S\phi U, X) - \beta g(\phi SU, X) + k\beta g(\phi U, X) = 0$. Therefore, as $g(-2S\phi U + k\phi U - \phi SU, \xi) = 0$, we get

$$-2S\phi U + k\phi U - \phi SU = 0. \tag{4.11}$$

And, in particular,

$$g(SU, \phi U) = 0. \tag{4.12}$$

If we take $X = U$, $Y \in \mathcal{C}$ in (4.1) we have

$$\begin{gathered} g(\phi SU, SY)\xi - \eta(SY)\phi SU - g(\phi S^2Y, U)\xi + k\eta(SY)\phi U \\ -g(\phi SU, Y)S\xi + g(\phi SY, U)S\xi = 0 \end{gathered} \tag{4.13}$$

and its scalar product with ϕU yields $-\eta(SY)g(SU,U)+k\eta(SY)=0$ for any $Y\in\mathcal{C}$. Taking $Y=U$ we get $-\beta g(SU,U)+k\beta=0$. Then

$$g(SU,U)=k. \tag{4.14}$$

The scalar product of (4.11) and ϕU gives $-2g(S\phi U,\phi U)+k-g(SU,U)=0$. Bearing in mind (4.14) we obtain

$$g(S\phi U,\phi U)=0. \tag{4.15}$$

The scalar product of (4.13) and U gives $-\beta g(\phi SU,Y)+\beta g(\phi SY,U)=0$ for any $Y\in\mathcal{C}$. That is, $g(\phi SU,Y)+g(S\phi U,Y)=0$, for any $Y\in\mathcal{C}$. Taking $Y=\phi U$ we obtain $g(SU,U)+g(S\phi U,\phi U)=0$. From (4.14) and (4.15), this yields $k=0$, which is impossible and we finish the proof.

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