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Research Article

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p-harmonic functions by way of intrinsic mean value properties

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Abstract: Let Ω be a bounded domain in $\mathbb{R}^n$. Under appropriate conditions on Ω, we prove existence and uniqueness of continuous functions solving the Dirichlet problem associated to certain nonlinear mean value properties in Ω with respect to balls of variable radius. We also show that, when properly normalized, such functions converge to the p-harmonic solution of the Dirichlet problem in Ω for $p \geqslant 2$. Existence is obtained via iteration, a fundamental tool being the construction of explicit universal barriers in Ω.

Keywords: p-laplacian, p-harmonic functions, Dirichlet problem, mean value properties, p-harmonious functions, approximation of solutions

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1 Introduction

1.1 Background

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The aim of this paper is twofold. First, we discuss existence and uniqueness of continuous solutions of the Dirichlet problem in Ω associated to certain nonlinear mean value properties in balls of variable radius $B(x, \rho(x)) \subset \Omega$. Then we study the convergence of such functions to the solution of the p-harmonic Dirichlet problem in Ω.

A primary motivation of our work is the mean value property for harmonic functions, saying that a continuous function u in a domain $\Omega \subset \mathbb{R}^n$ is harmonic if and only if

$$u(x) = \fint_{B(x,\rho)} u(y)\,dy \tag{1.1}$$

for each $x \in \Omega$ and each $\rho > 0$ such that $0 < \rho < \operatorname{dist}(x, \partial\Omega)$. The mean value property plays a relevant role in geometric function theory and is indeed the fundamental tool of the interplay between classical potential theory, probability and Brownian motion.

A theorem due to Volterra (for regular domains) and Kellogg (in the general case) says that if $\Omega \subset \mathbb{R}^n$ is bounded, $u \in C(\overline{\Omega})$ and for each $x \in \Omega$ there is a radius $\rho = \rho(x)$ with $0 < \rho \leqslant \operatorname{dist}(x, \partial\Omega)$ such that (1.1) holds, then u is harmonic in Ω (see [13, 23]). Therefore, under appropriate hypotheses, the mean value property for a single radius (depending on the point) implies harmonicity.

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In the recent decades, substantial efforts have been devoted to determine the stochastic structure of certain nonlinear PDEs, a crucial step being the identification of the corresponding (nonlinear) mean value properties. In this paper, we will focus on the *p-laplacian*, which for $1 < p < \infty$ is the divergence-form differential operator given by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Weak solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of $\Delta_p u = 0$ are said to be *p-harmonic* functions. Observe that the theory is nonlinear unless $p = 2$, in which case we recover harmonic functions. We refer to [16] for background and basic properties of p-harmonic functions.

Unfortunately, the nature of the connections between p-harmonic functions and mean value properties is more delicate when $p \neq 2$. We start with some basic facts in the smooth case. If $u \in C^2$ and $\nabla u \neq 0$, then a direct computation gives

$$\Delta_p u = |\nabla u|^{p-2}\Big(\Delta u + (p-2)\frac{\Delta_\infty u}{|\nabla u|^2}\Big), \tag{1.2}$$

where

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i,x_j} \tag{1.3}$$

is the so called *∞-laplacian* in $\mathbb{R}^n$. Then (1.2) shows that, in the smooth case and away from the critical points, the p-laplacian can be understood as a linear combination of the usual laplacian and the normalized ∞-laplacian.

By using the viscosity characterization of p-harmonic functions [12], Manfredi, Parviainen and Rossi characterized p-harmonicity in terms of nonlinear mean value properties in [19]. Namely, a function $u \in C(\Omega)$ is p-harmonic in $\Omega \subset \mathbb{R}^n$ if and only if u satisfies the *asymptotic p-mean value property*

$$u(x) = \frac{p-2}{n+p}\Big(\frac{1}{2}\sup_{B(x,\varepsilon)} u + \frac{1}{2}\inf_{B(x,\varepsilon)} u\Big) + \frac{n+2}{n+p} \fint_{B(x,\varepsilon)} u(y)\,dy + o(\varepsilon^2) \tag{1.4}$$

in a viscosity sense for each $x \in \Omega$. If $n = 2$, then this characterization holds also in the classical sense [2, 17], while for $n \geqslant 3$ the question of whether p-harmonic functions satisfy (1.4) in the classical sense is still open. Note that if $p = 2$, then (1.4) is actually equivalent to u being harmonic.

From a probabilistic point of view, the influential work of Peres, Schramm, Sheffield and Wilson [21] established a game-theoretic interpretation of the ∞-laplacian, and the functional equation

$$u_\varepsilon(x) = \frac{1}{2}\sup_{B(x,\varepsilon)} u_\varepsilon + \frac{1}{2}\inf_{B(x,\varepsilon)} u_\varepsilon$$

appears as a *dynamic programming principle* of a two-player zero-sum tug-of-war game. A similar interpretation for the p-laplacian, $p \in [2,\infty]$, was considered in [22]. Manfredi, Parviainen and Rossi gave a systematic twist to the theory, from both an analytic and probabilistic point of view [19, 20]. In particular, in [20] the term *p-harmonious* was introduced to denote (not necessarily continuous) solutions of the functional equation

$$u_\varepsilon(x) = \frac{p-2}{n+p}\Big(\frac{1}{2}\sup_{B(x,\varepsilon)} u_\varepsilon + \frac{1}{2}\inf_{B(x,\varepsilon)} u_\varepsilon\Big) + \frac{n+2}{n+p} \fint_{B(x,\varepsilon)} u_\varepsilon(y)\,dy \tag{1.5}$$

for each $x \in \Omega$. Note, however, that (1.5) raises some technical problems, coming from the fact that the balls $B(x,\varepsilon)$ eventually escape the domain. Manfredi, Parviainen and Rossi [20] extended a given $f \in C(\partial\Omega)$ to the strip $\{x \in \mathbb{R}^n \setminus \Omega : \operatorname{dist}(x,\partial\Omega) \leqslant \varepsilon\}$ and proved that, if $\Omega \subset \mathbb{R}^n$ is bounded and satisfies a so called *boundary regularity condition*, then there is a unique p-harmonious function u_ε having f as boundary values (in the extended sense). Furthermore, $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$, where u is the unique p-harmonic function solving the Dirichlet problem in Ω with boundary data f. It should be remarked that domains satisfying a uniform exterior cone condition (see Definition 1.4 below) verify the boundary regularity condition, in the sense of [20]; see also [1, 8, 18] for further approaches.

1.2 Main results

In this paper, we deal with a modified version of (1.5) in which the balls $B(x,\varepsilon)$ are replaced by balls of variable radius $B(x,\rho(x))$, where $0<\rho(x)<\operatorname{dist}(x,\partial\Omega)$. We want to emphasize that the variable radius setting is natural for at least two reasons: it is closely related to the classical theory (remember the Volterra–Kellogg theorem) and it is intrinsic, in the sense that no extension of the domain is needed (this explains the term *intrinsic* in the title).

Theorem 1.5 below is an existence and uniqueness result for the Dirichlet problem associated to intrinsic mean value properties. It extends the existence result in [20] to the variable radius setting, substantially relaxes the geometrical restrictions of [3] and, as an additional feature, the solution is constructively obtained via iteration of the averaging operators $\mathcal{T}_{\rho,p}$ (see (1.6)).

Theorem 1.7 is an approximation result showing that, when properly normalized, solutions of intrinsic mean value properties with fixed continuous boundary data converge to the solution of the p-harmonic Dirichlet problem with the same boundary data. The combination of Theorem 1.5 and Theorem 1.7 provides therefore an intrinsic and constructive method of obtaining solutions of p-harmonic Dirichlet problems which might be of interest from a computational point of view.

The fundamental tool to prove Theorem 1.5 and Theorem 1.7 is the construction of explicit barriers which neither depend on p nor on the admissible radius function ρ and work simultaneously for the operators $\mathcal{T}_{\rho,p}$ and the p-laplacian (Theorem 1.9). We believe that this construction has an independent interest which might be useful in other situations.

Even though the motivation for studying such problems is partly probabilistic (namely the search of "natural" stochastic processes associated to the p-laplacian), our arguments and techniques are entirely analytic.

Before stating the main results, let us introduce some necessary definitions.

Definition 1.1. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. We say that a function $\rho:\overline{\Omega}\to(0,\infty)$ is an *admissible radius function* in Ω if the following conditions hold:
(i) $0<\rho(x)<\operatorname{dist}(x,\partial\Omega)$ for every $x\in\Omega$.
(ii) $\rho(x)=0$ if and only if $x\in\partial\Omega$.
Hereafter, we will write $B_\rho(x):=B(x,\rho(x))$.

It is important to remark at this point that the main existence and convergence results (Theorems 1.5 and 1.7 below) of this paper require the admissible radius function to be continuous in Ω. However, this assumption, in spite of its essential role in the proof of the local equicontinuity [3, 4], is not directly involved in the proofs presented in this paper, which are focused on the behavior of the iterates near the boundary (see Section 3).

Definition 1.2. Let $\Omega\subset\mathbb{R}^n$ be a bounded domain and let ρ be an admissible radius function in Ω. Let $\mathcal{S}_\rho$ and $\mathcal{M}_\rho$ be the operators in $L^\infty(\overline{\Omega})$ defined by

$$\mathcal{S}_\rho u(x):=\begin{cases}\dfrac{1}{2}\sup\limits_{B_\rho(x)}u+\dfrac{1}{2}\inf\limits_{B_\rho(x)}u & \text{if } x\in\Omega,\\ u(x) & \text{if } x\in\partial\Omega,\end{cases}$$

and

$$\mathcal{M}_\rho u(x):=\begin{cases}\displaystyle\fint_{B_\rho(x)}u(y)\,dy & \text{if } x\in\Omega,\\ u(x) & \text{if } x\in\partial\Omega,\end{cases}$$

for every $u\in L^\infty(\overline{\Omega})$. In addition, for a fixed $p\in[2,\infty)$, we define the operator $\mathcal{T}_{\rho,p}$ in $L^\infty(\overline{\Omega})$ as the following linear combination of $\mathcal{S}_\rho$ and $\mathcal{M}_\rho$:

$$\mathcal{T}_{\rho,p}:=\frac{p-2}{n+p}\mathcal{S}_\rho+\frac{n+2}{n+p}\mathcal{M}_\rho. \tag{1.6}$$

Note that $\mathcal{T}_{\rho,2}=\mathcal{M}_\rho$ and that, formally, $\mathcal{T}_{\rho,\infty}=\mathcal{S}_\rho$. As the following proposition says, $\mathcal{T}_{\rho,p}$ preserves the class $C(\overline{\Omega})$, provided the admissible radius function ρ is continuous.

Proposition 1.3 ([4, Proposition 4.1]). *If $\rho \in C(\overline{\Omega})$, then $\mathcal{T}_{\rho,p} : C(\overline{\Omega}) \to C(\overline{\Omega})$ for each $p \in [1,\infty]$.*

From now on, we will take $C(\overline{\Omega})$ as the natural function space where the operators $\mathcal{T}_{\rho,p}$ are defined. In addition, $\mathcal{T}_{\rho,p}$ satisfies the following properties:
(i) *Affine invariance*: if $a, b \in \mathbb{R}$ and $u \in C(\overline{\Omega})$, then $\mathcal{T}_{\rho,p}(au+b) = a\mathcal{T}_{\rho,p}u + b$.
(ii) *Monotonicity*: if $u, v \in C(\overline{\Omega})$ such that $u \leqslant v$, then $\mathcal{T}_{\rho,p}u \leqslant \mathcal{T}_{\rho,p}v$.
(iii) *Non-expansiveness*: if $u, v \in C(\overline{\Omega})$, then

$$\|\mathcal{T}_{\rho,p}u - \mathcal{T}_{\rho,p}v\|_\infty \leqslant \|u - v\|_\infty.$$

(iv) $\inf_{B_\rho(x)} u \leqslant \mathcal{T}_{\rho,p}u(x) \leqslant \sup_{B_\rho(x)} u$ for every $x \in \Omega$.
It is easy to check that the k-th iteration of $\mathcal{T}_{\rho,p}$, denoted by $\mathcal{T}^k_{\rho,p}$, also satisfies the above four properties. We point out that, in the constant radius case $\rho(x) \equiv \varepsilon$, an operator satisfying the conditions (i), (ii) and (iv) above has been called an *average* in [8].

Let $f \in C(\partial\Omega)$. In this paper, we are interested in existence and uniqueness of solutions of the Dirichlet problem

$$\begin{cases} \mathcal{T}_{\rho,p}u = u & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

We will also discuss assumptions under which normalized solutions of (1.7) converge to the corresponding solution of the Dirichlet problem for the p-laplacian.

Notice that (1.7) is equivalent to the problem of finding a fixed point of $\mathcal{T}_{\rho,p}$ among all continuous functions with prescribed continuous boundary data f. Given $f \in C(\partial\Omega)$, we define $\mathcal{K}_f$ as the set of all *norm-preserving continuous extensions of f to $\overline{\Omega}$*:

$$\mathcal{K}_f := \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = f \text{ and } \|u\|_{\infty,\Omega} = \|f\|_{\infty,\partial\Omega}\}. \tag{1.8}$$

By Proposition 1.3 and the non-expansiveness of the operator, it follows that $\mathcal{T}_{\rho,p}(\mathcal{K}_f) \subset \mathcal{K}_f$. Furthermore, if u satisfies (1.7), then $\|u\|_{\infty,\Omega} = \|f\|_{\infty,\partial\Omega}$ by the comparison principle (Theorem 3.4). Therefore, the Dirichlet problem (1.7) has a solution in $C(\overline{\Omega})$ if and only if $\mathcal{T}_{\rho,p}$ has a fixed point in $\mathcal{K}_f$.

In order to state the main theorems of this work, we need to impose a certain geometrical condition on the boundary of the domain.

Definition 1.4 (Uniform exterior cone condition). Let $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$. We denote by $K_{\alpha,r}$ the truncated circular cone

$$K_{\alpha,r} = \{x \in \mathbb{R}^n : x_1 \leqslant -|x|\cos\alpha \text{ and } |x| \leqslant r\}.$$

We say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the *uniform exterior cone condition* if there exist constants $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$ such that for every $\xi \in \partial\Omega$ there is a rotation $R \in \mathrm{SO}(n)$ in $\mathbb{R}^n$ such that

$$\xi + R(K_{\alpha,r}) \subset \mathbb{R}^n \setminus \Omega.$$

Remark. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then Ω is Lipschitz if and only if both Ω and $\mathbb{R}^n \setminus \Omega$ satisfy the uniform exterior cone condition (see [9]).

We now state the first main result of this paper (compare with [3], where the same result was proven under the assumption that Ω is strictly convex and the admissible radius function is 1-Lipschitz); see [18, 20] for previous versions in the constant radius case.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the uniform exterior cone condition and let $p \in [2,\infty)$. Suppose that $\rho \in C(\overline{\Omega})$ is a continuous admissible radius function in Ω satisfying*

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leqslant \rho(x) \leqslant \Lambda \operatorname{dist}(x, \partial\Omega)$$

for all $x \in \Omega$, where

$$\beta \geqslant 1, \quad 0 < \Lambda < 1 - \Big(\frac{p-2}{n+p}\Big)^{1/\beta}, \quad 0 < \lambda \leqslant \Lambda\Big(\frac{\operatorname{diam}\Omega}{2}\Big)^{1-\beta}. \tag{1.9}$$

Then, for any $f \in C(\partial\Omega)$, there exists a unique solution $u_\rho \in C(\overline{\Omega})$ to the Dirichlet problem

$$\begin{cases} \mathcal{T}_{\rho,p} u_\rho = u_\rho & \text{in } \Omega, \\ u_\rho = f & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $\mathcal{T}_{\rho,p}$ is the averaging operator defined in (1.6). *Furthermore, for any norm-preserving continuous extension $u \in C(\overline{\Omega})$ of f, the sequence of iterates $\{\mathcal{T}^k_{\rho,p} u\}_k$ converges uniformly to u_ρ in $\overline{\Omega}$.*

By letting $\beta = 1$, we obtain the following corollary as an immediate consequence.

Corollary 1.6. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the uniform exterior cone condition and let $p \in [2, \infty)$. Suppose that $\rho \in C(\overline{\Omega})$ is a continuous admissible radius function in Ω satisfying*

$$\lambda \operatorname{dist}(x, \partial\Omega) \leqslant \rho(x) \leqslant \Lambda \operatorname{dist}(x, \partial\Omega)$$

for all $x \in \Omega$, where

$$0 < \lambda \leqslant \Lambda < \frac{n+2}{n+p}.$$

Then, for any $f \in C(\partial\Omega)$, there exists a unique solution $u_\rho \in C(\overline{\Omega})$ to the Dirichlet problem (1.10). *Furthermore, for any norm-preserving continuous extension $u \in C(\overline{\Omega})$ of f, the sequence of iterates $\{\mathcal{T}^k_{\rho,p} u\}_k$ converges uniformly to u_ρ in $\overline{\Omega}$.*

The next theorem is our second main result of the paper. It says that, when considering a family of admissible radius functions going to zero in an appropriate way, the corresponding solutions given by Theorem 1.5 converge uniformly to the p-harmonic solution of the Dirichlet problem.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the uniform exterior cone condition and let $p \in [2, \infty)$. Suppose that $\{\rho_\varepsilon\}_{0<\varepsilon\leqslant 1}$ is a collection of continuous admissible radius functions in Ω satisfying*

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leqslant \frac{\rho_\varepsilon(x)}{\varepsilon} \leqslant \Lambda \operatorname{dist}(x, \partial\Omega)$$

for all $x \in \Omega$ and every $0 < \varepsilon \leqslant 1$, where β, λ and Λ are as in (1.9). *Given any continuous boundary data $f \in C(\partial\Omega)$, let u_ε be the solution of*

$$\begin{cases} \mathcal{T}_{\rho_\varepsilon,p} u_\varepsilon = u_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega. \end{cases}$$

Then $u_\varepsilon \to u_0$ uniformly in $\overline{\Omega}$, where u_0 is the unique p-harmonic function in Ω solving

$$\begin{cases} \Delta_p u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases}$$

The fundamental tool for the proofs of Theorem 1.5 and Theorem 1.7 is provided by Theorem 1.9 below.

Definition 1.8. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\xi \in \partial\Omega$. We say that a function $w_\xi \in C(\overline{\Omega})$ is a $\mathcal{T}_{\rho,p}$*-barrier* at ξ if $w_\xi > 0$ in $\overline{\Omega} \setminus \{\xi\}$, $w_\xi(\xi) = 0$ and $w_\xi \geqslant \mathcal{T}_{\rho,p} w_\xi$ in Ω. If a barrier exists, we say that ξ is a $\mathcal{T}_{\rho,p}$*-regular* point. Moreover, a bounded domain $\Omega \subset \mathbb{R}^n$ is $\mathcal{T}_{\rho,p}$*-regular* if every point on $\partial\Omega$ is $\mathcal{T}_{\rho,p}$-regular.

Theorem 1.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the uniform exterior cone condition with constants $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$ as in Definition 1.4. Choose γ such that*

$$0 < \gamma < \frac{8(\sin\alpha)^{n-2}}{n(13\pi^2 + 4\pi)}. \tag{1.11}$$

Then, for each $\xi \in \partial\Omega$, there exists a function $w_\xi \in C(\overline{\Omega})$ such that $w_\xi(\xi) = 0$, $w_\xi > 0$ in $\overline{\Omega} \setminus \{\xi\}$,

$$w_\xi(x) \geqslant \fint_{B(x,\varrho)} w_\xi(y)\,dy, \quad w_\xi(x) \geqslant \frac{1}{2} \sup_{B(x,\varrho)} w_\xi + \frac{1}{2} \inf_{B(x,\varrho)} w_\xi \tag{1.12}$$

for every ball $B(x, \varrho) \subset \Omega$, and

$$\mathcal{L}(|x - \xi|) \leqslant w_\xi(x) \leqslant \gamma^{-2} |x - \xi|^\gamma \tag{1.13}$$

for every $x \in \overline{\Omega}$, where $\mathcal{L}(t) := \alpha^{2-n} \min\{t, r\}^\gamma$. In addition, for each $p \in [2, \infty]$ and any admissible radius function ρ in Ω, w_ξ is, simultaneously, a $\mathcal{T}_{\rho,p}$-barrier and a barrier for the p-laplacian at ξ in Ω.

1.3 Further remarks

It is worth recalling that the Dirichlet problem (1.7) for $p = \infty$ was studied by Le Gruyer and Archer in [14] in the context of metric spaces. There, functions satisfying $\mathcal{S}_\rho u = u$ were originally called *harmonious* and studied in connection to extension problems of continuous functions in metric spaces. It follows, as a particular case of results in [14], that if $\Omega \subset \mathbb{R}^n$ is a bounded and convex domain and ρ is 1-Lipschitz, then the Dirichlet problem

$$\begin{cases} \mathcal{S}_\rho u = u & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.14}$$

has a unique solution for each $f \in C(\overline{\Omega})$. One of the most important features of the operator $\mathcal{S}_\rho$ (with 1-Lipschitz ρ) is that it preserves the concave modulus of continuity: if $\widehat{\omega}_u$ is the lowest concave modulus of continuity of u in Ω, then $\widehat{\omega}_{\mathcal{S}_\rho u} \leqslant \widehat{\omega}_u$. This invariance property allows the use of Schauder's fixed point theorem to prove existence in (1.14). Unfortunately, the operators $\mathcal{T}_{\rho,p}$ do not preserve in general the modulus of continuity, and thus Schauder's Theorem is no longer available and different strategies are required to obtain fixed points.

The existence part in Theorem 1.5 is obtained from the equicontinuity and subsequent uniform convergence in $\overline{\Omega}$ of the iterates $\{\mathcal{T}^k_{\rho,p}u\}_k$ for $u \in C(\overline{\Omega})$. When $p = 2$, the fact that the sequence $\{\mathcal{M}^k_\rho u\}_k$ converges uniformly in $\overline{\Omega}$ to the solution of the (harmonic) Dirichlet problem in Ω with boundary data $f = u|_{\partial\Omega}$, was already observed by Lebesgue, in the case that Ω is regular and $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ (see [15], and also [6] for a more general approach in this direction). A significative difference between Lebesgue's setting and the methods of this paper is that in Lebesgue's note existence is taken for granted and the convergence of the iterates is obtained as a consequence, while we actually use the convergence of the iterates to prove existence. As for equicontinuity, it is worth mentioning that boundary equicontinuity turns out to be a much more delicate matter than interior equicontinuity. In [3], boundary equicontinuity was established under the assumption that Ω is strictly convex and ρ is 1-Lipschitz. Our approach here is based on the construction of explicit barriers for $\mathcal{T}_{\rho,p}$, having the additional advantage that they work for domains satisfying the uniform exterior cone condition. We would like to point out that, since the operators $\mathcal{T}_{\rho,p}$ are not local, some steps in Perron's method (like Poisson's modification) do not work in our setting and we need ad hoc arguments to prove existence. Our approach gives, in particular, a more constructive proof of the existence of solutions to the Dirichlet problem for the p-laplacian in domains satisfying a uniform exterior cone condition.

We would also like to stress that the technical constraint for the admissible radius function expressed in (1.9) restricts the validity of our main existence and convergence results (Theorems 1.5 and 1.7) to the range $p \in [2, \infty)$. Moreover, these results are not uniform on p, in the sense that the relationship between p and ρ established in (1.9) does not allow to pass to the limit as $p \to \infty$. This is independent of the fact that the barriers constructed in Theorem 1.9 are uniform both in $p \geqslant 2$ and ρ. In addition to this, we must point out that the definition of $\mathcal{T}_{\rho,p}$ implies that there is no comparison principle as in Theorem 3.4 for $p \in (1, 2)$, which is one of the key steps in the proofs.

The rest of the paper is organized as follows: In Section 2 we show the existence of $\mathcal{T}_{\rho,p}$-barriers for domains satisfying the uniform exterior cone condition (Theorem 1.9). Then, in Sections 3 and 4, we use these barriers to prove existence of fixed points of $\mathcal{T}_{\rho,p}$ (Theorem 1.5) and their convergence to p-harmonic functions (Theorem 1.7), respectively. For the sake of convenience and, whenever the role of $p \in [2, \infty)$ causes no confusion, we will write $\mathcal{T}_\rho$ instead of $\mathcal{T}_{\rho,p}$ in what follows.

2 Barriers for $\mathcal{T}_\rho$

Our goal is to construct a $\mathcal{T}_\rho$-barrier at each boundary point and, consequently, to show that each point on the boundary is $\mathcal{T}_\rho$-regular. Fix $\xi \in \partial\Omega$. Recalling the definition of the uniform exterior cone condition, there exist constants $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$ and a rotation $R_\xi \in SO(n)$ such that

$$\xi + R_\xi(K_{\alpha,r}) \subset \mathbb{R}^n \setminus \Omega,$$

where

$$K_{\alpha,r} = \{x \in \mathbb{R}^n : x_1 \leqslant -|x| \cos \alpha \text{ and } |x| \leqslant r\}. \tag{2.1}$$

We observe that after a translation and a rotation we can assume that $\xi = 0$ and $R_\xi = \mathrm{Id}$, in which case we define a bigger domain $\Omega_{\alpha,r} = \mathbb{R}^n \setminus K_{\alpha,r}$ so that $\Omega \subset \Omega_{\alpha,r}$. Then our aim is to construct a function w in $\Omega_{\alpha,r}$ such that its restriction to Ω, that is, $w|_\Omega$, verifies $w \geqslant \mathcal{T}_\rho w$ for every (not necessarily continuous) admissible radius function ρ in Ω.

We split the construction of such function in two steps. First, we construct the barrier at 0 for the complement of an unbounded cone along the negative x_1-axis. Second, we adapt the argument to work for the complement of a truncated cone.

2.1 Barrier for the complement of a whole cone

Let $\alpha \in (0, \frac{\pi}{2})$ and define

$$\Omega_\alpha := \{x \in \mathbb{R}^n : x_1 > -|x| \cos \alpha\}.$$

We will use *polar* coordinates with respect to the x_1-axis, that is, we assign a pair (R, θ) to each $x \in \mathbb{R}^n$, where $R = |x|$ and $\theta = \arccos(\frac{x_1}{|x|}) \in [0, \pi)$ is the angle between x and the positive x_1-axis. Then

$$\Omega_\alpha = \{x \in \mathbb{R}^n : 0 \leqslant \theta < \pi - \alpha\}.$$

Before stating the main result of this section, we define an auxiliary function $\phi : (-\pi, \pi) \to [0, \infty)$ as the solution of the differential equation

$$\begin{cases} \phi''(\theta) + (n-2)\phi'(\theta) \cot \theta = 1, \\ \phi(0) = \phi'(0) = 0, \end{cases} \tag{2.2}$$

which has the integral form

$$\phi(\theta) = \int_0^{|\theta|} \int_0^t \Big(\frac{\sin s}{\sin t}\Big)^{n-2} \, ds \, dt \tag{2.3}$$

for every $\theta \in (-\pi, \pi)$ (see [10, Lemma 2.4]). We review some of the properties of the auxiliary function ϕ in the following lemma.

Lemma 2.1. *The function $\phi : (-\pi, \pi) \to [0, \infty)$ defined in* (2.3) *satisfies the following assertions:*
(i) $\phi \in C^2(-\pi, \pi)$.
(ii) *ϕ is increasing in $(0, \pi)$ and convex in $(-\pi, \pi)$.*
(iii) *For every $|\theta| \leqslant \pi - \alpha$,*

$$0 \leqslant \phi(\theta) \leqslant \frac{\pi^2}{8} + \frac{\pi}{2(\sin \alpha)^{n-2}} \quad \text{and} \quad \phi'(\theta) \leqslant \frac{\pi}{(\sin \alpha)^{n-2}}. \tag{2.4}$$

Proof. It is easy to check that $\phi \in C^2(-\pi, \pi)$. Hereafter, we restrict the analysis to the interval $[0, \pi)$. By differentiation of (2.3), we obtain

$$\phi'(\theta) = \frac{1}{(\sin \theta)^{n-2}} \int_0^\theta (\sin t)^{n-2} \, dt \geqslant 0,$$

so ϕ is increasing in $(0, \pi)$. Next, since ϕ satisfies (2.2), we have

$$\phi''(\theta) = \frac{(\sin \theta)^{n-1} - (n-2) \cos \theta \int_0^\theta (\sin t)^{n-2} \, dt}{(\sin \theta)^{n-1}}$$

for $0 < \theta < \pi$. Observe that if $\frac{\pi}{2} \leqslant \theta < \pi$, then $\cos \theta \leqslant 0$, and so $\phi'' \geqslant 0$ in $[\frac{\pi}{2}, \pi)$. For $0 \leqslant \theta \leqslant \frac{\pi}{2}$ define

$$\psi(\theta) = (\sin \theta)^{n-1} - (n-2) \cos \theta \int_0^\theta (\sin t)^{n-2} \, dt$$

and observe that $\psi(0) = 0$ and

$$\psi'(\theta) = \cos\theta(\sin\theta)^{n-2} + (n-2)\sin\theta \int_0^\theta (\sin t)^{n-2}\,dt \geqslant 0$$

for $0 \leqslant \theta \leqslant \frac{\pi}{2}$. Therefore, ϕ is convex in $(0,\pi)$.

To show (2.4), note first that

$$\int_0^\theta (\sin t)^{n-2}\,dt \leqslant \theta \max_{0\leqslant t\leqslant\theta}\{(\sin t)^{n-2}\}$$

for each $0 < \theta < \pi$, so

$$\phi'(\theta) \leqslant \begin{cases} \theta & \text{if } 0 \leqslant \theta \leqslant \dfrac{\pi}{2}, \\ \dfrac{\theta}{(\sin\theta)^{n-2}} & \text{if } \dfrac{\pi}{2} \leqslant \theta < \pi. \end{cases} \tag{2.5}$$

Since ϕ is convex in $(0,\pi)$, we obtain that ϕ' is increasing in $(0,\pi)$. Recalling that $\alpha \in (0, \frac{\pi}{2})$, we obtain that

$$\phi'(\theta) \leqslant \phi'(\pi-\alpha) \leqslant \frac{\pi}{(\sin\alpha)^{n-2}}$$

for every $0 \leqslant \theta \leqslant \pi - \alpha$, which is the second inequality in (2.4).

On the other hand, for $\frac{\pi}{2} \leqslant \theta < \pi$ we get

$$\int_{\frac{\pi}{2}}^\theta \frac{t}{(\sin t)^{n-2}}\,dt \leqslant \Big(\theta - \frac{\pi}{2}\Big) \max_{\frac{\pi}{2}\leqslant t\leqslant\theta}\Big\{\frac{1}{(\sin t)^{n-2}}\Big\} = \frac{\theta - \frac{\pi}{2}}{(\sin\theta)^{n-2}}.$$

Integrating (2.5), we obtain

$$0 \leqslant \phi(\theta) \leqslant \frac{\pi^2}{8} + \frac{\theta - \frac{\pi}{2}}{(\sin\theta)^{n-2}}$$

for every $\frac{\pi}{2} \leqslant \theta < \pi$. In particular, since ϕ is increasing,

$$\phi(\theta) \leqslant \phi(\pi-\alpha) \leqslant \frac{\pi^2}{8} + \frac{\frac{\pi}{2} - \alpha}{(\sin(\pi-\alpha))^{n-2}} \leqslant \frac{\pi^2}{8} + \frac{\pi}{2(\sin\alpha)^{n-2}},$$

and the first inequality in (2.4) follows. □

Lemma 2.2. *For $\alpha \in (0, \frac{\pi}{2})$ let*

$$\Omega_\alpha := \{x \in \mathbb{R}^n : x_1 > -|x|\cos\alpha\}$$

and let $U : \overline{\Omega}_\alpha \to \mathbb{R}$ be the function defined by

$$\begin{cases} U(x) = |x|^\gamma (A - \phi(\theta)), \\ \quad \theta = \arccos\Big(\dfrac{x_1}{|x|}\Big), \end{cases} \tag{2.6}$$

where $\phi : (-\pi,\pi) \to [0,\infty)$ is the auxiliary function defined in (2.3) *and $A > 0$, $\gamma \in (0, \frac{1}{2}]$, are constants satisfying*

$$\frac{\pi^2}{8} + \frac{3\pi^2 + \pi}{2(\sin\alpha)^{n-2}} \leqslant A \leqslant \frac{1}{\gamma(\gamma+n-2)}. \tag{2.7}$$

Then $U \in C^2(\Omega_\alpha) \cap C(\overline{\Omega}_\alpha)$, $U(0) = 0$, $U > 0$ in $\overline{\Omega}_\alpha \setminus \{0\}$,

$$U(x) \geqslant \fint_{B(x,\varrho)} U(y)\,dy, \qquad U(x) \geqslant \frac{1}{2}\sup_{B(x,\varrho)} U + \frac{1}{2}\inf_{B(x,\varrho)} U \tag{2.8}$$

for every ball $B(x,\varrho) \subset \Omega_\alpha$, and

$$\alpha^{2-n}|x|^\gamma \leqslant U(x) \leqslant \gamma^{-2}|x|^\gamma \tag{2.9}$$

for every $x \in \overline{\Omega}_\alpha$. In particular, for each $p \in [2,\infty]$ and any admissible radius function ρ in Ω_α, U is, simultaneously, a $\mathcal{T}_{\rho,p}$-barrier and a barrier for the p-laplacian at 0 in Ω_α.

2.2 Proof of Theorem 2.2

The regularity of U is a direct consequence of its construction. To see (2.9), we recall (2.4) together with (2.7) to get that, for every $0 \leqslant \theta \leqslant \pi - \alpha$,

$$0 < \frac{3\pi^2}{2(\sin\alpha)^{n-2}} \leqslant A - \phi(\theta) \leqslant \frac{1}{\gamma(\gamma+n-2)}.$$

Then (2.9) follows.

In order to show (2.8), let us recall from [10, Lemma 2.4] the expression of the laplacian of U in the polar coordinates $x \leftrightarrow (R,\theta)$:

$$\Delta U = R^{\gamma-2}\big[-\phi''(\theta) - (n-2)\phi'(\theta)\cot\theta + \gamma(\gamma+n-2)(A-\phi(\theta))\big],$$

which together with (2.2) gives

$$\Delta U = -R^{\gamma-2}\big[1 - \gamma(\gamma+n-2)(A-\phi(\theta))\big]. \tag{2.10}$$

Since $\phi \geqslant 0$ and A and γ satisfy (2.7), it turns out that $\Delta U \leqslant 0$. That is, U is superharmonic and the first inequality in (2.8) follows by the mean value property for superharmonic functions.

Before proving the second inequality in (2.8), we first note that, since U is rotationally invariant with respect to the x_1-axis, the problem is actually bidimensional. Therefore, we replace $x \in \mathbb{R}^n$ by the complex number $z = Re^{i\vartheta}$, where $R = |x|$ and $\cos\vartheta = \frac{x_1}{|x|}$, and we assume that Ω_α lies in the complex plane, so

$$\Omega_\alpha = \{z = Re^{i\vartheta} : R > 0,\ |\vartheta| < \pi - \alpha\}.$$

Then the second inequality in (2.8) is equivalent to

$$U(z_0) \geqslant \frac{1}{2}\sup_{B(z_0,r)} U + \frac{1}{2}\inf_{B(z_0,r)} U \tag{2.11}$$

for each $z_0 = R_0 e^{i\vartheta_0}$ and $0 < r < R_0$ such that $B(z_0,r) \subset \Omega_\alpha$. Here we assume, by symmetry, that

$$0 \leqslant \vartheta_0 < \pi - \alpha.$$

Observe that $\overline{B}(z_0,r)$ lies in the cone $\{Re^{i\vartheta} : |\vartheta - \vartheta_0| \leqslant t_m\}$, where

$$t_m = \arcsin\Big(\frac{r}{R_0}\Big).$$

Given $|t| \leqslant t_m$, elementary computations show that the ray $\{Re^{i(\vartheta_0+t)} : R > 0\}$ intersects $\partial B(z_0,r)$ at two points $R_+(t)e^{i(\vartheta_0+t)}$ and $R_-(t)e^{i(\vartheta_0+t)}$, where

$$R_\pm(t) = R_0\bigg(\cos t \pm \sqrt{\Big(\frac{r}{R_0}\Big)^2 - \sin^2 t}\bigg). \tag{2.12}$$

By Lemma 2.1, ϕ is increasing and even, $\phi \geqslant 0$ and $\phi(0) = 0$. It follows that $\sup_{B(z_0,r)} U$ must be of the form $R_+^\gamma(t)(A - \phi(\vartheta_0 - t))$ for some $0 \leqslant t \leqslant t_m$. Then

$$\sup_{B(z_0,r)} U + \inf_{B(z_0,r)} U \leqslant R_+^\gamma(t)(A - \phi(\vartheta_0 - t)) + R_-^\gamma(t)(A - \phi(\vartheta_0 + t)),$$

and, since $U(z_0) = R_0^\gamma(A - \phi(\vartheta_0))$ by definition, the desired inequality (2.11) will follow from the next lemma.

Lemma 2.3. *Let $A > 0$ and $\gamma \in (0, \frac{1}{2}]$ satisfy (2.7). For $z_0 = R_0 e^{i\vartheta_0}$ and $0 < r < R_0$ such that $B(z_0,r) \subset \Omega_\alpha$, the inequality*

$$R_+(t)^\gamma(A - \phi(\vartheta_0 - t)) + R_-(t)^\gamma(A - \phi(\vartheta_0 + t)) \leqslant 2R_0^\gamma(A - \phi(\vartheta_0)) \tag{2.13}$$

holds for every $|t| \leqslant \arcsin(\frac{r}{R_0})$, where $R_\pm(t)$ were defined in (2.12).

Proof. Let us denote

$$\lambda_{\pm} = \lambda_{\pm}(t) = \frac{1}{2}\Big(\frac{R_{\pm}(t)}{R_0}\Big)^{\gamma}$$

for simplicity. Then (2.13) is equivalent to

$$F(t) := \frac{\phi(\vartheta_0) - (\lambda_+\phi(\vartheta_0 - t) + \lambda_-\phi(\vartheta_0 + t))}{1 - (\lambda_+ + \lambda_-)} \leqslant A$$

for every $0 \leqslant t \leqslant \arcsin(\frac{r}{R_0})$. We show that the previous inequality holds true. Observe that after a rearrangement of the terms we can write

$$F(t) = \phi(\vartheta_0) + \frac{\lambda_+ + \lambda_-}{1 - (\lambda_+ + \lambda_-)}\Big[\phi(\vartheta_0) - \frac{\lambda_+}{\lambda_+ + \lambda_-}\phi(\vartheta_0 - t) - \frac{\lambda_-}{\lambda_+ + \lambda_-}\phi(\vartheta_0 + t)\Big].$$

Let us focus on the term in brackets. From the convexity of ϕ we can estimate the term in brackets as follows:

$$\phi(\vartheta_0) - \frac{\lambda_+}{\lambda_+ + \lambda_-}\phi(\vartheta_0 - t) - \frac{\lambda_-}{\lambda_+ + \lambda_-}\phi(\vartheta_0 + t) \leqslant \phi(\vartheta_0) - \phi\Big(\vartheta_0 - \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-}t\Big) \leqslant \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-}t\phi'(\vartheta_0).$$

Thus,

$$F(t) \leqslant \phi(\theta_0) + \frac{\lambda_+ - \lambda_-}{1 - (\lambda_+ + \lambda_-)}t\phi'(\theta_0).$$

Notice that, since the function ϕ is increasing in $(0, \pi)$ and $\vartheta_0 \geqslant 0$ by assumption, we have $\phi'(\vartheta_0) \geqslant 0$. Next, using Lemma A.1 (see Section A), we get

$$\lambda_+ \pm \lambda_- = \frac{R_+(t)^{\gamma} + R_-(t)^{\gamma}}{2R_0^{\gamma}} \leqslant \frac{1}{2}\Big(1 + \frac{r}{R_0}\Big)^{\gamma} \pm \frac{1}{2}\Big(1 - \frac{r}{R_0}\Big)^{\gamma},$$

which together with

$$t \leqslant \arcsin\Big(\frac{r}{R_0}\Big) \leqslant \frac{\pi r}{2R_0}$$

yields

$$F(t) \leqslant \phi(\vartheta_0) + \frac{\pi}{2} \cdot \frac{\frac{r}{2R_0}[(1 + \frac{r}{R_0})^{\gamma} - (1 - \frac{r}{R_0})^{\gamma}]}{1 - \frac{1}{2}[(1 + \frac{r}{R_0})^{\gamma} + (1 - \frac{r}{R_0})^{\gamma}]}\phi'(\vartheta_0).$$

By Lemma A.2 together with the fact that $\gamma \in (0, \frac{1}{2}]$, we get

$$F(t) \leqslant \phi(\vartheta_0) + 2\pi\phi'(\vartheta_0) \leqslant \frac{\pi^2}{8} + \frac{3\pi^2 + \pi}{2(\sin\alpha)^{n-2}},$$

where in the second inequality we have recalled estimates (2.4). Then the result follows from the choice of A in (2.7). □

Remark. We want to emphasize that, in the proof of Theorem 2.2, the definition of ϕ as solution of the differential equation (2.2) is used exclusively to show the first inequality in (2.8), while for the second inequality we only need to require the convexity of ϕ in $(-\pi, \pi)$ and the fact that ϕ is increasing in $[0, \pi)$.

The following proposition says that the function U is also p-superharmonic for each $p \in [2, \infty]$.

Proposition 2.4. *Let U be the function defined in* (2.6) *with $A > 0$ and $\gamma \in (0, \frac{1}{2}]$ as in* (2.7). *Then $\Delta_p U \leqslant 0$ in Ω_α for each $p \in [2, \infty]$.*

Proof. From the representation (1.2) and the fact that $p \geqslant 2$, it is enough to check that $\Delta U \leqslant 0$ and $\Delta_\infty U \leqslant 0$.

The choices of A and γ in the expression of ΔU in (2.10) easily give that $\Delta U \leqslant 0$. We also need the expression of $\Delta_\infty U$ in polar coordinates (see [7]):

$$\Delta_\infty U = -R^{3\gamma-4}[\gamma^3(1-\gamma)(A-\phi) + \gamma(1-2\gamma)(A-\phi)(\phi')^2 + (\phi')^2\phi''].$$

Observe that, since $\gamma \in (0, \frac{1}{2}]$, $A - \phi > 0$ and $\phi'' \geqslant 0$, the term in brackets is positive, so $\Delta_\infty U \leqslant 0$. □

2.3 Barrier for the complement of a truncated cone

Let $\alpha \in (0, \frac{\pi}{2})$, $r > 0$ and define

$$\Omega_{\alpha,r} = \mathbb{R}^n \setminus K_{\alpha,r},$$

where $K_{\alpha,r}$ is as in (2.1). Note that $\Omega_{\alpha,r}$ is the complement of a truncated cone and that $\Omega_{\alpha,r} \supset \Omega_\alpha$. Let $U : \Omega_\alpha \to \mathbb{R}$ be the function defined in (2.6) for $A > 0$ and $\gamma \in (0, \frac{1}{2}]$ as in (2.7). Then, from the first inequality in (2.9), it follows that

$$m = \inf\{U(x) : x \in \Omega_\alpha \setminus B(0,r)\} \geqslant \alpha^{2-n} r^\gamma > 0.$$

Lemma 2.5. *Let U be as in (2.6) and define $w : \Omega_{\alpha,r} \to \mathbb{R}$ by*

$$w(x) = \begin{cases} \min\{U(x), m\} & \text{if } x \in \Omega_{\alpha,r} \cap B(0,r), \\ m & \text{if } x \in \Omega_{\alpha,r} \setminus B(0,r). \end{cases} \tag{2.14}$$

Then $w \in C(\overline{\Omega}_{\alpha,r})$, $w(0) = 0$, $w > 0$ in $\overline{\Omega_{\alpha,r}} \setminus \{0\}$,

$$w(x) \geqslant \fint_{B(x,\varrho)} w(y)\,dy, \qquad w(x) \geqslant \frac{1}{2} \sup_{B(x,\varrho)} w + \frac{1}{2} \inf_{B(x,\varrho)} w \tag{2.15}$$

for every ball $B(x,\varrho) \subset \Omega_{\alpha,r}$, and

$$\mathcal{L}(|x|) \leqslant w(x) \leqslant \gamma^{-2}|x|^\gamma \tag{2.16}$$

for every $x \in \overline{\Omega}_{\alpha,r}$, where

$$\mathcal{L}(t) := \alpha^{2-n} \min\{t, r\}^\gamma. \tag{2.17}$$

In particular, for each $p \in [2,\infty]$ and any admissible radius function in $\Omega_{\alpha,r}$, w is, simultaneously, a $\mathcal{T}_{\rho,p}$-barrier and a barrier for the p-laplacian at 0 in $\Omega_{\alpha,r}$.

Proof. Since $U \in C(\overline{\Omega}_\alpha)$, the continuity of w only needs to be checked at $\Omega_\alpha \cap \partial B(0,r)$. Fix $x_0 \in \Omega_\alpha \cap \partial B(0,r)$. Then $U(x_0) \geqslant m$. From the continuity of U it follows that

$$\lim_{x \to x_0} w(x) = \min\{U(x_0), m\} = m = w(x_0).$$

The inequalities in (2.16) follow from the definition of w and from (2.9). To prove (2.15), choose any ball $B(x,\rho) \subset \Omega_{\alpha,r}$. We distinguish two cases:

(1) If $w(x) < m$, then observe that $B(x,\varrho) \subset \Omega_\alpha$ and that $w \leqslant U$ in $B(x,\varrho)$. It follows from Theorem 2.2 that

$$\fint_{B(x,\varrho)} w(y)\,dy \leqslant \fint_{B(x,\varrho)} U(y)\,dy \leqslant U(x) = w(x)$$

and

$$\frac{1}{2} \sup_{B(x,\varrho)} w + \frac{1}{2} \inf_{B(x,\varrho)} w \leqslant \frac{1}{2} \sup_{B(x,\varrho)} U + \frac{1}{2} \inf_{B(x,\varrho)} U \leqslant U(x) = w(x).$$

(2) If $w(x) = m$, then (2.15) follows immediately since $w \leqslant m$.

From (2.15) it is immediate that $\mathcal{T}_{\rho,p} w \leqslant w$ for every admissible radius function ρ in $\Omega_{\alpha,r}$ and each $p \in [2,\infty]$. Finally, the fact that w is p-superharmonic is a consequence of the p-superharmonicity of U, the invariance of p-superharmonic functions by rotations and the pasting lemma [11, Lemma 7.9]. □

2.4 Proof of Theorem 1.9

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the uniform exterior cone condition with constants $\alpha \in (0, \frac{\pi}{2})$ and $r > 0$. From (1.11) it follows in particular that $\gamma \in (0, \frac{1}{2}]$ and

$$\frac{1}{\gamma(\gamma + n - 2)} > \frac{1}{\gamma n} > \frac{13\pi^2 + 4\pi}{8(\sin\alpha)^{n-2}} \geqslant \frac{\pi^2}{8} + \frac{3\pi^2 + \pi}{2(\sin\alpha)^{n-2}},$$

which allows to choose $A > 0$ so that (2.7) holds and, subsequently, to construct U and w as in (2.6) and (2.14), respectively.

Recalling Definition 1.4, there exists, for every $\xi \in \partial\Omega$, a rotation $R_\xi \in \mathrm{SO}(n)$ in $\mathbb{R}^n$ such that

$$\xi + R_\xi(K_{\alpha,r}) \subset \mathbb{R}^n \setminus \Omega$$

or, equivalently,

$$R_\xi^\top(\Omega - \xi) \subset \Omega_{\alpha,r}.$$

Then we define $w_\xi : \overline{\Omega} \to \mathbb{R}$ by

$$w_\xi(x) = w(R_\xi^\top(x - \xi)), \tag{2.18}$$

where w is the barrier function given by (2.14). Recalling Lemma 2.5, we observe that w_ξ is non-negative in $\overline{\Omega}$ and $w_\xi(x) = 0$ if and only if $x = \xi$. On the other hand, if ρ is an admissible radius function in Ω, since

$$\xi + R_\xi(B_\rho(x)) = B(\xi + R_\xi(x), \rho(x)) \subset \Omega_{\alpha,r},$$

we obtain

$$\mathcal{T}_\rho w_\xi(x) = \mathcal{T}_\rho w(R_\xi^\top(x - \xi)) \leqslant w(R_\xi^\top(x - \xi)) = w_\xi(x).$$

Thus (1.12) follows from (2.15) and, in particular, w_ξ is a $\mathcal{T}_\rho$-barrier at $\xi \in \partial\Omega$. The fact that w_ξ is also a barrier for the p-laplacian follows in a similar way. Finally, from (2.16) we get (1.13) for every $x \in \overline{\Omega}$, where $\mathcal{L}$ is given by (2.17). This finishes the proof of the theorem.

3 Existence of solutions

We split the section in two parts. In the first part, we show the equicontinuity of the sequence $\{\mathcal{T}_\rho^k u\}$ in $\overline{\Omega}$. In the second part, we establish existence and uniqueness of the Dirichlet problem for $\mathcal{T}_\rho$.

3.1 Equicontinuity results

At this point, we refer to [4, Theorem 4.5] for the equicontinuity of the sequence $\{\mathcal{T}_\rho^k u\}_k$ at interior points of Ω, where $u \in \mathcal{K}_f$ with $\mathcal{K}_f$ being the set of continuous extensions of f defined in (1.8) (see also [3, Proposition 2.6]).

Theorem 3.1 ([4, Theorem 4.5]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $p \in [2, \infty)$. Suppose that $\rho \in C(\overline{\Omega})$ is a continuous admissible radius function in Ω satisfying*

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leqslant \rho(x) \leqslant \Lambda \operatorname{dist}(x, \partial\Omega)$$

for all $x \in \Omega$, where

$$\beta \geqslant 1, \quad 0 < \Lambda < 1 - \left(\frac{p-2}{n+p}\right)^{1/\beta}, \quad 0 < \lambda \leqslant (\operatorname{diam}\Omega)^{1-\beta}\Lambda.$$

Then, for any $u \in C(\overline{\Omega})$, the sequence of iterates $\{\mathcal{T}_\rho^k u\}_k$ is locally uniformly equicontinuous in Ω.

Therefore, it only remains to show that, given a function $u \in \mathcal{K}_f$, the sequence of iterates $\{\mathcal{T}_\rho^k u\}_k$ is equicontinuous at each $\mathcal{T}_\rho$-regular point of the boundary.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C(\partial\Omega)$. For any $u \in \mathcal{K}_f$, the sequence of iterates $\{\mathcal{T}_\rho^k u\}_k$ is equicontinuous at each $\mathcal{T}_\rho$-regular point of $\partial\Omega$.*

Proof. Since $u \in \mathcal{K}_f$, we have that u is uniformly continuous in $\overline{\Omega}$, that is, for each $\eta > 0$ there exists $\delta > 0$ small enough such that $|u(x) - u(y)| < \eta$ for every $x, y \in \overline{\Omega}$ satisfying $|x - y| < \delta$. Fix $C = C_{u,\eta} = 2\|u\|_\infty/\mathcal{L}(\delta)$, where $\mathcal{L}$ is the non-decreasing continuous function defined in (2.17). Then

$$|u(x) - u(y)| \leqslant C\mathcal{L}(|x - y|) + \eta$$

for every $x, y \in \overline{\Omega}$. Therefore, if $\xi \in \partial\Omega$ is $\mathcal{T}_\rho$-regular, recalling (1.13), we obtain that

$$|u(x) - f(\xi)| \leqslant Cw_\xi(x) + \eta$$

for every $x \in \overline{\Omega}$, where w_ξ is a $\mathcal{T}_\rho$-barrier at ξ. Let $k \in \mathbb{N}$. By the affine invariance and the monotonicity of $\mathcal{T}_\rho^k$, we have

$$|\mathcal{T}_\rho^k u(x) - f(\xi)| = |\mathcal{T}_\rho^k(u - f(\xi))(x)| \leqslant \mathcal{T}_\rho^k(Cw_\xi + \eta)(x) \leqslant Cw_\xi(x) + \eta$$

for every $x \in \overline{\Omega}$, where in the second inequality we have used that $w_\xi \geqslant \mathcal{T}_\rho w_\xi$. Finally, by taking limits, it turns out that

$$0 \leqslant \limsup_{x\to\xi} |\mathcal{T}_\rho^k u(x) - f(\xi)| \leqslant C \limsup_{x\to\xi} w_\xi(x) + \eta = \eta$$

for each $k \in \mathbb{N}$ and every $\eta > 0$. Thus the sequence of iterates $\{\mathcal{T}_\rho^k u\}_k$ is equicontinuous at ξ and the proof is finished. □

Remark. The proof of the above result only requires the affine invariance and the monotonicity of $\mathcal{T}_\rho$. On the other hand, the proof does not require any further assumption on the admissible radius function, such as for example the continuity. Thus, the equicontinuity estimates obtained in the previous result are independent of the particular choice of ρ in the definition of the operator $\mathcal{T}_\rho$.

In view of Theorem 3.1 and Theorem 3.2, we have proved the following theorem.

Theorem 3.3. *Under the assumptions in Theorem 3.1, assume in addition that Ω is $\mathcal{T}_\rho$-regular. If $u \in \mathcal{K}_f$, then the sequence of iterates $\{\mathcal{T}_\rho^k u\}_k$ is equicontinuous in $\overline{\Omega}$.*

3.2 Existence and uniqueness

We start with the following comparison principle that uses a standard argument (see also [3, Proposition 4.1]).

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let ρ be an admissible radius function in Ω. Assume that $u, v \in C(\overline{\Omega})$ satisfy $u \leqslant \mathcal{T}_\rho u$, $v \geqslant \mathcal{T}_\rho v$ in Ω and $u \leqslant v$ on $\partial\Omega$. Then $u \leqslant v$ in Ω.*

Proof. Let $m = \max_{\overline{\Omega}}(u - v)$. We show that $m \leqslant 0$ by contradiction: suppose that $m > 0$ and let

$$A := \{x \in \Omega : u(x) - v(x) = m\}.$$

Since $u - v$ is upper semicontinuous in $\overline{\Omega}$ and $u - v \leqslant 0$ on the boundary, A is a nonempty closed subset of Ω. The contradiction will then follow by proving that A is also open, so $A = \Omega$ and $u(x) - v(x) = m > 0$ for every $x \in \Omega$.

To see that A is open, we choose any $a \in A$ and we show that $B_\rho(a) \subset A$. Recalling that u and v are sub- and super-solutions of $\mathcal{T}_\rho$, we obtain that

$$\mathcal{T}_\rho u(a) \geqslant u(a) = m + v(a) \geqslant m + \mathcal{T}_\rho v(a).$$

By the definition of $\mathcal{T}_\rho$,

$$\frac{p-2}{n+p}\mathcal{S}_\rho u(a) + \frac{n+2}{n+p}\mathcal{M}_\rho u(a) \geqslant \frac{p-2}{n+p}(m + \mathcal{S}_\rho v(a)) + \frac{n+2}{n+p}(m + \mathcal{M}_\rho v(a)).$$

Hence, by the monotonicity of $\mathcal{S}_\rho$ and $\mathcal{M}_\rho$ and since $p \in [2, \infty)$, it turns out that

$$\mathcal{M}_\rho u(a) = m + \mathcal{M}_\rho v(a).$$

Recalling the definition of $\mathcal{M}_\rho$, we obtain

$$m = \fint_{B_\rho(a)} (u - v).$$

Since m is defined as the maximum in $\overline{\Omega}$ of $u - v$, we have $u(x) - v(x) = m$ for every $x \in B_\rho(a)$. Then $B_\rho(a) \subset A$, and so A is an open set. Therefore, since Ω is connected, we obtain $A = \Omega$ and $u - v \equiv m > 0$ in Ω, which contradicts the assumption $u \leqslant v$ on $\partial\Omega$. □

The uniqueness of fixed points follows immediately as a corollary.

Corollary 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $f \in C(\partial\Omega)$. Suppose that $u, v \in \mathcal{K}_f$ are fixed points of $\mathcal{T}_\rho$. Then $u = v$ in $\overline{\Omega}$.*

In order to show the existence of fixed points for $\mathcal{T}_\rho$ in $\mathcal{K}_f$, we will make use of the following technical result, which can be stated in the more general context of Banach spaces.

Lemma 3.6. *Let $(X, \|\cdot\|)$ be a Banach space, let $\emptyset \neq K \subset X$ be any closed subset and let $T : K \to K$ be a non-expansive operator. Fix $x \in K$. If $y \in K$ is any limit point of the sequence $\{T^k x\}_k$, then*

$$\lim_{k\to\infty} \|T^{k+1}x - T^k x\| = \|Ty - y\|. \tag{3.1}$$

Proof. Observe first that, since T is non-expansive, the sequence of non-negative real numbers

$$\{\|T^{k+1}x - T^k x\|\}_k$$

is non-increasing, and thus every subsequence converges to the same limit. Next, take any convergent subsequence $\{T^{k_j}x\}_j$ and denote the limit by $y \in K$. The triangle inequality and the non-expansiveness of T yield

$$|\|T^{k_j+1}x - T^{k_j}x\| - \|Ty - y\|| \leqslant \|(T^{k_j+1}x - Ty) - (T^{k_j}x - y)\| \leqslant 2\|T^{k_j}x - y\|$$

for each $j \in \mathbb{N}$. Then (3.1) follows after taking limits as $j \to \infty$. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Since Ω is $\mathcal{T}_\rho$-regular by assumption, the sequence of iterates $\{\mathcal{T}_\rho^k u\}_k$ is equicontinuous at each point in $\overline{\Omega}$ for any $u \in \mathcal{K}_f$, by Theorem 3.3. Then the Arzelà–Ascoli theorem yields the existence of at least one subsequence converging uniformly to a function $v \in \mathcal{K}_f$. Furthermore, $\mathcal{T}_\rho^\ell v$ is also a limit point of $\{\mathcal{T}_\rho^k u\}_k$ for each $\ell = 0, 1, 2, \ldots$. Therefore, since $\mathcal{K}_f$ is a closed subset of $C(\overline{\Omega})$ and $\mathcal{T} = \mathcal{T}_\rho : \mathcal{K}_f \to \mathcal{K}_f$ is non-expansive, Lemma 3.6 implies that

$$\|\mathcal{T}^{\ell+1}v - \mathcal{T}^\ell v\|_\infty = \lim_{k\to\infty} \|\mathcal{T}^{k+1}u - \mathcal{T}^k u\|_\infty =: d \geqslant 0 \tag{3.2}$$

for every $\ell = 0, 1, 2, \ldots$. In consequence, if $d = 0$, we have in particular that $\mathcal{T}_\rho v = v$, so v is a fixed point of $\mathcal{T}_\rho$.

In order to show that actually $d = 0$, let us assume on the contrary that $d > 0$ and argue by contradiction. Let $\ell \in \mathbb{N}$ to be fixed later. Since $\mathcal{T}^{\ell+1}v - \mathcal{T}^\ell v$ is a continuous function vanishing on $\partial\Omega$, we can choose an interior point $x_0 \in \Omega$ such that

$$|\mathcal{T}^{\ell+1}v(x_0) - \mathcal{T}^\ell v(x_0)| = d.$$

We assume that $\mathcal{T}^{\ell+1}v(x_0) - \mathcal{T}^\ell v(x_0) = d$ since otherwise the proof goes in an analogous way. Recalling the definition of $\mathcal{T} = \mathcal{T}_\rho$ and $\mathcal{M} = \mathcal{M}_\rho$, it turns out that

$$d = \frac{p-2}{n+p}[\mathcal{S}(\mathcal{T}^\ell v)(x_0) - \mathcal{S}(\mathcal{T}^{\ell-1}v)(x_0)] + \frac{n+2}{n+p}\mathcal{M}(\mathcal{T}^\ell v - \mathcal{T}^{\ell-1}v)(x_0). \tag{3.3}$$

From (3.2) and the non-expansiveness of $\mathcal{S}$ and $\mathcal{M}$, it follows that

$$\mathcal{S}(\mathcal{T}^\ell v)(x_0) - \mathcal{S}(\mathcal{T}^{\ell-1}v)(x_0) \leqslant d \quad \text{and} \quad \mathcal{M}(\mathcal{T}^\ell v - \mathcal{T}^{\ell-1}v)(x_0) \leqslant d,$$

which together with (3.3) implies that $\mathcal{M}(\mathcal{T}^\ell v - \mathcal{T}^{\ell-1}v)(x_0) = d$.

Equivalently,

$$\fint_{B_\rho(x_0)} (\mathcal{T}^\ell v(y) - \mathcal{T}^{\ell-1}v(y))\, dy = d,$$

and by (3.2), the integrand must be equal to d in $B_\rho(x_0)$. In particular, $\mathcal{T}^\ell v(x_0) - \mathcal{T}^{\ell-1}v(x_0) = d$, so we can repeat this argument iteratively until we finally get that

$$\mathcal{T}^\ell v(x_0) = v(x_0) + \ell d.$$

Recalling (1.8) and the fact that $\mathcal{T} = \mathcal{T}_\rho : \mathcal{K}_f \to \mathcal{K}_f$, we get

$$\|f\|_\infty \geqslant \mathcal{T}^\ell v(x_0) = v(x_0) + \ell d \geqslant -\|f\|_\infty + \ell d.$$

Hence, choosing ℓ such that

$$\ell > \frac{2\|f\|_\infty}{d},$$

we obtain the desired contradiction.

Finally, to see that the sequence of iterates $\{\mathcal{T}_\rho^k u\}_k$ actually converges uniformly to the unique fixed point $v \in \mathcal{K}_f$, suppose on the contrary that there exist $\eta > 0$ and a subsequence $\{\mathcal{T}_\rho^{k_j} u\}_j$ such that $\|\mathcal{T}_\rho^{k_j} u - v\|_\infty \geqslant \eta$ for each $j \in \mathbb{N}$. We can assume that this subsequence converges uniformly to a function $w \in \mathcal{K}_f$ (otherwise, by equicontinuity and the Arzelà–Ascoli theorem, we could take a further subsequence), which would be a limit point of $\{\mathcal{T}_\rho^k u\}_k$ in $\mathcal{K}_f$, and thus a fixed point of $\mathcal{T}_\rho$. Then the contradiction follows by uniqueness and the fact that $\|w - v\|_\infty \geqslant \eta$. □

4 Convergence to *p*-harmonic functions

In this section, we study the convergence of solutions u_ρ to (1.10) as the admissible radius function converges to zero in Ω. Before moving into details, it is worth recalling that one of the main connections between mean value properties and p-harmonic functions arises from the asymptotic expansion for the p-laplacian of a twice-differentiable function ϕ at a non-critical point x. This expansion can be expressed in terms of the average operator $\mathcal{T}_\rho$ as follows:

$$\mathcal{T}_\rho \phi(x) = \phi(x) + \frac{\rho(x)^2}{2(n+p)} \Delta_p^N \phi(x) + o(\rho(x)^2) \quad (\rho(x) \to 0),$$

where $\Delta_p^N \phi$ stands for the *normalized p-laplacian* of ϕ defined by

$$\Delta_p^N \phi := \Delta\phi + (p-2)\frac{\Delta_\infty \phi}{|\nabla\phi|^2}.$$

Heuristically speaking, if the fixed points $\mathcal{T}_\rho u_\rho = u_\rho$ converge to a function u_0 as $\rho \to 0$, then it is reasonable to expect that this function is p-harmonic. Indeed, this is one of the key ideas required in the proof of Theorem 1.7.

To this end, first we need to impose appropriate conditions in order to ensure that $\rho(x)$ converges to zero in a uniform way. Given a bounded domain $\Omega \subset \mathbb{R}^n$, let us consider a collection of continuous admissible radius functions $\{\rho_\varepsilon\}_{0<\varepsilon\leqslant 1}$ satisfying

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leqslant \frac{\rho_\varepsilon(x)}{\varepsilon} \leqslant \Lambda \operatorname{dist}(x, \partial\Omega) \tag{4.1}$$

for all $x \in \Omega$ and every $0 < \varepsilon \leqslant 1$, where β, λ and Λ are as in (1.9). Since Ω is bounded, $\|\rho_\varepsilon\|_\infty$ decreases as fast as, at least, a constant multiple of ε. In consequence, $\rho_\varepsilon(x) = O(\varepsilon)$ uniformly for every $x \in \Omega$, and the asymptotic expansion for $\mathcal{T}_{\rho_\varepsilon}$ becomes

$$\mathcal{T}_{\rho_\varepsilon} \phi(x) = \phi(x) + \frac{\varepsilon^2}{2(n+p)} \Big(\frac{\rho_\varepsilon(x)}{\varepsilon}\Big)^2 \Delta_p^N \phi(x) + o(\varepsilon^2) \quad (\varepsilon \to 0) \tag{4.2}$$

for every $x \in \Omega$.

On the other hand, ρ_ε is an admissible radius function satisfying the hypothesis of Theorem 1.5 for each $0 < \varepsilon \leqslant 1$. Therefore, assuming that Ω satisfies the uniform exterior cone condition, Theorem 1.5 yields, for any fixed $f \in C(\partial\Omega)$, a function $u_\varepsilon \in C(\overline{\Omega})$ satisfying

$$\begin{cases} \mathcal{T}_\varepsilon u_\varepsilon = u_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

for each $0 < \varepsilon \leqslant 1$, where $\mathcal{T}_\varepsilon := \mathcal{T}_{\rho_\varepsilon}$.

The strategy to prove Theorem 1.7 is inspired by the method for convergence of numerical schemes established by Barles and Souganidis in the 90s [5] and in a more recent result by del Teso, Manfredi and Parviainen on the convergence of dynamic programming principles for the p-laplacian [8]. The steps in the proof can be split into two parts. First, by taking pointwise limits as $\varepsilon \to 0$, we define semicontinuous functions

$$\underline{u}(x) := \liminf_{y\to x,\, \varepsilon\to 0} u_\varepsilon(y) \leqslant \limsup_{y\to x,\, \varepsilon\to 0} u_\varepsilon(y) =: \overline{u}(x), \tag{4.4}$$

and, using the asymptotic expansion (4.2), we show that $\underline{u}$ and $\overline{u}$ are p-superharmonic and p-subharmonic, respectively. In the second part, we prove that $\overline{u} \leqslant \underline{u}$ in $\overline{\Omega}$ with the aid of the comparison principle for p-subharmonic and p-superharmonic functions [12, Theorem 2.7], so both functions coincide with u_0, the unique p-harmonic function satisfying

$$\begin{cases} \Delta_p u_0 = 0 & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega. \end{cases}$$

We remind that, by [12, Theorem 2.5], the concepts of p-subharmonic function and viscosity p-subsolution coincide. Hence, for the sake of simplicity, hereafter we will use the viscosity characterization.

Definition 4.1. Let $p \in [2, \infty]$.
(i) We say that an upper semicontinuous function u in Ω is *p-subharmonic* if $u \not\equiv -\infty$ and for every $x \in \Omega$ and any $\phi \in C^2(\Omega)$ such that $\nabla\phi(x) \neq 0$ and $u - \phi < u(x) - \phi(x) = 0$ in $\Omega \setminus \{x\}$, we have that $\Delta_p \phi(x) \geqslant 0$.
(ii) We say that a lower semicontinuous function u in Ω is *p-superharmonic* if $u \not\equiv \infty$ and for every $x \in \Omega$ and any $\phi \in C^2(\Omega)$ such that $\nabla\phi(x) \neq 0$ and $u - \phi > u(x) - \phi(x) = 0$ in $\Omega \setminus \{x\}$, we have that $\Delta_p \phi(x) \leqslant 0$.
(iii) $u \in C(\Omega)$ is *p-harmonic* if it is both p-subharmonic and p-superharmonic.

Proposition 4.2. *Let $p \in [2, \infty)$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the uniform exterior cone condition, let $f \in C(\partial\Omega)$ and let $u_\varepsilon \in \mathcal{K}_f$ be the unique solution of* (4.3) *provided by Theorem 1.5 for* $0 < \varepsilon \leqslant 1$. *Let $\underline{u}$ and $\overline{u}$ be the functions defined in* (4.4). *Then $\underline{u}$ is p-superharmonic and $\overline{u}$ is p-subharmonic in Ω.*

Proof. We show that $\overline{u}$ is p-subharmonic. Fix any $x \in \Omega$ and $\phi \in C^2(\Omega)$ such that

$$\nabla\phi(x) \neq 0 \quad \text{and} \quad \overline{u} - \phi < \overline{u}(x) - \phi(x) = 0$$

in $\Omega \setminus \{x\}$. That is, x is a strict global maximum of $\overline{u} - \phi$ in $\overline{\Omega}$. We need to check that $\Delta_p\phi(x) \geqslant 0$.

From the definition of $\overline{u}$, we pick sequences $\varepsilon_j \to 0$ and $z_j \to x$ such that $u_{\varepsilon_j}(z_j) \to \overline{u}(x)$. For each j, let $y_j \in \overline{\Omega}$ such that y_j is a maximum of $u_{\varepsilon_j} - \phi$ with $\nabla\phi(y_j) \neq 0$. We claim that $y_j \to x$ as $j \to \infty$. Otherwise, there would be a further subsequence (still denoted by $\{y_j\}_j$) converging to $x' \neq x$. Then

$$\overline{u}(x') - \phi(x') \geqslant \limsup_{j\to\infty}(u_{\varepsilon_j}(y_j) - \phi(y_j)) \geqslant \limsup_{j\to\infty}(u_{\varepsilon_j}(z_j) - \phi(z_j)) = \overline{u}(x) - \phi(x),$$

so we obtain a contradiction to the fact that x is a strict global maximum of $\overline{u} - \phi$. Then $y_j \to x$ and $u_{\varepsilon_j} - u_{\varepsilon_j}(y_j) \leqslant \phi - \phi(y_j)$ in $\overline{\Omega}$. In consequence, by the monotonicity and the affine invariance of $\mathcal{T}_{\varepsilon_j}$, we obtain

$$0 = \mathcal{T}_{\varepsilon_j} u_{\varepsilon_j}(y_j) - u_{\varepsilon_j}(y_j) \leqslant \mathcal{T}_{\varepsilon_j}\phi(y_j) - \phi(y_j)$$

for every $j \in \mathbb{N}$. Notice that the left-hand side of the inequality is equal to zero due to the fact that $\mathcal{T}_\varepsilon u_\varepsilon = u_\varepsilon$ for every $0 < \varepsilon \leqslant 1$. Recall (4.2), rearrange terms and divide by ε_j^2 to obtain

$$\Big(\frac{\rho_{\varepsilon_j}(y_j)}{\varepsilon_j}\Big)^2 \Delta_p^N \phi(y_j) \geqslant o(1) \quad (j \to \infty).$$

Assume for a moment that $\Delta_p^N\phi(x) < 0$. Then $\Delta_p^N\phi(y_j) < 0$ for every $j \in \mathbb{N}$ large enough. Recalling (4.1), we get

$$(\lambda \operatorname{dist}(y_j, \partial\Omega)^\beta)^2 \Delta_p^N \phi(y_j) \geqslant o(1) \quad (j \to \infty).$$

Hence, taking limits as $j \to \infty$, we obtain

$$(\lambda \operatorname{dist}(x, \partial\Omega)^\beta)^2 \Delta_p^N \phi(x) \geqslant 0,$$

and thus $\Delta_p^N\phi(x) \geqslant 0$. □

In the next proposition, we give a uniform boundary equicontinuity estimate for $\{u_\varepsilon\}_{0<\varepsilon\leqslant 1}$. This estimate is crucial to prove that the functions $\underline{u}$ and $\overline{u}$ attach the right values near the boundary.

Proposition 4.3. *Let $p \in [2,\infty)$, let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the uniform exterior cone condition (with constants α and r), let $\gamma \in (0, \frac{1}{2}]$ be as in (1.11), and let $f \in C(\partial\Omega)$. Under the assumptions of Theorem 1.7, let $u_\varepsilon \in \mathcal{K}_f$ be the unique solution of (4.3) for each $0 < \varepsilon \leqslant 1$. Then, for each $\eta > 0$, there exists a constant $C > 0$ depending only on Ω, f and η such that*

$$|u_\varepsilon(x) - f(\xi)| \leqslant C\gamma^{-2}|x-\xi|^\gamma + \eta \tag{4.5}$$

for every $x \in \overline{\Omega}$ and $\xi \in \partial\Omega$.

Proof. By uniform continuity, there exists $\delta > 0$ small enough such that $|f(\zeta) - f(\xi)| < \eta$ for every $\xi, \zeta \in \partial\Omega$ satisfying $|\zeta - \xi| < \delta$. Fix $C = C_{f,\eta} = 2\|f\|_\infty/\mathcal{L}(\delta)$, where $\mathcal{L}$ is the function defined in (2.17). Then

$$|f(\zeta) - f(\xi)| \leqslant C\mathcal{L}(|\zeta - \xi|) + \eta,$$

and from (1.13) it follows that

$$|f(\zeta) - f(\xi)| \leqslant Cw_\xi(\zeta) + \eta \tag{4.6}$$

for every $\xi, \zeta \in \partial\Omega$, where w_ξ is the $\mathcal{T}_\rho$-barrier at ξ defined in (2.18). From the fact that w_ξ is a $\mathcal{T}_\varepsilon$-barrier at ξ (Theorem 1.9) and from the comparison principle (Theorem 3.4), it follows that

$$|u_\varepsilon(x) - f(\xi)| \leqslant Cw_\xi(x) + \eta$$

for each $x \in \overline{\Omega}$, which together with (1.13) implies (4.5). □

Remark. If u is the p-harmonic function in Ω with boundary data f, then (4.5) holds with u_ε replaced by u. This follows from (4.6), the fact that w_ξ is p-superharmonic and the comparison principle for the p-laplacian.

Proof of Theorem 1.7. We know that the semicontinuous functions $\underline{u}$ and $\overline{u}$ defined in (4.4) satisfy $\underline{u} \leqslant \overline{u}$ by construction. We claim that the theorem follows from the reverse inequality. Indeed, suppose that $\underline{u}$ and $\overline{u}$ agree over the whole domain. This allows to define a continuous function $u_0 = \underline{u} = \overline{u} \in \mathcal{K}_f$ as the pointwise limit

$$u_0(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x)$$

for each $x \in \overline{\Omega}$. Then, by Theorem 4.2, u_0 is both p-subharmonic and p-superharmonic, so u_0 is p-harmonic in Ω and $u_0|_{\partial\Omega} = f$.

Our strategy to show that $\underline{u} \geqslant \overline{u}$ relies on the uniform equicontinuity estimate from Theorem 4.3 to show that $\underline{u}$ and $\overline{u}$ take the right values near the boundary. The desired inequality will then follow as a consequence of the comparison principle for p-subharmonic and p-superharmonic functions.

Fix an arbitrary small $\eta > 0$ and choose $C > 0$ as in Theorem 4.3. By the definition of $\overline{u}$, we have that

$$\overline{u}(x) - f(\xi) = \limsup_{y\to x,\, \varepsilon\to 0} (u_\varepsilon(y) - f(\xi)) \leqslant C\gamma^{-2}|x-\xi|^\gamma + \eta$$

for $x \in \overline{\Omega}$, $\xi \in \partial\Omega$ and $\varepsilon > 0$, where in the inequality we have used estimate (4.5). Taking limits as $x \to \xi$, we get

$$\limsup_{x\to\xi} (\overline{u}(x) - f(\xi)) \leqslant \eta$$

for arbitrary small $\eta > 0$. Repeating an analogous argument for $\underline{u}$, we obtain

$$\limsup_{x\to\xi} \overline{u}(x) \leqslant f(\xi) \leqslant \liminf_{x\to\xi} \underline{u}(x)$$

for every $\xi \in \partial\Omega$. Since $\underline{u}$ and $\overline{u}$ are p-superharmonic and p-subharmonic, respectively, by the comparison principle [12, Theorem 2.7], we finally obtain that $\overline{u} \geqslant \underline{u}$ in Ω. □

A Auxiliary lemmas

Lemma A.1. *Let $T \in [0, \frac{\pi}{2}]$ and $\gamma > 0$. Then*

$$(\cos t + \sqrt{\sin^2 T - \sin^2 t})^\gamma \pm (\cos t - \sqrt{\sin^2 T - \sin^2 t})^\gamma \leqslant (1 + \sin T)^\gamma \pm (1 - \sin T)^\gamma$$

whenever $|t| \leqslant T$.

Proof. Let $a \in [0, 1]$ and define $\varphi_\pm : [a, 1] \to \mathbb{R}$ by

$$\varphi_\pm(x) = (x + \sqrt{x^2 - a^2})^\gamma \pm (x - \sqrt{x^2 - a^2})^\gamma.$$

Direct computation shows that

$$\varphi'_\pm(x) = \frac{\gamma}{\sqrt{x^2 - a^2}} \varphi_\mp(x) \geqslant 0.$$

Therefore, $\varphi_\pm$ is positive and increasing in $[a, 1]$. In particular, $\varphi_\pm(x) \leqslant \varphi_\pm(1)$ for every $x \in [a, 1]$. Then the result follows by letting $a = \cos T$ and performing the change of variables $x = \cos t$. □

Lemma A.2. *Let $\gamma \in (0, 1)$. Then*

$$\frac{\frac{x}{2}[(1 + x)^\gamma - (1 - x)^\gamma]}{1 - \frac{1}{2}[(1 + x)^\gamma + (1 - x)^\gamma]} \leqslant \frac{2}{1 - \gamma} \tag{A.1}$$

for all $x \in (0, 1]$.

Proof. Let us recall the Taylor series of $f(x) = (1 + x)^\gamma$:

$$(1 + x)^\gamma = 1 + \sum_{k=1}^{\infty} \binom{\gamma}{k} x^k \tag{A.2}$$

for $|x| \leqslant 1$, where

$$\binom{\gamma}{k} = (-1)^{k-1} \frac{\gamma(1 - \gamma)(2 - \gamma) \cdots (k - 1 - \gamma)}{k!}$$

for each $k \in \mathbb{N}$. Observe that, since $\gamma \in (0, 1)$, we have that

$$\binom{\gamma}{2k - 1} > 0 \quad \text{and} \quad \binom{\gamma}{2k} < 0 \qquad \text{for each } k \in \mathbb{N}.$$

We can rewrite the left-hand side in (A.1) by replacing (A.2):

$$\frac{\frac{x}{2}[(1 + x)^\gamma - (1 - x)^\gamma]}{1 - \frac{1}{2}[(1 + x)^\gamma + (1 - x)^\gamma]} = \frac{\sum_{k=1}^{\infty} \binom{\gamma}{2k-1} x^{2k}}{-\sum_{k=1}^{\infty} \binom{\gamma}{2k} x^{2k}}.$$

Hence, (A.1) follows from the fact that

$$\sum_{k=1}^{\infty} \left[\binom{\gamma}{2k - 1} + \frac{2}{1 - \gamma} \binom{\gamma}{2k} \right] x^{2k} \leqslant 0$$

for every $x \in (0, 1)$. In fact, every coefficient in the above series is nonpositive, that is,

$$\binom{\gamma}{2k - 1} + \frac{2}{1 - \gamma} \binom{\gamma}{2k} = \binom{\gamma}{2k - 1} \left[1 + \frac{2}{1 - \gamma} \cdot \frac{\gamma - 2k + 1}{2k} \right] \leqslant 0$$

for every $k \in \mathbb{N}$. □

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References

[1] Á. Arroyo, J. Heino and M. Parviainen, Tug-of-war games with varying probabilities and the normalized $p(x)$-Laplacian, *Commun. Pure Appl. Anal.* **16** (2017), no. 3, 915–944.
[2] Á. Arroyo and J. G. Llorente, On the asymptotic mean value property for planar p-harmonic functions, *Proc. Amer. Math. Soc.* **144** (2016), no. 9, 3859–3868.
[3] Á. Arroyo and J. G. Llorente, On the Dirichlet problem for solutions of a restricted nonlinear mean value property, *Differential Integral Equations* **29** (2016), no. 1–2, 151–166.
[4] Á. Arroyo and J. G. Llorente, A priori Hölder and Lipschitz regularity for generalized p-harmonious functions in metric measure spaces, *Nonlinear Anal.* **168** (2018), 32–49.
[5] G. Barles and P. E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Asymptotic Anal.* **4** (1991), no. 3, 271–283.
[6] C. Caratheodory, On Dirichlet's Problem, *Amer. J. Math.* **59** (1937), no. 4, 709–731.
[7] D. DeBlassie and R. G. Smits, The p-harmonic measure of a small spherical cap, *Matematiche (Catania)* **71** (2016), no. 1, 149–171.
[8] F. del Teso, J. J. Manfredi and M. Parviainen, Convergence of dynamic programming principles for the p-Laplacian, *Adv. Calc. Var.* (2020), DOI 10.1515/acv-2019-0043.
[9] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Classics Appl. Math. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2011.
[10] W. K. Hayman and P. B. Kennedy, *Subharmonic Functions. Vol. I*, London Math. Soc. Monogr. 9, Academic Press, London, 1976.
[11] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Dover, Mineola, 2006.
[12] P. Juutinen, P. Lindqvist and J. J. Manfredi, On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation, *SIAM J. Math. Anal.* **33** (2001), no. 3, 699–717.
[13] O. D. Kellogg, Converses of Gauss's theorem on the arithmetic mean, *Trans. Amer. Math. Soc.* **36** (1934), no. 2, 227–242.
[14] E. Le Gruyer and J. C. Archer, Harmonious extensions, *SIAM J. Math. Anal.* **29** (1998), no. 1, 279–292.
[15] H. Lebesgue, Sur le problême de Dirichlet, *Comptes Rendus (Paris)* **154** (1912), 335–337.
[16] P. Lindqvist, *Notes on the Stationary p-Laplace Equation*, Springer Briefs Math., Springer, Cham, 2019.
[17] P. Lindqvist and J. Manfredi, On the mean value property for the p-Laplace equation in the plane, *Proc. Amer. Math. Soc.* **144** (2016), no. 1, 143–149.
[18] H. Luiro, M. Parviainen and E. Saksman, On the existence and uniqueness of p-harmonious functions, *Differential Integral Equations* **27** (2014), no. 3–4, 201–216.
[19] J. J. Manfredi, M. Parviainen and J. D. Rossi, An asymptotic mean value characterization for p-harmonic functions, *Proc. Amer. Math. Soc.* **138** (2010), no. 3, 881–889.
[20] J. J. Manfredi, M. Parviainen and J. D. Rossi, On the definition and properties of p-harmonious functions, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **11** (2012), no. 2, 215–241.
[21] Y. Peres, O. Schramm, S. Sheffield and D. B. Wilson, Tug-of-war and the infinity Laplacian, *J. Amer. Math. Soc.* **22** (2009), no. 1, 167–210.
[22] Y. Peres and S. Sheffield, Tug-of-war with noise: A game-theoretic view of the p-Laplacian, *Duke Math. J.* **145** (2008), no. 1, 91–120.
[23] V. Volterra, Alcune osservazioni sopra propietà atte ad individuare una funzione, *Atti Real. Acad. Lincei Roma* **18** (1909), 263–266.