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A priori Hölder and Lipschitz regularity for generalized p-harmonious functions in metric measure spaces

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Abstract

Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space and let $\Omega \subset \mathbb{X}$ be a bounded domain. For each $x \in \Omega$, we choose a radius $0 < \varrho(x) \leq \operatorname{dist}(x, \partial\Omega)$ and let B_x be the closed ball centered at x with radius $\varrho(x)$. If $\alpha \in \mathbb{R}$, consider the following operator in $C(\overline{\Omega})$,

$$\mathcal{T}_\alpha u(x) = \frac{\alpha}{2}\left(\sup_{B_x} u + \inf_{B_x} u\right) + (1-\alpha)\fint_{B_x} u \, d\mu.$$

Under appropriate assumptions on α, $\mathbb{X}$, μ and the radius function ϱ we show that solutions $u \in C(\overline{\Omega})$ of the functional equation $\mathcal{T}_\alpha u = u$ satisfy a local Hölder or Lipschitz condition in Ω. The motivation comes from the so called p-harmonious functions in euclidean domains.

1 Introduction

The main goal of this paper is to provide a priori regularity estimates for functions satisfying certain nonlinear mean value properties in metric measure spaces. Our main motivation are classical harmonic functions and the so called p-harmonious functions in $\mathbb{R}^n$. First of all, let us recall some basic facts about harmonic functions in euclidean space and their connections to the mean value property.

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It is well known that a continuous function u in a domain $\Omega \subset \mathbb{R}^n$ is harmonic if and only if it satisfies the *mean value property*

$$u(x) = \fint_{B(x,\varrho)} u\, dm \tag{1.1}$$

for each $x \in \Omega$ and each $\varrho > 0$ such that $0 < \varrho < \operatorname{dist}(x, \partial\Omega)$, where m denotes n-dimensional Lebesgue measure. The mean value property plays a relevant role in Geometric Function Theory and is indeed the fundamental key of the interplay between harmonic functions, Probability and Brownian motion.

The so called *restricted* mean value property problems ask how many radii ϱ in (1.1) are enough to guarantee harmonicity. One of the most representative results in this direction is a classical theorem due to Volterra (for regular domains) and Kellogg's (in the general case): if Ω is bounded, $u \in C(\overline{\Omega})$ and if for each $x \in \Omega$ there is a radius $\varrho = \varrho(x)$, with $0 < \varrho \leq \operatorname{dist}(x, \partial\Omega)$, such that (1.1) holds, then u is harmonic in Ω (see [23], [10]). Therefore, under appropriate hypothesis, the mean value property for a single radius (depending on the point) implies harmonicity. See [17] for a detailed account of this and other results related to the mean value property.

The question of what are the natural stochastic processes associated to some nonlinear differential operators, like the p-laplacian or the ∞-laplacian, has attracted an increasing attention in the last years. If $1 < p < \infty$ the *p-laplacian* is the divergence form differential operator given by

$$\triangle_p u = \operatorname{div}(\nabla u |\nabla u|^{p-2})$$

and weak solutions of the equation $\triangle_p u = 0$ are called *p-harmonic* functions. Suppose that $u \in C^2$ and that $\nabla u \neq 0$. Then direct computation gives

$$\triangle_p u = |\nabla u|^{p-2} \left(\triangle u + (p-2) \frac{\triangle_\infty u}{|\nabla u|^2} \right) \tag{1.2}$$

where

$$\triangle_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i, x_j}$$

is the so called *∞-laplacian* in $\mathbb{R}^n$. So, at least in the smooth case and away from the critical points, the p-laplacian can be understood as a sort of linear combination of the usual laplacian and the normalized ∞-laplacian. Observe that we recover the usual laplacian when $p = 2$.

Let us briefly explain now the connection between the p-laplacian and the mean value property. First, we recall that if $u \in C^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$, then the following asymptotic mean value property holds for any $x \in \Omega$:

$$\lim_{\varrho \to 0} \frac{1}{\varrho^2} \left(\fint_{B(x,\varrho)} u\, dm - u(x) \right) = \frac{\triangle u(x)}{2(n+2)} \tag{1.3}$$

On the other hand, if $\nabla u(x) \neq 0$ then the following mid-range asymptotic mean value property also holds (see [18], [14]):

$$\lim_{\varrho\to 0}\frac{1}{\varrho^2}\left[\frac{1}{2}\left(\sup_{B(x,\varrho)} u + \inf_{B(x,\varrho)} u\right) - u(x)\right] = \frac{\triangle_\infty u(x)}{2|\nabla u(x)|^2} \tag{1.4}$$

Therefore, taking

$$\alpha = \frac{p-2}{p+n} \tag{1.5}$$

it follows from (1.3) and (1.4) that if $u \in C^2(\Omega)$ is p-harmonic in Ω then u satisfies the *asymptotic p-mean value property*

$$\lim_{\varrho\to 0}\frac{1}{\varrho^2}\left[\frac{\alpha}{2}\left(\sup_{B(x,\varrho)} u + \inf_{B(x,\varrho)} u\right) + (1-\alpha)\fint_{B(x,\varrho)} u\,dm - u(x)\right] = 0 \tag{1.6}$$

at those x's such that $\nabla u(x) \neq 0$. When $p \neq 2$ and $n \geq 3$ it is an open question whether p-harmonic functions satisfy the asymptotic p-mean value property at any point, one of the obstacles being that p-harmonic functions are only $C^{1,\beta}$ for some $0 < \beta < 1$ ([22], [12]), but not C^2 in general. More information has been recently obtained when $n = 2$: it turns out that planar p-harmonic functions always satisfy the asymptotic p-mean value property at any point. (In [13] the result was proven for a certain interval of p's and in [3] for the whole range $1 < p < \infty$).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that for each $x \in \Omega$, a radius $\varrho = \varrho(x) > 0$ is given so that $0 < \varrho \leq \operatorname{dist}(x, \partial\Omega)$ and let $B_x = \overline{B}(x,\varrho)$ be the closed ball centered at x of radius ϱ. Inspection of formula (1.6) suggests the definition of the following operators in $C(\overline{\Omega})$:

$$\begin{aligned}
\mathcal{M}u(x) &= \fint_{B_x} u\;dm, \\
\mathcal{S}u(x) &= \frac{1}{2}\left(\sup_{B_x} u + \inf_{B_x} u\right), \\
\mathcal{T}_\alpha u(x) &= \alpha\,\mathcal{S}u(x) + (1-\alpha)\mathcal{M}u(x).
\end{aligned}$$

The operators $\mathcal{T}_\alpha$ have recently come out in different contexts. When $\alpha = 1$ and a radius function in Ω is given, functions u satisfying $\mathcal{S}u = u$ have been called *harmonious functions* in the literature. The connection between harmonious functions and extension problems was studied in [11], in the more general context of metric spaces. Existence and uniqueness of the Dirichlet problem for harmonious functions was also discussed there. The influential papers [20] and [21] opened the path to an stochastic interpretation of the p-laplacian and the ∞-laplacian, via the Dynamic Programming Principle (corresponding essentially to the functional equation $T_\alpha u = u$) for certain tug-of-war games. See also [18] and [19], where the game-stochastic approach was continued and developed, in the case $0 \le \alpha < 1$, or $p \ge 2$.

If $p \ge 2$, α is as in (1.5) and $r(x) = \varepsilon$ is constant then (not necessarily continuous) functions u satisfying $\mathcal{T}_\alpha u = u$ were called *p-harmonious functions* in [19]. Note that the range $0 \le \alpha \le 1$ corresponds to the range $2 \le p \le +\infty$. In order to pose the Dirichlet problem for such p-harmonious functions, the authors in [19] needed to extend a given $f \in C(\partial\Omega)$ to the strip $\{x \in \mathbb{R}^n \setminus \Omega : \mathrm{dist}(x, \partial\Omega) \le \varepsilon\}$ and proved that, if $\Omega \subset \mathbb{R}^n$ is bounded and satisfies some regularity assumptions then there is a unique p-harmonious function u_ε having f as boundary values (in the extended sense). Furthermore, $\{u_\varepsilon\} \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$, where u is the unique p-harmonic function solving the Dirichlet problem in Ω with boundary data f. See also [15] for an analytic approach, still in the constant radius case.

Continuous functions u satisfying $\mathcal{T}_\alpha u = u$ in the variable radius case were considered in [2] and the existence and uniqueness of the Dirichlet problem for such a class of functions was established there under certain assumptions on the domain, the parameter α and the radius function.

Our main concern in this paper is to provide Hölder and Lipschitz regularity estimates for continuous solutions of the functional equation $\mathcal{T}_\alpha u = u$ in metric measure spaces, depending on the regularity of the radius function ϱ (see Theorem 5.1 below). In the constant radius case, the local Lipschitz regularity of p-harmonious functions for $p \ge 2$ was obtained in [16]. As for the case $\alpha = 1$ (or $p = \infty$), not much is known. Unfortunately, our methods cannot be extended to cover the case $\alpha = 1$.

2 Preliminary definitions and main results

2.1 Metric measure spaces and admissible radius functions

Let $(\mathbb{X}, d)$ be a metric space. We say that $(\mathbb{X}, d)$ is *proper* if every closed and bounded subset of $\mathbb{X}$ is compact. $(\mathbb{X}, d)$ is a *geodesic space* if for any two points x, $y \in \mathbb{X}$ there is a curve γ connecting x and y whose length is equal to $d(x, y)$.

A *metric measure space* $(\mathbb{X}, d, \mu)$ is a metric space endowed with a Borel positive regular measure μ. In what follows, we will only consider measures μ such that $0 < \mu(B) < \infty$ for every ball $B \subset \mathbb{X}$.

Definition 2.1. Let $(\mathbb{X}, d, \mu)$ be a metric measure space. We say that μ is *doubling* (equivalently, $(\mathbb{X}, d, \mu)$ is a *doubling metric measure space*) if there exists a constant $D_\mu \ge 1$ such that

$$\mu\left(B(x, 2r)\right) \le D_\mu\, \mu(B(x, r)) \tag{2.1}$$

for any $x \in \mathbb{X}$ and each $r > 0$.

The following property will play a central role in what follows.

Definition 2.2. Let $\delta \in (0, 1]$. A metric measure space $(\mathbb{X}, d, \mu)$ satisfies the *δ-annular decay property* if there exists a constant $D_\delta \geq 1$ such that

$$\mu\left(B(x,R) \setminus B(x,r)\right) \leq D_\delta \left(\frac{R-r}{R}\right)^\delta \mu(B(x,R)), \tag{2.2}$$

for each $x \in \mathbb{X}$ and $0 < r \leq R$. For $\delta = 1$, this property is also known as the *strong annular decay property.*

We will also use the following definition when studying the continuity properties of the operator $\mathcal{M}$.

Definition 2.3. We say that a (Borel, regular) measure μ in a metric space $\mathbb{X}$ is *ring-continuous* if, for each $x \in \mathbb{X}$ the function

$$r \longmapsto \mu(B(x,r))$$

is continuous in $(0, +\infty)$.

As a canonical example, $\mathbb{R}^n$ endowed with the euclidean distance and Lebesgue n-dimensional measure satisfies the strong annular decay property. The δ-annular decay property was introduced in manifolds by Colding and Minicozzi ([6]) and, independently, in metric spaces by Buckley ([5]). It is easy to check that the δ-annular decay property implies the doubling property. Conversely, in [5] it is proved in particular that a geodesic metric space $(\mathbb{X}, d, \mu)$ with a doubling measure μ satisfies a δ-annular decay condition for some $\delta \in (0, 1]$, where δ only depends on the doubling constant. In the context of harmonicity in metric measure spaces, the δ-annular decay property has already been used in [1]. See also [4] for a local version.

Remark 2.4. Let $(\mathbb{X}, d, \mu)$ be a metric measure space. The following implications hold:

$$\begin{array}{ccc} \delta\text{-Annular Decay} & \Longrightarrow & \text{Ring-continuous} \\ \Downarrow & & \\ \text{Doubling Property} & & \end{array}$$

In addition, by [7] and [8], if $(\mathbb{X}, d, \mu)$ is geodesic then

$$\text{Doubling Property} \quad \Longrightarrow \quad \text{Ring-continuous.}$$

Moreover, by [5], if $(\mathbb{X}, d, \mu)$ is geodesic then

$$\text{Doubling Property} \quad \Longrightarrow \quad \delta\text{-Annular Decay.}$$

We introduce some basic concepts that will be useful in the following sections: given any subset $G \subset \mathbb{X}$, we denote by $\operatorname{dist}(x, G)$ the infimum of all distances

$d(x, y)$ where $y \in G$. Moreover, if G is bounded, let $\ell(G)$ be the largest distance to the boundary for points in G:

$$\ell(G) := \sup_{x \in G} \{\operatorname{dist}(x, \partial G)\} \leq \frac{1}{2} \operatorname{diam} G. \tag{2.3}$$

Given two subsets $A, B \subset \mathbb{X}$, we denote by $A \triangle B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference of A and B. If $A, B, C \subset \mathbb{X}$, it follows that

$$A \triangle B = (A \triangle C) \triangle (C \triangle B) \subset (A \triangle C) \cup (C \triangle B).$$

If $A, B \subset \mathbb{X}$ are two measurable subsets, then

$$|\mu(A) - \mu(B)| \leq \mu(A \triangle B).$$

and, from the triangle inequality,

$$\mu(A \triangle B) \leq \mu(A \triangle C) + \mu(C \triangle B). \tag{2.4}$$

A *modulus of continuity* in a bounded domain $\Omega \subset \mathbb{X}$ is a non-decreasing continuous function $\omega : [0, \operatorname{diam} \Omega] \to [0, \infty)$ such that $\omega(0) = 0$. We will often require ω to be concave too. If $G \subset \Omega$ and $u \in C(\overline{G})$, we will denote by $\omega_{u,G}$ a concave modulus of continuity such that

$$|u(x) - u(y)| \leq \omega_{u,G}(d(x, y)) \tag{2.5}$$

for all $x, y \in G$.

Definition 2.5. Let $\Omega \subset \mathbb{X}$ be a fixed bounded open domain in a proper metric space $\mathbb{X}$. We say that a non-negative function $\varrho \in C(\overline{\Omega})$ is an *admissible radius function in* Ω if $0 < \varrho(x) \leq \operatorname{dist}(x, \partial\Omega)$ for each $x \in \Omega$, and $\varrho(x) = 0$ if and only if $x \in \partial\Omega$. Whenever $G \Subset \Omega$, we define

$$\varrho_G := \inf_G \varrho > 0. \tag{2.6}$$

Also, we introduce the following notation for closed balls in Ω with radii given by ϱ:

$$B_x := \overline{B}(x, \varrho(x))$$

for each $x \in \Omega$. Since the balls B_x are not necessarily contained in G, we define

$$\widetilde{G} := \bigcup_{x \in G} B_x. \tag{2.7}$$

Following the notation in (2.5), we denote by $\omega_{\varrho,\Omega}$ a concave modulus of continuity for ϱ in Ω. Since $|\varrho(x) - \varrho(y)| \leq \operatorname{diam} \Omega$ for each x, $y \in \Omega$, we can also assume that $\omega_{\varrho,\Omega}(\operatorname{diam} \Omega) \leq \operatorname{diam} \Omega$. As we will see in the next sections, a distinguished case occurs when the admissible radius function is L-Lipschitz, that is,

$$|\varrho(x) - \varrho(y)| \leq L\, d(x, y),$$

for each $x, y \in \Omega$, in which case we can simply take $\omega_{\varrho,\Omega}(t) = Lt$. For technical reasons, we need to define another concave modulus of continuity for ϱ (that will be denoted by $\widehat{\omega}_\varrho$) as follows: if $\omega_{\varrho,\Omega}(t) \le t$ for all $t \in [0, \operatorname{diam}\Omega]$ then we set $\widehat{\omega}_\varrho(t) := t$. Otherwise, we define

$$\widehat{\omega}_\varrho(t) := \frac{\operatorname{diam}\Omega}{\omega_{\varrho,\Omega}(\operatorname{diam}\Omega)}\, \omega_{\varrho,\Omega}(t). \tag{2.8}$$

Note that, defined in this way, $\widehat{\omega}_\varrho(t)$ is a concave modulus of continuity for ϱ in Ω satisfying

$$\max\{t, \omega_{\varrho,\Omega}(t)\} \le \widehat{\omega}_\varrho(t) \le \operatorname{diam}\Omega = \widehat{\omega}_\varrho(\operatorname{diam}\Omega) \tag{2.9}$$

for each $t \in [0, \operatorname{diam}\Omega]$. Consequently, successive compositions of ω_ϱ with itself will produce a sequence of continuous functions $\widehat{\omega}_\varrho^{(n)} : [0, \operatorname{diam}\Omega] \to [0, \operatorname{diam}\Omega]$ given by

$$\widehat{\omega}_\varrho^{(n)}(t) := \widehat{\omega}_\varrho\left(\widehat{\omega}_\varrho^{(n-1)}(t)\right),$$

for $n \in \mathbb{N}$, where $\widehat{\omega}_\varrho^{(0)}(t) = t$.

Remark 2.6. We will hereafter make use of some of the concepts introduced in this subsection (like the family of balls $\{B_x : x \in \Omega\}$ and the operator on sets $\widetilde{(\cdot)}$) without any explicit mention of their dependence on the choice of the admissible radius function ϱ, which is assumed to be fixed.

2.2 Main results

Let $(\mathbb{X}, d, \mu)$ be a metric measure space. Assume that an admissible radius function ϱ in a domain $\Omega \subset \mathbb{X}$ is given. If $u \in C(\overline{\Omega})$, $x \in \Omega$ and $\alpha \in \mathbb{R}$ we define:

$$\mathcal{M}u(x) := \fint_{B_x} u\, d\mu, \tag{2.10}$$

$$\mathcal{S}u(x) := \frac{1}{2}\left(\sup_{B_x} u + \inf_{B_x} u\right), \tag{2.11}$$

$$\mathcal{T}_\alpha u(x) := \alpha\, \mathcal{S}u(x) + (1-\alpha)\mathcal{M}u(x). \tag{2.12}$$

We are interested in studying the fixed points of the operators $\mathcal{T}_\alpha$, which can be seen as functions satisfying an specific nonlinear mean value property. For that reason, we give the following fundamental definition.

Definition 2.7. Let $(\mathbb{X}, d, \mu)$ be a metric measure space, $\Omega \subset \mathbb{X}$ a domain and ϱ an admissible radius function in Ω. Let $\alpha \in \mathbb{R}$. A function $u \in C(\overline{\Omega})$ is said to satisfy the *α-mean value property* in Ω if it is a solution of the functional equation

$$\mathcal{T}_\alpha u = u.$$

The case $\alpha = 0$ is interesting enough by itself. Harmonicity in a metric measure space $\mathbb{X}$ in connection to the mean value property has been recently introduced in [9] and [1] in the following way: a locally integrable function in a domain $\Omega \Subset \mathbb{X}$ is said *strongly harmonic* in Ω if it satisfies the mean value property in any ball compactly contained in Ω. The following regularity result has been obtained in [1]:

Theorem ([1, Thm. 4.2])**.** *If $(\mathbb{X}, d, \mu)$ is a doubling metric measure space satisfying a δ-annular decay condition for some $\delta \in (0, 1]$ then every locally bounded, strongly harmonic function u in a domain $\Omega \subset \mathbb{X}$ is locally δ-Hölder continuous in Ω. In particular, if $\delta = 1$ then u is locally Lipschitz continuous in Ω.*

(See also Lemma 2.3 in [2], where the local Hölder continuity of functions satisfying the mean value property for a single radius is obtained if $\mathbb{X} = \mathbb{R}^n$, μ is doubling and the radius function is 1-Lipschitz).

We have obtained the following generalizations for functions satisfying the 0-mean value property in the sense of Definition 2.7 with respect to some admissible radius function.

Corollary 3.11. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property for some $\delta \in (0, 1]$. Suppose that there is $\gamma \in (0, 1]$ such that ϱ is a γ-Hölder continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then any $u \in L^\infty(\Omega)$ satisfying the 0-mean value property in Ω with respect to the radius admissible function ϱ (that is, $\mathcal{M}u = u$) is locally $\gamma\delta$-Hölder continuous in Ω. In particular, if $\delta = \gamma = 1$ then u is locally Lipschitz continuous in Ω.*

As for the general case $\alpha \neq 0$, our main result requires certain rigid control of the radius function.

Theorem 5.1. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space satisfying the δ-annular decay condition for some $\delta \in (0, 1]$ and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is a Lipschitz admissible radius function in Ω with Lipschitz constant $L \geq 1$ such that*

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$, where $0 < \lambda \leq \ell(\Omega)^{1-\beta}\varepsilon$. Assume also that

$$|\alpha| < L^{-1},$$
$$0 < \varepsilon < 1 - L\,|\alpha|\,,$$

and choose β so that

$$1 \leq \beta < \frac{\log \dfrac{1}{L\,|\alpha|}}{\log \dfrac{1}{1-\varepsilon}}.$$

Then any $u \in C(\overline{\Omega})$ verifying the α-mean value property in Ω with respect to ϱ (that is, $\mathcal{T}_\alpha u = u$) is locally δ-Hölder continuous in Ω. In particular, if $\delta = 1$ then u is locally Lipschitz continuous in Ω.

In the particular case $\beta = 1$ we get the following corollary.

Corollary 5.2. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space satisfying the δ-annular decay condition for some $\delta \in (0, 1]$ and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is a Lipschitz admissible radius function in Ω with Lipschitz constant $L \geq 1$ such that*

$$\lambda \operatorname{dist}(x, \partial\Omega) \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$, where $0 < \lambda \leq \varepsilon$. Assume also that

$$|\alpha| < L^{-1},$$
$$0 < \ \varepsilon \ < 1 - L\,|\alpha|\,.$$

Then any $u \in C(\overline{\Omega})$ verifying the α-mean value property in Ω with respect to ϱ (that is, $\mathcal{T}_\alpha u = u$) is locally δ-Hölder continuous in Ω. In particular, if $\delta = 1$ then u is locally Lipschitz continuous in Ω.

We obtain further regularity for solutions of the α-mean value property assuming that they are continuous in $\overline{\Omega}$. This explains the *a priori* in the title. However, the existence part is not discussed here. Compare with [2], where existence and uniqueness of the Dirichlet problem are established if $\mathbb{X} = \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ is bounded and strictly convex and μ is Lebesgue measure. In this particular case, the connection between p-harmonious functions and the α-mean value property has already been pointed out at the introduction, where α and p are related by (1.5) (note that the intervals $1 < p < \infty$ and $2 \leq p < \infty$ correspond, respectively, to the intervals $-\frac{1}{n+1} < \alpha < 1$ and $0 \leq \alpha < 1$). This explains the term *generalized p-harmonious* in the title, even though the link between p and α is missing in the general metric space case.

3 Basic Estimates for $\mathcal{M}$ and $\mathcal{S}$

3.1 Continuity of $\mathcal{M}$

We will first look at the continuity and regularity of the function

$$x \longmapsto \mathcal{M}u(x) = \fint_{B_x} u \; d\mu \tag{3.1}$$

where an admissible radius function ϱ in a domain $\Omega \subset \mathbb{X}$, a measure μ and a bounded, continuous function u in Ω are given. The following Lemma is a preliminary result in this direction.

Lemma 3.1. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space. If B_1 and B_2 are two balls contained in $\mathbb{X}$, then*

$$\left| \fint_{B_1} u \; d\mu - \fint_{B_2} u \; d\mu \right| \leq 2 \left\| u \right\|_\infty \frac{\mu(B_1 \triangle B_2)}{\max \left\{ \mu(B_1), \mu(B_2) \right\}}, \tag{3.2}$$

for each $u \in L^\infty(\mathbb{X})$.

Proof. We can assume that $\mu(B_1) \geq \mu(B_2)$, then

$$\mu(B_1) \left(\fint_{B_1} u \; d\mu - \fint_{B_2} u \; d\mu \right) = \int_{B_1} u \; d\mu - \int_{B_2} u \; d\mu + (\mu(B_2) - \mu(B_1)) \fint_{B_2} u \; d\mu,$$

and estimating this, we obtain

$$\begin{aligned} \mu(B_1) \left| \fint_{B_1} u \; d\mu - \fint_{B_2} u \; d\mu \right| &\leq \left| \int_{B_1} u \; d\mu - \int_{B_2} u \; d\mu \right| + \left\| u \right\|_\infty \left| \mu(B_2) - \mu(B_1) \right| \\ &\leq \int_{B_1 \triangle B_2} |u| \; d\mu + \left\| u \right\|_\infty \mu(B_1 \triangle B_2) \\ &\leq 2 \left\| u \right\|_\infty \mu(B_1 \triangle B_2). \end{aligned}$$

□

The following corollary follows from Lemma 3.1 and the fact that $||\mathcal{M}v||_\infty \leq ||v||_\infty$ if $v \in L^\infty(\Omega)$.

Corollary 3.2. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space. Let $\Omega \subset \mathbb{X}$ be a domain and ϱ an admissible radius function in Ω. Then, for each $u \in L^\infty(\Omega)$ and all x, $y \in \Omega$ we have*

$$\left| \mathcal{M}^n u(x) - \mathcal{M}^n u(y) \right| \leq 2 \left\| u \right\|_\infty \frac{\mu(B_x \triangle B_y)}{\max \left\{ \mu(B_x), \mu(B_y) \right\}}. \tag{3.3}$$

The importance of Corollary 3.2 lies in the fact that the continuity of $\mathcal{M}u$ can be transferred from the continuity of the function

$$x \longmapsto \mu(B_x) = \mu(B(x, \varrho(x))), \tag{3.4}$$

without any dependence of the function u. To see that, consider any $x, y \in \Omega$ and $r_1, r_2 > 0$ and recall (2.4). Then

$$\mu(B(x, r_1) \triangle B(y, r_2)) \leq \mu(B(x, r_1) \triangle B(x, r_2)) + \mu(B(x, r_2) \triangle B(y, r_2)). \tag{3.5}$$

Now suppose that μ is ring-continuous (recall Definition 2.3). Then, since $B(x, r_1) \subset B(x, r_2)$ or $B(x, r_2) \subset B(x, r_1)$, the first term in the right hand side of (3.5) is equal to $|\mu(B(x, r_1)) - \mu(B(x, r_2))|$. For the second term, we recall the following result due to Gaczkowski and Górka:

Lemma ([9, Theorem 2.1])**.** *Let $(\mathbb{X}, d, \mu)$ be a metric measure space such that μ is ring-continuous. Then for each $x \in \mathbb{X}$ and each $r > 0$,*

$$\lim_{y \to x} \mu\left(B(x,r) \triangle B(y,r)\right) = 0. \tag{3.6}$$

Moreover, the function $x \mapsto \mu(B(x,r))$ is continuous (w.r.t. d) for each fixed $r > 0$.

Remark 3.3. The converse is not true (see Example 2 in [1]).

Therefore, replacing $r_1 = \varrho(x)$ and $r_2 = \varrho(y)$ in (3.5) we get the following proposition.

Proposition 3.4. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space such that μ is ring-continuous. Suppose that $\Omega \subset \mathbb{X}$ is a domain and ϱ is a continuous admissible radius function in Ω. Then, $\mathcal{M} : L^\infty(\Omega) \to C(\Omega)$.*

Remark 3.5. By definition, the continuous admissible radius function ϱ vanishes on the boundary of the domain Ω, thus $\mu(B_x)$ tends to zero as x approaches the boundary of Ω. In consequence, estimates obtained from (3.3) are local, that is, they only make sense on compact subsets $K \subset \Omega$.

3.2 Estimates for $\mathcal{M}$

Let $\Omega \subset \mathbb{X}$ be a given domain in a metric measure space $(\mathbb{X}, d, \mu)$ and let $K \subset \Omega$ be a compact subset. In this section we will construct moduli of continuity $\mathcal{W}_{\mu,K}$ depending on μ, ϱ and K such that

$$\frac{\mu(B_x \triangle B_y)}{\max\{\mu(B_x), \mu(B_y)\}} \leq \frac{1}{2}\, \mathcal{W}_{\mu,K}(d(x,y)), \tag{3.7}$$

for every $x, y \in K$. Hence, by (3.3), we would have

$$|\mathcal{M}^n u(x) - \mathcal{M}^n u(y)| \leq \|u\|_\infty\, \mathcal{W}_{\mu,K}(d(x,y)), \tag{3.8}$$

for each $n \in \mathbb{N}$.

Lemma 3.6. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space satisfying the δ-annular decay property (2.2) for some $\delta \in (0,1]$ and $D_\delta \geq 1$. Suppose that ϱ is a L-Lipschitz admissible radius function in a domain $\Omega \subset \mathbb{X}$ for some $L \geq 1$. Then, for any compact set $K \subset \Omega$ and each x, $y \in K$ we have*

$$\frac{\mu(B_x \triangle B_y)}{\max\{\mu(B_x), \mu(B_y)\}} \leq 4\, L\, D_\delta \left(\frac{d(x,y)}{\varrho_K}\right)^\delta. \tag{3.9}$$

Proof. Since ϱ is L-Lipschitz by assumption, $|\varrho(x) - \varrho(y)| \leq L\, d(x,y)$. Then:

i) if $d(x,y) > \dfrac{\varrho_K}{2L}$, then $D_\delta \left(\dfrac{2L\, d(x,y)}{\varrho_K} \right)^\delta > 1$, and

$$\begin{aligned} \mu(B_x \triangle B_y) &\le 2 \max \{\mu(B_x), \mu(B_y)\} \\ &< 2\, D_\delta \left(\frac{2L\, d(x,y)}{\varrho_K} \right)^\delta \max \{\mu(B_x), \mu(B_y)\} . \end{aligned}$$

ii) If $d(x,y) \le \dfrac{\varrho_K}{2L}$ then, since $L \ge 1$, we get that $|\varrho(x) - \varrho(y)| \le \varrho_K/2$ and, in particular, $\varrho(y) \ge \varrho(x)/2$ and $\varrho(x) \ge \varrho(y)/2$. As a consequence, the following inclusions hold:

$$\begin{aligned} B_x \setminus B_y \subset B_x \setminus B(x, \varrho(y) - d(x,y)), \\ B_y \setminus B_x \subset B_y \setminus B(y, \varrho(x) - d(x,y)). \end{aligned}$$

Thus, by (2.2) and the fact that $\varrho(x), \varrho(y) \ge \varrho_K$ for $x, y \in K$, we obtain

$$\begin{aligned} \mu(B_x \setminus B_y) \le D_\delta \left(\frac{\varrho(x) - \varrho(y) + d(x,y)}{\varrho_K} \right)^\delta \max \{\mu(B_x), \mu(B_y)\} , \\ \mu(B_y \setminus B_x) \le D_\delta \left(\frac{\varrho(y) - \varrho(x) + d(x,y)}{\varrho_K} \right)^\delta \max \{\mu(B_x), \mu(B_y)\} . \end{aligned}$$

Using the L-Lipschitz assumption on ϱ and adding these two quantities we get

$$\mu(B_x \triangle B_y) \le 2\, D_\delta \left(\frac{(L+1)\, d(x,y)}{\varrho_K} \right)^\delta \max \{\mu(B_x), \mu(B_y)\} ,$$

which implies (3.9).

□

Remark 3.7. Note that if x, y are as in the statement of Lemma 3.6 then only the pointwise inequality $|\varrho(x) - \varrho(y)| \le L\, d(x,y)$ is really used in the proof.

Lemma 3.8. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property (2.2) for some $\delta \in (0,1]$ and $D_\delta \ge 1$. Suppose that ϱ is a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then, for any compact set $K \subset \Omega$ and each x,$y \in K$ we have*

$$\frac{\mu(B_x \triangle B_y)}{\max \{\mu(B_x), \mu(B_y)\}} \le C \left(\frac{\widehat{\omega}_\varrho (d(x,y))}{\varrho_K} \right)^\delta , \tag{3.10}$$

where $C = C(D_\delta, \mu) > 0$ and $\widehat{\omega}_\varrho$ is as in (2.9).

Proof. Since ϱ is a continuous function by assumption, for each pair of points $x, y \in K$, we need distinguish two cases depending on the values of $|\varrho(x) - \varrho(y)|$: if $|\varrho(x) - \varrho(y)| \le d(x,y)$, this case was already studied in Lemma 3.6 with $L = 1$, then (3.10) follows from (3.9) and (2.9).

Otherwise, $|\varrho(x) - \varrho(y)| > d(x,y)$. We can assume directly that

$$d(x,y) < \varrho(x) - \varrho(y), \tag{3.11}$$

since the other case is analogous. Then $B_y \subset B_x$ and

$$B_x \triangle B_y = B_x \setminus B_y \subset B(y, \varrho(x) + d(x,y)) \setminus B(y, \varrho(y)).$$

Consequently, the δ-annular decay (2.2) yields

$$\mu(B_x \triangle B_y) \le D_\delta \left(\frac{\varrho(x) - \varrho(y) + d(x,y)}{\varrho(x) + d(x,y)} \right)^\delta \mu(B(y, \varrho(x) + d(x,y))). \tag{3.12}$$

On the other hand, since the δ-annular decay property implies that μ is doubling with some constant $D_\mu \ge 1$, using the inclusion $B(y, \varrho(x) + d(x,y)) \subset B(y, 2\varrho(x))$, it turns out that

$$\mu(B(y, \varrho(x) + d(x,y))) \le D_\mu^2 \, \mu(B_x).$$

Therefore, replacing this in (3.12) we reach

$$\mu(B_x \triangle B_y) \le D_\mu^2 \, D_\delta \left(\frac{\varrho(x) - \varrho(y) + d(x,y)}{\varrho(x) + d(x,y)} \right)^\delta \mu(B_x).$$

Since $d(x,y) \ge 0$, $\varrho(x) \ge \varrho_K$, $\mu(B_x) \ge \mu(B_y)$ and (3.11),

$$\mu(B_x \triangle B_y) \le D_\mu^2 \, D_\delta \left(2 \, \frac{\varrho(x) - \varrho(y)}{\varrho_K} \right)^\delta \max\left\{ \mu(B_x), \mu(B_y) \right\}.$$

Recalling (2.9) the proof is completed. □

Theorem 3.9. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property (2.2) for some $\delta \in (0,1]$ and $D_\delta \ge 1$. Suppose that ϱ is a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then, for any $u \in L^\infty(\Omega)$, any compact set $K \subset \Omega$, any $x, y \in K$ and each $n \in \mathbb{N}$ we have*

$$|\mathcal{M}^n u(x) - \mathcal{M}^n u(y)| \le \|u\|_\infty \, \mathcal{W}_{\mu,K}(d(x,y)),$$

where $\mathcal{W}_{\mu,K} : [0, \operatorname{diam}\Omega] \to \mathbb{R}$ is given by

$$\mathcal{W}_{\mu,K}(t) = C \varrho_K^{-\delta} \, (\widehat{\omega}_\varrho(t))^\delta, \tag{3.13}$$

and $C = C(D_\delta, \mu) > 0$. In particular, the sequence $\{\mathcal{M}^n u\}_n$ is locally uniformly equicontinuous in Ω.

Corollary 3.10. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property (2.2) for some $\delta \in (0,1]$ and $D_\delta \ge 1$. Suppose that ϱ is a γ-Hölder continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$, for some $\gamma \in (0,1)$. Then,* *i) for any $u \in L^\infty(\Omega)$, any compact set $K \subset \Omega$, any $x, y \in K$ and each $n \in \mathbb{N}$ we have*

$$|\mathcal{M}^n u(x) - \mathcal{M}^n u(y)| \leq \|u\|_\infty \, \mathcal{W}_{\mu,K}(d(x,y)),$$

where $\mathcal{W}_{\mu,K} : [0, \operatorname{diam}\Omega] \to \mathbb{R}$ is given by

$$\mathcal{W}_{\mu,K}(t) = C\, \varrho_K^{-\delta}\, t^{\gamma\,\delta}, \tag{3.14}$$

and $C = C(D_\delta, D_\mu, L)$ where $L > 0$ is the Hölder coefficient of ϱ. In particular, the sequence $\{\mathcal{M}^n u\}_n$ is locally uniformly equicontinuous in Ω.

ii) the operator $\mathcal{M}$ sends $L^\infty(\Omega)$ to the space $\Lambda_{\gamma\delta,\mathrm{loc}}(\Omega)$ of locally $\gamma\delta$-Hölder continuous functions in Ω, that is

$$\mathcal{M} : L^\infty(\Omega) \to \Lambda_{\gamma\delta,\mathrm{loc}}(\Omega).$$

Corollary 3.11. *Let $(\mathbb{X}, d, \mu)$ be a prper metric measure space satisfying the δ-annular decay property for some $\delta \in (0,1]$. Suppose that there is $\gamma \in (0,1]$ such that ϱ is a γ-Hölder continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then any $u \in L^\infty(\Omega)$ satisfying the 0-mean value property in Ω with respect to the radius admissible function ϱ (that is, $\mathcal{M}u = u$) is locally $\gamma\delta$-Hölder continuous in Ω. In particular, if $\delta = \gamma = 1$ then u is locally Lipschitz continuous in Ω.*

3.3 Estimates for $\mathcal{S}$

The following lemma was proven in [11] under the assumption that the admissible radius function is 1-Lipschitz. Note that, since the operator $\mathcal{S}$ does not depend on any measure, we state it in the context of a metric space $(\mathbb{X}, d)$.

Lemma 3.12. *Let $(\mathbb{X}, d)$ be a geodesic metric space and let ϱ be a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then, for any $u \in C(\overline{\Omega})$, any compact subset $K \subset \Omega$ and each x, $y \in K$ we have*

$$|\mathcal{S}u(x) - \mathcal{S}u(y)| \leq \omega_{u,\widetilde{K}}\left(\widehat{\omega}_\varrho(d(x,y))\right).$$

where $\widetilde{K}$, $\omega_{u,\widetilde{K}}$ and $\widehat{\omega}_\varrho$ are as in (2.7) , (2.5) and (2.8). We have, in particular

$$\omega_{\mathcal{S}u,K}(t) \leq \omega_{u,\widetilde{K}}\left(\widehat{\omega}_\varrho(t)\right). \tag{3.15}$$

Proof. Recalling the definition of $\mathcal{S}u$, (2.11), and the elementary formulas

$$\sup_{i\in I} x_i - \sup_{j\in J} y_j = \sup_{i\in I} \inf_{j\in J} (x_i - y_j),$$
$$\inf_{i\in I} x_i - \inf_{j\in J} y_j = \sup_{j\in J} \inf_{i\in I} (x_i - y_j),$$

we can write

$$\mathcal{S}u(x) - \mathcal{S}u(y) = \frac{1}{2} \sup_{s\in B_x} \inf_{t\in B_y} (u(s) - u(t)) + \frac{1}{2} \sup_{t\in B_y} \inf_{s\in B_x} (u(s) - u(t)). \quad (3.16)$$

Note that it may happen that $B_x \not\subset K$ or $B_y \not\subset K$. However, by (2.7), the inclusion $B_x \cup B_y \subset \widetilde{K}$ holds. Then,

$$\sup_{s\in B_x} \inf_{t\in B_y} (u(s) - u(t)) \le \sup_{s\in B_x} \inf_{t\in B_y} \omega_{u,\widetilde{K}}(d(s,t)) \le \omega_{u,\widetilde{K}} \left(\sup_{s\in B_x} \inf_{t\in B_y} d(s,t) \right).$$

Replacing this term (the other term is analogous) in (3.16) and using that $\omega_{u,\widetilde{K}}$ is concave, we get

$$\mathcal{S}u(x) - \mathcal{S}u(y) \le \omega_{u,\widetilde{K}} \left(\frac{1}{2} \sup_{s\in B_x} \inf_{t\in B_y} d(s,t) + \frac{1}{2} \sup_{t\in B_y} \inf_{s\in B_x} d(s,t) \right).$$

Thus, we need to show that, for any $x, y \in \Omega$,

$$\frac{1}{2} \sup_{s\in B_x} \inf_{t\in B_y} d(s,t) + \frac{1}{2} \sup_{t\in B_y} \inf_{s\in B_x} d(s,t) \le \widehat{\omega}_\varrho(d(x,y)). \quad (3.17)$$

From [11, p.282] we get:

$$\begin{aligned} \sup_{t\in B_y} \inf_{s\in B_x} d(s,t) &\le \max\{d(x,y) + \varrho(x) - \varrho(y), 0\} \\ \sup_{s\in B_x} \inf_{t\in B_y} d(s,t) &\le \max\{d(x,y) + \varrho(y) - \varrho(x), 0\} \end{aligned} \quad (3.18)$$

Finally, (3.17) follows from (3.18) and (2.9). Therefore, this together with (3.16) finishes the proof.

□

4 Iteration of $\mathcal{T}_\alpha$

As a direct consequence of Proposition 3.4 and Lemma 3.12 we have the following result.

Proposition 4.1. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space. Suppose that $\Omega \subset \mathbb{X}$ is a bounded domain and let ϱ be a continuous admissible radius function in Ω. If $\alpha \in \mathbb{R}$ then $\mathcal{T}_\alpha : C(\overline{\Omega}) \to C(\overline{\Omega})$.*

As in the case $\alpha = 0$ in which $\mathcal{T}_\alpha$ reduces to $\mathcal{M}$, to go beyond this result we need to take into consideration stronger hypothesis on the measure μ.

Lemma 4.2. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is an admissible radius function in Ω and assume that, for every compact set $K \subset \Omega$, a modulus of continuity $\mathcal{W}_{\mu,K}$ is given satisfying (3.8). Then, if $|\alpha| \le 1$, and $u \in C(\overline{\Omega})$, the estimate*

$$\omega_{\mathcal{T}_\alpha u,K}(t) \le |\alpha|\, \omega_{u,\widetilde{K}}(\widehat{\omega}_\varrho(t)) + (1-\alpha)\, \|u\|_\infty \, \mathcal{W}_{\mu,K}(t), \quad (4.1)$$

holds for all $t \in [0, \operatorname{diam}\Omega]$.

Proof. Let $x, y \in K$. Then, since $\mathcal{T}_\alpha = \alpha\mathcal{S} + (1-\alpha)\mathcal{M}$, we get

$$|\mathcal{T}_\alpha u(x) - \mathcal{T}_\alpha u(y)| \leq |\alpha| \, |\mathcal{S}u(x) - \mathcal{S}u(y)| + (1-\alpha) \, |\mathcal{M}u(x) - \mathcal{M}u(y)| \, ,$$

and (4.1) is obtained by taking into consideration the estimates (3.15) and (3.8). ☐

The key point for this subsection is the iteration of formula (4.1). Note that, in order to obtain estimates for $\mathcal{T}_\alpha u$ on the compact set K, we need to control u on $\widetilde{K} \supset K$, where $\widetilde{K}$ is given by (2.7). Thus, when iterating (4.1), we need to guarantee some control on the sequence of sets given by successive application of the $\widetilde{(\cdot)}$ operation over the compact set K. For that reason, we need to assume that the domain $\Omega \subset \mathbb{X}$ is bounded and we impose the following restriction on ϱ:

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega), \tag{4.2}$$

for each $x \in \Omega$, where $0 < \lambda \leq \ell(\Omega)^{1-\beta}\varepsilon$, $0 < \varepsilon < 1$ and $\beta \geq 1$. We also introduce the following exhaustion of Ω:

$$K_m := \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) \geq (1-\varepsilon)^m\} \, , \tag{4.3}$$

for $m \in \mathbb{N}$, where ε is the constant appearing in (4.2). Hence, $K_1 \subset K_2 \subset \cdots \Subset \Omega$ and $\lim\limits_{m\to\infty} K_m = \Omega$ in the sense that, for every $x \in \Omega$, there exists large enough $m_0 = m_0(x) \in \mathbb{N}$ such that $x \in K_m$ for all $m \geq m_0$. Moreover, by (2.7) and (4.2), it is easy to check that

$$\widetilde{K_m} \subset K_{m+1}, \tag{4.4}$$

for $m \in \mathbb{N}$. From (4.2), we can also control from below the values of ϱ on K_m:

$$\varrho_{K_m} \geq \lambda \left(\inf_{K_m} \operatorname{dist}(x, \partial\Omega) \right)^\beta \geq \lambda(1-\varepsilon)^{m\beta}, \tag{4.5}$$

where ϱ_{K_m} is as in (2.6). Replacing K by K_m in (4.1) and iterating it we can control the oscillation of $\mathcal{T}_\alpha^n$, for $n \in \mathbb{N}$, as the next lemma shows.

Lemma 4.3. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space, $\Omega \subset \mathbb{X}$ a bounded domain and let ϱ be a continuous admissible radius function in Ω. Suppose that, for every compact sect $K \subset \Omega$, a modulus of continuity $\mathcal{W}_{\mu,K}$ is given satisfying (3.8). Then, for $|\alpha| \leq 1$ and $u \in C(\overline{\Omega})$, the estimate*

$$\omega_{\mathcal{T}_\alpha^n u, K_m}(t) \leq$$
$$|\alpha|^n \, \omega_{u, K_{m+n}} \left(\widehat{\omega}_\varrho^{(n)}(t) \right) + (1-\alpha) \, \|u\|_\infty \sum_{j=0}^{n-1} |\alpha|^j \, \mathcal{W}_{\mu, K_{m+j}} \left(\widehat{\omega}_\varrho^{(j)}(t) \right) \tag{4.6}$$

holds for each n, $m \in \mathbb{N}$ and every $t \in [0, \operatorname{diam}\Omega]$.

Proof. Since $\widetilde{K_m} \subset K_{m+1}$, we get from (4.1)

$$\omega_{\mathcal{T}_\alpha u, K_m}(t) \leq |\alpha|\, \omega_{u,K_{m+1}}\left(\widehat{\omega}_\varrho(t)\right) + (1-\alpha)\, \|u\|_\infty\, \mathcal{W}_{\mu,K_{m+1}}(t)$$

for each $t \in [0, \operatorname{diam}\Omega]$. Now, iteration of this inequality gives (4.6). □

To get equicontinuity of the sequence $\{\mathcal{T}_\alpha^n u\}_n$, we need to add some extra condition.

Lemma 4.4. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space with a continuous admissible radius function ϱ in a bounded domain $\Omega \subset \mathbb{X}$. Suppose that, for every compact sect $K \subset \Omega$, a modulus of continuity $\mathcal{W}_{\mu,K}$ is given satisfying (3.8). Assume also that*

$$|\alpha| \limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_j}(\operatorname{diam}\Omega)\right)^{1/j} < 1 \tag{4.7}$$

Then for any $u \in C(\overline{\Omega})$, the sequence $\{\mathcal{T}_\alpha^n u\}_n$ is locally uniformly equicontinuous in Ω.

Proof. Fix $m \in \mathbb{N}$. Regarding the first term in the right-hand side of (4.6) we note that, since $\widehat{\omega}_\varrho(t) \leq \operatorname{diam}\Omega$ for each $t \in [0, \operatorname{diam}\Omega]$, then

$$|\alpha|^n\, \omega_{u,K_{m+n}}\left(\widehat{\omega}^{(n)}(t)\right) \leq |\alpha|^n\, \omega_{u,\Omega}(\operatorname{diam}\Omega) \xrightarrow[n\to\infty]{} 0.$$

Thus,

$$\left\{t \mapsto |\alpha|^n\, \omega_{u,\Omega}\left(\widehat{\omega}_\varrho^{(n)}(t)\right)\right\}_n \xrightarrow[n\to\infty]{} 0 \tag{4.8}$$

uniformly in $[0, \operatorname{diam}\Omega]$ as $n \to \infty$. Consequently there exists a common modulus of continuity $\mathcal{F}_1$ for the sequence (4.8). Now we focus on the series in (4.6). Note that

$$\mathcal{W}_{\mu,K_{m+j}}\left(\widehat{\omega}_\varrho^{(j)}(t)\right) \leq \mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)$$

for all $t \in [0, \operatorname{diam}\Omega]$. Then, since

$$\limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)\right)^{1/j} = \limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)\right)^{1/(m+j)},$$

it follows from (4.7) that

$$|\alpha| \limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)\right)^{1/j} < 1,$$

so the root test implies that the series

$$\sum_{j=0}^{\infty} |\alpha|^j\, \mathcal{W}_{\mu,K_{m+j}}\left(\widehat{\omega}_\varrho^{(j)}(t)\right) < \infty$$

converges uniformly in $[0, \operatorname{diam}\Omega]$. In particular, there exists another modulus of continuity for the series, say $\mathcal{F}_2$. Summarizing:

$$\omega_{\mathcal{T}_\alpha^n u, K_m}(t) \leq \mathcal{F}_1(t) + (1-\alpha)\left\|u\right\|_\infty \mathcal{F}_2(t).$$

Since m is arbitrary and the right-hand side of the previous inequality does not depend on $n \in \mathbb{N}$, the proof is finished. □

Theorem 4.5. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space satisfying the δ-annular decay property (2.2) for some $\delta \in (0,1]$. Let $|\alpha| < 1$ and suppose that ϱ is a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$ satisfying (4.2) with $0 < \lambda \leq \ell(\Omega)^{1-\beta}\varepsilon$. Assume also that*

$$0 < \varepsilon < 1 - |\alpha|, \tag{4.9}$$

$$1 \leq \beta < \frac{\log \dfrac{1}{|\alpha|}}{\log \dfrac{1}{1-\varepsilon}}. \tag{4.10}$$

Then, for any $u \in C(\overline{\Omega})$, the sequence of iterates $\{\mathcal{T}_\alpha^n u\}_n$ is locally uniformly equicontinuous in Ω.

Proof. We only need to check that the assumptions in Lemma 4.4 are satisfied. By Theorem 3.9, for any compact set $K \subset \Omega$, we can choose $\mathcal{W}_{\mu,K}$ as in (3.13) for any compact set $K \subset \Omega$. Thus, after replacing K by K_j and t by $\operatorname{diam}\Omega$ and recalling that $\widehat{\omega}_\varrho(\operatorname{diam}\Omega) = \operatorname{diam}\Omega$, we get,

$$\left(\mathcal{W}_{\mu,K_j}(\operatorname{diam}\Omega)\right)^{1/j} = \left(C(\operatorname{diam}\Omega)^\delta\right)^{1/j} \varrho_{K_j}^{-\delta/j},$$

and by (4.5),

$$\left(\mathcal{W}_{\mu,K_j}(\operatorname{diam}\Omega)\right)^{1/j} \leq \left(\frac{C(\operatorname{diam}\Omega)^\delta}{\lambda^\delta}\right)^{1/j} (1-\varepsilon)^{-\delta\,\beta}.$$

Taking limits we get

$$\limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_j}(\operatorname{diam}\Omega)\right)^{1/j} \leq (1-\varepsilon)^{-\delta\,\beta}.$$

On the other hand, by (4.9) we have $|\alpha| < 1-\varepsilon \leq (1-\varepsilon)^{\delta\,\beta}$ so condition (4.7) follows and the sequence $\{\mathcal{T}_\alpha^n u\}_n$ is locally uniformly equicontinuous in Ω by Lemma 4.4. □

5 Regularity of solutions

In this section we give regularity results for functions $u \in C(\overline{\Omega})$ satisfying the α-mean value property (that is, solutions of the functional equation $\mathcal{T}_\alpha u = u$)

with respect to an admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. When $\alpha = 0$, then $\mathcal{T}_0 = \mathcal{M}$ and the regularity of such solutions was already obtained in Corollary 3.11. However, the case $\alpha \neq 0$ is more delicate and stronger assumptions on the radius function ϱ are needed, as we have already seen in Section 4.

We focus our attention on inequality (4.6). Since the continuous function u is assumed to be a fixed point of the operator $\mathcal{T}_\alpha$, after replacing $\mathcal{T}_\alpha^n u$ by u, we are allowed to pass to the limit when $n \to \infty$. From (4.6) we get

$$\omega_{u,K_m}(t) \leq (1-\alpha)\,\|u\|_\infty \sum_{j=0}^{\infty} |\alpha|^j\, \mathcal{W}_{\mu,K_{m+j}}\left(\widehat{\omega}_\varrho^{(j)}(t)\right), \tag{5.1}$$

for $t \in [0, \operatorname{diam}\Omega]$, where $m \in \mathbb{N}$ is fixed. Therefore, the series in (5.1) will provide the information about the regularity of the solution u. The following is our main regularity result.

Theorem 5.1. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space satisfying the δ-annular decay condition for some $\delta \in (0,1]$ and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is a Lipschitz admissible radius function in Ω with Lipschitz constant $L \geq 1$ such that*

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$, where $0 < \lambda \leq \ell(\Omega)^{1-\beta}\varepsilon$ and $\ell(\Omega)$ is given by (2.3). Assume also that

$$\begin{aligned} |\alpha| &< L^{-1}, \\ 0 < \ \varepsilon \ &< 1 - L\,|\alpha|, \end{aligned}$$

and choose β so that

$$1 \ \leq \ \beta < \frac{\log \dfrac{1}{L\,|\alpha|}}{\log \dfrac{1}{1-\varepsilon}}. \tag{5.2}$$

Then any $u \in C(\overline{\Omega})$ verifying the α-mean value property in Ω with respect to ϱ (that is, $\mathcal{T}_\alpha u = u$) is locally δ-Hölder continuous in Ω. In particular, if $\delta = 1$ then u is locally Lipschitz continuous in Ω.

Proof. By assumption, ϱ is L-Lipschitz, therefore we have $\widehat{\omega}_\varrho(t) = \min\{Lt, \operatorname{diam}\Omega\}$ Iterating we get the inequality $\widehat{\omega}_\varrho^{(j)}(t) \leq L^j t$ for each $t \in [0, \operatorname{diam}\Omega]$ and each $j \in \mathbb{N}$. Moreover, since μ satisfies the δ-annular decay property (2.2), from (3.14) together with (4.5) we get

$$\mathcal{W}_{\mu,K_{m+j}}(t) \leq \frac{C\,t^\delta}{\lambda^\delta (1-\varepsilon)^{(m+j)\beta\,\delta}}$$

for some constant $C = C(D_\delta, D_\mu, L) \geq 1$. Replacing all this in (5.1) we obtain the following estimate:

$$\omega_{u,K_m}(t) \leq \frac{C(1-\alpha)\,\|u\|_\infty}{\lambda^\delta (1-\varepsilon)^{m\,\beta\,\delta}} \left(\sum_{j=0}^{\infty} \left(\frac{L^\delta\,|\alpha|}{(1-\varepsilon)^{\beta\,\delta}} \right)^j \right) t^\delta.$$

Now observe that (5.2) implies the convergence of the above series and, consequently, the desired Hölder regularity estimate. □

In the particular case that $\beta = 1$ we obtain the following corollary.

Corollary 5.2. *Let $(\mathbb{X}, d, \mu)$ be a proper, geodesic metric measure space satisfying the δ-annular decay condition for some $\delta \in (0, 1]$ and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is a Lipschitz admissible radius function in Ω with Lipschitz constant $L \geq 1$ such that*

$$\lambda\,\mathrm{dist}(x, \partial\Omega) \leq \varrho(x) \leq \varepsilon\,\mathrm{dist}(x, \partial\Omega),$$

for all $x \in \Omega$, where $0 < \lambda \leq \varepsilon$. Assume also that

$$|\alpha| < L^{-1},$$
$$0 < \ \varepsilon \ < 1 - L\,|\alpha|\,.$$

Then any $u \in C(\overline{\Omega})$ verifying the α-mean value property in Ω with respect to ϱ (that is, $\mathcal{T}_\alpha u = u$) is locally δ-Hölder continuous in Ω. In particular, if $\delta = 1$ then u is locally Lipschitz continuous in Ω.

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