On the Dirichlet Problem for solutions of a restricted nonlinear mean value property

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Abstract

We give an analytic proof of the solution of Dirichlet Problem for continous functions satisfying a nonlinear mean value problem related to the p-laplace operator and certain stochastic games.

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1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^d$ and a function $r : \Omega \to (0, +\infty)$, we say that r is an **admissible radius function** in Ω if

$$0 < r(x) \leq dist(x, \partial\Omega)$$

for all $x \in \Omega$. In this paper we will be mainly interested in admissible radius functions satisfying the lipschitz condition

$$|r(x) - r(y)| \leq |x - y| \quad \text{for } x, y \in \Omega \tag{1.1}$$

When no confusion arises, we will also use the notation r_x instead of $r(x)$ and B_x instead of $B(x, r(x))$, the (open) ball centered at x of radius $r(x)$.

Suppose that μ is a fixed positive measure in $\mathbb{R}^d$. Given an admissible radius function r in Ω we define the operators S and M in $C(\overline{\Omega})$, the space of continous functions in $\overline{\Omega}$, by

$$Su(x) = \frac{1}{2}\Big(\sup_{B_x} u + \inf_{B_x} u\Big) \tag{1.2}$$

$$Mu(x) = \fint_{B_x} u\, d\mu \tag{1.3}$$

Now let $0 \le \alpha < 1$. This paper is concerned with the operator T_α obtained as a convex combination of S and M:

$$T_\alpha = \alpha S + (1-\alpha)M \tag{1.4}$$

in particular with the existence and uniqueness of the Dirichlet problem

$$\begin{cases} T_\alpha u &= u \ \text{ in } \ \Omega \\ u &= f \ \text{ on } \ \partial\Omega \end{cases} \tag{1.5}$$

in the space $C(\overline{\Omega})$, for a given boundary data $f \in C(\partial\Omega)$.

The motivation for this problem comes from the study of the so called *p-harmonious* functions. Let us first recall the relation between the usual mean value property and harmonic functions. It is well known that a continous function u in a domain $\Omega \subset \mathbb{R}^d$ is harmonic if and only if it satisfies the *mean value property*

$$u(x) = \fint_{B(x,r)} u\, dm \tag{1.6}$$

for each $x \in \Omega$ and all r with $0 < r < dist(x, \partial\Omega)$. A classical theorem of Kellogg ([7]) says that if Ω is bounded, $u \in C(\overline{\Omega})$ and for each $x \in \Omega$ there is an admissible radius function $r(x)$ such that (1.6) holds, then u is harmonic in Ω. In other words, assuming continuity up to the boundary, the mean value property for a single radius(depending on the point) implies harmonicity or, with the notation introduced above, if $\mu = m$ is Lebesgue measure in $\mathbb{R}^d$, $r : \Omega \to (0, +\infty)$ is an admissible radius function in Ω and M is the operator given by (1.3) then $Mu = u$ implies that u is harmonic. Observe that this case corresponds to $\alpha = 0$ in (1.4). Kellogg's theorem is one of the most representative results in the so called *restricted mean value property* problems in classical function theory (see the excellent survey [14] for these and other similar questions).

The other extreme case, $\alpha = 1$, has been object of increasing attention in the last years. If S is the operator given by (1.2), associated to some admissible radius function in Ω, then functions satisfying $Su = u$ are called *harmonious* functions. The functional equation $Su = u$ appears in different contexts, related to the problem of extending a continous function on a closed subset to the whole space respecting its modulus of continuity([9]),

as a *Dynamic Programming Principle* in tug-of-war games ([15], [16]), as a mean value property related to the *infinity laplacian* ([11], [8]) and also in connection with problems of image processing ([3]).

We briefly explain why nonlinear mean value properties are connected to some distinguished nonlinear differential operators.

In the linear case, it follows from Taylor's formula that if $u \in C^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$, then

$$\lim_{r \to 0} \frac{1}{r^2} \Big(\fint_{B(x,r)} u \, dm - u(x) \Big) = \frac{\triangle u(x)}{2(d+2)}$$

for $x \in \Omega$. On the other hand,

$$\lim_{r \to 0} \frac{1}{r^2} \Big[\frac{1}{2} \big(\sup_{B(x,r)} u + \inf_{B(x,r)} u \big) - u(x) \Big] = \frac{\triangle_\infty u(x)}{2|\nabla u(x)|^2}$$

where

$$\triangle_\infty u = \sum_{i,j=1}^{d} u_{x_i} u_{x_j} u_{x_i, x_j}$$

is the so called *infinity laplacian* of u (see [2], [8]). Another important differential operator is the *p*-laplacian:

$$\triangle_p u = div(\nabla u |\nabla u|^{p-2})$$

where $1 < p < \infty$. Direct computation shows that

$$\triangle_p u = |\nabla u|^{p-2} \Big(\triangle u + (p-2) \frac{\triangle_\infty u}{|\nabla u|^2} \Big)$$

which means that the *p*-laplacian can be interpreted as a sort of average between the usual laplacian and the infinity laplacian. It therefore makes sense to consider averages of the operators M and S as in (1.4). We may wonder whether the mean value property

$$u(x) = \frac{\alpha}{2} \Big(\sup_{B(x,r)} u + \inf_{B(x,r)} u \Big) + (1-\alpha) \fint_{B(x,r)} u \, dm \qquad (1.7)$$

is related to the *p*-laplacian for some specific value of α, possibly depending on d and p. If $p \geq 2$, it turns out that this is actually the case for the value

$$\alpha = \frac{p-2}{d+p} \qquad (1.8)$$

See [11], [12] for the precise interpretation of this relation.

The aim of this paper is to study the existence and uniqueness of the Dirichlet problem for the operator T_α under appropriate assumptions on the domain Ω and the radius function $r(x)$.

In the case of the operator S it follows, as a particular case of results in [9], that if $\Omega \subset \mathbb{R}^d$ is a bounded and convex domain and r is an admissible radius function in Ω satisfying (1.1) then the Dirichlet problem

$$\begin{cases} Su &= u \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{cases} \tag{1.9}$$

has a unique solution for each $f \in C(\overline{\Omega})$, where S is the operator given by (1.2), associated to the radius function r.

On the other hand, if $r(x) = \epsilon$ is constant then (not necessarily continuous) functions satisfying $T_\alpha u = u$ for the value of α given by (1.8) have been called p-harmonious functions in [12]. Let $f \in C(\partial\Omega)$ and $p \geq 2$. Since the balls $B(x, \epsilon)$ will eventually leave Ω if x is close to $\partial\Omega$, the authors in [12] extend f continuously to the strip

$$\{x \in \mathbb{R}^d \setminus \Omega \, : \, dist(x, \partial\Omega) \leq \epsilon\}$$

and they prove that there is a unique p-harmonious function u_ϵ having f as boundary values (in this extended sense). Furthermore, if Ω satisfies some regularity assumptions, then $u_\epsilon \to u$ uniformly in $\overline{\Omega}$, where u is the unique solution of the Dirichlet problem

$$\begin{cases} \triangle_p u &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{cases}$$

(See [12], Theorem 1.6). In order to prove existence and uniqueness of the p-harmonious Dirichlet problem, the authors in [12] use the interpretation of the functional equation $T_\alpha u = u$ as a dynamic programming principle for a tug-of-war game. See also [10] for a purely analytic proof.

Motivated by the classical results on the restricted mean value property for the operator M (as Kellogg's theorem) and the results in [9] for the operator S, we consider the operator T_α in the setting of admissible radius functions in domains instead of the constant radius case. Furthermore, this makes possible to avoid extending the boundary function outside the domain.

We recall that a convex domain $\Omega \subset \mathbb{R}^d$ is **strictly convex** if $\partial\Omega$ does not contain any segment or, equivalently, if for any x, $y \in \partial\Omega$ the open segment (x, y) is contained in Ω. For technical reasons that will become apparent in section 2 we also introduce the operator

$$H_\alpha = \frac{1}{2}(I + T_\alpha) \tag{1.10}$$

We state now the main result of the paper.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be an strictly convex bounded domain in $\mathbb{R}^d$, $0 \le \alpha < 1$, $0 < \epsilon < 1 - \alpha$ and λ, β such that*

$$1 \le \beta < \frac{\log(\frac{1}{\alpha})}{\log(\frac{1}{1-\epsilon})} \tag{1.11}$$

$$0 < \lambda < \min\{\epsilon, \frac{1}{\beta}\}\Big(\frac{2}{diam(\Omega)}\Big)^{\beta-1} \tag{1.12}$$

Suppose that $r : \Omega \to (0, +\infty)$ is an admissible radius function in Ω satisfying (1.1) together with the following conditions:

$$\lambda\, dist(x, \partial\Omega)^\beta \le r(x) \le \epsilon\, dist(x, \partial\Omega) \tag{1.13}$$

for all $x \in \Omega$. If S and M are the operators given by (1.2) and (1.3), associated to the admissible radius function $r(x)$ and to d-dimensional Lebesgue measure $\mu = m$ and $T_\alpha = \alpha S + (1-\alpha)M$ then for any $f \in C(\partial\Omega)$ the Dirichlet problem

$$\begin{cases} T_\alpha u & = u \ \ in \ \Omega \\ u & = f \ \ on \ \partial\Omega \end{cases} \tag{1.14}$$

has a unique solution $\widetilde{u} \in C(\overline{\Omega})$. Furthermore, if $u_0 \in C(\overline{\Omega})$ is any continuous extension of f to $\overline{\Omega}$ then $H_\alpha^k u_0 \to \widetilde{u}$ as $k \to \infty$, uniformly in $\overline{\Omega}$, where H_α is the operator given by (1.10).

Remark. If $\Omega \subset \mathbb{R}^d$ is bounded and convex and β, λ are as in (1.11) and (1.12) then $r(x) = \lambda\, dist(x, \partial\Omega)^\beta$ is an admissible radius function in Ω satisfying the requirements in Theorem 1. Note that the fact that r satisfies the lipschitz condition (1.1) is a consequence of (1.12) and Proposition 2.2.

The key ingredients in the proof of Theorem 1 are to show that the sequence of iterates $\{T_\alpha^k u_0\}$ is equicontinuous in $\overline{\Omega}$ together with a regularity result in metric fixed point theory due to Ishikawa(Theorem 4.1). It should be pointed out that the arguments necessary to prove the interior equicontinuity (section 2) and the boundary equicontinuity(section 3) are different in nature. Section 2 works for general doubling measures in $\mathbb{R}^d$ but requires certain rigid assumptions on the radius function. As for section 3, we have adapted a clever argument in [6] for the usual (linear) mean value operator M(when $\mu = m$ is Lebesgue measure) to the nonlinear operator T_α. It turns out that the trick in [6] of using strict convexity to get equicontinuity of the sequence of iterates at the boundary also works in our (nonlinear) situation. The arguments in section 3 work for general admissible radius functions but they require μ to be Lebesgue measure. However, for tentative future developments of the theory, we have preferred to state section 2 in the general doubling measure case. Note that in our setting we cannot assume the a priori existence of the solution as it happens for the usual mean value property.

2 Interior estimates and interior equicontinuity.

We start the section with an auxiliary result about doubling measures.

Lemma 2.1. *Let μ be a doubling measure on $\mathbb{R}^d$. There are constants $C > 0$ and $0 < \theta \le 1$ depending only on μ and d such that, if $a \in \mathbb{R}^d$ and $0 < r \le R$, then*

$$\frac{\mu\big(B(a,R) \setminus B(a,r)\big)}{\mu(B(a,R))} \le C\Big(\frac{R-r}{R}\Big)^{\theta} \tag{2.1}$$

Proof. If $a \in \mathbb{R}^d$ is fixed and $0 < t < s$, denote $A_{t,s} = B(a,s) \setminus B(a,t)$. The lemma is a consequence of the following

  Claim: there exists $0 < \delta < 1$ only depending on μ and d such that if $0 < t < s$ then

$$\mu\big(A_{\frac{t+s}{2},s}\big) \le \delta\mu\big(A_{t,s}\big) \tag{2.2}$$

  To prove the claim, choose first a finite number of points $\{\xi_k\}_{k=1}^{N}$ on the unit sphere S^{d-1} in such a way that the spherical caps $C_k = B(\xi_k, \frac{s-t}{s}) \cap S^{d-1}$ cover S^{d-1} with finite overlapping(overlapping number only depending on d).Now define the interior and exterior spherical sectors as

$$Q_k^{in} = a + \{\rho\eta \ : \rho \in \big(t, \frac{t+s}{2}\big), \eta \in C_k\}$$
$$Q_k^{ex} = a + \{\rho\eta \ : \rho \in \big(\frac{t+s}{2}, s\big), \eta \in C_k\}$$

  By the doubling property, there is a constant $D \ge 1$ such that , for each k, we have
$$\mu(Q_k^{ex}) \le D\,\mu(Q_k^{in})$$

Summing up over k we get
$$\mu\big(A_{\frac{t+s}{2},s}\big) \le D\,\mu\big(A_{t,\frac{t+s}{2}}\big)$$

In particular
$$\mu\big(A_{\frac{t+s}{2},s}\big) \le \frac{D}{D+1}\mu\big(A_{t,s}\big)$$

and (2.2) follows by taking $\delta = \frac{D}{D+1}$. Now, to prove the lemma from the claim choose an integer m such that

$$(1 - 2^{-m})R < r \le (1 - 2^{-(m+1)})R$$

and apply the claim iteratively to the spherical shells $A_{(1-2^{-k})R,R}$ for $0 \leq k \leq m$ to obtain

$$\mu\big(A_{r,R}\big) \leq \mu\big(A_{(1-2^{-m})R,R}\big) \leq \delta^m \mu(B(a,R)) \tag{2.3}$$

Then (2.1) follows from (2.3) by taking $\delta = 2^{-\theta}$ and $C = \delta^{-1}$. $\qquad \square$

Lemma 2.2. *Let μ be a doubling measure on $\mathbb{R}^d$. There are constants $C > 0$ and $0 < \theta \leq 1$ depending only on μ and d such that if $x, y \in \mathbb{R}^d$, $r_x > 0$, $r_y > 0$ and*

$$|r_x - r_y| \leq |x - y| \leq \frac{r_x}{2} \tag{2.4}$$

then

$$\frac{\mu\big(B(x,r_x) \setminus B(y,r_y)\big)}{\mu\big(B(x,r_x)\big)} + \frac{\mu\big(B(y,r_y) \setminus B(x,r_x)\big)}{\mu\big(B(y,r_y)\big)} \leq C\Big(\frac{|x-y|}{r_x}\Big)^{\theta}$$

Proof. Observe that, from (2.4), $r_y \geq \dfrac{r_x}{2}$ and

$$B(x,r_x) \setminus B(y,r_y) \subset B(x,r_x) \setminus B(x,r_y - |x-y|)$$
$$B(y,r_y) \setminus B(x,r_x) \subset B(y,r_y) \setminus B(y,r_x - |x-y|)$$

The conclusion follows from (2.4) and Lemma 2.1 with the choices $R = r_x$, $r = r_y - |x-y|$ and $R = r_y$, $r = r_x - |x-y|$. $\qquad \square$

The following lemma provides an interior estimate of the modulus of continuity of Mu where M is the operator given by (1.3).

Lemma 2.3. *Let μ be a doubling measure on $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ a domain, r an admissible radius function in Ω satisfying (1.1) and $u \in L^{\infty}(\Omega)$. There are constants $C > 0$ and $0 < \theta \leq 1$, only depending on μ and d, such that, if x, $y \in \Omega$, then*

$$|Mu(x) - Mu(y)| \leq C\|u\|_{\infty}\Big(\frac{|x-y|}{r_x}\Big)^{\theta} \tag{2.5}$$

Proof. Let x, y, $B_x = B(x,r_x)$ and $B_y = B(y,r_y)$. We can assume that $|x-y| \leq \dfrac{r_x}{2}$ since otherwise the conclusion is trivial. A simple computation gives

$$Mu(x) - Mu(y) = \fint_{B_x} u \, d\mu - \fint_{B_y} u \, d\mu =$$

$$\frac{1}{\mu(B_x)} \int_{B_x \setminus B_y} u \, d\mu - \frac{1}{\mu(B_y)} \int_{B_y \setminus B_x} u \, d\mu + \frac{\mu(B_y) - \mu(B_x)}{\mu(B_x)\mu(B_y)} \int_{B_x \cap B_y} u \, d\mu$$

7

In particular

$$|Mu(x) - Mu(y)| \leq$$
$$||u||_\infty \Big[\frac{\mu(B_x \setminus B_y)}{\mu(B_x)} + \frac{\mu(B_y \setminus B_x)}{\mu(B_y)} + \frac{\mu(B_x \setminus B_y) + \mu(B_y \setminus B_x)}{max\{\mu(B_x), \mu(B_y)\}} \Big] \leq$$
$$2||u||_\infty \Big[\frac{\mu(B_x \setminus B_y)}{\mu(B_x)} + \frac{\mu(B_y \setminus B_x)}{\mu(B_y)} \Big]$$

so (2.5) follows from Lemma 2.2. $\qquad\qquad\qquad\qquad\qquad$ $\square$

We recall now that a *concave modulus of continuity* is a non-decreasing concave function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$. If $\Omega \subset \mathbb{R}^d$ is convex and $u \in C(\Omega)$ we will denote by $\omega_{u,\Omega}$ the (lowest) concave modulus of continuity of u in Ω so, in particular

$$|u(x) - u(y)| \leq \omega_{u,\Omega}(|x - y|)$$

for $x, y \in \Omega$.

Consider the operators S, M and T_α given by (1.2), (1.3) and (1.4) respectively, where $0 \leq \alpha < 1$. Suppose now that $\Omega \subset \mathbb{R}^d$ is convex, r is an admissible radius function in Ω satisfying the lipschitz condition (1.1), $u \in C(\overline{\Omega})$ and G is a proper convex subdomain of Ω. If $x,y \in G$ then the proof of Proposition 3.2 in [9] actually shows that

$$|Su(x) - Su(y)| \leq \omega_{u,\widetilde{G}}(|x - y|) \tag{2.6}$$

where $\widetilde{G}$ is the convex hull of $\bigcup_{x \in G} B_x$ and $\omega_{u,\widetilde{G}}$ stands for the (concave) modulus of continuity of u in $\widetilde{G}$ (see [9]).

Proposition 2.1. *Let μ be a doubling measure on $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ a bounded, convex domain domain, $r : \Omega \to (0, +\infty)$ an admissible radius function in Ω satisfying (1.1), $u \in C(\overline{\Omega})$ and $G \Subset \Omega$ a proper convex sub-domain of Ω. There are constants $C > 0$ and $0 < \theta \leq 1$ only depending on μ and d such that, if $x, y \in G$ then*

$$|T_\alpha u(x) - T_\alpha u(y)| \leq \alpha \omega_{u,\widetilde{G}}(|x - y|) + (1 - \alpha)C||u||_\infty \Big(\frac{|x - y|}{r_x} \Big)^\theta \tag{2.7}$$

where

$$\widetilde{G} = co\big(\bigcup_{x \in G} B_x \big)$$

In particular, if $r_x \geq t_1 > 0$ for all $x \in G$ then

$$\omega_{T_\alpha u, G}(t) \leq \alpha \omega_{u,\widetilde{G}}(t) + (1 - \alpha)C||u||_\infty t_1^{-\theta} t^\theta \tag{2.8}$$

for $0 \leq t \leq diam(G)$.

Proof. Combine (2.6) and Lemma 2.3. $\qquad\square$

The next proposition justifies the choice of the normalization constant λ in (1.12) and (1.13).It implies in particular that (admissible) radius functions of the form $r(x) = \lambda\, dist(x, \partial\Omega)^\beta$ satisfy the lipschitz condition (1.1).

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex domain and let $\beta \geq 1$. Then for every $x, y \in \Omega$, the following inequality holds*

$$|dist(x, \partial\Omega)^\beta - dist(y, \partial\Omega)^\beta| \leq \beta\Big(\frac{diam(\Omega)}{2}\Big)^{\beta-1}|x - y|$$

In particular, the function $\lambda_{\Omega,\beta}\, dist(\cdot, \partial\Omega)^\beta$ is lipschitz with constant 1 in Ω, where

$$\lambda_{\Omega,\beta} = \frac{1}{\beta}\Big(\frac{2}{diam(\Omega)}\Big)^{\beta-1} \tag{2.9}$$

Proof. Use the mean value theorem applied to the function $t \to t^\beta$ together with the fact that $dist(x, \partial\Omega) \leq \frac{1}{2}diam(\Omega)$ for each $x \in \Omega$. $\qquad\square$

Let $\Omega \subset \mathbb{R}^d$ a bounded, convex domain and $0 < \epsilon < 1$. For $n \in \mathbb{N}$ define

$$\Omega_n = \{x \in \Omega : dist(x, \partial\Omega) > (1 - \epsilon)^n\} \tag{2.10}$$

Observe that there is $n_0 = n_0(\epsilon, \Omega)$ such that $diam(\Omega_n) \geq \frac{1}{2}diam(\Omega)$ if $n \geq n_0$.

Proposition 2.3. *Let $0 \leq \alpha < 1$, $0 < \epsilon < 1 - \alpha$ and suppose that $\beta \geq 1$ and $\lambda > 0$ satisfy (1.12). Let μ be a doubling measure on $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ a bounded, convex domain and r an admissible radius function in Ω satisfying (1.1) and (1.13) for all $x \in \Omega$. Then there are constants $C > 0$ and $0 < \theta \leq 1$ depending only on μ and d such that for any $u \in C(\overline{\Omega})$ and each $0 \leq t \leq \frac{1}{2}diam(\Omega)$ we have*

$$\omega_{T_\alpha u, \Omega_n}(t) \leq \alpha\omega_{u,\Omega_{n+1}}(t) + (1 - \alpha)C\|u\|_\infty\lambda^{-\theta}(1 - \epsilon)^{-n\beta\theta}t^\theta \tag{2.11}$$

where $n \geq n_0$ and Ω_n is as in (2.10).

Proof. (2.11) is consequence of (2.8), condition (1.13) and Proposition 2.1 applied to the convex subdomain Ω_n with the choice $t_1 = \lambda(1 - \epsilon)^{n\beta}$. $\qquad\square$

We now iterate (2.11).

Proposition 2.4. *Let $0 \leq \alpha < 1$, $0 < \epsilon < 1 - \alpha$ and β, λ satisfying (1.11) and (1.12). Let μ be a doubling measure on $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ a bounded, convex domain and r an admissible radius function in Ω satisfying (1.1) and (1.13) for all $x \in \Omega$. Let $G \Subset \Omega$ be a convex domain with $diam(G) \geq \frac{1}{2} diam(\Omega)$. Then there are constants $0 < \theta \leq 1$, only depending on μ and d and $A > 0$, depending on Ω, α, β, ϵ, λ, μ, d and G such that for any $u \in C(\overline{\Omega})$, and any $k \geq 1$ we have*

$$\omega_{T_\alpha^k u, G}(t) \leq \alpha^k \omega_{u,\Omega}(t) + A||u||_\infty t^\theta \qquad (2.12)$$

for $0 \leq t \leq \frac{1}{2} diam(\Omega)$, where $T_\alpha^k u$ stands for the k-th. iterate of the operator T_α.

Proof. Choose $n \geq n_0$ such that $G \subset \Omega_n$, where Ω_n is given by (2.10). We will show that

$$\omega_{T_\alpha^k u, \Omega_n}(t) \leq \alpha^k \omega_{u,\Omega}(t) + (1 - \alpha) \frac{C||u||_\infty (1 - \epsilon)^{-n\beta\theta} \lambda^{-\theta}}{1 - \alpha(1 - \epsilon)^{-\beta}} t^\theta \qquad (2.13)$$

Given G as in the statement of the proposition, fix n so that $G \subset \Omega_n$. Then (2.12) follows from (2.13) by choosing

$$A = (1 - \alpha) \frac{C(1 - \epsilon)^{-n\beta\theta} \lambda^{-\theta}}{1 - \alpha(1 - \epsilon)^{-\beta}}$$

To prove (2.13) we iterate (2.11) to obtain

$$\omega_{T_\alpha^k u, \Omega_n}(t) \leq \alpha^k \omega_{u,\Omega_{n+k}}(t) + B||u||_\infty t^\theta \sum_{j=0}^{k-1} \alpha^j (1 - \epsilon)^{-\beta\theta j} \qquad (2.14)$$

where $B = (1 - \alpha)C\lambda^{-\theta}(1 - \epsilon)^{-n\beta\theta}$. Then, (2.13) follows from (2.14), (1.11) and the fact that $\omega_{u,\Omega_{n+k}}(\cdot) \leq \omega_{u,\Omega}(\cdot)$. $\qquad \square$

The following proposition is the analogous of Proposition 2.4 for the operator H_α given by (1.10).

Proposition 2.5. *Let α, β, ϵ, λ, μ, θ, Ω, r, and G be as in Proposition 2.4. Then there are constants $0 < \theta \leq 1$, only depending on μ and d and $A > 0$, depending on Ω, α, β, ϵ, λ, μ, d and G such that for any $u \in C(\overline{\Omega})$, and any $k \geq 1$ we have*

$$\omega_{H_\alpha^k u, G}(t) \leq \left(\frac{1 + \alpha}{2}\right)^k \omega_{u,\Omega}(t) + A||u||_\infty t^\theta \qquad (2.15)$$

for $0 \leq t \leq \frac{1}{2} diam(\Omega)$, where $H_\alpha^k u$ stands for the k-th. iterate of the operator H_α.

Proof. Since

$$H_\alpha u(x) - H_\alpha u(y) = \frac{1}{2}(u(x) - u(y)) + \frac{1}{2}(T_\alpha u(x) - T_\alpha u(y))$$

it follows that the analogue of (2.11) for the operator H_α reads

$$\omega_{H_\alpha u, \Omega_n}(t) \leq \frac{1}{2}\omega_{u,\Omega_n}(t) + \frac{\alpha}{2}\omega_{u,\Omega_{n+1}}(t)$$
$$+ \frac{1-\alpha}{2}C||u||_\infty \lambda^{-\theta}(1-\epsilon)^{-n\beta\theta}t^\theta \qquad (2.16)$$

Now we iterate (2.16) to obtain

$$\omega_{H_\alpha^k u, \Omega_n}(t) \leq 2^{-k}\sum_{j=0}^{k}\binom{k}{j}\alpha^j \omega_{u,\Omega_{n+j}}(t)$$
$$+ B'||u||_\infty t^\theta \sum_{j=0}^{k-1}2^{-j}\left[1 + \alpha(1-\epsilon)^{-\beta\theta}\right]^j \qquad (2.17)$$

where $B' = \dfrac{1-\alpha}{2}C\lambda^{-\theta}(1-\epsilon)^{-n\beta\theta}$ so, from (2.17) and (1.11) we get

$$\omega_{H_\alpha^k u, \Omega_n}(t) \leq \left(\frac{1+\alpha}{2}\right)^k \omega_{u,\Omega}(t) + (1-\alpha)\frac{C||u||_\infty(1-\epsilon)^{-n\beta\theta}\lambda^{-\theta}}{1 - \alpha(1-\epsilon)^{-\beta}}t^\theta \quad (2.18)$$

and (2.15) follows from (2.18) as in Proposition 2.4. $\qquad\square$

For a fixed $u \in C(\overline{\Omega})$ we deduce as a consequence the equicontinuity of the sequences $\{T_\alpha^k u\}$ and $\{H_\alpha^k u\}$.

Proposition 2.6. *Let α, β, ϵ, λ,μ, Ω and r be as in Proposition 2.4. Then for any $u \in C(\overline{\Omega})$ and each $x \in \Omega$, the sequences $\{T_\alpha^k u\}_k$ and $\{H_\alpha^k u\}_k$ are equicontinuous at x.*

Proof. Choose a proper subdomain $G \Subset \Omega$ containing x and apply Propositions 2.4 and 2.5.

$\qquad\square$

3 Equicontinuity at the boundary

In this section we assume that μ is Lebesgue measure on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex domain. For $u \in C(\overline{\Omega})$ let

$$G_u = \{(x, u(x)) : x \in \overline{\Omega}\} \subset \mathbb{R}^{d+1}$$

be the graph of u and define $\Gamma_u = co(G_u)$ to be the convex hull of G_u.

Proposition 3.1. *Let $\mu = m$ be Lebesgue measure on $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ a bounded, convex domain, r an admissible radius function in Ω, $0 \le \alpha < 1$ and S, M, T_α and H_α the operators given by (1.2), (1.3), (1.4) and (1.11) respectively. If $u \in C(\overline{\Omega})$ then*

$$G_{Su} \cup G_{Mu} \cup G_{T_\alpha u} \cup G_{H_\alpha u} \subset \Gamma_u \tag{3.1}$$

In particular, for each $k \in \mathbb{N}$,

$$G_{T_\alpha^k u} \subset \Gamma_{T_\alpha^k u} \subset \Gamma_{T_\alpha^{k-1} u} \subset \cdots \subset \Gamma_u \tag{3.2}$$

$$G_{H_\alpha^k u} \subset \Gamma_{H_\alpha^k u} \subset \Gamma_{H_\alpha^{k-1} u} \subset \cdots \subset \Gamma_u \tag{3.3}$$

Proof. Since

$$(x, T_\alpha u(x)) = \alpha(x, Su(x)) + (1 - \alpha)(x, Mu(x))$$

$$(x, H_\alpha u(x)) = \frac{1}{2}(x, u(x)) + \frac{1}{2}(x, T_\alpha u(x))$$

and Γ_u is convex, to prove (3.1) it is enough to prove that $G_{Su} \subset \Gamma_u$ and $G_{Mu} \subset \Gamma_u$. Fix $x \in \Omega$.

Let us first show that $G_{Su} \subset \Gamma_u$. It is enough to show that there is $h \in \mathbb{R}^n$, with $|h| \le r_x$ so that

$$\sup_{B_x} u + \inf_{B_x} u = u(x + h) + u(x - h) \tag{3.4}$$

Indeed, if (3.4) is true then

$$(x, Su(x)) = \frac{1}{2}(x + h, u(x + h)) + \frac{1}{2}(x - h, u(x - h)) \in \Gamma_u$$

We may assume that $\sup_{B_x} u = 1$ and $\inf_{B_x} u = -1$ (otherwise replace u by $1 + \frac{2}{M - m}(u - M)$, where $\sup_{B_x} u = M$ and $\inf_{B_x} u = m$). Then we must show that there is $h \in \overline{B}(0, r_x)$ so that $u(x + h) + u(x - h) = 0$. Define the continuous function v in $\overline{B}(0, r_x)$ as

$$v(y) = u(x + y) + u(x - y)$$

and choose h_+, $h_- \in \overline{B}(0, r_x)$ such that $u(x + h_+) = 1$, $u(x + h_-) = -1$. Then

$$v(h_+) = u(x + h_+) + u(x - h_+) = 1 + u(x - h_+) \ge 0$$

$$v(h_-) = u(x + h_-) + u(x - h_-) = u(x + h_-) - 1 \le 0$$

so by continuity there must be $h \in \overline{B}(0, r_x)$ such that $v(h) = 0$. This proves (3.4) and therefore the inclusion $G_{Su} \subset \Gamma_u$.

We prove now that $G_{Mu} \subset \Gamma_u$. Observe that

$$\int_{B_x} u \, dm = \frac{1}{2} \int_{B(0,r_x)} [u(x+y) + u(x-y)] dm(y) \qquad (3.5)$$

which implies, by continuity, that there is $h \in B(0, r_x)$ such that

$$\fint_{B_x} u \, dm = \frac{1}{2}[u(x+h) + u(x-h)]$$

Then

$$(x, Mu(x)) = \frac{1}{2}\big((x+h, u(x+h)) + (x-h, u(x-h))\big)$$

and this shows that $G_{Mu} \subset \Gamma_u$.

Now, from (3.1) we have

$$\Gamma_{T_\alpha^k u} = co(G_{T_\alpha^k u}) \subset co(\Gamma_{T_\alpha^{k-1}u}) = \Gamma_{T_\alpha^{k-1}u}$$

which implies (3.2). The argument for (3.3) is analogous. $\qquad \square$

Remark. The fact that $\mu = m$ is Lebesgue measure has been used in identity (3.5).

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, strictly convex domain, $u \in C(\overline{\Omega})$ and $\Gamma_u = co(G_u)$. Then, for each $\xi \in \partial\Omega$*

$$\Gamma_u \cap (\{\xi\} \times \mathbb{R}) = \{(\xi, u(\xi))\} \qquad (3.6)$$

Proof. If $(\xi, t) \in \Gamma_u$ then there are $\lambda_1, \cdots, \lambda_m \geq 0$, with $\sum_1^m \lambda_i = 1$ and there exist $x_1, \cdots, x_m \in \overline{\Omega}$ such that

$$(\xi, t) = \sum_{i=1}^m \lambda_i(x_i, u(x_i)) \qquad (3.7)$$

From the strict convexity we deduce that the convex combination in (3.6) must be trivial in the sense that, say, $\lambda_1 = 1$, $\lambda_2 = \cdots = \lambda_m = 0$. Then $x_1 = \xi$, $t = u(\xi)$ and (3.5) follows. $\qquad \square$

Proposition 3.2. *Let $\mu = m$ be Lebesgue measure on $\mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ a bounded, strictly convex domain, r an admissible radius function in Ω, $0 \leq \alpha < 1$ and S, M, T_α and H_α the operators given by (1.2), (1.3), (1.4) and (1.11) respectively. Then for any $u \in C(\overline{\Omega})$ and each $\xi \in \partial\Omega$, the sequences $\{T_\alpha^k u\}_k$ and $\{H_\alpha^k u\}_k$ are equicontinuous at ξ.*

Proof. Fix $u \in C(\overline{\Omega})$. Let $\xi \in \partial\Omega$ and suppose that $\{T_\alpha^k u\}$ is not equicontinuous at ξ. Then there are $\epsilon > 0$ and sequences $\{k_j\} \subset \mathbb{N}$, $\{x_j\} \subset \overline{\Omega}$ with $k_j \uparrow \infty$, $x_j \to \xi$ such that

$$|T_\alpha^{k_j} u(x_j) - u(\xi)| \geq \epsilon$$

We can assume (otherwise we could take a further subsequence) that $T_\alpha^k u(x_j) \to t \in \mathbb{R}$ and that $|t - u(\xi)| \geq \frac{\epsilon}{2}$. By Proposition 3.1, $(x_j, T_\alpha^k u(x_j)) \in \Gamma_u$ which is a closed set, so $(\xi, t) \in \Gamma_u$. The contradiction then follows from Lemma 3.1. Therefore $\{T_\alpha^k u\}_k$ is equicontinuous at each point of $\partial\Omega$. The same argument provides equicontinuity of $\{H_\alpha^k u\}_k$ at each point of $\partial\Omega$. $\qquad\square$

4   Proof of Theorem 1

The uniqueness part follows from the next comparison principle, which holds under much more general assumptions.

Proposition 4.1. *Let μ be a positive Borel measure in $\mathbb{R}^d$ with the property that $\mu(B) > 0$ for every ball $B \subset \mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $r : \Omega \to (0, +\infty)$ an admissible one-radius function in Ω, $0 \leq \alpha < 1$ and let S, M, T_α be the operators given by (1.2), (1.3) and (1.4) respectively. Suppose that u and $v \in C(\overline{\Omega})$ satisfy $T_\alpha u = u$, $T_\alpha v = v$ and that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Proof. The argument is standard in comparison results. Let u and v as in the statement of the proposition. Let $m = \max_{\overline{\Omega}}(u - v)$. We will show that $m \leq 0$. Suppose, on the contrary, that $m > 0$ and define

$$A = \{x \in \Omega \,:\, (u - v)(x) = m\}$$

Then A is a nonempty, closed subset of Ω. Take $a \in A$. We will see that $B_a \subset A$ so A is also open. Indeed,

$$u(a) = \alpha S u(a) + (1 - \alpha) M u(a) = \alpha(m + Sv(a)) + (1 - \alpha)(m + Mv(a))$$

Since $u \leq v + m$ in B_a we must have in particular that $Mu(a) = m + Mv(a)$. Therefore

$$\int_{B_a} (v + m - u) \, d\mu = 0 \tag{4.1}$$

The integrand in (4.1) is continuous and nonnegative so by continuity and the hypothesis on μ it follows that $v + m - u \equiv 0$ in B_a. This proves that A is open. Therefore, by connectedness $A = \Omega$ and $v - u \equiv m > 0$ in Ω, which contradicts the assumption $u \leq v$ on $\partial\Omega$. Then $m \leq 0$ and the proposition follows. $\qquad\square$

To prove the existence part we will need a result from metric fixed point theory. Let $(X, ||.||)$ be a Banach space and $K \subset X$. A self-mapping $T : K \to K$ of K is *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||$$

for each x, $y \in K$. The following result will be a key ingredient in the proof of existence. It is a particular case of a more general result from Ishikawa ([5] , see also [4], Theorem 9.4).

Theorem 4.1. (Ishikawa) Let X be a Banach space, $K \subset X$ a bounded, closed and convex subset of X and let $T : K \to K$ be a nonexpansive self-mapping of K. Define $H = \dfrac{1}{2}(I + T)$. Then

$$\lim_{k \to \infty} ||H^{k+1}x - H^k x|| = 0 \tag{4.2}$$

for each $x \in K$.

Condition (4.2) has been named *asymptotic regularity* by some authors. Let us see how to prove the existence in Theorem 1. Take $X = (C(\overline{\Omega}), ||.||_\infty)$ and fix $f \in C(\partial\Omega)$. Define

$$K = \{u \in X : u|_{\partial\Omega} = f \, , \, ||u||_\infty = ||f||_\infty\}$$

Then K is a nonempty bounded, closed, convex subset of X. Observe that S, M, T_α and H_α are all nonexpansive self-mappings of K. To prove the existence of the solution of the Dirichlet problem (1.5) it is enough to show that T_α has a fixed point in K.

Choose $u_0 \in K$. Then the sequence $\{H_\alpha^k u_0\}$ is pointwise bounded and also equicontinuous at each point of the compact set $\overline{\Omega}$ by Propositions 2.6 and 3.2. Therefore, by Arzéla-Ascoli theorem there are a subsequence $\{k_j\}$, with $k_j \uparrow \infty$ and $\widetilde{u} \in K$ such that

$$\lim_{j \to \infty} H_\alpha^{k_j} u_0 = \widetilde{u} \tag{4.3}$$

uniformly in $\overline{\Omega}$. Then

$$\lim_{j \to \infty} H_\alpha^{k_j+1} u_0 = H_\alpha \widetilde{u} \tag{4.4}$$

and, from Theorem 4.1 applied to H_α, we get

$$\lim_{j \to \infty} ||H_\alpha^{k_j+1} u_0 - H_\alpha^{k_j} u_0||_\infty = 0 \tag{4.5}$$

so, from (4.3), (4.4) and (4.5) we deduce that $H_\alpha \widetilde{u} = \widetilde{u}$. Since T_α and H_α have the same fixed points, we get $T_\alpha \widetilde{u} = \widetilde{u}$ which proves the existence part

in Theorem 1. To see that, actually, $H_\alpha^k u_0 \to \widetilde{u}$ suppose, on the contrary, that there are $\epsilon > 0$ and a subsequence $\{m_j\}$ such that

$$||H_\alpha^{m_j} u_0 - \widetilde{u}||_\infty \geq \epsilon \qquad (4.6)$$

By equicontinuity, a subsequence of $\{H_\alpha^{m_j} u_0\}$ would converge to some $v \in K$ and, as before, $T_\alpha v = v$ so, by uniqueness, $v = \widetilde{u}$, which contradicts (4.6). This proves that $H_\alpha^k u_0 \to \widetilde{u}$ and finishes the proof of Theorem 1.

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