EISENSTEIN SERIES WITH CLASS NUMBER COEFFICIENTS CONSTRUCTED FROM THE WEIL REPRESENTATION

by
Keaton Stubis

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy

Baltimore, Maryland
August, 2023

Abstract

We generalize the work found in the paper [KY10] of Kudla and Yang to the case of Hilbert modular forms that come from the Weil representation associated to a 1-dimensional quadratic space. We also provide a computation of the level groups of certain forms produced this way. To achieve our results, we compute closed-form formulas for quadratic Gauss sums over local fields of characteristic 0. This includes the case of both odd and even residue characteristic.

Thesis Readers

Dr. David Savitt (Primary Adviser)
Professor
Department of Mathematics
Johns Hopkins University

Dr. Yiannis Sakellaridis
Professor
Department of Mathematics
Johns Hopkins University

Acknowledgements

I would like to thank my adviser David Savitt for our many discussions on the thesis material and his support. I would also like to thank my parents Mark and Qin, my sister Halley, and my friends Eric Bobrow and Kalyani Kansal for their help and support with this thesis.

Contents

Chapter 1

Introduction

This thesis started off as a computation related to The Ghost Conjecture and certain “generalized Hurwitz class numbers". A single step in this calculation proved especially difficult and ended up becoming what is essentially this thesis. We will not remark on the original problem and limit our discussion to the calculation that resulted from it.

Our calculation is based on a classical result of Zagier's found in [HZ76] Theorem 2, so we will take a moment to introduce it, mainly though the lens of the treatment in [BS20]. This result states that there is a (nonholomorphic) modular form of weight 3/2 and level $\Gamma_0(4)$ whose (holomorphic) Fourier coefficients are exactly the Hurwitz class numbers $H(m)$. We start by recalling the properties of Hurwitz class numbers and Zagier's modular form of weight 3/2.

The Hurwitz class number $H(m)$ is defined for any integer $m \geq 0$. $H(m) = 0$ if $-m$ is not a square mod 4 (alternatively: if $m \not\equiv 0, 3 \bmod 4$). $H(0) = -1/12$. In all other cases, one defines $H(m)$ to be the number of equivalence classes of binary quadratic forms $ax^2 + bxy + cy^2$ $(a, b, c \in \mathbb{Z})$ of discriminant $b^2 - 4ac = -m$ up to certain multiplicities. Specifically, we can consider the action of $SL_2(\mathbb{Z})$ on these quadratic forms via basechange. $H(m)$ is then the number of orbits under this action where each orbit is weighted by $2/\#stab$ where $\#stab$ is the size of its stabilizer. In practice, the weights will almost all be 1, with a weight of 1/2 occurring iff the orbit contains a multiple of x^2+y^2 and a weight of $1/3$ occurring iff the orbit contains a multiple of x^2+xy+y^2. The first handful of Hurwitz class numbers are

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$H(m)$	$\frac{-1}{12}$	0	0	$\frac{1}{3}$	$\frac{1}{2}$	0	0	1	1	0	0	1	$\frac{4}{3}$

The Hurwitz class numbers are quite close to the class numbers of imaginary quadratic number fields. For example, if we were to count the orbits for $H(m)$ but use trivial weights by weighting every orbit by 1, we would get the class number of $\mathbb{Q}[\sqrt{-m}]$. A further connection is given by the following formula for $H(m)$, which is a restatement of the one on p.2301 of [KY10]. The formula assumes that m is either 0 or 3 mod 4.

$$H(m)=\frac{2h(\mathbb{Q}[\sqrt{-m}])}{w(\mathbb{Q}[\sqrt{-m}])}\rho_h(\mathbb{Q}[\sqrt{-m}]),\text{ with }\rho_h(\mathbb{Q}[\sqrt{-m}])=\sum_{c|f}c\prod_{p|c}(1-\left(\frac{-m}{p}\right)p^{-1})$$

Here, h is the class number and w is the number of roots of unity in the field. We informally refer to ρ_h as a "Hurwitzification factor" due to its role in converting a normal class number into a Hurwitz class number. f is the positive integer determined by the equation $-m=f^2Disc(\mathbb{Q}[\sqrt{-m}])$ (note that our assumption on the value of m mod 4 is required for f to be integral). c varies over all positive divisors of f and p varies over all prime divisors of c. Finally, $\left(\frac{-m}{p}\right)$ is the Legendre symbol.

We can now address Zagier's result. For a variable $\tau=u+iv$ in the upper half plane $\mathcal{H}$ and $q=e^{2\pi i\tau}$, Zagier's modular form is given by the following equation. Note that we choose to separate out a term from each sum, which we will call the $m=0$ term of that sum.

$$Z(\tau)=-\frac{1}{12}+\sum_{m=1}^{\infty}H(m)q^m+\frac{1}{8\pi\sqrt{v}}+\frac{1}{4\sqrt{\pi}}\sum_{m=1}^{\infty}m\int_{t=4\pi m^2v}^{\infty}t^{-3/2}e^{-t}dt\cdot q^{-m^2}$$

Note the terms are separated into two parts. First there are the holomorphic terms which form a standard Fourier series where the coefficients are the Hurwitz class numbers. Second are the nonholomorphic terms, where the integrals are incomplete gamma functions (so called because the integral doesn't start from 0). Zagier proves that this (nonholomorphic) function transforms like a modular form of weight 3/2 and level $\Gamma_0(4)$. We will recall the exact definition of this later, but he shows that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $4/c$ and some choice of the squareroot of $c\tau + d$,

$$Z\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-3/2} Z(\tau)$$

This implies a great number of nontrivial relations between the Hurwitz class numbers $H(m)$.

Since Zagier's original modular form, there have been many generalizations. We quickly review them, since our work will nestle in among them. In [Coh75], Cohen gives generalizations of $Z(\tau)$ to arbitrary weights $\kappa + 1/2$ for integral $\kappa \geq 1$. In [Su16], Ren uses the framework of [HI13] to further generalize this work to the case of Hilbert modular forms over a totally real number field K.

Finally, there is a certain kind of Eisenstein series $E(\tau, s, \Phi)$ that is studied in [KY10], among other places. (Although we mainly refer to [KRY06] sections 5 and 8, [KRY04] (see equation (6.20) for a definition), and [KY10] for our approach to these ideas.) Here, τ is the main argument to the function, and one may treat s, Φ as parameters that determine which Eisenstein series one gets. Following the construction in [KRY06], one obtains vast quantities of Hilbert-Siegel[1] modular forms of either integral or half-integral weight. There are also expressions provided to calculate the Fourier series of these modular forms. However, these formulas involve numerous tricky integrals and are quite difficult to evaluate in most cases. In addition, it is not always clear what the levels of these modular forms are. [KY10] works to remedy this by showing how to extract explicit Fourier expansions in the classical case of $K = \mathbb{Q}$ with modular forms defined on $\mathcal{H}$. Their arguments also give some access to the level group.

Of particular note is that [KY10] not only implies that Zagier's modular form

[1]Although the construction allows for general Hilbert-Siegel modular forms, we will only work in the particular case of standard Hilbert modular forms. So, we will not make further reference to Siegel modular forms.

$Z(\tau)$ arises as such an Eisenstein series, but their proposition 6.5 tells us that for $\mu \in \{0, 1/2\}$ there are Eisenstein series called $E(\tau, 1/2, \Phi^{3/2,\mu})$ (or $E^{3/2,\mu}(\tau, 1/2)$ for short) such that the Fourier expansions of $-12E^{3/2,\mu}(4\tau, 1/2)$ consist of exactly the even m and odd m terms of $Z(\tau)$ for $\mu = 0, 1/2$ respectively. In particular, this implies that $-12(E^{3/2,0}(4\tau, 1/2) + E^{3/2,1/2}(4\tau, 1/2)) = Z(\tau)$. It is not surprising that the quantities $E^{3/2,\mu}(\tau, 1/2)$ are modular forms (very general arguments tell us that if one takes any subset of the terms of the Fourier series in arithmetic progression, we get another modular form). What is interesting is that these forms arise from very nice choices of the parameters s, Φ and this realization of them as Eisenstein series allows one to determine levels for them.

We now move onto the problem we are trying to solve. Fix a totally real number field K of degree $n > 1$. Our goal is to show that certain quantities showing up in Mizumoto's trace formula ([Miz84], Theorem 3) are the coefficients of a modular form. The quantities under consideration will be labeled $H(m)$ for $m \in O_K$. $H(m) = 0$ unless $m \gg 0$ (m totally positive) and $-m$ is a square mod $4O_K$. If these conditions hold, then $H(m)$ is given by

$$H(m) = \frac{2h(K[\sqrt{-m}])}{w(K[\sqrt{-m}])}\rho_h(K[\sqrt{-m}]), \text{ with}$$
$$\rho_h(K[\sqrt{-m}]) = \sum_{\mathfrak{c}|\mathfrak{f}} N(\mathfrak{c}) \prod_{\mathfrak{p}|\mathfrak{c}} (1 - \left(\frac{-m}{\mathfrak{p}}\right) N(\mathfrak{p})^{-1}) \quad (1.1)$$

Here, h is the class number and w is the number of roots of unity. We will call $\rho_h(K[\sqrt{-m}])$ a "Hurwitzification factor" in analogy with before. Under our conditions on $H(m)$ being nonzero, there is an integral ideal $\mathfrak{f}$ obeying $-m = Disc(K[\sqrt{-m}]/K)\mathfrak{f}^2$, where $Disc(L/K)$ denotes the relative discriminant. $\mathfrak{c}$ iterates over all integral divisors of $\mathfrak{f}$, and $\mathfrak{p}$ iterates over prime divisors of $\mathfrak{c}$. Finally, $\left(\frac{-m}{\mathfrak{p}}\right)$ is the quadratic character associated to the extension $K[\sqrt{-m}]$ and is given explicitly by

$$\left(\frac{-m}{\mathfrak{p}}\right) = \begin{cases} 0 & \mathfrak{p} \text{ ramifies in } K[\sqrt{-m}] \\ 1 & \mathfrak{p} \text{ splits} \\ -1 & \mathfrak{p} \text{ is inert} \end{cases}$$

As is heavily implied, the quantity $H(m)$ reduces to the Hurwitz class numbers if we allow $K = \mathbb{Q}$, and for this reason we call $H(m)$ a generalized Hurwitz class number. Our goal is to prove the following result.

Theorem 1.2. *Given $\mu \in \frac{1}{2}O_K/O_K$, there is a Hilbert modular form of parallel weight 3/2 whose Fourier coefficients are nearly the generalized Hurwitz class numbers. It is one of the Eisenstein series described in [KY10] and is given by*

$$E^{3/2,\mu}(4\vec{\tau}, 1/2) = \mathbb{1}_{O_K}(\mu) + \frac{1}{\zeta_K(-1)} \frac{2^{n-1}}{h_K} \sum_{\substack{m' \gg 0 \\ m' \in -(2\mu)^2 + 4O_K}} \frac{H(m')}{Q_{K[\sqrt{-m'}]}} e^{2\pi i \vec{m'} \cdot \vec{\tau}}$$

where $\mathbb{1}_{O_K}(\mu)$ is an indicator function for μ to be integral and $Q_{K[\sqrt{-m'}]} \in \{1, 2\}$ is given in terms of regulators by $2^{n-1}/Q_{K[\sqrt{-m'}]} = Reg(K[\sqrt{-m'}])/Reg(K)$.[2]

There are several points worth note about the above formula. The first is that this is a generalization of the pair of modular forms in [KY10] proposition 6.5. There are 2^n elements of $\frac{1}{2}O_K/O_K$, and indeed we get 2^n distinct Eisenstein series whose Fourier coefficients are pairwise disjoint. Adding them all produces the Eisenstein series

$$E^{3/2}(4\vec{\tau}, 1/2) = 1 + \frac{1}{\zeta_K(-1)} \frac{2^{n-1}}{h_K} \sum_{m' \gg 0} \frac{H(m')}{Q_{K[\sqrt{-m'}]}} e^{2\pi i \vec{m'} \cdot \vec{\tau}}$$

This particular Fourier series is one of the series found in Theorem 10.3 of [Su16]. For all $m' > 0$, $Q_{\mathbb{Q}[\sqrt{-m'}]} = 1$, and so in the case $K = \mathbb{Q}$ the above formula reduces to (a constant multiple of) the holomorphic part of $Z(\tau)$.

This leads into the second thing of note, which is that $E^{3/2,\mu}$ is holomorphic for all $K \supsetneq \mathbb{Q}$. We are not choosing to drop any nonholomorphic terms that may naturally arise. Rather, whenever $K \supsetneq \mathbb{Q}$, the nonholomorphic terms that could show up in our Fourier series vanish. It is rather surprising that the formula becomes nicer in the general case, so we will briefly attempt to explain how this phenomenon comes about. One way to prove the modularity of Zagier's $Z(\tau)$ is to first cleverly pick the right

[2] We will give some more explicit formulas for $Q_{K[\sqrt{-m'}]}$ later.

Eisenstein series $E(\tau, s, \Phi)$. Being an Eisenstein series, it is easy to prove modularity. Then, one has to compute the Fourier expansion of $E(\tau, s, \Phi)$ and show that you get $Z(\tau)$. During this computation, it turns out that each of the nonholomorphic terms comes about from an indeterminate expression. Specifically, for each nonholomorphic term the simple pole of the Riemann zeta function at $s = 0$ ends up canceling with a zero from something called a local Whittaker function, giving rise to an incomplete gamma function. However, when we do similar computations in the case of $K \supsetneq \mathbb{Q}$ the analogous calculation ends up seeing the pole of the Dedekind zeta function ζ_K at 0 multiplied by a local Whittaker function for each Archimedean place of K. Each Whittaker function will have a simple zero, and so if K has degree n, the Whittaker functions will in total contribute a zero of order n. However, ζ_K has a simple pole and hence the indeterminate expression will evaluate to 0.

Finally, we note that our Fourier coefficients above originally come about as L functions which were rewriten in terms of $H(m)$ using the formula

$$\frac{2^{n-1}}{h_K} \frac{H(m)}{Q_{K[\sqrt{-m'}]}} = L\left(0, \left(\frac{-m}{\cdot}\right)\right) \rho_h(K[\sqrt{-m}])$$

In fact, it seems to be common practice to prefer using L functions to express Fourier coefficients of such series, since they generalize more readily than class numbers. Even more strongly, one may observe that the quantity $L\left(0, \left(\frac{-m}{\cdot}\right)\right) \rho_h(K[\sqrt{-m}])$ is itself a possible way to generalize the notion of Hurwitz class number, different from the one we chose. Of course, we have already given our reasons for preferring our choice of $H(m)$.

Finally, we summarize the steps we will follow to prove this result. We are mainly trying to follow the outline given in [KY10] to evaluate the Eisenstein series $E(\tau, s, \Phi)$ in the case of Hilbert modular forms. [KY10] is generally an outline for those who are already somewhat familiar with the concepts involved, so we will take the time to elaborate on all of the necessary details and attempt to collect together most of what one needs to understand the argument into one place. [KY10] also only performs the arguments in the case $K = \mathbb{Q}$, so we rely on [HI13], [Su16], and [KRY06] to help fill in details in the case of general K (although, again, when using [KRY06] we only use the case of Hilbert modular forms and not Hilbert-Siegel modular forms). Although most of the steps directly generalize form [KY10], due to difficulties in tracking down arguments for some steps as well as differing notation and different coordinate systems between different references, I have chosen to include proofs for as many arguments as possible. I hope this will help make the paper a somewhat self-contained collection of the relevant ideas.

The actual core of our argument in [KY10] is to build an Eisenstein series $E(\tau, s, \Phi)$ out of a sum of functions $\Phi(g', s)$ which are called sections and collectively form a certain representation $I(s, \chi)$ of a metaplectic group. As such, we start by discussing metaplectic groups and build up many of their relevant properties. We next discuss the Weil representation, which is a representation of the metaplectic group on a space of Schwartz functions. The Weil representation requires us to fix a quadratic space V. Spaces V of any dimension may be used, although the results we need will all come from dimension 1. The Weil representation then gives rise to the representation $I(s, \chi)$ through a map called λ. This gives us our functions Φ, and from them we get an Eisenstein series.

Depending on which functions $\Phi \in I(s, \chi)$ we choose, we can get a whole host of Eisenstein series. We will be interested in a particular series called $E^{l,\mu}$. This is in fact the desired series with Hurwitz class number Fourier coefficients. However, actually computing the Fourier series will be a grueling task, which boils down to computing certain integrals called local Whittaker functions.

This takes us to one of the main steps that does not easily generalize from [KY10]. The local Whittaker functions involve general quadratic Gauss sums over local fields. Although it is not strictly necessary to compute the Gauss sums in order to evaluate the local Whittaker functions in the case $dim(V) = 1$, we opt to evaluate them anyway for the sake of getting closer to being able to evaluate the case $dim(V) > 1$.

As for the Gauss sums themselves, let $K_\mathfrak{p}$ be a completion of K at an (even or odd) finite place. For an unramified additive character[3] $\psi' : K \to \mathbb{C}$, a quadratic character $\chi : O_{K_\mathfrak{p}}^\times \to \pm 1$ (extend χ to 0 outside of $O_{K_\mathfrak{p}}^\times$) and $a \in K^\times$, $b \in K$, we will compute closed forms for the Gauss sums

$$\gamma(a,b) = \int_{O_{K_\mathfrak{p}}} \psi(ax^2 + bx)dx, \quad \gamma(\chi, a) = \int_{O_{K_\mathfrak{p}}} \chi(x)\psi(ax)dx$$

which we call quadratic form and quadratic character Gauss sums, respectively. The most difficult case is a quadratic form Gauss sum at an even place, the formula for which is given in corollary 6.35. Although the two types of Gauss sum are effectively the same thing in odd residue characteristic, they are not as obviously related for residue characteristic 2. Additionally, since computing the Gauss sums requires building up a significant body of work, we choose to take the time to include proofs of some fun facts at little extra cost. Namely, we show a correspondence between quadratic form and quadratic character Gauss sums in the case of residue characteristic 2 that mimics the correspondence in odd residue characteristic in proposition 6.83. We also show that in residue characteristic 2, for $a \in (1/4)O_{K_\mathfrak{p}}^\times$, $b = 0$, the Gauss sum $\gamma(a, 0)$ is a "multiplicative character of second degree in a" (essentially a multiplicative version of a quadratic form) and classify it up to isomorphism in proposition 6.71. After computing the Gauss sums, we use them to compute the local Whittaker functions, which will in turn give us our desired Fourier series. As already alluded to, I have not managed to perform all of the computations in this paper in a way that works for $dim(V) > 1$. However, a theme throughout is that I will attempt to do these computations in as much generality as possible. This results in some propositions being stronger than needed, but allows us to move closer to the case of general V.

[3]That is, trivial on O_K but nontrivial on $\mathfrak{p}^{-1}$.

Chapter 2

Notation and Setup

2.1 Notation

We start with some notation and conventions that will recur throughout. Let $K \supsetneq \mathbb{Q}$ be a totally real number field of degree n, O_K be its ring of integers, and ∂ its different. For a finite place $\mathfrak{p}$ of K, let π be some fixed choice of uniformizer, p denote the integer prime it lies over, e be the ramification index, f be the inertial degree, and $q = p^f$ be the size of the residue field. If a finite prime $\mathfrak{p}$ lies over 2, we call it even. Otherwise we call it odd. Use $v_\pi(x)$ for the π-adic valuation of x. Although we will often use $\bar{z}$ to denote complex conjugation, sometimes (especially when it comes to valuations) we will prefer to let it denote reduction mod 2. That is, $\overline{v_\pi(x)} \in \{0, 1\}$ such that $\overline{v_\pi(x)} \equiv v_\pi(x) \bmod 2$. We will point out any such uses of the notation.

Let $K_\mathfrak{p}$ denote the completion at $\mathfrak{p}$. For any sort of global object x, we will use $x_\mathfrak{p}$ to denote its local version or local component at $\mathfrak{p}$. If the object comes with a subscript such as x_n, then we will write the local version as $x_{n,\mathfrak{p}}$. All notation in the first paragraph carries over to here as appropriate. Let $\partial_\mathfrak{p}$ denote the different of $K_\mathfrak{p}$. We will often use $r_\mathfrak{p}$ to denote the valuation of this local different. That is, let $\partial_\mathfrak{p} = (\pi^{r_\mathfrak{p}})$. We may omit the subscripts if it is clear we are working at some place $\mathfrak{p}$. Additionally, for the entirety of chapter 6 we will be focused on a fixed local field. As such, for just chapter 6, K will instead be used to denote one of these completions $K_\mathfrak{p}$.

Let $\mathbb{A}_K$ be the ring of adeles. Let $\psi : \mathbb{A}_K/K \to \mathbb{C}^\times$ be the standard additive character given by

$$\psi(x) = \prod_{\mathfrak{p}<\infty} e^{-2\pi i\{\mathrm{tr}(x_\mathfrak{p})\}} \prod_{\mathfrak{p}|\infty} e^{2\pi i x_\mathfrak{p}}, \tag{2.1}$$

where $\{\}$ denotes the fractional part, the trace is from $K_\mathfrak{p}$ to $\mathbb{Q}_p$, and the product is over all primes of K.

We also recall the definition of smoothness of a function f on $\mathbb{A}_K$. Write the function as $f(x, y)$ where x consists of all Archimedean places and y consists of all non-Archimedean places. Then f is smooth if it is infinitely differentiable with respect to x and is locally constant in y. That is, for each pair x_0, y_0 there exists an open set V of the finite adeles such that $y_0 \in V$ and $f(x_0, V) = f(x_0, y_0)$. Furthermore, we require uniformity in that our choice of V will be valid for all x in an open neighborhood of x_0.

Let $G = \mathrm{SL}_2$. For $G(K) = \mathrm{SL}_2(K)$, let $P = NM$ be the standard Borel subgroup given by

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in K \right\}, \quad M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in K^\times \right\}.$$

Obviously the definitions above also can define the groups G, P, N, M with entries in the adeles or at any completion $K_\mathfrak{p}$ as well. For a place $\mathfrak{p}$, we use $\mathcal{K}_\mathfrak{p}$ to denote a certain choice of maximal compact subgroup of $G(K_\mathfrak{p})$. We let $\mathcal{K}_\mathfrak{p} = SO_2(\mathbb{R})$ for infinite places and for finite places let

$$\mathcal{K}_\mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K_\mathfrak{p}) \middle| b \in \partial^{-1}, c \in \partial,\ a, d \in O_{K_\mathfrak{p}} \right\}$$

We use $\mathcal{K}_{0,\mathfrak{p}}(N)$ to denote the further subgroup whose bottom left entry is in $N\partial$. (This is in reference to the notation $\Gamma_0(N)$.) $\mathcal{K}$ is then used to denote the maximal compact subgroup of $G(\mathbb{A}_K)$ which is a product over all places of $\mathcal{K}_\mathfrak{p}$. On the other hand, $\mathcal{K}_0(N)$ will denote product of $\mathcal{K}_{0,\mathfrak{p}}(N)$ over all *finite* places.

Following [KY10], we will write

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This is the opposite of what is done in [KRY06], who take w to be the inverse of the above matrix. Some care must then be taken when comparing with their work.

We will denote a point in the n-fold product of upper halfplanes $\mathcal{H}^n$ as a vector $\vec{\tau}$. Furthermore, given a number $m \in K$, we will let $\vec{m} \in \mathbb{R}^n$ denote the image of m under the Minkowski embedding. The main purpose of this notation will be in computing the complex number $\vec{m} \cdot \vec{\tau}$, as an alternative to the more common notation $\mathrm{tr}(m\tau)$. For $\vec{\tau} = (u_1 + iv_1, u_2 + iv_2, \ldots) \in \mathcal{H}^n$, we let

$$g_{\vec{\tau}} = (n(u_1)m(\sqrt{v_1}), n(u_2)m(\sqrt{v_2}), \ldots) \in P(\mathbb{R})^n$$

which when used as a linear fractional transformation takes $\vec{i} = (i, i, \ldots)$ to $\vec{\tau}$. Further, for $\vec{\theta} = (\theta_1, \ldots) \in \mathbb{R}^d$ let $k(\vec{\theta}) = (k(\theta_1), k(\theta_2), \ldots)$, where $k(\theta)$ is a (clockwise!) rotation matrix by θ.

Similarly to the notation in [Su16], if we ever need to choose a branch cut for an exponential a^b with $a, b \in \mathbb{C}$, we will default to taking $a^b = e^{b\ln(a)}$ for $-\pi < Im(\ln(a)) \leq \pi$.

2.2 Modular Forms

We quickly recall the definition of a modular form, modular forms of half-integral weight, Hilbert modular forms, and then Hilbert modular forms of half-integral weight. See [Fre90] for details on Hilbert modular forms. See the start of [Shi87] and proposition 1.2 of [Shi85] for the definition of the half-integral weight case.[1]

Standard modular forms: Let $\Gamma \subset SL_2(\mathbb{Q}) \subset SL_2(\mathbb{R})$ be a subgroup commensurable with $SL_2(\mathbb{Z})$ (two groups are commensurable if their intersection has finite

[1]Shimura uses a slightly different normalization on his modular forms than the rest of our sources. In our chosen notation, the functions Shimura considers are of the form $f(\tau/2)$ where f is a modular form. This explains Shimura's slightly different level group.

index in each of them). Let k be an integer. A modular form of weight k level Γ is a function $f(\tau) : \mathcal{H} \to \mathbb{C}$ that is holomorphic, bounded as $Im(\tau) \to \infty$ and obeys the modularity condition

$$f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \ \tau \in \mathcal{H}$$

and $\gamma\tau$ is given by the Mobius transformation

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}$$

We use $j(\gamma, \tau) = (c\tau + d)$ to refer to the automorphy factor in the (weight 1) modularity condition.

Modular forms of half-integral weight: The theta function

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 \tau}, \ \tau \in \mathcal{H}$$

obeys a modularity condition of the form

$$\theta(\gamma\tau) = j_\theta(\gamma, \tau)\theta(\tau), \quad \gamma \in \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| 4|c \right\}$$

where j_θ is an automorphy factor satisfying the relation $j_\theta(\gamma, \tau)^4 = j(\gamma, \tau)^2$. This makes $\theta(\tau)$ into a modular form of level $\Gamma_0(4)$ and "weight 1/2", since we have $j_\theta(\gamma, \tau) = \pm\sqrt{\pm 1}\sqrt{c + d\tau}$. Note that we couldn't simply define $j_\theta(\gamma, \tau) = \sqrt{j(\gamma, \tau)}$ since this is not a factor of automorphy due to the signs not working out. For $\Gamma \subset \Gamma_0(4)$ and a (non-integral) half-integer k, a modular form of level Γ weight k is a function $f(\tau) : \mathcal{H} \to \mathbb{C}$ that is holomorphic, bounded as $Im(\tau) \to \infty$ and obeys the modularity condition

$$f(\gamma\tau) = j_\theta(\gamma, \tau)(c\tau + d)^{k-1/2} f(\tau), \quad \gamma \in \Gamma, \ \tau \in \mathcal{H}$$

Note that if one multiplies a modular form of weight k_1 by a form of weight k_2, one does not get a form of weight $k_1 + k_2$, since $j_\theta^2 \neq j$. An alternate (non-equivalent!) way that people sometimes define a modular form of weight k is by the transformation property

$$f(\gamma\tau) = j_\theta(\gamma, \tau)^{2k} f(\tau), \quad \gamma \in \Gamma, \ \tau \in \mathcal{H}$$

Each definition has its advantages, although the forms we will be building fit the latter of the two. Forms of weight 1/2 are the same in both definitions, so for those there can be no confusion. When we later need to specify that a form f is of weight 3/2 with respect to the second definition, we will instead say that $f * \theta$ is of weight 2. This allows one to unambiguously specify which conventions are being used.

Hilbert modular forms: Given our totally real number field K of degree $n > 1$, let $\iota_1 \dots \iota_n$ denote the embeddings of K into $\mathbb{R}$. Given $x \in K$, temporarily let $x_i = \iota_i(x)$. Then, we may embed $SL_2(K)$ as a subset of $SL_2(\mathbb{R})^n$ using the Minkowski embedding

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right)$$

Under this embedding, $SL_2(O_K)$ is a discrete subgroup. Let $\Gamma \subset SL_2(K) \subset SL_2(\mathbb{R})^n$ be a subgroup commensurable with $SL_2(O_K)$ (under our embedding). Let $\vec{k} \in \mathbb{Z}^n$. A Hilbert modular form of level Γ weight $\vec{k}$ is a function $f(\vec{\tau}) : \mathcal{H}^n \to \mathbb{C}$ that is holomorphic and obeys the modularity condition

$$f(\gamma\vec{\tau}) = \prod_{i=1}^{n} (c_i\tau_i + d_i)^{k_i} \ f(\vec{\tau}), \quad \gamma \in \Gamma, \ \vec{\tau} \in \mathcal{H}^n$$

and $\gamma\vec{\tau}$ is given by the component-wise Mobius transformation

$$(\gamma\vec{\tau})_i = \frac{a_i\tau_i + b_i}{c_i\tau_i + d_i}$$

We do not ask for boundedness as $Im(\vec{\tau}) \to \infty$, since this is implied by the other conditions when $n > 1$ by the Koecher principle. If all components of $\vec{k}$ are the same integer k, then we say the Hilbert modular form has parallel weight k.

Hilbert modular forms of half-integral weight: Given $x \in K$, we use $\vec{x}$ to denote the vector $(x_1, x_2, \dots x_n)$. Let ∂ be the different ideal of K. The theta function

$$\theta_K(\vec{\tau}) = \sum_{n \in O_K} e^{2\pi i \overrightarrow{n^2} \cdot \vec{\tau}}, \ \vec{\tau} \in \mathcal{H}^n$$

obeys a modularity condition of the form

$$\theta_K(\gamma\vec{\tau}) = j_\theta(\gamma,\vec{\tau})\theta_K(\vec{\tau}),$$
$$\gamma \in \Gamma[\partial^{-1}, 4\partial] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K) \;\middle|\; b \in \partial^{-1}, c \in 4\partial,\ a, d \in O_K \right\} \quad (2.2)$$

where j_θ satisfies the relation $j_\theta(\gamma,\tau)^4 = \prod_{i=1}^n (c_i\tau_i + d_i)^2$. This makes $\theta(\tau)$ into a modular form of level $\Gamma[\partial^{-1}, 4\partial]$ and "parallel weight 1/2". The appearance of the different ideal may be surprising, but things clear up quickly if one attempt any direct calculations.

Example 2.3. *Fix $t \in \partial^{-1}$ and let $\gamma_0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \Gamma[\partial^{-1}, 4\partial]$. Then we have $\gamma_0\vec{\tau} = (\tau_1 + t_1, \dots \tau_n + t_n)$. Thus,*

$$\theta_K(\gamma_0\vec{\tau}) = \sum_{n\in O_K} e^{2\pi i \overrightarrow{n^2}\cdot(\vec{\tau}+\vec{t})},\ \vec{\tau} \in \mathcal{H}^n$$

However, for $n \in O_K, t \in \partial^{-1}$ we have

$$e^{2\pi i \overrightarrow{n^2}\cdot\vec{t}} = e^{2\pi i Tr(n^2 t)} = 1$$

since the trace of any element of ∂^{-1} is an integer. So, we get $\theta_K(\gamma_0\vec{\tau}) = \theta_K(\vec{\tau})$.

For $\Gamma \subset \Gamma[\partial^{-1}, 4\partial]$ and a weight vector $\vec{k}$ all of whose components are (non-integral) half-integers, a modular form of level Γ weight $\vec{k}$ is a function $f(\vec{\tau}) : \mathcal{H}^n \to \mathbb{C}$ that is holomorphic and obeys the modularity condition

$$f(\gamma\vec{\tau}) = j_\theta(\gamma,\tau)\prod_{i=1}^n (c_i\tau_i + d_i)^{k_i - 1/2} f(\vec{\tau}), \quad \gamma \in \Gamma,\ \vec{\tau} \in \mathcal{H}^n$$

If all entries of $\vec{k}$ are the same half-integer k, we say f is of parallel weight k. However, we will instead prefer to use a different (again non-equivalent) definition. For this definition, we only have forms of parallel weight k. The transformation rule is

$$f(\gamma\vec{\tau}) = j_\theta(\gamma,\tau)^{2k} f(\vec{\tau}), \quad \gamma \in \Gamma,\ \vec{\tau} \in \mathcal{H}^n$$

Chapter 3

The Metaplectic Group

3.1 Motivation

We start by describing the classical treatment of the metaplectic group and why this group is relevant to our construction (the answer boiling down to the fact that our modular forms will have half-integral weight, although in some sense this only pushes the question back to one I do not know the answer to). We will not use this description of the metaplectic group in our arguments, but it still makes for decent motivation.

When we were discussing (non-Hilbert) modular forms of half-integral weight, we ran into the complication that we could not simply use $\sqrt{c\tau + d}$ as a weight 1/2 factor of automorphy since the signs don't work out. A rather simple way around this is instead of having the group $SL_2(\mathbb{R})$ acting on the upper half plane, one can instead consider the group of pairs

$$Mp_2(\mathbb{R}) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, f(z) \right), \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),\ f(z)^2 = c\tau + d$$

Here, $f(z)$ must be a holomorphic function on $\mathcal{H}$.

So, an element of this classical metaplectic group is just an element of SL_2 along with a choice of which sign to use on the automorphy factor. One could go further and simply think of it as the group of all possible automorphy factors. Composition in this group is given simply by composing the chosen automorphy factors. That is,

$$(g_1, f_1(z)) * (g_2, f_2(z)) = (g_1 g_2, f_1(g_2 z) f_2(z))$$

This group therefore gives a natural way of working with automorphy factors of weight 1/2 without needing to choose signs. Hence, the fact that we wish to create a modular form of weight 3/2 makes the appearance of the metaplectic group actually quite natural as a starting point. We close with two remarks.

First, it is clear that Mp_2 is a double cover of SL_2. Topologically, we have that $SL_2(\mathbb{R})$ is homeomorphic to $S^1 \times \mathbb{R}^2$. As such it admits a unique nontrivial topological double cover. This cover is Mp_2.

Secondly, we connect this group back to the automorphy factor coming from the θ function. To do this, consider the 4-fold cover of $SL_2(\mathbb{R})$

$$H = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, f(z) \right), \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),\ f(z)^4 = (c\tau + d)^2$$

Then, our choice of weight 1/2 automorphy factor $\tilde{j}$ actually gives us a section of the covering map taking $\Gamma_0(4) \mapsto H$. Indeed, the existence of such a section is equivalent to finding a way to "choose the signs" on $\sqrt{c\tau + d}$. It is worth noting that such a section does not exist on a larger subgroup than $\Gamma_0(4)$, which is why one will run into various divisibility by 4 criteria when looking at the level of a half-integral weight modular form.

3.2 Important functions

Fix a completion $K_{\mathfrak{p}}$ where $\mathfrak{p}$ is either finite or infinite. In order to define general metaplectic groups, we will need a collection of specialized functions, some more esoteric than others. We start by collecting a set of such functions related to the place $\mathfrak{p}$ and state some important properties of each.

3.2.1 The Hilbert Symbol

We start with the Hilbert symbol. The (quadratic) local Hilbert symbol is a commutative function $\langle \cdot, \cdot \rangle_{\mathfrak{p}} : K_{\mathfrak{p}}^{\times} \times K_{\mathfrak{p}}^{\times} \to \pm 1$ is given by

$$\langle a, b \rangle_{\mathfrak{p}} = \begin{cases} 1 & ax^2 + by^2 = z^2 \text{ admits a solution in } K_{\mathfrak{p}}^3 - \{(0,0,0)\} \\ -1 & \text{otherwise} \end{cases}$$

For example, if $K_{\mathfrak{p}} = \mathbb{R}$, then the Hilbert symbol is -1 iff both arguments are negative. In the case of a non-archimedean field, the Hilbert symbol is related to the quadratic residue symbol.

The Hilbert symbol vanishes when either input is a square, owing to the solution $(x, y, z) = (1/a, 0, 1)$ of $a^2x^2 + by^2 = z^2$. A less trivial property is that the Hilbert symbol is bimultiplicative, from which it follows that the Hilbert symbol is actually a function $\langle \cdot, \cdot \rangle_{\mathfrak{p}} : K_{\mathfrak{p}}^{\times}/K_{\mathfrak{p}}^{\times 2} \times K_{\mathfrak{p}}^{\times}/K_{\mathfrak{p}}^{\times 2} \to \pm 1$. It also obeys the identities

$$\langle a, 1-a \rangle_{\mathfrak{p}} = \langle a, -a \rangle_{\mathfrak{p}} = 1$$

Whenever $\mathfrak{p}$ is odd and $a, b \in O_{K_{\mathfrak{p}}}^{\times}$, we have $\langle a, b \rangle_{\mathfrak{p}} = 1$. As such, when $a, b \in \mathbb{A}_K^{\times}$ are members of the Idele group, we see that the local Hilbert symbols $\langle a, b \rangle_{\mathfrak{p}}$ will be 1 for all but finitely many places. This allows one to define the global Hilbert symbol

$$\langle a, b \rangle_{\mathbb{A}} := \prod_{\mathfrak{p} \leq \infty} \langle a, b \rangle_{\mathfrak{p}}$$

Finally, whenever $a, b \in K$, we have $\langle a, b \rangle_{\mathbb{A}} = 1$, which may be thought of as a product formula on the local Hilbert symbols.

3.2.2 The Weil Constant

Next we discuss the Weil constant, also known as the Weil (local) index. Much of the following discussion comes from sections 1 of [HI13] and [Su16]. Let $\mathcal{S}(K_{\mathfrak{p}})$ denote the set of Schwartz functions on $K_{\mathfrak{p}}$. If $\mathfrak{p}$ is archimedean, this is defined to be the set of smooth functions of fast decay. If $\mathfrak{p}$ is finite, this is defined to be locally constant functions of compact support. All of the following definitions will depend on our choice of additive character ψ, which for us is the standard additive character given in equation (2.1). In particular, the Weil constant depends on our choice of ψ, although we suppress this in the notation.

For each Schwartz function $\phi \in \mathcal{S}(K_\mathfrak{p})$, the Fourier transform $\hat{\phi}$ is given by

$$\hat{\phi}(x) = q^{-r/2} \int_{K_\mathfrak{p}} \phi(y)\psi(xy)dy$$

where the measure $q^{-r/2}dy$ is the self dual Haar measure for the Fourier transform. For each $a \in K_\mathfrak{p}^\times$, there is then a constant called the Weil constant, which we denote $\gamma_w(a)$. It is defined by and satisfies the identity

$$\int_{K_\mathfrak{p}} \phi(x)\psi(ax^2)dx = \gamma_w(a)|2a|^{-1/2} \int_{K_\mathfrak{p}} \hat{\phi}(x)\psi\left(-\frac{x^2}{4a}\right) dx \tag{3.1}$$

for any $\phi \in \mathcal{S}(K_\mathfrak{p})$.[1] If $K_\mathfrak{p} \neq \mathbb{C}$, $\gamma_w(a)$ is a non-constant in a. The Weil constant $\gamma_w(a)$ is an eighth root of unity which only depends on the class of a in $K_\mathfrak{p}^\times/K_\mathfrak{p}^{\times 2}$. Furthermore, for any $a, b \in K_\mathfrak{p}^\times$, $\gamma_w(a)/\gamma_w(b)$ is always a fourth root of unity. [2]

It satisfies $\gamma_w(-a) = \overline{\gamma_w(a)}$, and for any $a, b \in K_\mathfrak{p}^\times$ we have

$$\frac{\gamma_w(a)\gamma_w(b)}{\gamma_w(1)\gamma_w(ab)} = \langle a, b\rangle_\mathfrak{p} \tag{3.2}$$

If $K_\mathfrak{p}$ is non-Archimedean over an odd prime and $a \in O_{K_\mathfrak{p}}^\times$, then $\gamma_w(a) = 1$. Combined with equation (3.2), we see that for an odd place and $a, b \in O_{K_\mathfrak{p}}^\times$, we have $\langle a, b\rangle_\mathfrak{p} = 1$.

For $a \in K^\times$, the Weil constant obeys the product formula

$$\prod_{\mathfrak{p}\leq\infty} \gamma_w(a) = 1 \tag{3.3}$$

In the case of an infinite place, the formula for γ_w is well known. (See [HI13] section 7 for $b = 1$ and section 1 for how γ_w varies with b.)

[1] In [KRY06], the the notation for this Weil constant is $\gamma(\eta)$, where η is an additive character on $K_\mathfrak{p}$. The two notations compare via $\gamma_w(a) = \gamma(\psi(ax))$. In [KRY06], η is taken to be $\psi((1/2)x)$.

[2] This implies that, depending on the local field, $\gamma_w(a)$ is always a primitive eighth root of unity or it is always a fourth root of unity. The former case happens iff $\mathfrak{p}$ is even and $K_\mathfrak{p}$ is an odd degree extension of $\mathbb{Q}_2$.

Fact 3.4. *If $K_{\mathfrak{p}} = \mathbb{C}$, then $\gamma_w(a) = 1$.*

If $K_{\mathfrak{p}} = \mathbb{R}$ and one has chosen an additive character with $\psi_{L,\mathfrak{p}}(t) = e^{2\pi ibt}$ where $b \in K_{\mathfrak{p}}^{\times}$, then

$$\gamma_w(a) = e^{2\pi i \cdot sign(ba)/8}$$

(By our choice of ψ, we are in the case $b = 1$.)

The case of finite places is harder, but we can make a first step in this direction.

Lemma 3.5. *Let $\mathfrak{p}$ be finite. Let $\phi_0(t) \in \mathcal{S}(K_{\mathfrak{p}})$ denote the characteristic function of $O_{K_{\mathfrak{p}}}$. Then*

$$\hat{\phi}_0(t) = q^{-r/2}\phi_0(\pi^r t)$$

Proof. This follows immediately from the definitions. □

Lemma 3.6. *Let $\mathfrak{p}$ be finite. Let $e_2 = v_\pi(2)$. In other words, it is the ramification index if $\mathfrak{p}$ is even and 1 otherwise. Write $a = u_a * \pi^{v_\pi(a)}$, where u_a is a unit. Let $\overline{v_\pi(a)} \in \{0, 1\}$ denote the value mod 2. Then,*

$$\gamma_w(a) = q^{(e_2 + \overline{r + v_\pi(a)})/2} \int_{O_{K_{\mathfrak{p}}}} \psi\left(\frac{u_a}{\pi^{2e_2 + r + \overline{r + v_\pi(a)}}}\, x^2\right) dx \tag{3.7}$$

Proof. Since the Weil constant is insensitive to square factors, we have $\gamma_w(a) = \gamma_w\left(\frac{u_a}{\pi^{2e_2+r+\overline{r+v_\pi(a)}}}\right)$. Equation (3.1) with $\phi = \phi_0$ now tells us

$$\int_{O_{K_{\mathfrak{p}}}} \psi\left(\frac{u_a}{\pi^{2e_2+r+\overline{r+v_\pi(a)}}} x^2\right) dx =$$
$$\gamma_w\left(\frac{u_a}{\pi^{2e_2+r+\overline{r+v_\pi(a)}}}\right)\left|2\frac{u_a}{\pi^{2e_2+r+\overline{r+v_\pi(a)}}}\right|^{-1/2} q^{-r/2}\int_{\pi^{-r}O_{K_{\mathfrak{p}}}} \psi\left(-\frac{\pi^{2e_2+r+\overline{r+v_\pi(a)}}x^2}{4u_a}\right) dx \tag{3.8}$$

Since $\frac{\pi^{2e_2+r+\overline{r+v_\pi(a)}}}{4u_a} * \pi^{-2r} \in \partial_{\mathfrak{p}}^{-1}$, the right side integrand is identically 1 and the result follows. □

This identifies the Weil constant with a particular integral called a Gauss sum. Once we compute the Gauss sum later, we will have a formula for the Weil constant for finite $\mathfrak{p}$. For now, we will only be able to calculate the Weil constant in simple cases.

Remark 3.9. *The integrand in equation (3.7) only depends on the value of u_a mod $\pi^{2e_2+\overline{r+v_\pi(a)}}$. It follows that $\gamma_w(a)$ only cares about the value of u_a mod $\pi^{2e_2+\overline{r+v_\pi(a)}}$ (which we could upper bound as mod π^{2e_2+1}, which is mod π for odd places!) In particular, it follows that for any finite place $\mathfrak{p}$, $\gamma_w(a)$ is a continuous function on $K_\mathfrak{p}^\times$. (Continuity on $K_\mathfrak{p}^\times$ also follows for infinite places by fact 3.4.)*

Example 3.10. *Let $K = \mathbb{Q}$ and $\mathfrak{p} = 2$. At this place, recall $\psi(t) = e^{-2\pi i\{t\}}$ and take $\pi = 2$. We have $q = 2, e_2 = 1, r = 0$. We may calculate*

$$\gamma_w(1) = 2^{1/2} \int_{O_{K_\mathfrak{p}}} \psi\left(\frac{1}{4}\, x^2\right) dx$$

The value of x^2 only matters mod 4 in the above integral. We know that mod 4, x^2 is either 0 or 1 depending on if x is even or odd. Hence, we get

$$\gamma_w(1) = 2^{1/2}\left(\frac{1}{2}\psi(0) + \frac{1}{2}\psi\left(\frac{1}{4}\right)\right) = 2^{1/2}\left(\frac{1}{2}(1-i)\right) = e^{-2\pi i/8}$$

Remark 3.11. *The Weil constant $\gamma_w(a)$ is not defined for $a = 0$, but it will notationally convenient to pretend that $\gamma_w(0) = 1$. We will adopt this convention, although one needs to be very careful to note that this $\gamma_w(0)$ does not play well with any of the nice formulas or properties enjoyed by the Weil constant.*

3.2.3 Local Factors

These next facts about quadratic spaces and "local factors" composed from Weil constants are mainly from the appendix to [RR93] and section 8 of [KRY06]. There is also a commonly used relative variant of the Weil constant. Per p. 367 of [RR93], for $a \in K_\mathfrak{p}^\times$ let

$$\gamma_w(a, b) := \frac{\gamma_w(ab)}{\gamma_w(b)}$$

For our purposes, a quadratic space (V, Q) is a pair consisting of a $K_\mathfrak{p}$ vector space V and a nondegenerate quadratic form Q on V. Let $(x,y)_Q = Q(x+y) - Q(x) - Q(y)$ denote the associated bilinear form. We will often use L to denote a sublattice of V and L^* to denote the dual lattice under $(x,y)_Q$. Let $det(V) \in K_\mathfrak{p}^\times / K_\mathfrak{p}^{\times 2}$ denote the determinant of the matrix of this bilinear form when written in any basis. Also associated to V is a character $\chi_V(x) := \langle x, (-1)^{dim(V)*(dim(V)-1)/2} det(V) \rangle_\mathfrak{p}$ (see [KRY06] lemma 8.5.6). Nondegeneracy of Q will be important so that $det(V) \neq 0$, as well as the relation $L^{**} = L$.

We will only end up needing the case $dim(V) = 1$, but list the more complicated formulas to make it easier to find where they were taken from [KRY06].

There are a number of so called local factors associated to a quadratic space. First, we have the Hasse invariant. In the case that V possesses a diagonalizable quadratic form $Q \sim \sum a_i x_i^2$, this is defined to be

$$h_\mathfrak{p}(V) = \prod_{i<j} \langle a_i, a_j \rangle_\mathfrak{p}$$

and is independent of the diagonalization. We will not define the Hasse invariant for non-diagonalizable forms, but will mention that non-diagonalizability can only happen if $\mathfrak{p}|2$. For a full definition, see definition A.6 and lemma A.7 of [RR93]. Note that in the case that V is 1-dimensional, $h_\mathfrak{p}(V)$ is vacuously 1.

Following [KRY06] equation (8.5.21), we may now define the local factor[3]

$$\begin{aligned} \gamma\left(\psi\left(\frac{1}{2}t\right) \circ V\right) : &= \gamma_w\left(det(V), \frac{1}{2}\right) \gamma_w\left(\frac{1}{2}\right)^{dim(V)} h_\mathfrak{p}(V) \\ &= \gamma_w\left(\frac{1}{2} det(V)\right) \gamma_w\left(\frac{1}{2}\right)^{dim(V)-1} h_\mathfrak{p}(V) \end{aligned} \tag{3.12}$$

In the case $dim(V) = 1$, this is simply $\gamma_w\left(\frac{1}{2}det(V)\right)$. There is one more local factor

[3]In actuality, this local factor is defined to be a Weil constant which can be associated to the function $\psi\left(\frac{1}{2}Q\right)$. However, this formula will suffice for our purposes, so we take it as the definition.

that will show up later. It is given by[4]

$$\gamma(V) := \gamma_w\left(\frac{1}{2}\right)\gamma\left(\psi\left(\frac{1}{2}t\right)\circ V\right)^{-1} = \gamma_w\left(-\frac{1}{2}det(V)\right)\gamma_w\left(\frac{1}{2}\right)^{2-dim(V)} h_{\mathfrak{p}}(V)$$

Under the further assumption $Q(x) = (1/2)x^2$, we have $det(V) = 1$ in which case $\gamma(V) = 1$.

3.2.4 j and x

Finally, we quickly define the j and x functions. x is given at the top of page 364 of [RR93]. j can be deduced from the comment after [KRY06] equation (8.5.17) and [RR93] definition 5.2.

Given a matrix in $G(K_{\mathfrak{p}})$, let

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} 0 & c = 0 \\ 1 & c \neq 0 \end{cases}, \qquad x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & c \neq 0 \\ d & c = 0 \end{cases}$$

Any time we will be using $x(g)$, we will only care about its output up to a square. This makes the next lemma, which is 5.1 from [RR93] particularly useful.

Lemma 3.13.

$$x(p_1 g p_2)K_{\mathfrak{p}}^{\times 2} = x(p_1)x(g)x(p_2)K_{\mathfrak{p}}^{\times 2}, \quad p_1, p_2 \in P(K_{\mathfrak{p}}), \;\; g \in G(K_{\mathfrak{p}})$$

Proof. We show $x(p_1 g) = x(p_1)x(g)$ and then $x(gp_2)K_{\mathfrak{p}}^{\times 2} = x(g)x(p_2)K_{\mathfrak{p}}^{\times 2}$. The first formula follows immediately from

$$\begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} * & * \\ dc' & dd' \end{pmatrix}$$

While proving $x(gp_2)K_{\mathfrak{p}}^{\times 2} = x(g)x(p_2)K_{\mathfrak{p}}^{\times 2}$, we may assume that $g \notin P(K_{\mathfrak{p}})$, since then the result would follow from the first case. We observe

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} d^{-1}a' & * \\ d^{-1}c' & * \end{pmatrix}$$

Since $g \notin P(K_{\mathfrak{p}})$, we know $c' \neq 0$ and the result follows immediately. $\square$

[4] In [KRY06] equation (8.5.21) there is an additional factor written $\chi_V(-1)$. This extra factor is missing here due to the choice of matrix w we made when introducing notation.

3.3 The Local Metaplectic Group and Coordinate Change

The following is based off of [KRY06] section 8.5 and [HI13] section 1. Although not used directly in this section, other major references on this topic are [Gel06] and [W$^+$64].

Warning 3.14. *Everything in [KRY06] section 8.5 makes the assumption ψ is an unramified (kernel is $O_{K_\mathfrak{p}}$) character, which is contradictory to the choice of ψ we have made. This is needed for some of their results, although many of the results (including all of the results we need) do not depend on this assumption. This becomes clear since there are no unramifiedness assumptions being made in the corresponding propositions of their sources. As such, whenever we cite work from [KRY06] section 8.5 we will either provide proof or cite the source that they used.*

For two matrices $g_1, g_2 \in G(K_\mathfrak{p})$, we define a function on them called the Leray cocycle. In our case of $G = SL_2$, [KRY06] Example 8.5.1 based off of [RR93] Corollary 4.3 gives us the following formula, which we take as the definition.

Definition 3.15. *Let $g_3 = g_1 g_2$ and write $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. The Leray cocycle is then given by*

$$c_L(g_1, g_2) := \gamma_w \left(\frac{1}{2} c_1 c_2 c_3 \right)$$

If $c_1 c_2 c_3 = 0$, then the Leray cocycle is taken to be 1. c_L depends on our earlier choice of character ψ (via the Weil constant) although we suppress this from the notation.

From the definition, it is easy to see c_L is trivial on $P \times G$ and $G \times P$. We will later show c_L is non-trivial on $\mathcal{K} \times \mathcal{K}$.

We may use this cocycle to define the (local) metaplectic group $G'_{K_\mathfrak{p}}$. Let $\mathbb{T}$ denote the group of complex numbers of norm 1. As a set, define

$$G'_{K_\mathfrak{p}} = G(K_\mathfrak{p}) \times \mathbb{T}$$

We will denote an element in this set as $[g, z]_L$, where $g \in G(K_\mathfrak{p})$ and $z \in \mathbb{T}$. By [KRY06] example 8.5.1, the composition law

$$[g_1, z_1]_L[g_2, z_2]_L = [g_1g_2, z_1z_2c_L(g_1, g_2)]_L$$

turns $G'_{K_\mathfrak{p}}$ into a group. The subgroup $[1, \mathbb{T}]_L$ is the center of this group, making $G'_{K_\mathfrak{p}}$ a central extension of $G(K_\mathfrak{p})$ by $\mathbb{T}$. Given a subgroup $H \subset G(K_\mathfrak{p})$, we will use the notation H' to denote the subgroup of $G'_{K_\mathfrak{p}}$ consisting of elements with first coordinate in H. We will mainly use this to define the groups $P'(K_\mathfrak{p})$ and $\mathcal{K}'_\mathfrak{p}$.

Remark 3.16. *For any $g \in G(K_\mathfrak{p})$, we have $c_L(g, [I, z]_L) = 1$. It follows that for any $g' = [g, z]_L \in G'_{K_\mathfrak{p}}$ we have $[g, z_1]_L[I, z_2]_L = [g, z_1z_2]_L$.*

Example 3.17. *Let $K = \mathbb{Q}$ and $\mathfrak{p} = 2$. As an example of a non-trivial multiplication, we have*

$$\left[\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}, 1\right]\left[\begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}, i\right] = \left[\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}, i\gamma_w(64)\right]$$

Using example 3.10, we may choose to simplify $i\gamma_w(64) = i\gamma_w(1) = e^{2\pi i/8}$.

Example 3.18. *Let $\mathfrak{p}$ be Archimedean so that $K_\mathfrak{p} = \mathbb{R}$. Define*

$$\epsilon : [0, 4\pi) \to \mathbb{T} \qquad \epsilon(\theta) = \begin{cases} e^{\frac{n\pi i}{2}} & \theta = n\pi \quad (n \in \mathbb{Z}) \\ e^{\frac{(2n+1)\pi i}{4}} & n\pi < \theta < (n+1)\pi \end{cases}$$

For example, we are letting

$$\epsilon(\theta) = \begin{cases} 1 & \theta = 0 \\ e^{2\pi i/8} & 0 < \theta < \pi \\ e^{2\pi i/4} & \theta = \pi \\ e^{2\pi i * 3/8} & \pi < \theta < 2\pi \\ e^{\pi i} & \theta = 2\pi \\ \dots \end{cases}$$

The set of pairs of the form

$$[k(\theta), \epsilon(\theta)]_L = \left[\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \epsilon(\theta)\right]_L$$

is a subgroup of $G'_{K_\mathfrak{p}}$ *which is isomorphic to* $\widetilde{SO_2(\mathbb{R})}$*, the double cover of* $SO_2(\mathbb{R})$*. Rather than prove this is a subgroup, we will verify an example calculation. We check that*

$$[k_{\pi/2}, \epsilon(\pi/2)][k_{3\pi/4}, \epsilon(3\pi/4)] = [k_{5\pi/4}, \epsilon(5\pi/4)]$$

This is equivalent to checking

$$\epsilon(\pi/2)\epsilon(3\pi/4)\gamma_w\left(\frac{1}{2} * -1 * \frac{-1}{\sqrt{2}} * \frac{1}{\sqrt{2}}\right) = \epsilon(5\pi/4) \quad \leftrightarrow \quad e^{\frac{\pi i}{4}} e^{\frac{\pi i}{4}} e^{2\pi i/8} = e^{\frac{3\pi i}{4}}$$

We would now like to define a global metaplectic group on the set $G'_{\mathbb{A}_K} = G(\mathbb{A}_K) \times \mathbb{T}$. The naive thing we would like to do is to simply define a global cocycle $c_L : G(\mathbb{A}_K) \times G(\mathbb{A}_K) \to \mathbb{T}$ as the product

$$c_L(g_1, g_2) = \prod_{\mathfrak{p}} c_{L,\mathfrak{p}}(g_1, g_2) \tag{3.19}$$

of local cocycles acting on g_1 and g_2 at each place. From this we can then define $[g_1, z_1]_L[g_2, z_2]_L = [g_1 g_2, z_1 z_2 c_L(g_1, g_2)]_L$. Unfortunately, this does not quite work. In order for the global c_L to be well defined, for any input pair g_1, g_2 we need the local cocycles to evaluate to $+1$ at almost all places. However, it is quite easy to cook up examples where this fails.

Example 3.20. *Fix any odd prime* $\mathfrak{p}$*. Since* $\gamma_w(a)$ *is not a constant function, choose* $u \in K_\mathfrak{p}$ *so that* $\gamma_w(u) \neq 1$*. Let*

$$g_{1,\mathfrak{p}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad g_{2,\mathfrak{p}} = \begin{pmatrix} 1 & 0 \\ 2u\pi^2 & 1 \end{pmatrix} \quad g_{1,\mathfrak{p}} g_{2,\mathfrak{p}} = \begin{pmatrix} 1 & 0 \\ 1 + 2u\pi^2 & 1 \end{pmatrix}$$

Then, we can calculate

$$c_{L,\mathfrak{p}}(g_{1,\mathfrak{p}}, g_{2,\mathfrak{p}}) = \gamma_w\left(u\pi^2(1 + 2u\pi^2)\right) = \gamma_w\left(u(1 + 2u\pi^2)\right)$$

By remark 3.9, we know that the quantity $u(1 + 2u\pi^2)$ *only matters mod* π*. Hence, we get* $\gamma_w(u) \neq 1$*.*

Thus, we can choose $g_1, g_2 \in G(\mathbb{A}_K)$ *whose local components look like the above at each odd place. Then, there will be infinitely many local cocycles that evaluate to -1.*

There are a number of ways to get around this issue. [HI13] section 8 uses an alternate construction that only considers products of finitely many local cocycles and their associated double covers, and then builds $G'_{\mathbb{A}_K}$ as a direct limit. We will follow a more explicit construction from [KRY06] sections 8.5.1 and 8.5.5, where we salvage the naive approach. We do this by changing coordinates to a new system called normalized coordinates. In this new system we will have a new cocycle c_N which is trivial on $\mathcal{K}_\mathfrak{p} \times \mathcal{K}_\mathfrak{p}$. Since in equation (3.19), $g_1, g_2 \in \mathcal{K}_\mathfrak{p}$ for almost all places, it would follow that almost all of the local cocycles would be $+1$ and the product would be well defined. Let us start this process by explaining what we mean by changing coordinates.

Definition 3.21. *Given a continuous map of sets $\epsilon : G(K_\mathfrak{p}) \to \mathbb{T}$, we make the definition*

$$[g, z]_\epsilon := [g, z\epsilon(g)]_L$$

The pairs $[g, z]_\epsilon$ inherit a group law from $G'_{K_\mathfrak{p}}$, which can be calculated as

$$\begin{aligned}
[g_1, z_1]_\epsilon [g_2, z_2]_\epsilon = [g_1, z_1\epsilon(g_1)]_L [g_2, z_2\epsilon(g_2)]_L &= [g_1g_2, z_1z_2c_L(g_1, g_2)_L\epsilon(g_1)\epsilon(g_2)]_L \\
&= [g_1g_2, z_1z_2c_L(g_1, g_2)\epsilon(g_1)\epsilon(g_2)\epsilon(g_1g_2)^{-1}]_\epsilon
\end{aligned}$$

Thus, we could choose to instead work with pairs $[g, z]_\epsilon$ and define multiplication using the new (cohomologous) cocycle $c_\epsilon = c_L(g_1, g_2)\epsilon(g_1)\epsilon(g_2)\epsilon(g_1g_2)^{-1}$. This yields an isomorphic way of working with the metaplectic group and is what we mean by changing coordinates.

3.4 Rao Coordinates

An example of another coordinate system that sees heavy use called Rao coordinates. This is the preferred coordinate system of many of our sources, including [HI13] so we take a moment to list its properties for future use.

Definition 3.22. *Rao coordinates are given in terms of Leray coordinates by [KRY06] equation (8.5.17). This is based off of [RR93] Section 5. We have*

$$[g, z]_R = [g, z\beta(g)]_L$$

where

$$\beta(g) = \gamma_w\left(x(g), 1/2\right)^{-1} \gamma_w(1/2)^{-j(g)}$$

The associated cocycle c_R is called the Rao cocycle. It is given by[5]

$$c_R(g_1, g_2) = \left\langle \frac{x(g_1)}{x(g_1 g_2)}, \frac{x(g_2)}{x(g_1 g_2)} \right\rangle_{\mathfrak{p}}$$

As examples, the change of coordinates formula tells us that

$$[I, z]_R = [I, z]_L, \quad [I, z]_R = [I, z\gamma_W(-1, 1/2)^{-1}]_L$$

Furthermore, for any $g \in G(K_{\mathfrak{p}})$, $c_R(g, I) = 1$ so we still have the relation $[g, z_1]_R[I, z_2]_R = [g, z_1 z_2]_R$, similarly to remark 3.16.

Remark 3.23. *A remarkable property of the Rao cocycle is that it is valued in ± 1. Hence, one actually has a subgroup $G(K_{\mathfrak{p}}) \times \pm 1 \subset G(K_{\mathfrak{p}}) \times \mathbb{T}$. This subgroup is a non-trivial double cover of $G(K_{\mathfrak{p}})$, which is also referred to in many contexts as the metaplectic group. Of course, in other coordinate systems, this double cover will not look as nice (see example 3.18). There will even be a double cover buried inside the global metaplectic group once we build it.*

Finally, we note that unlike Leray coordinates, Rao coordinates fail to be trivial on $P \times G$.

[5] Rao gives a formula for c_R when $G = SL_2$ in a remark on page 364, where he notes that in the case of $G = SL_2$, his cocycle is equal to Kubota's cocycle. The formula we have listed is a rearrangement of Rao's formula from [HI13] section 1.

3.5 The Weil Representation

In order to create the normalized coordinate system, we will need to choose an appropriate ϵ in definition 3.21. This will be done using an important tool - the Weil representation. The following is [KRY06] lemma 8.5.6, which gives an explicit formula for how $g' = [g, z]_L$ acts in this representation, and which we will take as its definition. This lemma is based off of Proposition 4.3 of [Kud96], which gives the action of an element $g' = [g, z]_R$ instead. However, the only difference in the formulas is a slightly different Weil constant in front. It is trivial to check that they become the same upon changing coordinates.

Definition 3.24. *Given a quadratic space (V, Q) we have an associated Weil representation ω_V of $G'_{K_\mathfrak{p}}$ acting on the space of Schwartz functions $\mathcal{S}(V)$. For a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\phi(t) \in \mathcal{S}(V)$ it is given by*

$$\omega_V([g, z]_L)\phi(t) = \chi_V(x(g)) \left(z\gamma_w \left(\frac{1}{2}\right)^{j(g)} \right)^{\overline{dim(V)}} \gamma \left(\psi \left(\frac{1}{2} t\right) \circ V \right)^{-j(g)} r_V(g)\phi(t)$$

where

$$r_V(g)\phi(t) = \int_{y \in cV} \psi \left(\frac{1}{2}(at, bt)_Q + (bt, cy)_Q + \frac{1}{2}(cy, dy)_Q \right) \phi\left(at + cy\right) d_g y$$

and $\overline{dim(V)} \in \{0, 1\}$ denotes the value mod 2. Here, we should interpret the domain as

$$cV = \begin{cases} V & c \neq 0 \\ 0 & c = 0 \end{cases}$$

and $d_g y$ is the unique Haar measure on cV that makes $r_V(g)$ unitary under the inner product

$$(\phi_1, \phi_2) = \int_V \phi_1(t)\overline{\phi_2(t)} dt$$

(In particular, this asserts that such a Haar measure exists.)

We are interested in the case $dim(V) = 1$, in which case the local factor out front simplifies slightly to

$$\omega_V([g,z]_L)\phi(t) = z\langle x(g), det(V)\rangle_{\mathfrak{p}}\, \gamma_w \left(\frac{1}{2}\right)^{j(g)} \gamma_w \left(\frac{1}{2}det(V)\right)^{-j(g)} r_V(g)\phi(t)$$

Finally, we remark that whenever $dim(V)$ is odd, the Weil representation obeys $\omega_V([g,z]_L)\phi(t) = z\omega_V([g,1]_L)\phi(t)$. This property is often referred to by saying that the representation is genuine.

3.6 Normalized Coordinates for Odd Primes

In this section, we will only consider the case of an odd prime $\mathfrak{p}$.

Consider the Weil representation associated to the quadratic space $(V = K_{\mathfrak{p}}, Q(x) = (1/2)x^2)$. We have $(x,y)_Q = xy$ and $det(V) = 1$. In this case, the local factors out front disappear and the Weil representation is given by

$$\omega_V([g,z]_L)\phi(t) = zr_V([g,z]_L)\phi(t) = z\int_{y\in cK_{\mathfrak{p}}} \psi\left(\frac{1}{2}abt^2 + bcty + \frac{1}{2}cdy^2\right)\phi\left(at+cy\right) d_g y$$

It will be helpful to consider this representation on generating set of $G'_{K_{\mathfrak{p}}}$ where its behavior is easier to understand. Let

$$\mathbf{n}(b) = \left[\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}, 1\right]_L \quad (b\in K_{\mathfrak{p}}), \qquad \mathbf{m}(a) = \left[\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}, 1\right]_L \quad (a\in K_{\mathfrak{p}}^{\times}),$$
$$\mathbf{w}_a = \left[\begin{pmatrix}0 & -a^{-1}\\ a & 0\end{pmatrix}, 1\right]_L$$

where the boldface indicates we are forming a matrix in the metaplectic group.

Proposition 3.25. *The Weil representation ω_V acts on a Schwartz function $\phi(t) \in \mathcal{S}(K_{\mathfrak{p}})$ via*

$$\omega_V([1,z]_L)\phi(t) = z\phi(t)$$
$$\omega_V(\boldsymbol{m}(a))\phi(t) = |a|^{1/2}\phi(at)$$
$$\omega_V(\boldsymbol{n}(b))\phi(t) = \psi\left(\frac{1}{2}bt^2\right)\phi(t)$$
$$\omega_V(\boldsymbol{w}_a)\phi(t) = |a|^{-1/2}\hat{\phi}(-a^{-1}t)$$

Proof. We will verify the second formula and leave the rest to the reader. The remaining three formulas are just as easy and follow by the same argument we are about to make. For the matrix $\mathbf{m}(a)$, we know

$$\omega_V(\mathbf{m}(a))\phi(t) = \int_{y\in\{0\}} \psi(0)\,\phi(at)\,d_g y = \phi(at)\mu(g)$$

where $\mu(g)$ is some unknown positive real number coming from the Haar measure. To disambiguate $\mu(g)$, we will use ϕ_0, which is the characteristic function of $O_{K_\mathfrak{p}}$. In order for r_V to be unitary, we must have

$$(r_V(\mathbf{m}(a))\phi_0, r_V(\mathbf{m}(a))\phi_0) = (\phi_0, \phi_0)$$

Which is the same as

$$\int_{K_\mathfrak{p}} \mu(g)^2\phi_0(at)^2 dt = \int_{K_\mathfrak{p}} \phi_0(t)^2 dt$$

Since ϕ is a characteristic function, we have

$$\mu(g)^2 \int_{a^{-1}O_{K_\mathfrak{p}}} dt = \int_{O_{K_\mathfrak{p}}} dt$$

from which we get $\mu(g)^2|a|^{-1} = 1$, concluding the proof of this case. □

The following proposition is commented on p. 322 of [KRY06].

Proposition 3.26. *Continue taking $\mathfrak{p}$ to be an odd prime and letting $\phi_0 \in \mathcal{S}(K'_\mathfrak{p})$ denote the characteristic function of $O_{K_\mathfrak{p}}$. Then there is a character $\tilde{\epsilon} : \mathcal{K}_\mathfrak{p} \to \mathbb{T}$ such that*

$$w(g')\phi_0(t) = \tilde{\epsilon}^{-1}(g')\phi_0(t) \tag{3.27}$$

for all $g' \in \mathcal{K}'_\mathfrak{p}$. $\tilde{\epsilon}^{-1}$ is genuine, which means that it obeys

$$\tilde{\epsilon}^{-1}([g,z]_L) = z\tilde{\epsilon}^{-1}([g,1]_L)$$

$\tilde{\epsilon}$ is given explicitly in all cases by

$$\tilde{\epsilon}_0([g,z]) = \begin{cases} \bar{z} & c\pi^{-r} \text{ is a unit} \\ \bar{z}\gamma_w(-2cd) & d \text{ is a unit} \end{cases} \tag{3.28}$$

Recall that by our convention $\gamma_w(0) = 1$. If $c\pi^{-r}, d$ are both units, then both formulas hold.

Proof. To show that ϕ_0 is an eigenfunction, we observe that for matrices in $\mathcal{K}_{\mathfrak{p}}$ we have

$$\omega_V(\mathbf{m}(a))\phi_0 = \omega_V(\mathbf{n}(b))\phi_0 = \omega_V(\mathbf{w}_{\pi^r})\phi_0 = \phi_0$$

We again check one of the cases and leave the rest to the reader. If $\mathbf{m}(a) \in \mathcal{K}'_{\mathfrak{p}}$, then a must be a unit, and so $|a| = 1$. The result follows. The other two cases are just as easy.

Since the above matrices along with $[1, z]_L$ generate $\mathcal{K}'_{\mathfrak{p}}$, it follows that ϕ_0 is an eigenfunction under all of $\mathcal{K}'_{\mathfrak{p}}$. Although ϕ_0 is fixed by so many elements, the eigenvalues will tend to be nontrivial due to the presence of the cocycle c_L.

Now that we know $\tilde{\epsilon}^{-1}$ is well-defined and is a character, the claim that $\tilde{\epsilon}^{-1}$ is genuine follows directly from the genuineness of the Weil representation.

Now we calculate $\tilde{\epsilon}^{-1}$, from which the formula for $\tilde{\epsilon}$ will follow. Since it is genuine, it suffices to verify our formula only for $g' = [g, 1]$. First we calculate $\tilde{\epsilon}^{-1}$ on $P'(K_{\mathfrak{p}}) \cap \mathcal{K}'_{\mathfrak{p}}$. Write

$$\left[\begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}, 1\right]_L = \left[\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}, 1\right]_L \left[\begin{pmatrix} 1 & bd \\ 0 & 1 \end{pmatrix}, 1\right]_L$$

and so we get

$$\omega\left(\left[\begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}, 1\right]_L\right)\phi_0 = \phi_0$$

Now we proceed in two cases. First, consider the case that $c\pi^{-r}$ is a unit. We compute the factorization of matrices (where at each step we always factor the rightmost matrix)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c^{-1}\pi^r & a\pi^{-r} \\ 0 & c\pi^{-r} \end{pmatrix} \begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & dc^{-1}\pi^r \end{pmatrix} =$$
$$\begin{pmatrix} c^{-1}\pi^r & a\pi^{-r} \\ 0 & c\pi^{-r} \end{pmatrix} \begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & 0 \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix} = g_1 g_2 g_3$$

It is easy to check that that as we perform each product, the Leray cocycle is always trivial. Therefore,

$$\omega\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 1\right]_L\right)\phi_0 = \omega\left([g_1, 1]_L\right)\omega\left([g_2, 1]_L\right)\omega\left([g_3, 1]_L\right)\phi_0$$

However, we know that all three matrices on the right side act trivially on ϕ_0, which concludes this case.

For the second case, assume that d is a unit. Then we use the factorization (where at each step we always factor the rightmost matrix)

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} = \begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & 0 \end{pmatrix}\begin{pmatrix} d^{-1}c\pi^{-r} & \pi^{-r} \\ -\pi^r & 0 \end{pmatrix} \\ &= \begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & 0 \end{pmatrix}\begin{pmatrix} 1 & -d^{-1}c\pi^{-2r} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \pi^{-r} \\ -\pi^r & 0 \end{pmatrix} = g_1g_2g_3g_4\end{aligned}$$

Out of the three factorizations we did, the Leray cocycles associated to the first and third are trivial due to the upper-triangular matrix involved. However, for the second factorization

$$\begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & 0 \end{pmatrix}\begin{pmatrix} d^{-1}c\pi^{-r} & \pi^{-r} \\ -\pi^r & 0 \end{pmatrix}$$

the associated Leray cocycle is $\gamma_w\left(-\frac{1}{2}d^{-1}c\right) = \gamma_w(-2cd)$ if $c \neq 0$ and 1 if $c = 0$. By our choice of $\gamma_w(0) = 1$ we may treat these as the same case. We get

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \gamma_w\left(-2cd\right)\right]_L = [g_1, 1]_L\,[g_2, 1]_L\,[g_3, 1]_L\,[g_4, 1]_L$$

All the matrices on the right side act trivially on ϕ_0. Hence, we get

$$w\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 1\right]_L\right)\phi_0 = \gamma_w\left(2cd\right)\phi_0$$

and so

$$\epsilon^{-1}([g, 1]_L) = \gamma_w\left(2cd\right)$$

$\square$

We can use the character $\tilde{\epsilon}$ to define a function ϵ on $\mathcal{K}_\mathfrak{p}$ simply by

$$\epsilon(g) := \tilde{\epsilon}([g, 1]_L) = \begin{cases} 1 & c \text{ is a unit} \\ \gamma_w\left(-2cd\right) & d \text{ is a unit} \end{cases} \tag{3.29}$$

Lemma 3.30. *Let $\mathfrak{p}$ be odd. For all $g_1, g_2 \in \mathcal{K}_\mathfrak{p}$,*

$$c_L(g_1, g_2)\epsilon(g_1)\epsilon(g_2)\epsilon(g_1g_2)^{-1} = 1$$

Proof. We know

$$\omega([g_1, 1]_L)\omega([g_2, 1]_L)\phi_0 = \omega([g_1g_2, c_L(g_1, g_2)]_L)\phi_0$$

From the definition of ϵ, this becomes

$$\epsilon(g_1)^{-1}\epsilon(g_2)^{-1}\phi_0 = \epsilon(g_1g_2)^{-1}c_L(g_1, g_2)\phi_0$$

and the result follows. □

As our notation suggests, our choice of ϵ for use in definition 3.21 will be the above function. The only remaining issue is that our ϵ is only defined on the set $\mathcal{K}_\mathfrak{p}$ and not all of $G(K_\mathfrak{p})$. This can be easily remedied using [KRY06] equation (8.5.10), which we do, which is the below proposition 3.31. First, we can see from the definition that ϵ is trivial on $P(K_\mathfrak{p}) \cap \mathcal{K}_\mathfrak{p}$, since $c = 0$ for such matrices.

Even more strongly, ϵ is left invariant to $P(K_\mathfrak{p}) \cap \mathcal{K}_\mathfrak{p}$. Let $p \in P(K_\mathfrak{p}) \cap \mathcal{K}_\mathfrak{p}$ and $g \in \mathcal{K}_\mathfrak{p}$. By lemma 3.30, we have

$$\epsilon(pg) = \epsilon(p)\epsilon(g)c_L(p, g) = \epsilon(g)$$

This suggests a way to extend ϵ to all of $G(K_\mathfrak{p})$.

Proposition 3.31. *For any $g \in G(K_\mathfrak{p})$, write $g = pk$ for $p \in P(K_\mathfrak{p})$ and $k \in \mathcal{K}_\mathfrak{p}$. Then, we may extend the definition of ϵ by setting*

$$\epsilon(g) := \epsilon(k)$$

This definition is well defined in that it doesn't depend on the decomposition $g = pk$.

Proof. The proof is a standard argument. If $g = p_1k_1 = p_2k_2$, then set $h = p_1^{-1}p_2 = k_1k_2^{-1}$. Then $h \in P(K_\mathfrak{p}) \cap \mathcal{K}_\mathfrak{p}$ and so $\epsilon(k_2) = \epsilon(h^{-1}k_1) = \epsilon(k_1)$. □

We now define normalized coordinates as in [KRY06] equation (8.5.11) and then discuss some properties they mention right after.

Definition 3.32. *For any odd place* $\mathfrak{p}$*, define normalized coordinates using the extension of* ϵ *above. Set*

$$[g, z]_N = [g, z\epsilon(g)]_L$$

with cocycle

$$c_N = c_L(g_1, g_2)\epsilon(g_1)\epsilon(g_2)\epsilon(g_1g_2)^{-1}$$

Lemma 3.33. c_N *is trivial on the sets* $\mathcal{K}_\mathfrak{p} \times \mathcal{K}_\mathfrak{p}$*,* $P(K_\mathfrak{p}) \times P(K_\mathfrak{p})$*, and* $P(K_\mathfrak{p}) \times \mathcal{K}_\mathfrak{p}$*.*

Proof. The first statement is lemma 3.30. The second follows since c_L and ϵ are both trivial on $P(K_\mathfrak{p})$. For the third, we calculate

$$c_N(p, k) = c_L(p, k)\epsilon(p)\epsilon(k)\epsilon(pk)^{-1} = \epsilon(k)\epsilon(k)^{-1} = 1$$

□

Corollary 3.34. *We may identify* $P(K_\mathfrak{p})$ *as a subgroup of* $P'_{K_\mathfrak{p}}$ *via the splitting map* $p \mapsto [p, 1]_N$*. This gives a group isomorphism* $P(K_\mathfrak{p}) \times \mathbb{T} \to P'_{K_\mathfrak{p}}$ *given by* $(p, z) \mapsto [p, z]_N$*, where the group on the left is the product group.*

Similarly, $k \mapsto [k, 1]_N$ *gives us a splitting map* $\mathcal{K}_\mathfrak{p} \to \mathcal{K}'_\mathfrak{p}$ *and* $\mathcal{K}_\mathfrak{p} \times \mathbb{T} \cong \mathcal{K}'_\mathfrak{p}$ *as groups.*

In particular, if $g \in P(K_\mathfrak{p}) \cap \mathcal{K}_\mathfrak{p}$*, then vanishing of* ϵ *on* $P(K_\mathfrak{p})$ *implies that under either splitting map we have* $g \mapsto [g, 1]_N = [g, 1]_L$*.*

We will later use this corollary to think of $P(K_\mathfrak{p})$ and $\mathcal{K}_\mathfrak{p}$ as subgroups of $G'_{K_\mathfrak{p}}$.

3.7 The Even Case

Rather than jump straight into defining the global metaplectic group, we will first perform an analogue of the above computations for even places. This is based on [KRY06] section 8.5.4. Throughout this section, $\mathfrak{p}$ will always denote an even prime.

When working with even primes, we will use the Weil representation associated to the quadratic space ($V = K_\mathfrak{p}, Q(x) = x^2$). This has $(x, y)_Q = 2xy$ and $det(V) = 2$. In this case, the Weil representation is given by

$$\omega_V([g, z]_L)\phi(t) = z\langle x(g), 2\rangle_\mathfrak{p}\, \gamma_w \left(\frac{1}{2}\right)^{j(g)} \gamma_w\,(1)^{-j(g)}\, r_V(g)\phi(t)$$

with

$$r_V(g)\phi(t) = \int_{y\in cK_\mathfrak{p}} \psi\left(abt^2 + 2bcty + cdy^2\right)\phi\left(at + cy\right) d_g y$$

Recall $\mathcal{K}_{0,\mathfrak{p}}(4) \subset \mathcal{K}_\mathfrak{p}$ is the subset of $\mathcal{K}_\mathfrak{p}$ with $c \in 4\partial$. Our goal is to show that although ϕ_0 is not an eigenfunction under all of $\mathcal{K}_\mathfrak{p}$, it is an eigenfunction under $\mathcal{K}_{0,\mathfrak{p}}(4)$. To this end, we will again look at the Weil representation on a generating set in order to get cleaner descriptions. In addition to the previous matrices we defined, let

$$n_-(c) := \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \qquad \mathbf{n}_-(c) := \left[\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, 1\right]_L$$

Proposition 3.35. *For the matrices in this proposition, assume that $\boldsymbol{m}(a), \boldsymbol{n}(b), \boldsymbol{n}_-(c) \in \mathcal{K}_{0,\mathfrak{p}}(4)$. Then, the Weil representation satisfies the following 6 statements, where the first three are for arbitrary $\phi(t) \in \mathcal{S}(K_\mathfrak{p})$ and the last three are related to $\phi_0(t)$ specifically.*

$$\omega_V([1, z]_L)\phi(t) = z\phi(t)$$

$$\omega_V(\boldsymbol{m}(a))\phi(t) = \langle a, 2\rangle_\mathfrak{p}\phi(at)$$

$$\omega_V(\boldsymbol{n}(b))\phi(t) = \psi(bt^2)\phi(t)$$

$$\omega_V(\boldsymbol{w}_{\pm\pi^r})\phi_0(t) = |2|^{1/2}\gamma_w\left(\frac{1}{2}\right)\gamma_w\,(1)^{-1}\,\phi_0\,(2t)$$

$$\omega_V(\boldsymbol{w}_{\pm\pi^r})\phi_0\,(2t) = |2|^{-1/2}\gamma_w\left(\frac{1}{2}\right)\gamma_w\,(1)^{-1}\,\phi_0(t)$$

$$\omega_V(\boldsymbol{n}_-(c))\phi_0(t) = \gamma_w(2c)\phi_0(t)$$

Also note that if we specialize statements 2 and 3 to the case $\phi(t) = \phi_0(t)$, the outputs become $\langle a, 2\rangle_\mathfrak{p}\phi_0(t)$ and $\phi_0(t)$, respectively.

Proof. The first statement is trivial and is only included for completeness. We will prove the second, fourth, and sixth statements. The omitted arguments are basically carbon copies of the ones we give.

For the second statement, our assumption $\mathbf{m}(a) \in \mathcal{K}_{0,\mathfrak{p}}(4)$ implies that a is a unit. We have $j(m(a)) = 0$, so we get

$$\omega_V(\mathbf{m}(a))\phi(t) = \langle a, 2\rangle_{\mathfrak{p}} \int_{y\in\{0\}} \psi(0)\,\phi(at)\,d_g y = \langle a, 2\rangle_{\mathfrak{p}}\phi(at)\mu(g)$$

for some positive real $\mu(g)$ depending on the Haar measure. We now compute $\mu(g)$ using the same exact method as before. If $r_V(g)$ is to be unitary, then in particular we must have

$$(r_V(\mathbf{m}(a))\phi_0(t), r_V(\mathbf{m}(a))\phi_0(t)) = (\phi_0(t), \phi_0(t))$$

Since a is a unit, we may use $\phi_0(at) = \phi_0(t)$ to get

$$\int_{K_{\mathfrak{p}}} \phi_0(t)^2\mu(g)^2 dt = \int_{K_{\mathfrak{p}}} \phi_0(t)^2 dt$$

from which we conclude $\mu(g)^2 = 1 \implies \mu(g) = 1$.

For the fourth statement, we get a leading factor of

$$\langle \pm 1, 2\rangle_{\mathfrak{p}}\gamma_w\left(\frac{1}{2}\right)\gamma(1)^{-1} = \gamma_w\left(\frac{1}{2}\right)\gamma(1)^{-1}$$

The Hilbert symbol disappears in the case of -1 due to the equation $2 * 1^2 - 1 * 1^2 = 1^2$. For the integral, we have

$$r_V(\mathbf{w}_{\pi^r})\phi_0(t) = \int_{y\in K_{\mathfrak{p}}} \psi(-2ty)\,\phi_0(\pm\pi^r y)\,d_g y = \int_{y\in\pi^{-r}O_{K_{\mathfrak{p}}}} \psi(-2ty)\,d_g y = \mu(g)\phi_0(2t)$$

where we have abused notation slightly in the last step. The idea is that since we are picking $\mu(g)$ to make this operation unitary, we may absorb any positive real factors into $\mu(g)$ beforehand and then choose $\mu(g)$ to make things unitary. In this case, a factor of q^r was absorbed.

To evaluate $\mu(g)$ we use the same trick and get

$$\int_{K_{\mathfrak{p}}} \phi_0(2t)^2\mu(g)^2 dt = \int_{K_{\mathfrak{p}}} \phi_0(t)^2 dt$$

This becomes

$$\mu(g)^2 \int_{(1/2)O_{K_\mathfrak{p}}} dt = 1 \implies \mu(g)^2 |1/2| = 1 \implies \mu(g) = |2|^{1/2}$$

For the sixth statement, if $c \neq 0$ we make use of the factorization

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & 0 \end{pmatrix} \begin{pmatrix} c\pi^{-r} & \pi^{-r} \\ -\pi^r & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & 0 \end{pmatrix} \begin{pmatrix} 1 & -c\pi^{-2r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \pi^{-r} \\ -\pi^r & 0 \end{pmatrix}$$

The Leray cocycles associated to these two factorizations are $\gamma_w\left(-\frac{1}{2}\pi^{2r}c\right) = \gamma_w(-2c)$ and 1 respectively. From this, we may conclude

$$\mathbf{n}_-(c)[1, \gamma_w(-2c)]_L = \mathbf{w}_{\pi^r}\mathbf{n}(-c)\mathbf{w}_{-\pi^r}$$

We may now use the previous statements in this proposition to conclude that

$$\omega_V(\mathbf{n}_-(c))\phi_0(t) = \gamma_w(2c)\omega_V(\mathbf{w}_{\pi^r})\omega_V(\mathbf{n}(-c))\omega_V(\mathbf{w}_{-\pi^r})\phi_0(t)$$

By statement 5 of this proposition, this becomes

$$= |2|^{1/2}\gamma_w(1)\,\gamma_w\left(\frac{1}{2}\right)^{-1}\gamma_w(2c)\omega_V(\mathbf{w}_{\pi^r})\omega_V(\mathbf{n}(-c))\phi_0(2t)$$

By statement 3, we have

$$= |2|^{1/2}\gamma_w(1)\,\gamma_w\left(\frac{1}{2}\right)^{-1}\gamma_w(2c)\omega_V(\mathbf{w}_{\pi^r})\psi(-ct^2)\phi_0(2t)$$

However, $\phi_0(2t)$ restricts t to lie in $(1/2)O_{K_\mathfrak{p}}$. Since c is a multiple of 4, it follows that $-ct^2$ will always be integral and $\psi(-ct^2)\phi_0(2t) = \phi_0(2t)$. We get

$$= |2|^{1/2}\gamma_w(1)\,\gamma_w\left(\frac{1}{2}\right)^{-1}\gamma_w(2c)\omega_V(\mathbf{w}_{\pi^r})\phi_0(2t) = \gamma_w(2c)\phi_0(t)$$

as desired. If $c = 0$ in this case, then $\mathbf{n}_-(c)$ is the identity element, so we clearly get $\phi_0(t)$ as our final answer. This fits with our convention $\gamma_w(0) = 1$. $\square$

Conjugating [KRY06] equation (8.5.26) gives us the following fact.

Fact 3.36. *Every element of $\mathcal{K}_{0,\mathfrak{p}}(4)$ may be written uniquely in the form $n(b)m(a)n_-(c)$ for $a \in O_{K_\mathfrak{p}}^\times$, $b \in \partial^{-1}$, $c \in 4\partial$.*

Corollary 3.37. *There is a genuine character* $\tilde{\epsilon}_2^{-1} : \mathcal{K}_{0,\mathfrak{p}}(4) \to \mathbb{T}$ *so that*

$$\omega_V(g')\phi_0(t) = \tilde{\epsilon}^{-1}\phi_0(t)$$

For $a, b, c \in O_{K_\mathfrak{p}}$, *it is given by*

$$\tilde{\epsilon}_2([n(b)m(a)n_-(c), z]_L) = \bar{z}\langle a, 2\rangle_\mathfrak{p}\gamma_w(-2c)$$

In terms of the matrix entries, this is

$$\tilde{\epsilon}_2\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right]_L\right) = \bar{z}\langle d, 2\rangle_\mathfrak{p}\gamma_w(-2cd)$$

Proof. The fact $\tilde{\epsilon}_2^{-1}$ is a genuine character follows from a similar argument to the odd case (see proposition 3.26). The first explicit formula follows immediately from proposition 3.35.

To get the formula directly in terms of the matrix entries, first note that since $4/c$ and our matrix is in $SL_2(O_{K_\mathfrak{p}})$, a and d must be units. This lets us factor the integral matrix as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^{-1} & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}$$

From this, we see

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = n(bd^{-1})m(d^{-1})n_-(cd^{-1})$$

We get that

$$\tilde{\epsilon}_2\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right]_L\right) = \bar{z}\langle d^{-1}, 2\rangle_\mathfrak{p}\gamma_w(-2cd^{-1})$$

Since γ_w and the Hilbert symbol are insensitive to squares, we may swap out that d^{-1} for d to get the desired formula. □

From this character, we get a function

$$\epsilon_2(g) = \tilde{\epsilon}_2([g, 1]_L) = \langle a, 2\rangle_\mathfrak{p}\gamma_w(-2cd)$$

Unlike in the odd case, there isn't an obvious extension of ϵ_2 to all of $G(K_\mathfrak{p})$, so we don't get a normalized coordinate system.

Corollary 3.38. *The splitting map $k \mapsto [k, \epsilon_2(k)]_L$ is a group homomorphism of $\mathcal{K}_{0,\mathfrak{p}}(4)$ into $\mathcal{K}'_{0,\mathfrak{p}}(4)$ and we get an isomorphism of groups $\mathcal{K}'_{0,\mathfrak{p}}(4) \cong \mathcal{K}_{0,\mathfrak{p}}(4) \times \mathbb{T}$.*

Via the above corollary, we can use the splitting map to think of $\mathcal{K}_{0,\mathfrak{p}}(4)$ as a subset of $G'_{K_\mathfrak{p}}$.

3.8 The Global Metaplectic Group

We may finally define the global metaplectic group using our earlier naive method. This follows [KRY06] section 8.5.5. The next definition is [KRY06] equation (8.5.31).

Definition 3.39. *Define the global cocycle $c_g : G(\mathbb{A}_K) \times G(\mathbb{A}_K) \to \mathbb{T}$ as the product*

$$c_g(g_1, g_2) = \prod_{\mathfrak{p}|\infty} c_{L,\mathfrak{p}}(g_1, g_2) \prod_{\mathfrak{p}|2} c_{L,\mathfrak{p}}(g_1, g_2) \prod_{\mathfrak{p}\ odd} c_{N,\mathfrak{p}}(g_1, g_2)$$

of local cocycles acting on g_1 and g_2 at each place. For any $g_1, g_2 \in G(\mathbb{A}_K)$ we will have $g_1, g_2 \in \mathcal{K}_\mathfrak{p}$ for almost all places and hence the global cocycle is well defined.

Let $G'_{\mathbb{A}_K} = G(\mathbb{A}_K) \times \mathbb{T}$ as a set and denote an element in this set as $[g, z]_g$, where $g \in G(\mathbb{A}_K)$ and $z \in \mathbb{T}$. Give this set the group composition law

$$[g_1, z_1]_g [g_2, z_2]_g = [g_1 g_2, z_1 z_2 c_g(g_1, g_2)]_g$$

We refer to this group as the (global) metaplectic group.

It will sometimes be convenient to refer to the $\mathfrak{p}$ components of some $g' = [g, z]_g$, which would lie in the local metaplectic groups. Our convention for this will be to set $g'_\mathfrak{p} = [g_\mathfrak{p}, 1]_L$ for $\mathfrak{p}$ even and to set $g'_\mathfrak{p} = [g_\mathfrak{p}, 1]_N$ for $\mathfrak{p}$ odd. Note that none of these depend on z, so we may also wish to introduce a z component, which we refer to as $z_{g'}$. Of course, our choice of local components is fairly arbitrary and depends on our choice of global coordinate system.

Definition 3.40. *Given a collection of elements $[g_\mathfrak{p}, z_\mathfrak{p}]$ in the local metaplectic groups (using normalizaed coordinates for an odd place and Leray coordinates otherwise) as* *long as $z_\mathfrak{p}$ is almost always 1, it makes sense to form their product as an element of $G'_{\mathbb{A}_K}$. Define*

$$\prod_{\mathfrak{p}\leq\infty}[g_\mathfrak{p}, z_\mathfrak{p}] = [\prod_{\mathfrak{p}\leq\infty} g_\mathfrak{p}, \prod_{\mathfrak{p}\leq\infty} z_\mathfrak{p}]_g$$

It should be clear that the local components of some g' (as given above this definition) multiply to g' in the sense that

$$g' = [1, z_{g'}]_g \prod_{\mathfrak{p}\leq\infty} g'_\mathfrak{p}$$

We call this the primary factorization of g'. In general, any product of local components (along with possibly a $[1, z]_g$ term) that multiplies to an element g' will be called a factorization of g'.

If we have two factorizations $g' = \prod_{\mathfrak{p}\leq\infty}[g_{\mathfrak{p},1}, z_{\mathfrak{p},1}] = \prod_{\mathfrak{p}\leq\infty}[g_{\mathfrak{p},2}, z_{\mathfrak{p},2}]$, we must clearly have $g_{\mathfrak{p},1} = g_{\mathfrak{p},2}$ for all places. Furthermore, we know that all but finitely many of the z components must be 1.

We conclude this section by noting that the local splitting maps we found earlier imply global splittings as well.

Proposition 3.41. *Let $\mathcal{K}_{0,f}(4) \subset G(\mathbb{A}_{K,finite})$ denote the set of all matrices where c (the bottom left entry) is divisible by 4. Let $\mathcal{K}'_{0,f}(4) \subset G'_{\mathbb{A}_{K,finite}}$ consist of all elements with first coordinate in $\mathcal{K}_0(4)$. One may think of these as subsets of $G(\mathbb{A}_K)$ and $G'_{\mathbb{A}_K}$ by letting all infinite components be trivial. Then, the splitting map $k \mapsto \left[k, \prod_{\mathfrak{p}|2}\epsilon_{2,\mathfrak{p}}(k)\right]_g$ is a homomorphism of $\mathcal{K}_{0,f}(4)$ into $\mathcal{K}'_{0,f}(4)$. From this homomorphism, we get $\mathcal{K}'_{0,f}(4) \cong \mathcal{K}_{0,f}(4) \times \mathbb{T}$.*

Proof. The global splitting map follows immediately from the local splitting maps we computed in corollaries 3.34 and 3.38. ☐

Something new we didn't have before is that there is also a splitting homomorphism of $G(K)$ into $G'_{\mathbb{A}_K}$. This is [KRY06] lemma 8.5.15.

Proposition 3.42. *(i) For $g \in G(K)$, $\epsilon_{\mathfrak{p}}(g) = 1$ for almost all $\mathfrak{p}$. Hence,*

$$\epsilon(g) := \prod_{\mathfrak{p} \text{ odd}} \epsilon_{\mathfrak{p}}(g)$$

is well defined.

(ii) For $g_1, g_2 \in G(K)$, $c_{L,\mathfrak{p}}(g_1, g_2)$=1 for almost all $\mathfrak{p}$ and we have a product formula

$$\prod_{\mathfrak{p} \leq \infty} c_{L,\mathfrak{p}}(g_1, g_2) = 1$$

(iii) A splitting homomorphism from $G(K)$ to $G'_{\mathbb{A}_K}$ is given by

$$g \mapsto [g, \epsilon(g)^{-1}]_g$$

It is trivial on $P(K)$ in that the map becomes

$$p \mapsto [p, 1]_g$$

(iv) We have

$$[g, \epsilon(g)^{-1}]_g = \prod_{\mathfrak{p}|2} [g_{\mathfrak{p}}, 1]_L \prod_{\mathfrak{p} \text{ odd}} [g_{\mathfrak{p}}, \epsilon(g_{\mathfrak{p}})^{-1}]_N \prod_{\mathfrak{p}|\infty} [g_{\mathfrak{p}}, 1]_L$$

At every place $\mathfrak{p}$, the element of $G'_{K_{\mathfrak{p}}}$ in this factorization is equal to $[g_{\mathfrak{p}}, 1]_L$.

Proof. For (i), equation (3.29) gives $\epsilon_{\mathfrak{p}}$ at odd places as

$$\tilde{\epsilon_{\mathfrak{p}}}(g) = \begin{cases} 1 & c\pi^{-r} \text{ is a unit} \\ \gamma_w(-2cd) & d \text{ is a unit} \end{cases}$$

Since $-2cd \in K$, it follows that $-2cd \in O_{K_{\mathfrak{p}}}^{\times}$ for almost all places. The comment just after equation (3.2) then implies that $\gamma_w(-2cd) = 1$ at almost all places, proving (i).

For (ii), we make a similar argument. Recall the definition $c_L(g_1, g_2) = \gamma_w\left(\frac{1}{2}c_1c_2c_3\right)$. Since $\frac{1}{2}c_1c_2c_3 \in K$, we have $\frac{1}{2}c_1c_2c_3 \in O_{K_{\mathfrak{p}}}^{\times}$ for almost all places. The comment after equation (3.2) then implies that the cocycle is 1 at almost all places. The product formula follows from the product formula for the Weil constant - equation (3.3).

For (iii), it suffices to show that for any $g_1, g_2 \in G(K)$, we have $c_g(g_1, g_2) = \epsilon(g_1)\epsilon(g_2)\epsilon(g_1 g_2)^{-1}$. This is because then

$$[g_1, \epsilon(g_1)^{-1}]_g [g_2, \epsilon(g_2)^{-1}]_g = [g_1 g_2, c_g(g_1, g_2)\epsilon(g_1)^{-1}\epsilon(g_2)^{-1}]_g = [g_1 g_2, \epsilon(g_1 g_2)^{-1}]_g$$

To show the desired identity, we write out

$$c_g(g_1, g_2) = \prod_{\mathfrak{p}|\infty} c_{L,\mathfrak{p}}(g_1, g_2) \prod_{\mathfrak{p}|2} c_{L,\mathfrak{p}}(g_1, g_2) \prod_{\mathfrak{p}\text{ odd}} c_{N,\mathfrak{p}}(g_1, g_2)$$

By part (ii), c_L is almost always 1, so it is okay to expand this as

$$\prod_{\mathfrak{p}|\infty} c_{L,\mathfrak{p}}(g_1, g_2) \prod_{\mathfrak{p}|2} c_{L,\mathfrak{p}}(g_1, g_2) \prod_{\mathfrak{p}\text{ odd}} c_{L,\mathfrak{p}}(g_1, g_2)\epsilon_{\mathfrak{p}}(g_1)\epsilon_{\mathfrak{p}}(g_2)\epsilon_{\mathfrak{p}}(g_1 g_2)^{-1}$$

We may collect terms and use the definition of the global ϵ to get

$$\epsilon(g_1)\epsilon(g_2)\epsilon(g_1 g_2)^{-1} \prod_{\mathfrak{p}} c_{L,\mathfrak{p}}(g_1, g_2)$$

By the product formula in part (ii), the desired result follows.

Finally, (iv) follows from the definition of normalized coordinates. $\square$

Chapter 4

Sections

4.1 General Properties

The intermediary between the metaplectic group and the modular forms we will be building is the notion of a section. We refer to [KRY04] page 902 (and for the general case [Kud97] page 558) for all of the main ideas of this section, to which we add some of our own exposition. (A definition of spherical may be found in [KY10] proposition 2.1.)

4.1.1 Preliminaries

For an idele class character χ (that is, a character of $K^\times \backslash \mathbb{A}_K$) and a complex number s, we obtain a character $\chi_{P'}$ of $P'_{\mathbb{A}_K}$. It is given by

$$\chi_{P'}\left(\left[\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, z\right]\right) = z\chi(a)|a|^s_{\mathbb{A}_K} \tag{4.1}$$

We may think of this character as a 1 dimensional representation we will call ρ. Although, this representation will only be a stepping stone and we will cease referring to it after this argument. For $p' \in P'_{\mathbb{A}_K}$ and $v \in \mathbb{C}^1$, ρ is given by

$$\rho\,(p')\,v = \chi_{P'}(p')v$$

We may now get to the representation we are actually interested in, which we denote by $I(s,\chi)$. It is defined through normalized induction as

$$I(s,\chi) = Ind_{P'_{\mathbb{A}_K}}^{G'_{\mathbb{A}_K}} \rho$$

By definition, $I(s,\chi)$ consists of all smooth functions Φ on $G'_{\mathbb{A}_K}$ satisfying

$$\Phi([n(b)m(a),z]_g g', s) = z\chi(a)|a|_{\mathbb{A}_K}^{s+1}\Phi(g',s), \quad b \in \mathbb{A}_K, a \in \mathbb{A}_K^\times \tag{4.2}$$

This is a left representation, with the action of $g' \in G'_{\mathbb{A}_K}$ given by $\Phi(x) \mapsto \Phi(xg')$.

Moreso than just functions in $I(s,\chi)$, we will be interested in sections. A section $\Phi(s)$ will denote a (possibly arbitrary) choice of function in each $I(s,\chi)$ as s varies over the entire complex plane.[1] We will need some sense of coherence of a section over different values of s. We will first define local sections and then address this with the notion of a standard section.

Write our idele class character χ as $\chi = \otimes_{\mathfrak{p}\leq\infty}\chi_{\mathfrak{p}}$. For a (finite or infinite) place $\mathfrak{p}$, let $I_{\mathfrak{p}}(s,\chi_{\mathfrak{p}})$ consist of smooth functions on $G'_{K_{\mathfrak{p}}}$ satisfying[2]

$$\Phi_{\mathfrak{p}}([n(b)m(a),z]_L g', s) = z\chi_{\mathfrak{p}}(a)|a|_{\mathfrak{p}}^{s+1}\Phi_{\mathfrak{p}}(g',s), \quad b \in K_{\mathfrak{p}}, a \in K_{\mathfrak{p}}^\times \tag{4.3}$$

$I_{\mathfrak{p}}(s,\chi)$ is a $G'_{K_{\mathfrak{p}}}$ representation where g' acts via $\Phi(x) \mapsto \Phi(xg')$. A local section will denote a (possibly arbitrary) choice of function in each $I_{\mathfrak{p}}(s,\chi)$ as s varies over the entire complex plane.

Equation (4.3) tells us that a function $\Phi_{\mathfrak{p}}$ enjoys some form of degree $s+1$ polynomial scaling in terms of the matrix entries. However, the exact nature in terms of the matrix entries is unclear. The following calculation of the Iwasawa decomposition will help clear things up for finite places.

[1] I find this notation somewhat confusing but it seems to be what people use. Essentially, these functions should take as input some element $g' \in G_{\mathbb{A}_K}$, but this argument is often omitted.

[2] Since $\epsilon_{\mathfrak{p}}(n(b)m(a)) = 1$ for all odd places, there is no difference between using Leray or Normalized coordinates in the formula above. We use Leray coordinates so as to handle all places simultaneously.

Lemma 4.4. *Let $\mathfrak{p}$ be a finite place and take matrix entries in $K_\mathfrak{p}$. Given a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we may write $g = pk$ for $p \in P(K_\mathfrak{p})$ and $k \in \mathcal{K}_\mathfrak{p}$. If $\mathfrak{p}$ is unramified, this may be done via an expression of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} M^{-1} & * \\ 0 & M \end{pmatrix} \begin{pmatrix} * & * \\ cM^{-1} & dM^{-1} \end{pmatrix}$$

where $M \in \{c, d\}$ such that $v_\pi(M) = min(v_\pi(c), v_\pi(d))$.

Proof. Observe that

$$\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} g = \begin{pmatrix} aM & bM \\ cM^{-1} & dM^{-1} \end{pmatrix}$$

By our choice of M, cM^{-1} and dM^{-1} are both integral and at least one is a unit. Hence we have the equality of ideals $(cM^{-1}, dM^{-1}) = (1)$ and we may choose integral elements x, y so that $x * dM^{-1} - y * cM^{-1} = 1$. Now we can confirm that

$$\begin{pmatrix} 1 & (x - aM)/(cM^{-1}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} g = \begin{pmatrix} x & y \\ cM^{-1} & dM^{-1} \end{pmatrix}$$

This implies the desired decomposition.

The second part follows immediately from the first after noting that $|\chi_\mathfrak{p}| = 1$. $\square$

Corollary 4.5. *(i) A function $\Phi_\mathfrak{p} \in I_\mathfrak{p}(s, \chi_\mathfrak{p})$ is completely determined by its values on $\mathcal{K}_\mathfrak{p} \subset \mathcal{K}'_\mathfrak{p}$.*

(ii) Take $\mathfrak{p}$ unramified. Given a $g' \in G'_{K_\mathfrak{p}}$, there is some $k' \in \mathcal{K}'_\mathfrak{p}$ so that

$$|\Phi_\mathfrak{p}(g')| = q^{(s+1)min(v_\pi(c), v_\pi(d))} |\Phi_\mathfrak{p}(k')|$$

Remark 4.6. *Since $\Phi_\mathfrak{p}$ is smooth, it in particular takes on only finitely many different absolute values on $\mathcal{K}'_\mathfrak{p}$. Hence, (ii) above gives a precise description of the growth/decay rate of $\Phi_\mathfrak{p}$. One may get bounds in the ramified case as well.*

4.1.2 Types of Sections

Definition 4.7. *Let $\mathcal{K}'$ denote the maximal compact subgroup of $G'_{\mathbb{A}_K}$ that is a product of all local $\mathcal{K}'_\mathfrak{p}$. A global section is called standard if its restriction to $\mathcal{K}'$ is independent of s. A local section is called standard if its restriction to $\mathcal{K}'_\mathfrak{p}$ is independent of s.*

Remark 4.8. *A local standard section is completely determined by its values on the compact subgroup $\mathcal{K}'_\mathfrak{p}$ (and even just by its values on $\mathcal{K}_\mathfrak{p}$). Similarly, a global section is determined by its values on $\mathcal{K}'$, which follows from the global Iwasawa decomposition $G'_{\mathbb{A}_K} = P'_{\mathbb{A}_K}\mathcal{K}'$.*

Proposition 4.9. *A function $\Phi_\mathfrak{p}$ with domain $\mathcal{K}'_\mathfrak{p}$ extends to a function in $I_\mathfrak{p}(s, \chi_\mathfrak{p})$ (with domain $G'_{K_\mathfrak{p}}$) iff $\Phi_\mathfrak{p}$ obeys $\Phi([n(b)m(a), z]_L k) = z\chi_\mathfrak{p}(a)\Phi(k)$ for $[n(b)m(a), 1]_L, k \in \mathcal{K}'_\mathfrak{p}$. Such an extension is unique when it exists.*

Let $\mathfrak{p}$ be an odd prime so that we have the splitting giving $\mathcal{K}_\mathfrak{p} \subset G'_{K_\mathfrak{p}}$. Then, the above holds for $\mathcal{K}_\mathfrak{p}$. That is, a function $\Phi_\mathfrak{p}$ with domain $\mathcal{K}_\mathfrak{p}$ extends to a function in $I_\mathfrak{p}(s, \chi_\mathfrak{p})$ (with domain $G'_{K_\mathfrak{p}}$) iff $\Phi_\mathfrak{p}$ obeys $\Phi([n(b)m(a), 1]_L k) = \chi_\mathfrak{p}(a)\Phi(k)$ for $[n(b)m(a), 1]_L, k \in \mathcal{K}_\mathfrak{p}$. Such an extension is unique when it exists.[3]

Proof. It is clear the stated hypothesis is needed, since it is just equation (4.3) restricted to $\mathcal{K}'_\mathfrak{p}$ or $\mathcal{K}_\mathfrak{p}$. Uniqueness follows from the previous remark. For existence, the natural way to try to extend $\Phi_\mathfrak{p}$ is by using equation (4.3). We do this below in the $\mathcal{K}_\mathfrak{p}$ case, although the same proof still works if one replaces all occurrences of $\mathcal{K}_\mathfrak{p}$ with $\mathcal{K}'_\mathfrak{p}$.

Given $g' \in G'_{K_\mathfrak{p}}$, write $g' = p'k$ for $p' = [n(b)m(a), z]_L \in P'(K_\mathfrak{p})$ and $k \in \mathcal{K}_\mathfrak{p}$. Then we would like to define the extension by setting

$$\Phi_\mathfrak{p}(g') = z\chi_\mathfrak{p}(a)|a|_\mathfrak{p}^{s+1}\Phi_\mathfrak{p}(k)$$

The obstruction to this is showing that this definition does not depend on the decomposition $g' = p'k$. To prove this, consider two different decompositions $g' = p'_1k_1 = p'_2k_2$. Then, let $h = p'^{-1}_1 p'_2 = k_1k_2^{-1} \in P'_{\mathbb{A}_K} \cap \mathcal{K}_\mathfrak{p}$, so that $p'_2k_2 = (p'_1h)(h^{-1}k_1)$. Let $p'_1 = [n(b_1)m(a_1), z_1]_L$ and $h = [n(b_h)m(a_h), z_h]_L$, so that $p'_2 = p'_1h = [n(b_2)m(a_1a_h), z_1z_h]_L$. We can see our construction is well defined if

$$z_1\chi_\mathfrak{p}(a_1)|a_1|_\mathfrak{p}^{s+1}\Phi_\mathfrak{p}(k_1) = z_1z_h\chi_\mathfrak{p}(a_1a_h)|a_1a_h|_\mathfrak{p}^{s+1}\Phi_\mathfrak{p}(h^{-1}k_1)$$

[3] Note the missing z term in this case! This is due to corollary 3.34.

If we are in the $\mathcal{K}_\mathfrak{p}$ case, then the fact $h \in P'_{\mathbb{A}_K} \cap \mathcal{K}_\mathfrak{p}$ along with corollary 3.34 imply $z_h = 1$. The desired formula then follows immediately since by hypotheiss

$$\Phi_\mathfrak{p}(h^{-1}k_1) = z_h^{-1}\chi_\mathfrak{p}(a_h^{-1})|a_h^{-1}|_\mathfrak{p}^{s+1}\Phi_\mathfrak{p}(k_1)$$

Now, we just need to show that $\Phi_\mathfrak{p}^0$ is a section. The fact that $\Phi_\mathfrak{p}^0$ obeys equation (4.3) will follow immediately from well-definedness. Namely, if we have some $g' = p_1'k$, then we get

$$\Phi_\mathfrak{p}^0(p_2'g') = \Phi_\mathfrak{p}^0(p_2'p_1'k) = \chi_{P',\mathfrak{p}}(p_2'p_1') = \chi_{P',\mathfrak{p}}(p_2')\chi_{P',\mathfrak{p}}(p_1') = \chi_{P',\mathfrak{p}}(p_2')\Phi_\mathfrak{p}^0(p_1'g')$$

$\square$

Corollary 4.10. *If $\chi_\mathfrak{p}$ is unramified (this is, it is trivial on $O_{K_\mathfrak{p}}^\times$), then there is a unique $\Phi_\mathfrak{p} \in I_\mathfrak{p}(s, \chi_\mathfrak{p})$ that is identically 1 on $\mathcal{K}_\mathfrak{p}$. This is called the spherical function.*

Similarly, there is a unique standard section $\Phi_\mathfrak{p}(s)$ that is identically 1 on $\mathcal{K}_\mathfrak{p}$, which we call the spherical section. We denote it by $\Phi_\mathfrak{p}^0(s)$.

An alternate description of sphericality is that this is the unique element of the representation $I_\mathfrak{p}(s, \chi)$ invariant under $\mathcal{K}_\mathfrak{p}$ and having $\Phi_\mathfrak{p}^0(1, s) = 1$.

Given a collection of sections $\Phi_\mathfrak{p}(s)$ at every place such that almost all of the sections are spherical, we may form their tensor product

$$\Phi(s) = \otimes_{\mathfrak{p} \leq \infty}\Phi_\mathfrak{p}(s)$$

This is a function via

$$\Phi(g', s) = z_{g'}\prod_{\mathfrak{p} \leq \infty}\Phi_\mathfrak{p}(g'_\mathfrak{p}, s) \tag{4.11}$$

where we used the primary factorization of g' in definition 3.40. The product is well defined in the sense that almost all terms are 1. This is because $g'_\mathfrak{p} \in \mathcal{K}_\mathfrak{p}$ almost everywhere, and almost all of our sections are spherical. One can also take such tensor products over smaller collections of local sections. For example, one could form a section Φ_f over finite places. Finally, we note that the above definition is invariant of the factorization of g' that is used. That is, if we swap out the z and local components of the primary factorization for any other factorization, we will get the same result for $\Phi(g', s)$.

The representation $I(s, \chi)$ is the restricted product $I(s, \chi) = \otimes'_{\mathfrak{p} \leq \infty} I_{\mathfrak{p}}(s, \chi)$ (see [Kud96] p.558). We will take this to mean that it is given by sums of functions

$$\Phi = \otimes_{\mathfrak{p} \leq \infty} \Phi_{\mathfrak{p}}$$

where almost all of the component functions are spherical. Smoothness of sections of $I(s, \chi)$ should make it unsurprising that they tend to look spherical at almost all places.

Remark 4.12. *It can be informative to verify that such a pure tensor is indeed an element of $I(s, \chi)$. Let $p' = \left[\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, z_{p'}\right] \in P'(\mathbb{A}_K)$. We can then check*

$$\Phi(p'g') = z_{p'} z_{g'} \prod_{\mathfrak{p} \leq \infty} \Phi(p'_{\mathfrak{p}} g'_{\mathfrak{p}})_{\mathfrak{p}} = z_{p'} z_{g'} \prod_{\mathfrak{p} \leq \infty} \chi_{\mathfrak{p}}(a_{\mathfrak{p}}) |a|_{\mathfrak{p}}^{s+1} \Phi(g'_{\mathfrak{p}})_{\mathfrak{p}} = z_{p'} \chi(a) |a|_{\mathbb{A}_K}^{s+1} \Phi(g')$$

A section $\Phi(s)$ is called factorizable if $\Phi(s) = \otimes_{\mathfrak{p}} \Phi_{\mathfrak{p}}(s)$ is a pure tensor with respect to the restricted product.

4.2 The λ mapping

For this section, take $\mathfrak{p}$ to be a finite place. Given a quadratic space (V, Q) of odd dimension over $K_{\mathfrak{p}}$ there is a way to use the Weil representation to associate a local section to any Schwartz function ϕ on V called the λ map. Although we provide proofs, the below properties of the λ mapping are found in [KRY06] pages 328-329 and [Kud96] section III.5. Afterwards, we use λ to prove a type of symmetry possessed by local sections.

4.2.1 Properties of λ

Definition 4.13. *Define a map $\lambda_V : \mathcal{S}(V) \to I_{\mathfrak{p}}(s_0, \chi_V)$ by*

$$\lambda_V(\phi) := \left(g' \mapsto (\omega_V(g')\phi(t))|_{t=0}\right)$$

where $s_0 = \frac{dim(V)}{2} - 1$.

$\lambda_V(\phi)$ then extends to a unique standard section, which we call the section associated to ϕ.

Of course, we need to show that $\lambda_V(\phi)$ is a member of $I_{\mathfrak{p}}(s_0, \chi_V)$. First we need a lemma where we look at some of the behavior of the Weil action.

Lemma 4.14.

$$\omega_V([n(b)m(a), z]_L)\phi(t) = z\chi_V(a)|a|^{dim(V)/2}\psi\left(\frac{1}{2}b(t,t)_Q\right)\phi(at)$$

Proof. This is similar to many calculations we have already done. First recall the definition of the Weil representation.

$$\omega_V([g,z]_L)\phi(t) = \chi_V(x(g))\left(z\gamma_w\left(\frac{1}{2}\right)^{j(g)}\right)^{\overline{dim(V)}}\gamma\left(\psi\left(\frac{1}{2}t\right)\circ V\right)^{-j(g)} r_V(g)\phi(t)$$

where

$$r_V(g)\phi(t) = \int_{y\in cV}\psi\left(\frac{1}{2}(at,bt)_Q + (bt,cy)_Q + \frac{1}{2}(cy,dy)_Q\right)\phi\left(at+cy\right)d_g y$$

Note $j(g) = 0$ here so we have

$$\begin{aligned}\omega_V([n(b)m(a), z]_L)\phi(t) &= z\chi_V(a^{-1})\int_{y\in\{0\}}\psi\left(\frac{1}{2}(at, a^{-1}bt)_Q\right)\phi\left(at\right)d_g y \\ &= z\chi_V(a^{-1})\psi\left(\frac{1}{2}b(t,t)_Q\right)\phi(at)\mu(g)\end{aligned}$$

for some positive real $\mu(g)$ coming from the Haar measure. For $r_V(g)$ to be unitary, we must have $(r_V([n(b)m(a), z]_L)\phi_L(t), r_V([n(b)m(a), z]_L)\phi_L(t)) = (\phi_L(t), \phi_L(t))$, where ϕ_L is the indicator function of some lattice $L \subset V$ of measure $\#L$. Or in other words

$$\mu(g)^2\int_V \psi\left(-\frac{1}{2}b(t,t)_Q\right)\phi(at)\psi\left(\frac{1}{2}b(t,t)_Q\right)\phi(at)dt = \#L$$

The integral is just over the set $a^{-1}L$, and so we get $\mu(g)^2|a|^{-dim(V)}\#L = \#L \implies \mu(g) = |a|^{dim(V)/2}$. Thus, we have

$$\omega_V([n(b)m(a), z]_L)\phi(t) = z\chi_V(a)|a|^{dim(V)/2}\psi\left(\frac{1}{2}b(t,t)_Q\right)\phi(at)$$

where we used the fact χ_V is quadratic to swap out $\chi_V(a^{-1})$ for $\chi_V(a)$. □

Proposition 4.15. *Assume $dim(V)$ is odd.*

(i) $\lambda_V(\phi)$ is indeed a function in $I_{\mathfrak{p}}(s_0, \chi_V)$.

(ii) The map λ_V intertwines the Weil action with the action of $G'_{K_{\mathfrak{p}}}$ on $I_{\mathfrak{p}}(s_0, \chi_V)$. That is, if $\lambda_V(\phi) = \Phi(x, s_0)$, then $\lambda_V(\omega_V(g')\phi) = \Phi(xg', s_0)$.

(iii) Take $dim(V) = 1$ and $\mathfrak{p}$ odd. If $det(V) \in O^\times_{K_{\mathfrak{p}}}$, then $\lambda_V(\phi_0)$ is identically 1 on $\mathcal{K}_{\mathfrak{p}} \subset G'_{K_{\mathfrak{p}}}$. Hence, the associated section to ϕ_0 is the spherical section $\phi^0_{\mathfrak{p}}$.

Proof. Points (i) and (ii) are stated on page 328 of [KRY06], which come from the start of section III.5 of [Kud96]. (iii) is a weaker version of Lemma 4.1 from [KY10]. We give elementary verifications for all of these points except for showing that the function $\lambda_V(\phi)$ is smooth.

Before anything else, we remark that the representation $I^G_P(\chi^\psi_V|det|^{m/2-(n+1)/2})$ given in section III.5 of [Kud96] is indeed the same as the $I(s, \chi_V)$ from [KRY06] that we use (although we must plug in $n = 1$ into their formula which corresponds to us taking $G = SL_2$). Using [Kud96] proposition 4.3 for the definition of χ^ψ_V, we see they use the normalized induction of the one dimensional representation

$$\left[\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}, z\right]_R \mapsto z\chi_V(a)\gamma_w(a, 1/2)^{-1}|a|^{dim(V)/2-1}$$

Definition 3.22 gives the coordinate change $[g, z]_R = [g, z\gamma_w\left(x(g), 1/2\right)^{-1}\gamma(1/2)^{-j(g)}]_L$. Since in the above formula g is upper triangular, we have $j(g) = 0$, and we see their representation is the same as

$$\left[\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}, z\gamma_w\left(a^{-1}, 1/2\right)^{-1}\right]_L \mapsto z\chi_V(a)\gamma_w(a, 1/2)^{-1}|a|^{dim(V)/2-1}$$

Since γ_w is insensitive to square factors, it follows that this is just the representation given by the character in equation (4.1), which is what we used to build $I(s, \chi_V)$. Now we move on to our elementary verification.

For (i), we must verify equation (4.3). We start with the definition

$$\lambda_V(\phi)([n(b)m(a), z]_L g') = \omega_V([n(b)m(a), z]_L g')\phi(t)|_{t=0}$$

Since we are working in Leray coordinates, c_L is trivial on $P \times G$ and so this equals

$$= \omega_V([n(b)m(a), z]_L)\omega_V(g')\phi(t)|_{t=0}$$

By the lemma, this is the same as

$$= \chi_V(a)|a|^{dim(V)/2}\psi\left(\frac{1}{2}b(t,t)_Q\right)(\omega_V(g')\phi)(at)\Big|_{t=0} = \chi_V(a)|a|^{dim(V)/2}(\omega_V(g')\phi)(0)$$

However, this is precisely the same as $\chi_V(a)|a|^{dim(V)/2}\lambda_V(\phi)$, proving (i).

(ii) follows immediately from the definitions. We may describe the section $\lambda_V(\phi)$ as a function using the notation

$$\lambda_V(\phi) = \left(x \mapsto (\omega_V(x)\phi(t))|_{t=0}\right)$$

We also clearly have

$$\lambda_V(\omega_V(g')\phi) = \left(x \mapsto (\omega_V(x)\omega_V(g')\phi(t))|_{t=0}\right) = \left(x \mapsto (\omega_V(xg')\phi(t))|_{t=0}\right)$$

Putting these two observations together, we have (ii).

Now we tackle (iii). The splitting $\mathcal{K}_\mathfrak{p} \subset G'_{K_\mathfrak{p}}$ is given by corollary 3.34 as $k \mapsto [k, \epsilon(k)]_L$. The element of $I(s_0, \chi_V)$ corresponding to ϕ_0 is given on $\mathcal{K}_\mathfrak{p}$ by

$$\lambda_V(\phi_0) = \left([k, \epsilon(k)]_L \mapsto (\omega_V([k, \epsilon(k)]_L)\phi_0(t))|_{t=0}\right)$$

Thus it suffices to check $(\omega_V([k, \epsilon(k)]_L)\phi_0(t))|_{t=0} = 1$. However, this follows by the definition of ϵ. Namely, by proposition 3.26 we have

$$\omega_V([k, 1]_L)\phi_0(t) = \epsilon(k)^{-1}\phi_0(t) \implies \omega_V([k, \epsilon(k)]_L)\phi_0(t) = \phi_0(t)$$

$\square$

4.2.2 Symmetries of Schwartz Functions

The following proposition tells us that Schwartz functions always have some amount of symmetry under the Weil action, which then carries over to the associated section. This will later be used to prove the level structure of a modular form.

Proposition 4.16. *(i) Given a Schwartz function $\phi \in \mathcal{S}(V)$, there exists a subgroup $\Gamma'_{\mathfrak{p}} \subset \mathcal{K}'_{0,\mathfrak{p}}(4)$ commensurable with $\mathcal{K}'_{0,\mathfrak{p}}(4)$ such that ϕ is an eigenfunction under the action of $\omega_V(k')$ for $k' \in \Gamma'_{\mathfrak{p}}$.*

(ii) If we denote the eigenvalue by $\epsilon_\phi(k')^{-1}$, then ϵ_ϕ^{-1} is a genuine character $\Gamma'_{\mathfrak{p}} \to \mathbb{T}$.

(iii) If $V = K$ and $\phi = \phi_0 = \mathbb{1}_{O_{K_{\mathfrak{p}}}}$, then we may take $\Gamma'_{\mathfrak{p}} = \mathcal{K}'_{0,\mathfrak{p}}(4)$ and take $\epsilon_\phi = \epsilon_{\mathfrak{p}}$ to be the character we defined previously.

(iv) If Φ is the associated section to ϕ, then for any $\Gamma'_{\mathfrak{p}}$ and ϵ_ϕ as above we have

$$\Phi(g'k', s) = \epsilon_\phi(k')^{-1}\Phi(g', s) \text{ for } k \in \Gamma'_{\mathfrak{p}}$$

Proof. Start by proving (i). Let $L \subset V$ denote some sublattice and L^* denote its dual under $(\cdot,\cdot)_Q$. It suffices to prove the result for indicator functions $\phi = \mathbb{1}_{\mu+L}$ $(\mu \in V)$ since every Schwartz function ϕ is a finite sum of such functions. For a given indicator function ϕ, we will show that one may take

$$\Gamma = \Gamma(\pi^N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \pi^N \right\}$$

for N sufficiently large, which will prove the claim.

To reduce the problem further, note that $\Gamma(\pi^N)$ is generated by its elements of the form $m(a), n(b), n_-(c)$. So, it suffices to show that ϕ is an eigenfunction under these elements. Since ϕ is an indicator function, we have for a sufficiently close to 1 that (for all t) $\phi(at) = \phi(t)$. Similarly, for all b sufficiently close to 0, we have $\psi\left(\frac{1}{2}b(t,t)_Q\right) = 1$. It follows from lemma 4.14 that for N large enough that ϕ is an eigenfunction under $m(a), n(b) \in \Gamma(\pi^N)$. This reduces the problem to checking that ϕ

is an eigenfunction under $\omega_V([n_-(c),1]_L)$ for c sufficiently small. In turn, it suffices to check ϕ is an eiegenfunction under the action of r_V.

To do this, we use of the following factorization which is derived from the one in the proof of proposition 3.35:

$$n_-(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\pi^{-r} \\ \pi^r & 0 \end{pmatrix} \begin{pmatrix} 1 & -c\pi^{-2r} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \pi^{-r} \\ -\pi^r & 0 \end{pmatrix}$$

Temporarily adopt the notation $\sim$ to denote that two functions of t are constant multiples of each other. Then, this factorization gives us

$$r_V(n_-(c))\phi(t) \sim r_V(w_{\pi^r})r_V(n(-c\pi^{-2r}))r_V(w_{-\pi^r})\phi(t)$$

Starting the evaluation from the right side, we have

$$r_V(w_{-\pi^r})\phi(t) \sim \int_{y\in V} \psi\left(-(t,y)_Q\right)\phi(-\pi^r y)dy$$

Note that we no longer have to worry about the measure $d_g y$ since we only care about everything up to a constant multiple. Plugging in $\phi = \mathbb{1}_{\mu+L}$ and lightly rearranging, we get the Fourier transform

$$r_V(w_{-\pi^r})\phi(t) \sim \int_{y\in V} \psi\left(-(t,y)_Q\right)\mathbb{1}_L(-\mu - \pi^r y)dy$$

Substitute $y_2 = -\mu - \pi^r y$ to get

$$r_V(w_{-\pi^r})\phi(t) \sim \psi\left(\frac{1}{\pi^r}(t,\mu)_Q\right)\int_{y_2\in L} \psi\left(\frac{1}{\pi^r}(t,y_2)_Q\right)dy_2$$

Just as before, the integrand is an additive character and hence the integral is 0 unless the integrand is identically 1. This gives us

$$r_V(w_{-\pi^r})\phi(t) \sim \psi\left(\frac{1}{\pi^r}(t,\mu)_Q\right)\mathbb{1}_{L^*}(t)$$

From this, we can conclude that

$$r_V(n(-c\pi^{-2r}))r_V(w_{-\pi^r})\phi(t) \sim \psi\left(-\frac{1}{2}c\pi^{-2r}(t,t)_Q\right)\psi\left(\frac{1}{\pi^r}(t,\mu)_Q\right)\mathbb{1}_{L^*}(t)$$

For c sufficiently close to 0, $\psi\left(-\frac{1}{2}c\pi^{-2r}(t,t)_Q\right)$ will be identically 1 on L^*. Take N large enough for this to happen, so that

$$r_V(n(-c\pi^{-2r}))r_V(w_{-\pi^r})\phi(t) \sim \psi\left(\frac{1}{\pi^r}(t,\mu)_Q\right)\mathbb{1}_{L^*}(t)$$

Finally, we apply the $r_V(w_{\pi^r})$ operator, so that the operators on the left hand side become (a constant multiple of) $r_V(n_-(c))\phi(t)$. This application of $r_V(w_{\pi^r})$ will undo the Fourier transform and give us our original indicator function back, which we check. From the definitions, we have

$$r_V(n_-(c))\phi(t) \sim \int_{y\in V}\psi\left(-(t,y)_Q\right)\psi\left((y,\mu)_Q\right)\mathbb{1}_{L^*}(\pi^r y)dy$$

Letting $y_{\text{new}} = \pi^r y_{\text{old}}$ alongside some simple rearranging gives

$$r_V(n_-(c))\phi(t) \sim \int_{y\in L^*}\psi\left(-\frac{1}{\pi^r}(y,t-\mu)_Q\right)dy \sim \mathbb{1}_{\mu+L}(t)$$

This concludes the proof of (i). Note that in general the group $\Gamma'_{\mathfrak{p}}$ will be able to be bigger than what we constructed.

(ii) follows easily. For whichever Γ we choose, ϵ_ϕ^{-1} is a character since it arises from eigenvalues. Genuineness follows from the definition of the Weil representation. The fact that this character outputs into $\mathbb{T}$ follows since the Weil representation is normalized to be unitary.

(iii) follows immediately from proposition 3.26 and corollary 3.37.

Finally, we prove (iv). Write $g' = [m(a)n(b),1]_L k'_g$ for $k'_g \in \mathcal{K}'_{\mathfrak{p}}$. From the definition of a standard section, we have

$$\Phi(g'k',s) = \chi_{\mathfrak{p}}(a)|a|_{\mathfrak{p}}^{s+1}\Phi(k'_g k',s) = \chi_{\mathfrak{p}}(a)|a|_{\mathfrak{p}}^{s+1}\Phi(k'_g k',s_0)$$

By the intertwining property of λ from (ii) of proposition 4.15, we get

$$\chi_{\mathfrak{p}}(a)|a|_{\mathfrak{p}}^{s+1}\Phi(k'_g k',s_0) = \chi_{\mathfrak{p}}(a)|a|_{\mathfrak{p}}^{s+1}\lambda_V(\omega_V(k')\phi)(k'_g) = \epsilon_\phi(k')^{-1}\chi_{\mathfrak{p}}(a)|a|_{\mathfrak{p}}^{s+1}\lambda_V(\phi)(k'_g)$$

From this point it is easy to reverse the previous steps to get to the desired result. We continue from the previous line to get

$$= \epsilon_\phi(k')^{-1}\chi_\mathfrak{p}(a)|a|_\mathfrak{p}^{s+1}\Phi(k'_g, s_0) = \epsilon_\phi(k')^{-1}\chi_\mathfrak{p}(a)|a|_\mathfrak{p}^{s+1}\Phi(k'_g, s) = \epsilon_\phi(k')^{-1}\Phi(g', s)$$

$\square$

4.3 The Archimedean Section Φ^l

For this section, take $\mathfrak{p}$ to be an Archimedean place. We follow [KY10] p.2280, with [KRY06] p.336-338 for the definition of Φ^l and [HI13] p.1988 for a formula for the subgroup $\widetilde{SO}_2(\mathbb{R})$ in our particular cases.

The local metaplectic group $G'_{K_\mathfrak{p}}$ admits a subgroup $\widetilde{SO}_2(\mathbb{R})$, which we identify with the group $\mathbb{R}/4\pi$ and is given in Rao coordinates (see remark 3.22 for all relevant details on Rao coordinates) as

$$k'(\theta) = \begin{cases} [k(\theta), 1]_R & -\pi < \theta \leq \pi \\ [k(\theta), -1]_R & \pi < \theta \leq 3\pi \end{cases} \tag{4.17}$$

where $k(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ is a (clockwise!) rotation by θ. This same group in Leray coordinates is given by example 3.18. Notice that we have $\mathcal{K}'_\mathfrak{p} = \widetilde{SO}_2(\mathbb{R})\mathbb{T}$, where $\mathbb{T} = \{[1, z]_R\}$.

For $l \in \frac{1}{2}\mathbb{Z}$, let ν_l denote the character of $\widetilde{SO}_2(\mathbb{R})$ given by

$$\nu_l(k'(\theta)) = e^{il\theta}$$

For $2l$ odd, we may extend ν_l to all of $\mathcal{K}'_\mathfrak{p}$ (that is, to all $z \in \mathbb{T}$) by letting[4]

$$\nu_l(k'(\theta)[I, z]_R) = ze^{il\theta} \tag{4.18}$$

It should be clear this is the unique such extension that is a genuine character.

[4] For $2l$ even, set $\nu_l(k'(\theta)[I, z]_R) = e^{il\theta}$. However, we will only be interested in the odd case.

Remark 4.19. *Since $k'(\theta) = k'(\theta+2\pi)[I,-1]_R$, checking this extension is well defined comes down to the calculation*

$$\nu_l(k'(\theta+2\pi)[I,-1]_R) = -e^{il\theta}e^{il*2\pi} = e^{il\theta} = \nu_l(k'(\theta))$$

Proposition 4.20. *Take $\mathfrak{p}$ Archimedean. Fix a non-integral choice of l. Set $\kappa = \pm 1$, taking $\kappa = 1$ if $2l$ is 1 mod 4 and $\kappa = -1$ if $2l$ is 3 mod 4. Set $\chi_\mathfrak{p}(x) = \langle x, \kappa\rangle_\mathfrak{p}$ (note that the Hilbert symbol only cares about the sign of κ). Then, there is a unique Archimedean section $\Phi^l_\mathfrak{p}(s) \in I_\mathfrak{p}(s, \chi_\mathfrak{p})$ that satisfies*

$$\Phi^l_\mathfrak{p}(g'k', s) = \nu_l(k')\Phi^l_\mathfrak{p}(g', s), \quad \Phi^l_\infty(1, s) = 1$$

Furthermore, the existence of this section forces our above choice of $\kappa \in \{\pm 1\}$. That is, there is no such section if we had taken $\chi_\mathfrak{p}(x) = \langle x, -\kappa\rangle_\mathfrak{p}$ instead.

Proof. By proposition 4.9, it suffices to verify that ν_l obeys equation (4.3) on the set $\mathcal{K}'_\mathfrak{p}$. That is, we need

$$\nu_l(p'g') = z\chi_\mathfrak{p}(a)\nu_l(g') \tag{4.21}$$

for $p', g' \in \mathcal{K}'_\mathfrak{p}$ and where p' has the Leray coordinate description $p' = [n(b)m(a), z]_L$.

Write g' in the form $g' = k'(\theta)[I, z_2]_R$. Now observe that $[n(b)m(a), 1]_L \in P'(K_\mathfrak{p}) \cap \mathcal{K}'_\mathfrak{p}$. However, since our place is Archimedean, we have $P'(K_\mathfrak{p}) \cap \mathcal{K}'_\mathfrak{p} = \{[I, z]_R, [-I, z]_R\}$.

We proceed by casework. Take $p' = [I, z_1]_R$ and note that $[I, z_1]_R = [I, z_1]_L$. Equation (4.21) becomes

$$\nu_l([I, z_1]_R k'(\theta)[I, z_2]_R) = z_1 * \chi_\mathfrak{p}(1)\nu_l(k'(\theta)[I, z_2]_R)$$

which is clearly true by how we defined ν_l.

Now take $p' = [-I, z_1]_R$ Note the equalities $k'(\pi) = [-I, 1]_R \implies [-I, z_1]_R = [I, z_1]_R k'(\pi)$ and $[-I, z_1]_R = [-I, z_1\gamma_w(-1, 1/2)^{-1}]_L = [-I, z_1 e^{2\pi i/4}]_L$. Equation (4.21) becomes

$$\nu_l([I, z_1]_R k'(\pi)k'(\theta)[I, z_2]_R) = z_1 e^{2\pi i/4} * \chi_\mathfrak{p}(-1)\nu_l(k'(\theta)[I, z_2]_R)$$

Using $k'(\pi)k'(\theta) = k'(\theta + \pi)$ and applying the definition of ν_l, we see this is equivalent to

$$z_1 z_2 e^{il\theta} e^{il\pi} = z_1 e^{2\pi i/4} \langle -1, \kappa \rangle_{\mathfrak{p}} z_2 e^{il\theta}$$

Simplifying, this holds iff

$$e^{2\pi i*(2l-1)/4} = \langle -1, \kappa \rangle_{\mathfrak{p}}$$

Since we are at an Archimedean place, the Hilbert symbol is -1 iff both arguments are negative. Thus, if $2l$ is 1 mod 4, we need $\kappa = 1$ and if $2l$ is 3 mod 4, we need $\kappa = -1$. □

Let

$$\Phi^l_\infty(s) := \otimes_{\mathfrak{p}|\infty} \Phi^l_{\mathfrak{p}}(s)$$

be a section of $\otimes_{\mathfrak{p}|\infty} I_{\mathfrak{p}}(s, \langle x, \kappa \rangle_{\mathfrak{p}})$. We will later be building a factorizable section with Φ^l_∞ as the Archimedean part. An Eisenstein series created from such a section (via the method we are about to describe) will then have a weight of l.

Chapter 5

Eisenstein Series

5.1 The Series $E(\Phi)$

The following construction and formulas are based off of a standard calculation that can all be found in any number of our sources. See for example [KY10] section 2 or for greater generality [Kud97] parts I,1,2,4.

Proposition 3.42 gives us the splitting map $\gamma \mapsto [\gamma, \epsilon(\gamma)^{-1}]_g$ for $\gamma \in G(K)$. Under this map, we have $p \mapsto [p, 1]_g$ for $p \in P(K)$. Every section $\Phi(s) \in I(s, \chi)$ descends to a function on $P(K)\backslash G'_{\mathbb{A}_K}$. This follows from the calculation

$$\Phi\left(\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, 1\right]_g g', s\right) = \chi(a)|a|_{\mathbb{A}_K}^{s+1}\Phi(g', s) = \Phi(g', s)$$

where we used that $a \in K$ and χ is an idele class character.

For a standard section $\Phi(s)$ and $g' \in G'_{\mathbb{A}_K}$, we then form the sum

$$E(g', s, \Phi) = \sum_{\gamma \in P(K)\backslash G(K)} \Phi(\gamma g', s)$$

This is the Eisenstein series associated to $\Phi(s)$. The comments around $(I.2)$ of [Kud97] tell us that this series is absolutely convergent for $Re(s) > 1$ and has a meromorphic extension (in the variable s) to all s.

This function has a Fourier expansion, which can be found by temporarily fixing

g', using the splitting $n(b) = [n(b), 1]_g$ and considering the function of $b \in \mathbb{A}_K$

$$E(n(b)g', s, \Phi) = \sum_{\gamma \in P(K)\backslash G(K)} \Phi(\gamma n(b)g', s)$$

For $b \in K$, we have $(P(K)\backslash G(K))n(b) = P(K)\backslash G(K)$, and so it follows the above function is periodic with respect to K and admits a Fourier series. Let ψ denote the (unnormalized) exponential on $\mathbb{A}_K$. We may write (for Fourier coefficients depending on g')

$$E(n(b)g', s, \Phi) = \sum_{m \in K} E_m(g', s, \Phi)\psi(mb),$$
$$\text{where } E_m(g') = \int_{K\backslash\mathbb{A}_K} E(n(b)g', s, \Phi)\psi(-mb)db \quad (5.1)$$

In the above integral, db is chosen so the total measure is 1. Taking $b = 0$ in the left sum lets us write

$$E(g', s, \Phi) = \sum_{m \in K} E_m(g', s, \Phi), \text{ where } E_m(g') = \int_{K\backslash\mathbb{A}_K} E(n(b)g', s, \Phi)\psi(-mb)db$$

which is called the Fourier series of $E(g', s, \Phi)$.

In the case our section is factorizable, we get a nice formula for E_m. Below, let δ_m be a discrete delta function, equal to 1 when $m = 0$ and equal to 0 otherwise.

Proposition 5.2. *If Φ is factorizable and $g' = [g, z'_g]_g$,*

$$E_m(g', s, \Phi) = \delta_m \Phi(g', s) + z_{g'}|Disc(K)|^{-1/2} \prod_{\mathfrak{p} \leq \infty} W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}})$$

where $W_{m,\mathfrak{p}}$ are the local Whittaker functions, given by

$$W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}}) = \int_{K_{\mathfrak{p}}} \Phi_{\mathfrak{p}}([wn(b), 1]_L g'_{\mathfrak{p}}, s)\psi_{\mathfrak{p}}(-mb)db$$

Proof. Start with the definition of E_m and plug in the definition of E to get

$$E_m(g') = \int_{K\backslash\mathbb{A}_K} \sum_{\gamma \in P(K)\backslash G(K)} \Phi(\gamma n(b)g', s)\psi(-mb)db$$

For the integral, we will use the fundamental domain $\left(\prod_{p<\infty} O_{K_\mathfrak{p}}\right) \times (\mathbb{R}^n/O_K)$. We now swap the integral and sum and use (iv) of proposition 3.42 to get

$$\sum_{\gamma \in P(K)\backslash G(K)} z_{g'} \prod_{\mathfrak{p}<\infty} \int_{O_{K_\mathfrak{p}}} \Phi_\mathfrak{p}([\gamma n(b), 1]_L g', s)\psi_\mathfrak{p}(-mb)db \cdot$$
$$\int_{\mathbb{R}^n/O_K} \prod_{\mathfrak{p}|\infty} \Phi_p([\gamma n(b), 1]_L g', s)\psi_\mathfrak{p}(-mb)db \quad (5.3)$$

To maintain the correct measure, we choose db so that each $O_{K_\mathfrak{p}}$ has measure 1 and $\mathbb{R}^n/O_K$ also has measure 1.

By the Bruhat decomposition, a set of coset representatives for $P(K)\backslash G(K)$ is given by I along with $\left\{ w \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \right\}_{d \in K}$. We will prove the case we need right after this proposition, but see [Kud97] equation (0.5) for the general statement. For now, we continue with two cases.

When $\gamma = I$ in the above sum, we get a summand of

$$z_{g'} \prod_{p<\infty} \int_{O_{K_\mathfrak{p}}} \Phi_p([n(b), 1]_L g', s)\psi_p(-mb)db \int_{\mathbb{R}^n/O_K} \prod_{p|\infty} \Phi_p([n(b), 1]_L g', s)\psi_p(-mb)db$$

By the definition of section, Φ_p is left invariant to $n(b)$. Hence, we get

$$z_{g'} \prod_{\mathfrak{p}<\infty} \int_{O_{K_\mathfrak{p}}} \Phi_\mathfrak{p}(g', s)\psi_\mathfrak{p}(-mb)db \int_{\mathbb{R}^n/O_K} \prod_{\mathfrak{p}|\infty} \Phi_\mathfrak{p}(g', s)\psi_\mathfrak{p}(-mb)db$$
$$= z_{g'}\Phi(g', s) \prod_{\mathfrak{p}<\infty} \int_{O_{K_\mathfrak{p}}} \psi_\mathfrak{p}(-mb)db \int_{\mathbb{R}^n/O_K} \prod_{\mathfrak{p}|\infty} \psi_\mathfrak{p}(-mb)db$$

Each of the integrals over $O_{K_\mathfrak{p}}$ are 0 unless the integrand is identically 1. Hence, we can only get a nonzero value if $m \in \partial^{-1}$. Assuming this to be the case, we get

$$= z_{g'}\Phi(g', s) \int_{\mathbb{R}^n/O_K} \prod_{\mathfrak{p}|\infty} \psi_\mathfrak{p}(-mb)db = z_{g'}\Phi(g', s) \int_{\mathbb{R}^n/O_K} e^{2\pi i \mathrm{tr}(-mb)}$$

Since we assumed $m \in \partial^{-1}$, it follows our integrand is an exponential that is periodic with respect to the lattice O_K. Hence, by general properties of Fourier series, the integral is 0 unless $m = 0$. This gives us the $\delta_m \Phi(g', s)$ term in our sum.

For the second case, we now consider all terms where $\gamma = w\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}$. This yields

$$\sum_{d\in K} z_{g'} \prod_{\mathfrak{p}<\infty} \int_{O_{K_\mathfrak{p}}} \Phi_\mathfrak{p}([wn(b+d),1]_L g', s)\psi_\mathfrak{p}(-mb)db \int_{\mathbb{R}^n/O_K} \cdot$$
$$\prod_{\mathfrak{p}|\infty} \Phi_\mathfrak{p}([wn(b+d),1]_L g', s)\psi_\mathfrak{p}(-mb)db \quad (5.4)$$

We may change variables using $b_{\text{new}} = b_{\text{old}} + d$ to get

$$\psi(md) \sum_{d\in K} \prod_{\mathfrak{p}<\infty} \int_{d+O_{K_\mathfrak{p}}} \Phi_\mathfrak{p}([wn(b),1]_L g', s)\psi_\mathfrak{p}(-mb)db \int_{d+\mathbb{R}^n/O_K} \cdot$$
$$\prod_{\mathfrak{p}|\infty} \Phi_\mathfrak{p}([wn(b),1]_L g', s)\psi_\mathfrak{p}(-mb)db \quad (5.5)$$

Since $md \in K$, we have $\psi(md) = 1$. As d varies, our domain of integration perfectly tessellates $\mathbb{A}_K$ and we get

$$\prod_{\mathfrak{p}<\infty} \int_{K_\mathfrak{p}} \Phi_\mathfrak{p}([wn(b),1]_L g', s)\psi_\mathfrak{p}(-mb)db \int_{\mathbb{R}^n} \Phi_\mathfrak{p}([wn(b),1]_L g', s)\psi_\mathfrak{p}(-mb)db$$

The last step is to rescale our measure to the “usual" measure on $\mathbb{R}^n$. This gives us a factor of 1 over the area of O_K, which is where the $|Disc(K)|^{-1/2}$ comes from. □

Lemma 5.6. *A set of coset representatives for $P(K)\backslash G(K)$ is given by I along with* $\left\{w\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix}\right\}_{d\in K}$.

Proof. Once can easily verify the product

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}1/c & a\\ 0 & c\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & d/c\\ 0 & 1\end{pmatrix}$$

Depending on whether $c = 0$, we see that our set of coset representatives is sufficient. To see that I is not equivalent to any other representative, it suffices to check that

$$P(K) \neq P(K)w\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix} \Leftrightarrow w\begin{pmatrix}1 & d\\ 0 & 1\end{pmatrix} \notin P(K)$$

To check the other representatives are not equivalent, we must verify

$$P(K)w\begin{pmatrix}1 & d1\\ 0 & 1\end{pmatrix} \neq P(K)w\begin{pmatrix}1 & d2\\ 0 & 1\end{pmatrix} \Leftrightarrow w\begin{pmatrix}1 & d1-d2\\ 0 & 1\end{pmatrix}w^{-1} \notin P(K)$$

However, this just reduces to

$$\begin{pmatrix} 1 & 0 \\ d2 - d1 & 1 \end{pmatrix} \notin P(K)$$

□

Although we didn't explicitly check claims of the analyticity of E, it is easy enough to directly check that $W_{m,\mathfrak{p}}$ is holomorphic where defined. Even more strongly, there is a sense in which W is meromorphic at positive real infinity, which we describe in the next lemma.

Lemma 5.7. *$W_{m,\mathfrak{p}}$ is holomorphic in s for $Re(s) > 1$. For each fixed $g'_{\mathfrak{p}}$, there is a meromorphic function $f_{g'}$ on the open disc of radius $1/q$ such that $W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}}) = f_{g'}(q^{-s})$ for all $Re(s) > 1$.*

Proof. First recall the definition

$$W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}}) = \int_{K_{\mathfrak{p}}} \Phi_{\mathfrak{p}}([wn(b), 1]_L g'_{\mathfrak{p}}, s)\psi_{\mathfrak{p}}(-mb)db$$

Write $g' = \left[\begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}, 1\right]_L * k(g')$ for some $k'(g') \in \mathcal{K}'_{\mathfrak{p}}$. This gives us

$$W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}}) = \int_{K_{\mathfrak{p}}} \Phi_{\mathfrak{p}} \left(\left[\begin{pmatrix} 0 & -u^{-1} \\ u & v + bu^{-1} \end{pmatrix}, 1\right]_L k'(g'), s\right) \psi_{\mathfrak{p}}(-mb)db$$

By lemma 4.4, we have the Iwasawa decomposition $\begin{pmatrix} 0 & -u^{-1} \\ u & v + bu^{-1} \end{pmatrix} = p'_1 k'_1$, with $p'_1 = \begin{pmatrix} M(g', b)^{-1} & * \\ 0 & M(g', b) \end{pmatrix}$, where $M(g', b) \in \{u, v + bu^{-1}\}$ such that $v_\pi(M(g', b)) = min(v_\pi(u), v_\pi(v + bu^{-1}))$. Letting the function $k'(g', b) := k'_1 k'(g')$, we have

$$W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}}) = \int_{K_{\mathfrak{p}}} \chi_{\mathfrak{p}}(M(g', b)^{-1})|M(g', b)^{-1}|_{\mathfrak{p}}^{s+1} \Phi_{\mathfrak{p}}\left(k'(g', b), s\right) \psi_{\mathfrak{p}}(-mb)db$$

Since we are dealing with a standard section, $\Phi_{\mathfrak{p}}\left(k'(g', b), s\right)$ has no s dependence. With a little cleanup, we get

$$W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}}) = \int_{K_{\mathfrak{p}}} \chi_{\mathfrak{p}}(M(g', b)^{-1}) q^{(s+1)min(v_\pi(u), v_\pi(v+bu^{-1}))} \Phi_{\mathfrak{p}}\left(k'(g', b)\right) \psi_{\mathfrak{p}}(-mb)db$$

Since $\Phi_{\mathfrak{p}}$ is smooth, its absolute value on $\mathcal{K}_{\mathfrak{p}}$ is bounded above by some constant C. This gives us the upper bound

$$|W_{m,\mathfrak{p}}(g'_{\mathfrak{p}}, s, \Phi_{\mathfrak{p}})| \leq C \int_{K_{\mathfrak{p}}} q^{(s+1)min(v_\pi(u), v_\pi(v+bu^{-1}))} db$$

To handle the piecewise nature of the remaining term, it is convenient to partition the domain as $K_{\mathfrak{p}} = \pi^{v_0} O_{K_{\mathfrak{p}}} \sqcup (K_{\mathfrak{p}} - \pi^{v_0} O_{K_{\mathfrak{p}}})$ for some constant v_0. We choose $v_0 = min(v_\pi(u^2), v_\pi(uv))$ since as long as $b \notin \pi^{v_0} O_{K_{\mathfrak{p}}}$ then we will have

$$v_\pi(b) < min(v_\pi(u^2), v_\pi(uv)) \implies min(v_\pi(u), v_\pi(v + bu^{-1})) = v_\pi(bu^{-1})$$

Looking at our two cases, we can first consider

$$\int_{\pi^{v_0} O_{K_{\mathfrak{p}}}} q^{(s+1)min(v_\pi(u), v_\pi(v+bu^{-1}))} db$$

By construction, $min(v_\pi(u), v_\pi(v)) \leq min(v_\pi(u), v_\pi(v + bu^{-1})) \leq v_\pi(u)$, so this is clearly just some finite degree Laurent polynomial in q^{-s}. The domain is compact, so this integral converges absolutely.

In the second case, we consider

$$\int_{K_{\mathfrak{p}} - \pi^{v_0} O_{K_{\mathfrak{p}}}} q^{(s+1)v_\pi(bu^{-1})} db$$

If we break up the integral by the valuation of b, we get

$$\begin{aligned}
\sum_{j=v_0+1}^{\infty} (q^j - q^{j-1}) q^{(s+1)(-j-v_\pi(u))} &= \left(1 - \frac{1}{q}\right) q^{-v_\pi(u)} \sum_{j=v_0+1}^{\infty} q^{-s(j+v_\pi(u))} \\
&= \left(1 - \frac{1}{q}\right) q^{-v_\pi(u)} \sum_{j=v_0+v_\pi(u)+1}^{\infty} q^{-sj}
\end{aligned}$$

Since (for the convergence of the Eisenstein series) we are assuming $s > 1$, this sum converges absolutely. The desired results clearly follow.

$\square$

Remark 5.8. *When $v_\pi(b)$ is sufficiently negative, the Iwasawa decomposition we used above can be replaced by an explicit decomposition such as*

$$\begin{pmatrix} 0 & -u^{-1} \\ u & v + bu^{-1} \end{pmatrix} = \begin{pmatrix} b^{-1}u & b^{-1}v - u^{-1} \\ 0 & bu^{-1} \end{pmatrix} \begin{pmatrix} 1 - b^{-1}uv & -b^{-1}v^2 \\ b^{-1}u^2 & 1 + b^{-1}uv \end{pmatrix}$$

Using this, one may perform a slightly more detailed analysis to show that as long as $m \neq 0$, the power series above will only have finitely many terms. The reason for this vanishing is essentially that as $v_\pi(b)$ gets more negative, the exponential $\psi_\mathfrak{p}$ will start to oscillate much faster than the other terms in the integral, and so will cause the integral to average to 0.

We will now discuss how to convert the adelic Eisenstein series above into a Hilbert modular form (and how the Fourier expansions of the two relate).

Let $\Phi_f(s) = \otimes_{\mathfrak{p}<\infty}\Phi_\mathfrak{p}(s)$ denote a factorizable section of $\otimes'_{\mathfrak{p}<\infty} I_\mathfrak{p}(s,\chi_\mathfrak{p})$. Let v denote the product of the imaginary parts of $\vec{\tau}$ and let $g'_{\vec{\tau}} = \prod_{j=1}^n [n(u_j)m(\sqrt{v_j}), 1]_L = [g_{\vec{\tau}}, 1]_g$ where $g_{\vec{\tau}}$ is I at finite places and $n(u_j)m(\sqrt{v_j})$ at the jth infinite place. Then define

$$E(\vec{\tau}, s, \Phi^l_\infty(s) \otimes \Phi_f(s)) := v^{-l/2} E(g'_{\vec{\tau}}, s, \Phi^l_\infty(s) \otimes \Phi_f(s))$$

Applying the earlier "Fourier series" formula here, we may write $E(\vec{\tau})$ as a similar looking sum, yielding

$$E(\vec{\tau}, s, \Phi) = \sum_{m\in K} E_m(\vec{\tau}, s, \Phi), \text{ where} \tag{5.9}$$

$$E_m(\vec{\tau}, s, \Phi) = \delta_m v^{-l/2}\Phi(g'_{\vec{\tau}}, s) + |Disc(K)|^{-1/2} \prod_{\mathfrak{p}|\infty} v_\mathfrak{p}^{-l/2} W_{m,\mathfrak{p}}(g'_{\vec{\tau}}, s, \Phi^l_\mathfrak{p}) \prod_{\mathfrak{p}<\infty} W_{m,\mathfrak{p}}(1, s, \Phi_\mathfrak{p})$$

We may simplify the formula with some notation. For archimedean places, let $W_{m,\mathfrak{p}}(\vec{\tau}, s, \Phi^l_\mathfrak{p}) = v_\mathfrak{p}^{-l/2} W_{m,\mathfrak{p}}(g'_{\vec{\tau},\mathfrak{p}}, s, \Phi^l_\mathfrak{p})$. For nonarchimedean places, let $W_{m,\mathfrak{p}}(s, \Phi_\mathfrak{p}) = W_{m,\mathfrak{p}}(1, s, \Phi_\mathfrak{p})$. With the new notation, we have

$$E_m(\tau, s, \Phi) = \delta_m v^{-l/2}\Phi(g'_{\vec{\tau}}, s) + |Disc(K)|^{-1/2} \prod_{\mathfrak{p}|\infty} W_{m,p}(\vec{\tau}, s, \Phi^l_\mathfrak{p}) \cdot \prod_{\mathfrak{p}<\infty} W_{m,\mathfrak{p}}(s, \Phi_\mathfrak{p}) \tag{5.10}$$

Remark 5.11. *The reason we leave the "Fourier series" in equation (5.9) the way we do is that $E(\vec{\tau}, s, \Phi)$ will actually not be holomorphic in τ for most choices of s. As such, it does not admit a standard Fourier series. However, the sum above will provide a close enough stand in. When we compute the terms, we will find a natural* *decomposition* $E_m = c_m(s, \Phi) * f(s, \Phi, v) * e^{2\pi i \vec{m}\cdot\vec{\tau}}$ *where* c_m *is a constant (in that it doesn't depend on* τ*) and* f *is some non-holomorphic part. Furthermore, in all cases we will care about the non-holomorphic terms will vanish and we will get a standard Fourier series for a holomorphic function* $E(\vec{\tau}, s, \Phi)$*.*

5.2 Constructing $E^{l,\mu}$

We discuss a method from section 6 of [KY10] that allows one to start from certain Schwartz functions and build them into an Eisenstein series using the tools we have introduced. We will then start to evaluate the Fourier series of this Eisenstein series in a particular case.

Start by fixing a non-integer $l \in \frac{1}{2}\mathbb{Z}$ and a character $\chi(x) = \langle x, 2\kappa\rangle_{\mathbb{A}}$ for some squarefree integral element $\kappa \in O_K$. From our discussion of the global Hilbert symbol, χ is a ± 1 valued character on $K\backslash\mathbb{A}_K$.[1] The local components of χ are just given by $\chi_{\mathfrak{p}} = \langle x, 2\kappa\rangle_{\mathfrak{p}}$.

Assume that κ is either totally positive or totally negative and that, similarly to before,

$$\mathrm{sign}(\kappa) \equiv 2l \bmod 4$$

Note that before we were effectively taking κ to denote $sign(\kappa)$. Regardless of whether one uses κ or $sign(\kappa)$, the Archimedean $\chi_{\mathfrak{p}}$ are unchanged, so this change doesn't affect any of our previous conversation.

Let (V, Q) be a quadratic space of odd dimension d, chosen so that $\chi_V = \chi$. Choose an integral O_K-lattice L - that is, such that $(L, L)_Q \in O_K$. Finally, take $\mu \in L^*$.

Using the above data, for each finite place p consider the quadratic space $(V_{\mathfrak{p}}, Q)$ obtained by completing V. Let $\hat{L} \subset V_{\mathfrak{p}}$ denote the corresponding completion of L. On this space, we then take $\phi_{\mathfrak{p},\mu}$ to be the characteristic function $\mathbb{1}_{\mu+\hat{L}}$, which will have

[1] In fact, as we vary κ we get every character with these properties.

an associated standard section we call $\Phi_{\mathfrak{p},\mu} \in I(s, \chi_{\mathfrak{p}})$.

Multiplying these standard sections for all finite places along with Φ^l_∞ at the infinite places yields a global section

$$\Phi^{l,\mu} = \prod_{\mathfrak{p}<\infty} \Phi_{\mathfrak{p},\mu} * \Phi^l_\infty \in I(s, \chi)$$

From this, we then get the Eisenstein series

$$E^{l,\mu}(\vec{\tau}, s) := E(\vec{\tau}, s, \Phi^{l,\mu})$$

Our goal is to compute the Fourier series of this function by evaluating the local Whittaker functions and then plugging the results into equation (5.10). We first handle Archimedean places, then describe how to reduce the Whittaker function at a finite place to an integral in terms of more elementary functions. By far the hardest part will be evaluating this integral, which we will end up devoting two sections of writing to.

5.3 Level Structure of $E^{l,\mu}$

We now analyze the Eisenstein series $E^{l,\mu}(\tau, s)$ and attempt to determine its level structure. The following arguments are based heavily on [KRY06] section 8.5.6 and [HI13], sections 7 and 8.

The following proposition gives a transformation law for the function $E^{l,\mu}(g', s)$ on $g' \in G'_{\mathbb{A}_K}$.

Proposition 5.12. *There is a group $\Gamma_f = \prod_{\mathfrak{p}<\infty} \Gamma_{f,\mathfrak{p}}$ commensurable with $\prod_{\mathfrak{p}<\infty} G(O_{K_\mathfrak{p}})$, and a genuine character $\epsilon_\mu^{-1} : \Gamma'_f \to \mathbb{T}$ such that*

$$E^{l,\mu}(g'k', s) = \epsilon_\mu^{-1}(k')\ E^{l,\mu}(g', s) \text{ for } k' \in \Gamma'_f$$

Here, Γ'_f consists of all metaplectic elements with first coordinate in Γ_f. We may think of it as a subset of $G'_{\mathbb{A}_K}$ by taking the Archimedean components to be I. Furthermore,

Γ_f may be chosen so that for all $k' \in \Gamma_{f,\mathfrak{p}}$

$$\omega_V(k')\phi_{\mu,\mathfrak{p}}(t) = \epsilon_{\mu,\mathfrak{p}}^{-1}(k')\phi_{\mu,\mathfrak{p}}(t)$$

where $\epsilon_\mu^{-1}(k') = z_{k'} \prod_{\mathfrak{p}<\infty} \epsilon_{\mu,\mathfrak{p}}^{-1}(k'_\mathfrak{p})$, the components of ϵ_μ^{-1} are genuine characters guaranteed by proposition 4.16, and the quantities $z_{k'}$ and $k'_\mathfrak{p}$ are from the primary decomposition of k'.

In particular, if $k'_\mathfrak{p} \in \Gamma'_{f,\mathfrak{p}} \subset \mathcal{K}'_{0,\mathfrak{p}}(4)$ and $\Phi_{\mu,\mathfrak{p}}$ is spherical, then the component $\epsilon_{\mu,\mathfrak{p}}^{-1}(k'_\mathfrak{p}) = 1$.

Proof. By the definition of $E^{l,\mu}$, we have

$$E^{l,\mu}(g'k', s) = \sum_{\gamma \in P(K)\backslash G(K)} z_{k'} \prod_{\mathfrak{p}<\infty} \Phi_{\mu,\mathfrak{p}}([\gamma, 1]_L g'_\mathfrak{p}[k_\mathfrak{p}, 1], s) \prod_{\mathfrak{p}|\infty} \Phi_\mathfrak{p}^l([\gamma, 1]_L g'_\mathfrak{p}, s)$$

The result then follows by application of proposition 4.16. Commensurability follows since the local sections are spherical for all but finitely many places. □

Remark 5.13. *The dependence of $E^{l,\mu}$ on Archimedean components is simpler to calculate and basically comes down to the relation $\Phi^l(g'k') = \Phi^l(g')\nu_l(k')$.*

The above transformation law will turn into the level structure of $E^{l,\mu}(\tau, s)$. For this calculation, it will be most convenient to define a new automorphy factor of half-integral weight and then show later how it matches up with the standard one we introduced earlier.

Definition 5.14. *Fix an Archimedean place $\mathfrak{p}$. Let $[g, z] \in G'(K_\mathfrak{p})$ act on the upper half-plane $\mathcal{H}$ through the usual action of $g \in SL_2(\mathbb{R})$. (In particular, we ignore z for this action, which is why we do not specify the coordinate system of $[g, z]$.) Let $j(g, \tau) = c\tau + d$ denote the usual weight 1 factor of automorphy of $SL_2(\mathbb{R})$ acting on $\mathcal{H}$. Then, [HI13] section 7 tells us there is a unique factor of automorphy $\tilde{j}$ of $G'(K_\mathfrak{p})$ acting on $\mathcal{H}$ such that*

$$\tilde{j}([g, z]_R, \tau)^2 = z^2 j(g, \tau)$$

$\tilde{j}$ is given by the formula

$$\tilde{j}\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right]_R, \tau\right) = \begin{cases} z\sqrt{d} & c = 0, d > 0 \\ -z\sqrt{d} & c = 0, d < 0 \\ z\sqrt{c\tau + d} & c \neq 0 \end{cases}$$

Per our conventions, we use the branch cut of $\sqrt{x}$ so that the output has a complex argument obeying $-\pi/2 < \theta \leq \pi/2$.

Let $G'_{\mathbb{A}_K,\infty}$ denote those elements $g' = [g, z]_g$ such that $g_{\mathfrak{p}} = I$ for all finite places. (That is, $G'_{\mathbb{A}_K,\infty}$ is a metaplectic cover of $SL_2(\mathbb{R})^n$. An element $g' = [g, z]_g$ of $G'_{\mathbb{A}_K,\infty}$ acts on $\mathcal{H}^n$ via $g'\vec{\tau} = g\vec{\tau}$. Writing $g' = z_{g',R}\prod_{j=1}^n [g_j, 1]_R$ (the subscript R is to emphasize that z comes from Rao coordinates), we define an automorphy factor for this action by

$$\tilde{j}_\infty(g', \vec{\tau}) = z_{g',R}\prod_{j=1}^n \tilde{j}([g_j, 1]_R, \tau_j)$$

Lemma 5.15. *$\tilde{j}_\infty$ is indeed a factor of automorphy.*

Proof. We do not check that the components $\tilde{j}$ are factors of automorphy, and continue to take this as fact from [HI13]. For a product $g'_1 g'_2$, we have the decomposition $g'_1 g'_2 = z_{g'_1 g'_2,R}\prod_{j=1}^n [g_{1,j}g_{2,j}, 1]_R$. This yields

$$\tilde{j}_\infty(g'_1 g'_2, \vec{\tau}) = z_{g'_1 g'_2,R}\prod_{j=1}^n \tilde{j}([g_{1,j}g_{2,j}, 1]_R, \tau_j)$$

Since $\tilde{j}$ is a factor of automorphy for each Archimedean place, we have

$$\tilde{j}_\infty(g'_1 g'_2, \vec{\tau}) = z_{g'_1 g'_2,R}\prod_{j=1}^n c_R(g_{1,j}, g_{2,j})^{-1}\prod_{j=1}^n \tilde{j}([g_{1,j}, 1]_R, g_{2,j}\tau_j)\prod_{j=1}^n \tilde{j}(g_{2,j}, 1]_R, \tau_j)$$

This simplifies to

$$\tilde{j}_\infty(g'_1 g'_2, \vec{\tau}) = z_{g'_1 g'_2,R}\prod_{j=1}^n c_R(g_{1,j}, g_{2,j})^{-1} z^{-1}_{g'_1,R} z^{-1}_{g'_2,R}\tilde{j}_\infty(g'_1, g'_2\vec{\tau})\tilde{j}_\infty(g'_2, \vec{\tau})$$

Therefore, it suffices to show that

$$z_{g'_1 g'_2,R}\prod_{j=1}^n c_R(g_{1,j}, g_{2,j})^{-1} z^{-1}_{g'_1,R} z^{-1}_{g'_2,R} = 1$$

However, this follows easily from $g_1'g_2' = z_{g_1'g_2',R} \prod_{j=1}^n [g_{1,j}g_{2,j}, 1]_R$. We expand the g_1' and g_2' on the left as products of local terms and expand the right using the Rao cocycle to get

$$z_{g_1',R} \prod_{j=1}^n [g_{1,j}, 1]_R \cdot z_{g_2',R} \prod_{j=1}^n [g_{2,j}, 1]_R = z_{g_1'g_2',R} \prod_{j=1}^n c_R(g_{1,j}, g_{2,j})^{-1} \prod_{j=1}^n [g_{1,j}, 1]_R \prod_{j=1}^n [g_{2,j}, 1]_R$$

Cancelling terms yields the desired relation. $\square$

Besides an automorphy factor, we will also need a formula for $E^{l,\mu}(\tau, s)$ that depends more naturally on $E^{l,\mu}(g', s)$ than the current way it is defined (where one must use the particular choice of matrix $g'_{\vec{\tau}}$).

Lemma 5.16. *(i) For all $\tau = u+iv \in \mathcal{H}$, one has $[n(u)m(\sqrt{v}), 1]_L = [n(u)m(\sqrt{v}), 1]_R$. In particular, this implies $g'_{\vec{\tau}} = \prod_{j=1}^n [n(u_j)m(\sqrt{v_j}), 1]_L = \prod_{j=1}^n [n(u_j)m(\sqrt{v_j}), 1]_R$*

(ii) $c_R(g_\tau, h) = 1$ for any $h \in SL_2(\mathbb{R})$.

(iii) $c_R(h, g_\tau) = 1$ for any $h \in SL_2(\mathbb{R})$.

Proof. Let $g_\tau = n(u)m(\sqrt{v})$. For (i), we have

$$[g_\tau, 1]_L = [g_\tau, \gamma_w\left(x(g_\tau), 1/2\right)\gamma_w(1/2)^{j(g_\tau)}]_R = [g_\tau, \gamma_w(1/(2\sqrt{v}))/\gamma_w(1/2)]_R = [g_\tau, 1]_R$$

This relation clearly implies the second part of (i).

For (ii), since g_τ is upper triangular, we have $x(g_\tau h) = x(h)/\sqrt{v}$. Hence

$$c_R(g_\tau, h) = \left\langle \frac{1/\sqrt{v}}{x(h)/\sqrt{v}}, \frac{x(h)}{x(h)/\sqrt{v}} \right\rangle_{\mathbb{R}}$$

Since the second entry is positive, the Hilbert symbol is automatically 1.

For (iii), write $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We may assume $c \neq 0$, since that case is covered by part (ii). Then, we have $x(h) = c$ and $x(hg_\tau) = c\sqrt{v}$. Hence

$$c_R(g_\tau, h) = \left\langle \frac{c}{c\sqrt{v}}, \frac{\sqrt{v}}{c\sqrt{v}} \right\rangle_{\mathbb{R}}$$

Since the first entry is positive, the Hilbert symbol is automatically 1. $\square$

Recall that $\vec{i} \in \mathcal{H}^n$ is the vector all of whose entries are i.

Proposition 5.17. *Fix a $\vec{\tau} \in \mathcal{H}^n$ and let $g' \in G'_{\mathbb{A}_K,\infty}$. Write $g' = z_{g',R} \prod_{j=1}^n [g_j, 1]_R$ as above. Then, for any g' such that $g'\vec{i} = \vec{\tau}$ (for example, $g'_{\vec{\tau}}$ is one such element), we have*

$$E^{l,\mu}(\vec{\tau}, s) = E^{l,\mu}(g', s)\tilde{j}_\infty(g', \vec{i})^{2l} z_{g',R}^{-(2l+1)}$$

Proof. As we have done so far, let the jth component of $\vec{\tau}$ be τ_j and write $\tau_j = u_j + iv_j$. We first check that our formula holds In the case $g' = g'_{\vec{\tau}}$. In this case, we have $\tilde{j}([g_j, 1]_R, i) = \sqrt{1/\sqrt{v_j}}$. Lemma 5.16 further tells us that $[g_j, 1]_R = [g_j, 1]_L$ and $z_{g',R} = 1$. It follows that setting $g' = g'_{\vec{\tau}}$ in the above formula gives

$$E^{l,\mu}(\vec{\tau}, s) = E^{l,\mu}v^{-l/2}(g'_{\vec{\tau}}, s)$$

which is precisely the definition of $E^{l,\mu}(\vec{\tau}, s)$. This shows that the desired formula holds for $g' = g'_{\vec{\tau}}$.

To prove the formula for arbitrary g', we note that we can always write

$$g' = g'_{\vec{\tau}} \left(\prod_{\mathfrak{p}|\infty} k'(\theta_\mathfrak{p}) \right) [1, z]_g$$

where the middle term consists of an element of $\widetilde{SO}_2(K_\mathfrak{p})$ at each Archimedean place.[2] Our desired formula is clearly invariant to z, since $E^{l,\mu}(g', s)$ scales with z^1, $\tilde{j}_\infty(g', \vec{i})^{2l}$ scales with z^{2l}, and $z_{g',R}^{-(2l+1)}$ scales with $z^{-(2l+1)}$. So, it suffices to prove the formula for $g' = g'_{\vec{\tau}} \left(\prod_{\mathfrak{p}|\infty} k'(\theta_\mathfrak{p}) \right)$.

Generally, in a matrix $k'(\theta_\mathfrak{p})$, one has $0 \leq \theta_\mathfrak{p} < 4\pi$. However, since $k'(\theta_\mathfrak{p} + 2\pi) = k'(\theta_\mathfrak{p})[1, -1]_g$, we may restrict ourselves to the range $-\pi < \theta_\mathfrak{p} \leq \pi$ and hence take $k'(\theta_\mathfrak{p}) = [k(\theta_\mathfrak{p}), 1]_R$ as per equation (4.17). Now write out the factorization

$$g'_{\vec{\tau}} = z_{g'_{\vec{\tau}},R} \prod_{j=1}^n [g_{\tau_j}, 1]_R$$

[2]Although we usually index the Archimedean places with $1 \leq j \leq n$, we don't in this formula to really emphasize that the k' are at different places.

Multiplying each side by $\prod_{\mathfrak{p}|\infty}[k(\theta_\mathfrak{p}),1]_R$ then gives us

$$g' = z_{g'_{\vec{\tau}},R}\prod_{j=1}^{n}[g_{\tau_j}k(\theta_\mathfrak{p}),1]_R$$

where we made use of part (ii) of lemma 5.16. However, the above equation is now exactly the factorization that defines $z_{g',R}$ and hence we conclude that $z_{g',R} = z_{g'_{\vec{\tau}},R}$. We may now write

$$E^{l,\mu}(g',s)\tilde{j}_\infty(g',\vec{i})^{2l}z_{g',R}^{-(2l+1)} = \\ E^{l,\mu}\left(g'_{\vec{\tau}}\left(\prod_{\mathfrak{p}|\infty}k'(\theta_\mathfrak{p})\right),s\right)\cdot\tilde{j}_\infty\left(g'_{\vec{\tau}}\left(\prod_{\mathfrak{p}|\infty}k'(\theta_\mathfrak{p})\right),\vec{i}\right)^{2l}\cdot z_{g'_{\vec{\tau}},R}^{-(2l+1)} \quad (5.18)$$

The $E^{l,\mu}$ term may be expanded via the definition of the section Φ^l. The $\tilde{j}_\infty$ term may be expanded since it is an automorphy factor. Continuing from the previous line, we have

$$= E^{l,\mu}\left(g'_{\vec{\tau}},s\right)\prod_{\mathfrak{p}|\infty}\nu_l(\theta_\mathfrak{p})\cdot\tilde{j}_\infty\left(g'_{\vec{\tau}},\vec{i}\right)^{2l}\tilde{j}_\infty\left(\left(\prod_{\mathfrak{p}|\infty}k'(\theta_\mathfrak{p})\right),\vec{i}\right)^{2l}\cdot z_{g'_{\vec{\tau}},R}^{-(2l+1)}$$

By the case $g' = g'_{\vec{\tau}}$ we already did, the terms containing $g'_{\vec{\tau}}$ combine to exactly form $E^{l,\mu}(\vec{\tau},s)$. We again plug in $k'(\theta_\mathfrak{p}) = [k(\theta_\mathfrak{p}),1]_R$ to get

$$= E^{l,\mu}(\vec{\tau},s)\cdot\prod_{\mathfrak{p}|\infty}\nu_l(\theta_\mathfrak{p})\cdot\tilde{j}_\infty\left(\left(\prod_{\mathfrak{p}|\infty}[k(\theta_\mathfrak{p}),1]_R\right),\vec{i}\right)^{2l}$$

We can now evaluate this using the definition of $\tilde{j}_\infty$. It may help to recall that $k(\theta) = \begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}$. For our $-\pi < \theta_\mathfrak{p} \leq \pi$ there is light casework on whether $\theta_\mathfrak{p} \in \{0,\pi\}$, although in any case we get

$$= E^{l,\mu}(\vec{\tau},s)\cdot\prod_{\mathfrak{p}|\infty}\nu_l(\theta_\mathfrak{p})\cdot\prod_{\mathfrak{p}|\infty}\left(e^{-i\theta_\mathfrak{p}/2}\right)^{2l} = E^{l,\mu}(\vec{\tau},s)$$

$\square$

For $\gamma \in SL_2(K)$, let $\gamma_{0,\infty}$ denote the image of γ_0 under the embedding $SL_2(K) \to SL_2(\mathbb{R})^n$. We may think of it as an element of $G(\mathbb{A}_K)$ by taking it to be I at all finite places. In particular, we have $[\gamma_{0,\infty},1]_g = \prod_{\mathfrak{p}|\infty}[\gamma_{0,\mathfrak{p}},1]_L$. Furthermore, let $\gamma_{0,f} = \prod_{\mathfrak{p}<\infty}\gamma_{0,\mathfrak{p}}$. The following proposition gives an automorphy factor for $E^{l,\mu}(\vec{\tau},s)$.

Proposition 5.19. *Let $\Gamma = SL_2(K) \cap (\Gamma_f \times SL_2(\mathbb{R})^n)$ for the group Γ_f guaranteed by proposition 5.12. For $\gamma_0 \in \Gamma$,*

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \epsilon_\mu([\gamma_{0,f}, \epsilon(\gamma_0)^{-1}]_g) \prod_{\mathfrak{p}|\infty} \beta_\mathfrak{p}(\gamma_{0,\mathfrak{p}}) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_R, \vec{\tau})^{2l} E^{l,\mu}(\vec{\tau}, s)$$

where ϵ is the character giving the splitting of $SL_2(K)$ given in proposition 3.42, ϵ_μ is from proposition 5.12, and $\beta_\mathfrak{p}(g) = \gamma_w\,(x(g), 1/2)^{-1}\,\gamma_w(1/2)^{-j(g)}$ is from definition 3.22.

This proposition along with proposition 5.12 implies that any $\gamma_0 \in SL_2(K)$ that takes all of the $\phi_{\mu,\mathfrak{p}}$ as eigenfunctions under the Weil representation will be a member of Γ.

Warning 5.20. *Although $E^{l,\mu}(\gamma_0\vec{\tau}, s)$ transforms like a modular form and for each $\vec{\tau}$ it is holomorphic in s, it will not be holomorphic in $\vec{\tau}$ for most (if any) choices of s and hence won't be a (holomorphic) modular form. However, when we evaluate our cases of interest later, it will be clear from their Fourier series formulas that they are holomorphic and hence are actual modular forms.*

Proof. Let $\gamma_{0,\infty}$ denote the image of γ_0 under the embedding $SL_2(K) \to SL_2(\mathbb{R})^n$. We may think of it as an element of $G(\mathbb{A}_K)$ by taking it to be I at all finite places. The idea is now to compute $E^{l,\mu}(\gamma_0\vec{\tau}, s)$ by taking $g' = [\gamma_{0,\infty}, 1]_g g'_{\vec{\tau}}$ in proposition 5.17, since we have $g'\vec{i} = \gamma_0\vec{\tau}$. The first step is to find $z_{g',R}$. We do this by writing

$$g' = [\gamma_{0,\infty}, 1]_g \cdot g'_{\vec{\tau}} = \prod_{\mathfrak{p}|\infty} [\gamma_{0,\mathfrak{p}}, 1]_L \prod_{\mathfrak{p}|\infty} [g_{\tau_\mathfrak{p}}, 1]_R$$

Using definition 3.22 we convert to Rao coordinates and then apply part (iii) of lemma 5.16 to get

$$g' = \prod_{\mathfrak{p}|\infty} \beta_\mathfrak{p}^{-1}(\gamma_{0,\mathfrak{p}}) \prod_{\mathfrak{p}|\infty} [\gamma_{0,\mathfrak{p}} g_{\tau_\mathfrak{p}}, 1]_R \implies z_{g',R} = \prod_{\mathfrak{p}|\infty} \beta_\mathfrak{p}^{-1}(\gamma_{0,\mathfrak{p}})$$

where β is the factor that is picked up under the coordinate change.

If we now take $g' = \gamma_0 g'_{\vec{\tau}}$ in proposition 5.17, we have

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = E^{l,\mu}([\gamma_{0,\infty}, 1]_g g'_{\vec{\tau}}, s) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_g g'_{\vec{\tau}}, i)^{2l} \cdot z_{g',R}^{-(2l+1)}$$

Note that $[\gamma_{0,f}, \epsilon^{-1}(\gamma_0)]_g[\gamma_{0,\infty}, 1]_g = \gamma_0$. By proposition 5.12, we get

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \chi_{l,\mu}([\gamma_{0,f}, \epsilon^{-1}(\gamma_0)]_g)E^{l,\mu}(\gamma_0 g'_{\vec{\tau}}, s) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_g g'_{\vec{\tau}}, i)^{2l} \cdot z_{g',R}^{-(2l+1)}$$

Because we can reindex the sum defining $E^{l,\mu}(g', s)$, it follows that $E^{l,\mu}(\gamma_0 g', s) = E^{l,\mu}(g', s)$ for all $\gamma_0 \in SL_2(K)$. So, we get

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \chi_{l,\mu}([\gamma_{0,f}, \epsilon^{-1}(\gamma_0)]_g)E^{l,\mu}(g'_{\vec{\tau}}, s) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_g g'_{\vec{\tau}}, i)^{2l} \cdot z_{g',R}^{-(2l+1)}$$

Since $\tilde{j}_\infty$ is a factor of automorphy, we get

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \epsilon_\mu([\gamma_{0,f}, \epsilon^{-1}(\gamma_0)]_g)E^{l,\mu}(g'_{\vec{\tau}}, s) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_g, \vec{\tau})^{2l}\tilde{j}_\infty(g'_{\vec{\tau}}, i)^{2l} \cdot z_{g',R}^{-(2l+1)}$$

By part (i) of lemma 5.16, $z_{g'_{\vec{\tau}},R} = 1$ and so we get

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \epsilon_\mu([\gamma_{0,f}, \epsilon^{-1}(\gamma_0)]_g)\tilde{j}_\infty([\gamma_{0,\infty}, 1]_g, \vec{\tau})^{2l} \cdot z_{g',R}^{-(2l+1)} E^{l,\mu}(\vec{\tau}, s)$$

For a final simplification, we can move some of the β terms into the $\tilde{j}_\infty$ term to get

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \epsilon_\mu([\gamma_{0,f}, \epsilon^{-1}(\gamma_0)]_g)\tilde{j}_\infty([\gamma_{0,\infty}, 1]_R, \vec{\tau})^{2l} \cdot z_{g',R}^{-1} E^{l,\mu}(\vec{\tau}, s)$$

$\square$

Unfortunately, we will need to wait until after our evaluation of the Gauss sum in order to make these automorphy factors more explicit and relate them back to the standard automorphy factors of half-integral weight. We do however have one more important remark we can make. The Eisenstein series only converges for $Re(s)$ sufficiently large, so all of our formulas we have proven currently only hold for such s.

Remark 5.21. *As previously stated, $E^{l,\mu}(\vec{\tau}, s)$ admits an analytic continuation to all s. The transformation law*

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \epsilon_\mu([\gamma_{0,f}, \epsilon^{-1}(\gamma_0)]_g) \prod_{\mathfrak{p}|\infty} \beta_{\mathfrak{p}}(\gamma_{0,\mathfrak{p}}) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_R, \vec{\tau})^{2l} E^{l,\mu}(\vec{\tau}, s)$$

also analytically continues. This is easily seen by noting that for fixed τ, the left and right sides are both holomorphic in s. Hence, as long as our calculations show the two sides are equal in some right halfplane, then they will be equal for all s.

5.4 Archimedean Local Whittaker Functions

Fix an infinite place so that $K_{\mathfrak{p}} = \mathbb{R}$. In this case, computing the local Whittaker function is mainly a calculus problem that was solved in [Shi82], where it is shown to depend on a certain integral called a confluent hypergeometric function. We follow the approach taken in [KY10] p.2281-2282. However, due to subtle differences in coordinates, we will get a slightly different result that instead matches with [KRY06] equation (5.7.14).

For this section we are working at a fixed place, so we write $\tau = u + iv$ to denote the appropriate component of $\vec{\tau}$. Furthermore, although $m \in K$, for this section we will use $m \in \mathbb{R}$ to denote the image of m under the embedding of K into $K_{\mathfrak{p}}$.

Recall that the Archimedean local Whittaker function is given by

$$W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^l) = v_{\mathfrak{p}}^{-l/2} \int_{\mathbb{R}} \Phi_{\mathfrak{p}}^l([wn(b), 1]_L g'_\tau, s) e^{-2\pi i m b} db$$

Temporarily introduce new quantities α, β given by

$$\alpha = \frac{l+s+1}{2}, \quad \beta = \frac{-l+s+1}{2}$$

Proposition 5.22.

$$W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^l) = e^{2\pi i/8} e^{2\pi i m u} v_{\mathfrak{p}}^{\beta} \xi(v, m; \alpha, \beta)$$

where $\tau = u + iv$ and

$$\xi(v, m; \alpha, \beta) := \int_{\mathbb{R}} (b + iv)^{-\alpha} (b - iv)^{-\beta} e^{-2\pi i m b} db$$

is the same as the function Shimura defines in [Shi82] equation (1.25).[3]

Proof. This comes mostly down to evaluating the function $\Phi^l_{\mathfrak{p}}([wn(b),1]_L g'_\tau, s)$. To do this, we multiply out

$$[wn(b),1]_L g'_\tau = \left[\begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix}, 1\right]_L \left[\begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}, 1\right]_L = \left[\begin{pmatrix} 0 & -1/\sqrt{v} \\ \sqrt{v} & (u+b)/\sqrt{v} \end{pmatrix}, 1\right]_L$$

We then use the decomposition

$$\begin{aligned}\left[\begin{pmatrix} 0 & -1/\sqrt{v} \\ \sqrt{v} & (u+b)/\sqrt{v} \end{pmatrix}, 1\right]_L &= \\ &\left[\begin{pmatrix} \frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}} & \frac{-(u+b)}{\sqrt{v}\sqrt{v^2+(u+b)^2}} \\ 0 & \frac{\sqrt{v^2+(u+b)^2}}{\sqrt{v}} \end{pmatrix}, 1\right]_L \left[\begin{pmatrix} \frac{u+b}{\sqrt{v^2+(u+b)^2}} & -\frac{v}{\sqrt{v^2+(u+b)^2}} \\ \frac{v}{\sqrt{v^2+(u+b)^2}} & \frac{u+b}{\sqrt{v^2+(u+b)^2}} \end{pmatrix}, 1\right]_L \end{aligned} \tag{5.23}$$

The second matrix should be in Rao coordinates so that we can evaluate equation (4.18). From definition 3.22, we have

$$[g,z]_L = [g, z\gamma_w\left(x(g),1/2\right)\gamma(1/2)^{j(g)}]_R$$

In our case, this becomes

$$\left[\begin{pmatrix} \frac{u+b}{\sqrt{v^2+(u+b)^2}} & -\frac{v}{\sqrt{v^2+(u+b)^2}} \\ \frac{v}{\sqrt{v^2+(u+b)^2}} & \frac{u+b}{\sqrt{v^2+(u+b)^2}} \end{pmatrix}, 1\right]_L = \left[\begin{pmatrix} \frac{u+b}{\sqrt{v^2+(u+b)^2}} & -\frac{v}{\sqrt{v^2+(u+b)^2}} \\ \frac{v}{\sqrt{v^2+(u+b)^2}} & \frac{u+b}{\sqrt{v^2+(u+b)^2}} \end{pmatrix}, \gamma_w(x(g)/2)\right]_R$$

In our case, $x(g) = v/\sqrt{v^2+(u+b)^2} > 0$ and so $\gamma_w(x(g)/2) = e^{2\pi i/8}$ by fact 3.4. By equation (4.3) for the definition of a section, we can now write our section as

$$\begin{aligned}\Phi^l_{\mathfrak{p}}([wn(b),1]_L g'_\tau, s) = e^{2\pi i/8}\chi_V &\left(\frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}}\right)\left(\frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}}\right)^{s+1} \\ &\times \Phi^l_{\mathfrak{p}}\left(\left[\begin{pmatrix} \frac{u+b}{\sqrt{v^2+(u+b)^2}} & -\frac{v}{\sqrt{v^2+(u+b)^2}} \\ \frac{v}{\sqrt{v^2+(u+b)^2}} & \frac{u+b}{\sqrt{v^2+(u+b)^2}} \end{pmatrix}, 1\right]_R, s\right) \end{aligned} \tag{5.24}$$

[3]The function Shimura defines is more general and specializes to the function ξ we give here. One does have to wade through some notational differences to check that Shimura's definition lines up. This is fairly straightforward using the fact that the case $G = SL_2$ and $K_{\mathfrak{p}} = \mathbb{R}$ we are working in corresponds to Case I, $m = 1$ in Shimura's work. (Although take care that we mean $m = 1$ for Shimura's variable called m, which is not the same as our m.) The only point requiring some added care is to note that equation (1.11) of [Shi82] gives the same exponentiation conventions on points in the upper or lower halfplanes that we are using here.

The matrix in $\Phi_{\mathfrak{p}}^{l}$ is now some $k'(\theta) \in \widetilde{SO_2}(\mathbb{R})$. We will write it as such from now on to save space. We can further simplify by noting that the χ_V term is trivial. To prove this we first write out

$$\chi_V\left(\frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}}\right) = \left\langle \frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}}, (-1)^{dim(V)*(dim(V)-1)/2} det(V)\right\rangle_{\mathfrak{p}}$$

and then recall that at an Archimedean place the Hilbert symbol is 1 as long as either argument (in this case the first one) is positive. From all this we may rewrite the equation as the more manageable

$$\Phi_{\mathfrak{p}}^{l}([wn(b),1]_L g'_\tau, s) = e^{2\pi i/8}\left(\frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}}\right)^{s+1} \nu_l\left(k'(\theta)\right)$$

To determine $\theta \in \mathbb{R}/4\pi$, we use (4.17)to see that $-\pi < \theta \leq \pi$. θ is now determined by the relation

$$\begin{pmatrix} \frac{u+b}{\sqrt{v^2+(u+b)^2}} & -\frac{v}{\sqrt{v^2+(u+b)^2}} \\ \frac{v}{\sqrt{v^2+(u+b)^2}} & \frac{u+b}{\sqrt{v^2+(u+b)^2}} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

From this it follows that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) = \frac{u+b-iv}{\sqrt{v^2+(u+b)^2}}$$

By our conventions on raising complex numbers to non-integer powers[4], we get that

$$\Phi_{\mathfrak{p}}^{l}([wn(b),1]_L g'_\tau, s) = e^{2\pi i/8}\left(\frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}}\right)^{s+1}\left(\frac{u+b-iv}{\sqrt{v^2+(u+b)^2}}\right)^{l}$$

Therefore, the Archimedean local Whittaker function is given by

$$W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^{l}) = e^{2\pi i/8} v_{\mathfrak{p}}^{-l/2}\int_{\mathbb{R}}\left(\frac{\sqrt{v}}{\sqrt{v^2+(u+b)^2}}\right)^{s+1}\left(\frac{u+b-iv}{\sqrt{v^2+(u+b)^2}}\right)^{l} e^{-2\pi i m b}db$$

If we apply the change of variables $b_{\text{new}} = b_{\text{old}} + u$, we get

$$e^{2\pi i/8}e^{2\pi i m u} v_{\mathfrak{p}}^{-l/2}\int_{\mathbb{R}}\left(\frac{\sqrt{v}}{\sqrt{v^2+b^2}}\right)^{s+1}\left(\frac{b-iv}{\sqrt{v^2+b^2}}\right)^{l} e^{-2\pi i m b}db$$

[4]Recall that at the end of our notation section, we take $a^b = e^{b\ln(a)}$ for $-\pi < Im(\ln(a)) \leq \pi$, which works perfectly with what we have here.

Again under our complex power conventions, we have the factorization $\sqrt{v^2+b^2} = (b+iv)^{1/2}(b-iv)^{1/2}$. This lets us rewrite the integral as

$$e^{2\pi i/8}e^{2\pi imu}v_{\mathfrak{p}}^{\beta}\int_{\mathbb{R}}(b+iv)^{-\alpha}(b-iv)^{-\beta}e^{-2\pi imb}db$$

□

The following proposition comes directly from [Shi82] equation (1.29) and the comment directly after, the definition of η is from [Shi82] equation (1.26). It is okay that the condition on s we give is far from tight, since the bound on s here will not affect the coefficients of the Fourier series we end up calculating.

Proposition 5.25. *If the real part of s is sufficiently large ($Re(s) > |l|+1$ suffices), then*

$$\xi(v,m;\alpha,\beta) = 2\pi(-i)^l\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\eta(2v,\pi m;\alpha,\beta)$$

where

$$\eta(v,m;\alpha,\beta) := \int_{x>|m|}e^{-vx}(x+m)^{\alpha-1}(x-m)^{\beta-1}dx$$

Proof. Specializing Shimura's results to our case gives the above on the condition that $Re(\alpha) > 0$ and $Re(\beta) > 1$. It is easy to check that $Re(s) > |l|+1$ is sufficient for these conditions to hold. □

The function η is closely related to the confluent hypergeometric function of the second kind. The following properties are from [LSL65] sections 9.10 and equation (9.11.6) of section 9.11. This is a slight rearrangement of how things are stated in [LSL65], but is better for our purposes.

Definition 5.26. *Let a, z be complex numbers of positive real part and let b be any complex number. The confluent hypergeometric function of the second kind is given by the integral*

$$\Psi(a,b;z) := \frac{1}{\Gamma(a)}\int_0^{\infty}e^{-zt}(t+1)^{b-a-1}t^{a-1}dt$$

$\Psi(a, b; z)$ is holomorphic in all three arguments and admits an analytic continuation that is entire in a, b but where z is restricted to complex numbers of argument between $-\pi$ and π.

Define the related function

$$\Psi_l(s, z) := \Psi\left(\frac{l+s+1}{2}, s+1; z\right) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zt}(t+1)^{\beta-1} t^{\alpha-1} dt$$

and note that since replacing l with $-l$ swaps α and β, we have

$$\Psi_{-l}(s, z) := \Psi\left(\frac{-l+s+1}{2}, s+1; z\right) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-zt}(t+1)^{\alpha-1} t^{\beta-1} dt$$

Lemma 5.27.

$$\Psi(0, b; z) = 1$$

Proof. This fact is stated on page 2281 of [KY10]. We provide an informal argument anyway for the fun of it. First, write

$$\Psi(0, b; z) = \lim_{a \to 0^+} \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}(t+1)^{b-a-1} t^{a-1} dt$$

In the limit, we have $1/\Gamma(a) \sim a$. On the other hand, for any $\epsilon > 0$ we have that

$$\lim_{a \to 0^+} \frac{1}{\Gamma(a)} \int_\epsilon^\infty e^{-zt}(t+1)^{b-a-1} t^{a-1} dt = \lim_{a \to 0^+} \frac{1}{\Gamma(a)} \int_\epsilon^\infty e^{-zt}(t+1)^{b-1} t^{-1} dt = 0$$

Hence, the behavior of the integrand only matters near $t = 0$ and so we can replace the integrand with $e^{-z*0}(0+1)^{b-a-1} t^{a-1} = t^{a-1}$. We then get

$$\lim_{a \to 0^+} \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}(t+1)^{b-a-1} t^{a-1} dt = \lim_{a \to 0^+} a \int_0^\infty t^{a-1} dt = 1$$

$\square$

Lemma 5.28. *For any α, β such that $\eta(v, m; \alpha, \beta)$ converges we have*

$$\eta(v, m; \alpha, \beta) = \begin{cases} (2m)^s e^{-mv} \Gamma(\beta) \Psi_{-l}(s, 2mv) & m > 0 \\ (2|m|)^s e^{-|m|v} \Gamma(\alpha) \Psi_l(s, 2|m|v) & m < 0 \\ v^{-s} \Gamma(s) & m = 0 \end{cases}$$

The conditions $Re(\alpha), Re(\beta) > 0$ are sufficient for $\eta(v, m; \alpha, \beta)$ to converge. Therefore, it suffices for $Re(s) > |l| - 1$.

Proof. It is trivial to check that $Re(s) > |l|-1$ implies that $Re(\alpha), Re(\beta) > 0$ and that this is sufficient for convergence, so we move onto calculating η. Start by substituting $x_{\text{new}} = x_{\text{old}} - m$ to get

$$\eta(v,m;\alpha,\beta) = e^{-mv} \int_{x>0} e^{-vx}(x+2m)^{\alpha-1}x^{\beta-1}dx$$

Now substitute $x_{\text{old}} = 2mx_{\text{new}}$ to get

$$\eta(v,m;\alpha,\beta) = (2m)^{\alpha+\beta-1}e^{-mv} \int_{x>0} e^{-2mvx}(x+1)^{\alpha-1}x^{\beta-1}dx$$

At this point, we can use the relation $\alpha+\beta-1 = s$ and recognize the integral as $\Psi_{-l}(s, 2mv)$.

In the case $m < 0$, rewrite

$$\eta(v,m;\alpha,\beta) = \int_{x>|m|} e^{-vx}(x-|m|)^{\alpha-1}(x+|m|)^{\beta-1}dx = \eta(v,|m|,\beta,\alpha)$$

This is then handled by the $m > 0$ case.

Finally, the $m = 0$ case immediately gives us

$$\eta(v,0;\alpha,\beta) = \int_{x>0} e^{-vx}x^{\alpha+\beta-2}dx$$

Recognize $\alpha+\beta-2 = s-1$ and substitute $x_{\text{new}} = vx_{\text{old}}$ to get

$$= v^{-s}\int_{x>0} e^{-x}x^{s-1}dx = v^{-s}\Gamma(s)$$

□

We may now put all of this work together to evaluate the Archimedean local Whittaker function. Since this contains all of the formulas we will take away to use later, we will use $m_{\mathfrak{p}}$ in the final formulas to emphasize we are using the embedding of m into $K_{\mathfrak{p}}$ (although we will continue to just write m in the proof). The first four parts of the following lemma are taken from Proposition 2.3 of [KY10].

Lemma 5.29. *Assume $l > 0$ and $l \neq 1$. Let $q^x = e^{2\pi i x\tau}$, $\tau = u + iv$, and let $(-i)^l = e^{-2\pi i l/4}$. Continue to let $\alpha = \frac{s+1+l}{2}$ and $\beta = \frac{s+1-l}{2}$. For $Re(s)$ sufficiently large ($Re(s) > l+1$ suffices) we have*

(i) For $m_{\mathfrak{p}} > 0$

$$W_{m,\mathfrak{p}}(\tau, s, \Phi^l_{\mathfrak{p}}) = 2\pi(-i)^{l-1/2} v^{\beta} (2\pi m_{\mathfrak{p}})^s \frac{\Psi_{-l}(s, 4\pi m_{\mathfrak{p}} v)}{\Gamma(\alpha)} q^{m_{\mathfrak{p}}}$$

(ii) For $m_{\mathfrak{p}} < 0$

$$W_{m,\mathfrak{p}}(\tau, s, \Phi^l_{\mathfrak{p}}) = 2\pi(-i)^{l-1/2} v^{\beta} (2\pi |m_{\mathfrak{p}}|)^s \frac{\Psi_{l}(s, 4\pi\ |m_{\mathfrak{p}}| v)}{\Gamma(\beta)} e^{-4\pi\ |m_{\mathfrak{p}}| v} q^{m_{\mathfrak{p}}}$$

(iii) For $m_{\mathfrak{p}} = 0$

$$W_{m,\mathfrak{p}}(\tau, s, \Phi^l_{\mathfrak{p}}) = 2\pi(-i)^{l-1/2} v^{\frac{1}{2}(1-l-s)} \frac{2^{-s}\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta)}$$

For fixed τ, m, the Archimedean $W_{m,\mathfrak{p}}$ is meromorphic in s and admits an analytic continuation to all s given by the equations above. For this extension, we have

(iv) At the special value $s = l-1$,

$$W_{m,\mathfrak{p}}(\tau, l-1, \Phi^l_{\mathfrak{p}}) = \begin{cases} 0 & m_{\mathfrak{p}} \leq 0 \\ \frac{(2\pi)^l(-i)^{l-1/2}}{\Gamma(l)} m_{\mathfrak{p}}^{l-1} q^{m_{\mathfrak{p}}} & m_{\mathfrak{p}} > 0 \end{cases}$$

(v) When $m_{\mathfrak{p}} \leq 0$, the zero at $s = l-1$ is a simple zero.

Proof. By proposition 5.22 we have

$$W_{m,\mathfrak{p}}(\tau, s, \Phi^l_{\mathfrak{p}}) = e^{2\pi i/8} e^{2\pi i m u} v_{\mathfrak{p}}^{\beta} \xi(v, m; \alpha, \beta)$$

By proposition 5.25 we have

$$W_{m,\mathfrak{p}}(\tau, s, \Phi^l_{\mathfrak{p}}) = e^{2\pi i/8} e^{2\pi i m u} v_{\mathfrak{p}}^{\beta} \cdot 2\pi(-i)^l \Gamma(\alpha)^{-1} \Gamma(\beta)^{-1} \eta(2v, \pi m; \alpha, \beta)$$

There isn't much simplification we can do, although we can combine the $e^{2\pi i/8}$ and

$(-i)^l$ terms. We now evaluate $\eta(2v, \pi m; \alpha, \beta)$ using lemma 5.28 to get

$$W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^l) =$$
$$e^{2\pi i m u} v_{\mathfrak{p}}^{\beta} \cdot 2\pi(-i)^{l+1/2}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1} \begin{cases} (2\pi m)^s e^{-2\pi m v}\Gamma(\beta)\Psi_{-l}(s, 4\pi m v) & m > 0 \\ (2\pi|m|)^s e^{-2\pi|m|v}\Gamma(\alpha)\Psi_l(s, 4\pi|m|v) & m < 0 \\ (2v)^{-s}\Gamma(s) & m = 0 \end{cases} \tag{5.30}$$

Parts (i),(ii),(iii) then follow immediately after some rearranging of terms.

Next we tackle (iv). When $s = l - 1$, we have $\alpha = l, \beta = 0$.

First we tackle the case $m < 0$. By definition, we have

$$\Psi_l(l-1, 4\pi\, |m_{\mathfrak{p}}|v) = \Psi\left(l, l; z\right) = \frac{1}{\Gamma(l)} \int_0^{\infty} e^{-zt}(t+1)^{-1} t^{l-1} dt$$

Hence, $0 < \Psi_l(l-1, 4\pi\, |m_{\mathfrak{p}}|v) < \infty$. It follows that when $m < 0$, the Whittaker function vanishes due to the $1/\Gamma(\beta)$ term since all other terms are finite and nonzero.

When $m = 0$, we have to similarly verify that $\Gamma(s)/\Gamma(\alpha) = \Gamma(l-1)/\Gamma(l) = 1/(l-1)$ won't cause any problems. Since we assume $l \neq 1$, it is clear that the Whittaker function again vanishes.

Finally, we tackle $m > 0$. In this case we calculate

$$\Psi_{-l}(l-1, 4\pi\, |m_{\mathfrak{p}}|v) = \Psi\left(0, l; z\right)$$

However, this is just 1 by lemma 5.27. Hence, in the case $m > 0$ we have

$$W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^l) = 2\pi(-i)^{l-1/2} v^{\beta} (2\pi m_{\mathfrak{p}})^{l-1} \frac{1}{\Gamma(l)} q^{m_{\mathfrak{p}}}$$

which gives us the desired formula.

During the proof of (iv) it should be clear that none of the factors in the formula for $W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^l)$ have zeros or poles at $s = l - 1$, except for the factor $1/\Gamma(\beta)$. $1/\Gamma(x)$ has a simple pole at 0, which proves (v). □

5.5 Simplifying $W_{m,\mathfrak{p}}$ for Finite Places

Throughout this section, let $\mathfrak{p}$ be a finite place. Our goal is to simplify the quantity

$$W_{m,\mathfrak{p}}(s, \Phi_{\mathfrak{p},\mu}) = \int_{K_\mathfrak{p}} \Phi_{\mathfrak{p},\mu}([wn(b), 1]_L, s)\psi_\mathfrak{p}(-mb)db$$

using a formula like equation (4.2) of [KY10]. We do this by following the procedure of [KY10] section 4 (see the end of this section for the equation). From definition 4.13, we know that for $s_0 = \frac{dim(V)}{2} - 1$ we may rewrite this as

$$W_{m,\mathfrak{p}}(s_0, \Phi_{\mathfrak{p},\mu}) = \int_{K_\mathfrak{p}} (\omega_V([wn(b), 1]_L)\phi_{\mathfrak{p},\mu}(t))|_{t=0}\psi_\mathfrak{p}(-mb)db$$

However, this formula only computes the Whittaker function for one value of s. Lemma 4.2 from [KY10] (or more generally lemma A.3 of [Kud97]) help extend our reach. Let Δs be a non-negative integer. Let $V_{\Delta s} = K_\mathfrak{p}^{2\Delta s}$ and give it a basis x_i, x_i' $(1 \leq i \leq \Delta s)$. We can make $V_{\Delta s}$ into a quadratic space under $Q_{\Delta s} = \sum_i x_i x_i'$. Let $\phi_{\Delta s}$ denote the characteristic function of $O_{K_\mathfrak{p}}^{2\Delta s}$.

Lemma 5.31. *If $\Phi_\mathfrak{p}$ is the standard section associated to some $\phi_\mathfrak{p}$ then*

$$\Phi_\mathfrak{p}(g', s_0 + \Delta s) = (\omega_{V \oplus V_{\Delta s}}(g')\phi_\mathfrak{p}(t)\phi_{\Delta s}(t))|_{t=0}$$

The end goal will be to use this to compute the Whittaker functions (and hence the Eisenstein series) for infinitely many values of s and then use analytic continuation to get values not covered by the above formula.[5] In our particular case, lemma 5.31 gives us

$$\Phi_{\mathfrak{p},\mu}(g', s_0 + \Delta s) = (\omega_{V \oplus V_{\Delta s}}([wn(b), 1]_L)\phi_{\mathfrak{p},\mu}(t)\phi_{\Delta s}(t))|_{t=0} \tag{5.32}$$

This may be further reduced to an explicit integral in terms of more elementary functions.

[5]As lemma 5.31 suggests, we end up computing the Whittaker function on the set $s_0 + \mathbb{N}$, which is not enough to apply analytic continuation on its own. However, our formula will also "work at positive real infinity", which will be enough to get analytic continuation to apply. We will make this rigorous using lemma 5.7.

Proposition 5.33. *Fix some basis $\mathscr{B}$ of V, letting us identify $V \cong K_{\mathfrak{p}}^{dim(V)}$. Take the Haar measure dy on $V \oplus K_{\mathfrak{p}}^{2\Delta s}$ such that in this basis $O_{K_{\mathfrak{p}}}^{dim(V)+2\Delta s}$ has measure 1. Let $det_{\mathscr{B}}(V)$ denote the determinant of the matrix of Q in this basis (as opposed to the basis-independent value in $K_{\mathfrak{p}}^{\times}/K_{\mathfrak{p}}^{\times 2}$ we have used thus far). Let $L \subset V$ denote the lattice chosen when constructing $E^{l,\mu}$ and let $Vol_{\mathscr{B}}(L)$ denote its volume under dy. Then,*

$$\Phi_{\mathfrak{p},\mu}(g', s_0+\Delta s) = \gamma(V)\frac{|det_{\mathscr{B}}(V)|_{\mathfrak{p}}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}} q^{-r(\Delta s+dim(V)/2)} \int_{y\in(\mu+L)\oplus O_{K_{\mathfrak{p}}}^{2\Delta s}} \psi\left(bQ \oplus Q_{\Delta s}(y)\right) dy$$

Here,

$$\gamma(V) = \gamma_w\left(\frac{1}{2}\right)\gamma\left(\psi\left(\frac{1}{2}t\right)\circ V\right)^{-1} = \gamma_w\left(-\frac{1}{2}det(V)\right)\gamma_w\left(\frac{1}{2}\right)^{2-dim(V)} h_{\mathfrak{p}}(V)$$

is the local factor we defined earlier.

Proof. We do this by evaluating equation (5.32). First we note

$$wn(b) = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix}$$

and yet again recall the Weil representation

$$\omega_V([g,z]_L)\phi(t) = \chi_V(x(g))\overline{\left(z\gamma_w\left(\frac{1}{2}\right)^{j(g)}\right)^{dim(V)}}\gamma\left(\psi\left(\frac{1}{2}t\right)\circ V\right)^{-j(g)} r_V(g)\phi(t)$$

where

$$r_V(g)\phi(t) = \int_{y\in cV}\psi\left(\frac{1}{2}(at,bt)_Q + (bt,cy)_Q + \frac{1}{2}(cy,dy)_Q\right)\phi\left(at+cy\right)d_g y$$

We will first focus on calculating

$$r_{V\oplus V_{\Delta s}}(wn(b))(\phi_{\mathfrak{p},\mu}(t)\phi_{\Delta s}(t))$$

By the formula for the Weil representation, this is

$$\int_{y\in V\oplus K_{\mathfrak{p}}^{2\Delta s}}\psi\left(-(t,y)_{Q\oplus Q_{\Delta s}} + \frac{b}{2}(y,y)_{Q\oplus Q_{\Delta s}}\right)\phi_{\mathfrak{p},\mu}(y)\phi_{\Delta s}(y)d_g y \tag{5.34}$$

Now decompose the coordinate y as $y = (y_0, y_1, y_1', \dots y_{\Delta s}, y_{\Delta s}')$, where $y_0 \in V$ and $y_i, y_i' \in K_{\mathfrak{p}}$ for $1 \le i \le \Delta s$. Do the same decomposition for t. Then, we may rewrite our integral as

$$\int_{y \in (\mu+L) \oplus O_{K_{\mathfrak{p}}}^{2\Delta s}} \psi\left(\left(-(t_0, y_0)_Q + \sum_i t_i y_i' + t_i' y_i\right) + b\left(-\frac{1}{2}(y_0, y_0)_Q + \sum_i y_i y_i'\right)\right) d_g y$$

We now write $d_g y = \mu(g) dy$, where dy is such that $O_{K_{\mathfrak{p}}}^{dim(V)+\Delta s}$ has measure 1 (for some choice of basis of V). We will now have to determine $\mu(g)$. As we have done in the past, we will do this by checking that $r_{V \oplus V_{\Delta s}}$ is unitary when it acts on the particular pairing $(\phi_{\mathfrak{p},0}(t)\phi_{\Delta s}(t), \phi_{\mathfrak{p},0}(t)\phi_{\Delta s}(t))$. We get the condition

$$\begin{aligned}1 = \mu(g)^2 \int_{t \in V \oplus K_{\mathfrak{p}}^{2\Delta s}} \Bigg[& \\ \int_{x \in L \oplus O_{K_{\mathfrak{p}}}^{2\Delta s}} & \psi\left(\left(-(t_0, x_0)_Q + \sum_i t_i x_i' + t_i' x_i\right) + b\left(-\frac{1}{2}(x_0, x_0)_Q + \sum_i x_i x_i'\right)\right) dx \\ \times \int_{y \in L \oplus O_{K_{\mathfrak{p}}}^{2\Delta s}} & \psi\left(\left((t_0, y_0)_Q + \sum_i -t_i y_i' - t_i' y_i\right) + b\left(\frac{1}{2}(y_0, y_0)_Q + \sum_i -y_i y_i'\right)\right) dy \Bigg] dt\end{aligned} \tag{5.35}$$

The giant integral can be written as a product of two pieces, for coordinates of index 0, and for coordinates of positive index.

Case 1: The coordinates of index 0 give us

$$\int_{t_0 \in V} \int_{x_0 \in L} \int_{y_0 \in L} \psi\left((t_0, y_0 - x_0)_Q + \frac{b}{2}((y_0, y_0)_Q - (x_0, x_0)_Q)\right) dy_0 dx_0 dt_0$$

Using the identity $(y_0, y_0)_Q - (x_0, x_0)_Q = (y_0 + x_0, y_0 - x_0)_Q$, we see the integral equals

$$\int_{t_0 \in V} \int_{x_0 \in L} \int_{y_0 \in L} \psi\left((t_0 + \frac{b}{2}(x_0 + y_0), y_0 - x_0)_Q\right) dy_0 dx_0 dt_0$$

It would be convenient to be able to rearrange the integrals at this point, but as things stand the triple integral does not converge absolutely. To get around this, note that the inner double integral is over a compact domain and does converge absolutely, so we can change variables to $x = x_0$, $y = x_0 + y_0$ yielding

$$\int_{t_0 \in V} \int_{x \in L} \int_{y \in L} \psi\left((t_0 + \frac{b}{2}y, -2x + y)_Q\right) dy dx dt_0$$

Slight rearrangement gives

$$\int_{t_0\in V}\int_{y\in L}\psi\left((t_0+\frac{b}{2}y,y)_Q\right)\int_{x\in L}\psi\left((t_0+\frac{b}{2}y,-2x)_Q\right)dxdydt_0$$

The inner integral over x is the integral of an additive character and hence vanishes unless the integrand is identically 1. Recalling the notation L^* for the dual lattice of L under $(\cdot,\cdot)_Q$ and the fact that ψ is trivial on $\pi^{-r}O_{K_\mathfrak{p}}$, the integral becomes

$$Vol_{\mathscr{B}}(L)\int_{t_0\in V}\int_{y\in L}\psi\left((t_0+\frac{b}{2}y,y)_Q\right)\mathbb{1}_{\pi^{-r}L^*}(-2t_0-by)dydt_0$$

Notice that the indicator function can only be satisfied if the two sets $2t_0+bL$ and $\pi^{-r}L^*$ have some overlap, and for this to happen we must at least have $2t_0\in bL\cup\pi^{-r}L^*$. Therefore, we may restrict t_0 to a compact domain. Choose a large enough that $\pi^{-a}L\supset 2t_0\in bL\cup\pi^{-r}L^*$. One may choose a arbitrarily large, but this is the minimum size that lets the following arguments run smoothly. The integral becomes

$$Vol_{\mathscr{B}}(L)\int_{t_0\in\pi^{-a}L}\int_{y\in L}\psi\left((t_0+\frac{b}{2}y,y)_Q\right)\mathbb{1}_{\pi^{-r}L^*}(-2t_0-by)dydt_0$$

Now that we have absolute convergence, we are justified in performing the change of variables $t=t_0+\frac{b}{2}y$, $y_{\text{new}}=y_{\text{old}}$. The integral becomes

$$Vol_{\mathscr{B}}(L)\int_{t\in\pi^{-a}L}\int_{y\in L}\psi\left((t,y)_Q\right)\mathbb{1}_{\pi^{-r}L^*}(-2t)dydt=$$
$$Vol_{\mathscr{B}}(L)\int_{t\in\pi^{-e-r}L^*}\int_{y\in L}\psi\left((t,y)_Q\right)dydt \quad (5.36)$$

As before, the inner integral is of an additive character, which will be trivial when $t\in\pi^{-r}L^*$. One gets

$$Vol_{\mathscr{B}}(L)^2\int_{t\in\pi^{-r}L^*}dt=q^{rdim(V)}Vol_{\mathscr{B}}(L)^2Vol(L^*)=q^{rdim(V)}Vol_{\mathscr{B}}(L)/|det_{\mathscr{B}}(V)|_\mathfrak{p}$$

where the last step is the general quadratic space identity $Vol_{\mathscr{B}}(L)Vol(L^*)=|det_{\mathscr{B}}(V)|_\mathfrak{p}^{-1}$, which holds for any lattice L.

Case 2: The coordinates of positive index in equation (5.35) give rise to the integral

$$\int_{t_i,t_i'\in K_{\mathfrak{p}}(1\le i\le \Delta s)}\int_{x_i,x_i',y_i,y_i'\in O_{K_{\mathfrak{p}}}(1\le i\le\Delta s)}\psi\Bigg(\left(\sum_i t_ix_i'+t_i'x_i\right)+\left(\sum_i -t_iy_i'-t_i'y_i\right)+ b\left(\sum_i x_ix_i'+\sum_i -y_iy_i'\right)\Bigg)dxdx'dydy'dtdt' \tag{5.37}$$

Note that there are $6\Delta s$ variables of integration. For the sake of space, we have condensed down the differentials by letting $dx=\prod_i dx_i$, etc... For some particular variable x_i, if we consider only the terms it shows up in, we have the integral

$$\int_{x_i\in O_{K_{\mathfrak{p}}}}\psi\left(t_i'x_i+bx_i'x_i\right)dx_i=\mathbb{1}_{\partial^{-1}}(t_i'+bx_i')$$

As we did previously, we choose a large enough so that $\pi^{-a}O_{K_{\mathfrak{p}}}\supset\partial^{-1}+bO_{K_{\mathfrak{p}}}$. Notice that in order for the indicator function to be satisfied for any x_i', it is necessary (though not necessarily sufficient) for $t_i'\in\pi^{-a}O_{K_{\mathfrak{p}}}$. Similarly, for a variable y_i', one has

$$\int_{y_i'\in O_{K_{\mathfrak{p}}}}\psi\left(-t_iy_i'-y_iy_i'\right)dy_i'=\mathbb{1}_{\partial^{-1}}(t_i+by_i)$$

Using this, we may integrate over the $2\Delta s$ variables of the form x_i or y_i'. This reduces integral (5.37) to

$$\int_{t_i,t_i'\in K_{\mathfrak{p}}(1\le i\le\Delta s)}\int_{x_i',y_i\in O_{K_{\mathfrak{p}}}(1\le i\le\Delta s)}\psi\left(\sum_i t_ix_i'-t_i'y_i\right)\mathbb{1}_{\partial^{-1}}(t_i'+bx_i')\mathbb{1}_{\partial^{-1}}(t_i+by_i)dx'dydtdt'$$

Since the inner integral vanishes for $t_i,t_i'\notin\pi^{-a}O_{K_{\mathfrak{p}}}$, we may restrict the integral to a compact domain. We get

$$\int_{\substack{t_i,t_i'\in\pi^{-a}O_{K_{\mathfrak{p}}}\\(1\le i\le\Delta s)}}\int_{\substack{x_i',y_i\in O_{K_{\mathfrak{p}}}\\(1\le i\le\Delta s)}}\psi\left(\sum_i t_ix_i'-t_i'y_i\right)\mathbb{1}_{\partial^{-1}}(t_i'+bx_i')\mathbb{1}_{\partial^{-1}}(t_i+by_i)dx'dydtdt'$$

We now change variables, fixing x_i',y_i and letting $t_{i,\text{new}}=t_{i,\text{old}}+by_i$ and $t'_{i,\text{new}}=t'_{i,\text{old}}+bx_i'$. This turns the integral into

$$\int_{t_i,t_i'\in\pi^{-a}O_{K_{\mathfrak{p}}}(1\le i\le\Delta s)}\int_{x_i',y_i\in O_{K_{\mathfrak{p}}}(1\le i\le\Delta s)}\psi\left(\sum_i t_ix_i'-t_i'y_i\right)\mathbb{1}_{\partial^{-1}}(t_i')\mathbb{1}_{\partial^{-1}}(t_i)dx'dydtdt'$$

Notice that when we make this substitution, the input to ψ does not appear to change. This is due to the cancellation

$$\sum_i (t_i - by_i)x_i' - (t_i' - bx_i')y_i = \sum_i t_i x_i' - by_i x_i' - t_i' y_i + bx_i' y_i = \sum_i t_i x_i' - t_i' y_i$$

Continuing with the integral, we may absorb the indicator functions into the domain, which gives

$$\int_{t_i, t_i' \in \partial^{-1}(1 \leq i \leq \Delta s)} \int_{x_i', y_i \in O_{K_\mathfrak{p}}(1 \leq i \leq \Delta s)} \psi\left(\sum_i t_i x_i' - t_i' y_i\right) dx' dy dt dt'$$

However, on the given domain ψ is identically 1 and so the integral evaluates to $q^{2r\Delta s}$.

After all of this, we may finally plug back into equation (5.35) and obtain

$$\mu(g)^2 * q^{rdim(V)} Vol_{\mathscr{B}}(L)/|det_{\mathscr{B}}(V)|_\mathfrak{p} * q^{2r\Delta s} = 1$$

From this, we can see that

$$\mu(g) = \frac{|det_{\mathscr{B}}(V)|_\mathfrak{p}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}} q^{-r(\Delta s + dim(V)/2)}$$

We are now in a position to determine the quantity we were originally after, which is the value of

$$(\omega_{V \oplus V_{\Delta s}}([wn(b), 1]_L)\phi_{\mathfrak{p},\mu}(t)\phi_{\Delta s}(t))|_{t=0}$$

By plugging $t = 0$ into equation (5.34) and using our value of $\mu(g)$, we see that it is equal to

$$\gamma(V)\frac{|det_{\mathscr{B}}(V)|_\mathfrak{p}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}} q^{-r(\Delta s + dim(V)/2)} \int_{y \in (\mu + L) \oplus O_{K_\mathfrak{p}}^{2\Delta s}} \psi\left(\frac{b}{2}(y, y)_{Q \oplus Q_{\Delta s}}\right) dy$$

Using the general formula $(x, x)_Q = 2Q(x)$ now yields the desired result.

□

From this proposition, we have the formula

$$W_{m,\mathfrak{p}}(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu}) = \gamma(V)\frac{|det_{\mathscr{B}}(V)|_\mathfrak{p}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}} q^{-r(\Delta s + dim(V)/2)} \int_{K_\mathfrak{p}} \int_{y \in (\mu + L) \oplus O_{K_\mathfrak{p}}^{2\Delta s}} \psi\left(b(Q \oplus Q_{\Delta s}(y) - m)\right) dy db \tag{5.38}$$

It will be convenient to replace the ramified ψ with an unramified additive character ψ'. The following equation is our analogue to [KY10] equation (4.2).

Lemma 5.39. *Let ψ' be any unramified additive character. Then*

$$W_{m,\mathfrak{p}}\left(s_0+\Delta s,\Phi_{\mathfrak{p},\mu}\right)= \\ \gamma(V)\frac{|det_{\mathscr{B}}(V)|_{\mathfrak{p}}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}}q^{r(1-\Delta s-dim(V)/2)}\int_{K_{\mathfrak{p}}}\int_{y\in(\mu+O_{K_{\mathfrak{p}}})\oplus O_{K_{\mathfrak{p}}}^{2\Delta s}}\psi'\left(b(Q\oplus Q_{\Delta s}(y)-m)\right)dydb \tag{5.40}$$

We have a particular choice of ψ' in mind, which we will introduce later.

Proof. Simply change variables via $b_{\text{old}}=b_{\text{new}}/\pi^r$. □

Since this new integral will become our entire focus, we give it a name.

$$I_{W,\mathfrak{p}}(\mu,m,\Delta s):=\int_{K_{\mathfrak{p}}}\int_{y\in(\mu+O_{K_{\mathfrak{p}}})\oplus O_{K_{\mathfrak{p}}}^{2\Delta s}}\psi'\left(b(Q\oplus Q_{\Delta s}(y)-m)\right)dydb$$

This has reduced our calculation to (a local factor and) an integral in terms of relatively elementary functions. However, even in simple cases when $dim(V)=1$, the computation will be quite tedious and require many pages of algebra to finish.

Remark 5.41. *Of some mild additional concern is that the double integral is no longer absolutely convergent now that we have replaced $\Phi_{\mathfrak{p}}$ with the inner integral. We may partially remedy this by noting the integral is equal to a limit of absolutely convergent integrals, namely*

$$I_{W,\mathfrak{p}}(\mu,m,\Delta s)=\lim_{k\to\infty}\int_{\pi^{-k}O_{K_{\mathfrak{p}}}}\int_{y\in(\mu+O_{K_{\mathfrak{p}}})\oplus O_{K_{\mathfrak{p}}}^{2\Delta s}}\psi'\left(b(Q\oplus Q_{\Delta s}(y)-m)\right)dydb$$

Chapter 6

Evaluating the Gauss Sum

6.1 Review

For the particular application we have in mind, we will need to evaluate the integral $I_{W,\mathfrak{p}}(\mu, m, \Delta s)$ in the case $dim(V) = 1$. These next two chapters will be dedicated to this computation. The current chapter will be for calculating quadratic Gauss sums, which will be used in the next chapter where we actually calculate the integral. Be aware that for the duration of chapter 6 we will be letting K denote a local field. We will redefine the relevant notation shortly.

In this section we will calculate explicit formulas for all quadratic Gauss sums over any local field of characteristic 0. The case of odd residue characteristic is fairly easy, so most of the computation is geared towards handling the case of characteristic 2. In particular, in characteristic 2, there are two different types of quadratic Gauss sum. One is of the form

$$\int \psi(ax^2)dx$$

whereas the other is of the form

$$\int \chi(x)\psi(ax)$$

where χ is some quadratic character on the local field. In general, there are 2^{n+1} such quadratic characters, where n is the degree of the local field. This is in stark contrast to the case of odd residue degree, where there is a unique quadratic character and the two types of Gauss sum we listed above are actually equal.

Finally, we note in the even residue characteristic case that $\int \psi(ax^2)dx$ acts like a "character of second degree" in a and we classify this character up to isomorphism. The classification will not be needed for our future computations, but our calculations come so close to it that we may as well.

First we review the relevant notation and conventions. Let K be a finite extension of the 2-adic field Q_2. Denote the degree of K with n, and let $n = ef$, where e is the ramification index and f is the degree of the residue field extension. Let O_K denote the ring of integers, and let $q = 2^f$ so that F_q is the residue field of O_K. Let π be a fixed uniformizer of O_K, and let the different ideal be $(\partial) = (\pi)^r$, where this equation defines r. Let K_0 be the maximal unramified subfield of K. Also, let the unit α be given by $\pi^e = 2\alpha$.

Given an integer $x \in O_K$, we will often let $x = x_0 + x_1\pi + x_2\pi^2 \ldots$ denote its π-adic expansion. Throughout, we will use the convention that the coefficients of such expansions will be the canonical multiplicative lifts coming from F_q. That is, our coefficients will be the $q-1$st roots of unity (and 0 of course), and whenever we refer to any π-adic coefficient, we will be necessarily mean such a value.

For a prime number $p \in \mathbb{Z}$ and $K = \mathbb{Q}_p$, we have the standard exponential function

$$\psi_p : \mathbb{Q}_p \to \mathbb{C}, \;\; x \mapsto e^{2\pi i x}$$

Here, we make sense of the exponential using the isomorphism $\mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Q}/\mathbb{Z}$. Given $x \in \mathbb{Q}_p$, the isomorphism gives us an element of $\mathbb{Q}/\mathbb{Z}$, which we can then exponentiate the usual way.

From this we may define a normalized exponential function ψ_* on K. Specifically, set

$$\psi_*(x) := e^{2\pi i \mathrm{tr}\left(\frac{x}{\pi^r}\right)},$$

where the trace is $\mathrm{tr}_{\mathbb{Q}_2}^{K}$. Note that due to the normalizing factor of π^r and the definition of the different, ψ_* is unramified ($\psi_*(x) = 1$ on O_K, but is nontrivial on $\frac{1}{\pi}O_K$).

6.2 Tracechanging

The fact that π is an arbitrarily chosen uniformizer makes computing the trace in the definition of ψ_* both unnatural and messy. The goal of this section is to construct a special unit $\tau \in O_K^\times$ that encodes data about this messiness. Then we will use τ to create a simple formula to help compute some traces of the form $\mathrm{tr}_{K_0}^{K}\left(\frac{x}{\pi^{r+a}}\right)$. Such a formula is the main goal of this section and will later be used to evaluate ψ_* by taking a further trace $\mathrm{tr}_{Q_2}^{K_0}(\mathrm{tr}_{K_0}^{K}\left(\frac{x}{\pi^{r+a}}\right)) = \mathrm{tr}_{Q_2}^{K}\left(\frac{x}{\pi^{r+a}}\right)$. Note that since ψ_* only cares about this trace mod Z_2 and $\mathrm{tr}_{Q_2}^{K_0}(O_{K_0}) = Z_2$, our formula in this section only needs to compute the trace mod O_{K_0}. We will end up taking advantage of this to simplify the final computation.

The first step is the following general lemma

Lemma 6.1. *Let L/K be a finite separable extension of fields, having basis $b_1, b_2, \ldots b_n$. Then,*

$$T : L \to K^n, \;\; x \mapsto (tr_K^L(b_1x), tr_K^L(b_2x), \ldots tr_K^L(b_nx))$$

is an isomorphism of K vector spaces.

Proof. Since L and K^n have equal dimension, it suffices to show ker(T)=0. So, let x be such that $T(x) = 0$, so that $\mathrm{tr}_K^L(b_ix) = 0$ for all i. Let $c_1, c_2, \ldots c_n$ be arbitrary elements of K. Then we have

$$\mathrm{tr}_K^L\left((\sum_{i=1}^{n} c_ib_i)x\right) = \sum_{i=1}^{n} c_i\mathrm{tr}_K^L(b_ix) = 0$$

Of course, $\sum_{i=1}^{n} c_ib_i$ can be any element of L, so we have $\mathrm{tr}_K^L(Lx) = 0$. Since our extension is separable, the trace map is nontrivial and we necessarily have $x = 0$. $\square$

Corollary 6.2. *There is a unique $\tau \in K$ such that*

$$tr_{K_0}^{K}\left(\frac{\tau}{\pi^{r+1}}\right) = \frac{1}{2}, \quad tr_{K_0}^{K}\left(\frac{\tau}{\pi^{r+i}}\right) = 0 \quad 2 \leq i \leq e.$$

Proof. Apply the previous lemma to K/K_0 with the basis $b_i = \frac{1}{\pi^{r+i}}$. □

We can use this defining property of τ to get a general formula for all of the traces we will need later on. However, let us first prove some useful properties of τ. In particular, it turns out that as defined, we will always have $\tau \in O_K^\times$. The main tool to show this is the following general fact related to different ideals.[1]

Fact 6.3.

$$tr_{K_0}^{K}\left(\frac{1}{\pi^{r+a}}O_K\right) = \frac{1}{2^{\lceil a/e \rceil}}O_{K_0}$$

Proposition 6.4. *For the τ given above we have $\tau \in O_K^\times$.*

Proof. Since we are working with local fields we have $O_K = O_{K_0}[\pi]$. This means that for $x \in O_K$, we may write $x = \sum_{i=0}^{e-1} c_i \pi^i$, where each $c_i \in O_{K_0}$. For such an x, we can use the defining properties of τ to get

$$\mathrm{tr}_{K_0}^{K}\left(x \cdot \frac{\tau}{\pi^{r+e}}\right) = \sum_{i=0}^{e-1} c_i \mathrm{tr}_{K_0}^{K}\left(\frac{\tau}{\pi^{r+e-i}}\right) = \frac{c_{e-1}}{2}$$

From this we see

$$\mathrm{tr}_{K_0}^{K}\left(O_K \cdot \frac{\tau}{\pi^{r+e}}\right) = \frac{O_{K_0}}{2}$$

This will immediately imply $\tau \in O_K$ when combined with Fact 3. In particular, we get $\lceil (e - v_\pi(\tau))/e \rceil = 1 \implies v_\pi(\tau) \geq 0$.

To further show τ is a unit, start with the formula

$$\mathrm{tr}_{K_0}^{K}\left(\frac{\tau}{\pi^{r+1}}\right) = \frac{1}{2}$$

If we were to have $\tau \in \pi O_K$, then Fact 3 would imply the above trace is an integer. Hence, τ is a unit. □

[1] If you change which of the traces in corollary 6.2 is the nonzero one, along with the valuation of the nonzero trace, you can get τ to have any valuation you want. I just chose the setup that makes τ a unit, for the sake of convenience.

Proposition 6.5. *For the given τ, we have*

$$tr_{K_0}^K\left(\frac{\tau}{\pi^{r+e+1}}\right) = \frac{1}{4\alpha_0},$$

where α_0 is defined by writing $\alpha = \sum_{i=0}^{e-1} \alpha_i \pi^i$ with $\alpha_i \in O_{K_0}$.

Proof.

$$\frac{1}{2} = \mathrm{tr}_{K_0}^K\left(\frac{\tau}{\pi^{r+1}}\right) = 2\mathrm{tr}_{K_0}^K\left(\frac{\alpha\tau}{\pi^{r+e+1}}\right)$$

Since $O_K = O_{K_0}[\pi]$, we may uniquely write $\alpha = \sum_{i=0}^{e-1} \alpha_i \pi^i$ with $\alpha_i \in O_{K_0}$. Then we get

$$\frac{1}{4} = \sum_{i=0}^{e-1} \alpha_i \mathrm{tr}_{K_0}^K\left(\frac{\tau}{\pi^{r+e+1-i}}\right) = \alpha_0 \mathrm{tr}_{K_0}^K\left(\frac{\tau}{\pi^{r+e+1}}\right)$$

The result follows. □

Though we will not need this fact, α_0 admits a very nice description. Let $x^e + 2a_{e-1}x^{e-1} + \ldots + 2a_0$ be the Eisenstein polynomial for π over K_0. Then, we have $\alpha = -\sum_{i=0}^{e-1} a_i \pi^i$. Thus, $\alpha_0 = -a_0$ comes directly from the constant coefficient of the Eisenstein polynomial.

We can now put together the main formula for this section.

Proposition 6.6. *Tracechanging formula: Let τ be as defined above, let α_0 be as in the previous proposition. Then, if $y \in O_K$ has π-adic expansion $y = \sum_{i=0}^{\infty} y_i \pi^i$, we have*

$$tr_{K_0}^K\left(\frac{\tau y}{\pi^{r+e+1}}\right) \equiv \frac{y_e}{2} + \frac{y_0}{4\alpha_0} \ mod\, O_{K_0}$$

and if $1 \le i \le e$ we have

$$tr_{K_0}^K\left(\frac{\tau y}{\pi^{r+i}}\right) \equiv \frac{y_{i-1}}{2} \ mod\, O_{K_0}$$

Proof. Applying fact 3, we get

$$\mathrm{tr}_{K_0}^K\left(\frac{\tau(\sum_{i=0}^{\infty} y_i\pi^i)}{\pi^{r+e+1}}\right) \equiv \mathrm{tr}_{K_0}^K\left(\frac{\tau(\sum_{i=0}^{e} y_i\pi^i)}{\pi^{r+e+1}}\right) \ \mathrm{mod}\ O_{K_0}$$

We may rewrite this as

$$\sum_{i=0}^{e} y_i \mathrm{tr}_{K_0}^{K}\left(\frac{\tau}{\pi^{r+e+1-i}}\right)$$

By the defining equations for τ and proposition 5, only the 0th and eth terms of the sum are nonzero and we get

$$\frac{y_e}{2} + \frac{y_0}{4\alpha_0}$$

The second part of the claim follows immediately from the first part by writing

$$\mathrm{tr}_{K_0}^{K}\left(\frac{\tau y}{\pi^{r+i}}\right) = \mathrm{tr}_{K_0}^{K}\left(\frac{\tau(y\pi^{e+1-i})}{\pi^{r+e+1}}\right)$$

and noting that the 0th π-adic coefficient of $y\pi^{e+1-i}$ is 0, and the eth coefficient is y_{i-1}. □

Corollary 6.7. *If $d \in \{0, 1\}$, then*

$$tr_{K_0}^{K}\left(\frac{\tau y}{\pi^{r+e+d}}\right) \equiv \frac{y_{e+d-1}}{2} + d\frac{y_0}{4\alpha_0} \ mod\ O_{K_0}$$

Proof. This follows from the proposition by just checking the two cases for d. The formula is a little unnatural in how it uses d, but it will help us keep casework clean later. □

In later applications, the value y above will be a product. The following easy lemma tells us how to deal with this.

Lemma 6.8. *Let $a \leq e$. Then,*

$$(xy)_a \equiv \sum_{i=0}^{a} x_i y_{a-i} \ mod\ 2$$

Proof. Although not entirely trivial, this can be checked easily by multiplying the π-adic representations of x and y. □

6.3 The Odd Case

Here, we briefly jump back to a local field over $\mathbb{Q}_p$ for p odd. We will compute the value of a quadratic Gauss sum. In the case of K unramified, the ideas and result are nothing new, but it will be a decent warmup for handling the more difficult case of $p = 2$ later.

First, define τ as per corollary 6.2. To be specific, we will take $e = 0$ in the corollary, so that τ is definde by the single property

$$\mathrm{tr}_{K_0}^{K}\left(\frac{\tau}{\pi^{r+1}}\right) = \frac{1}{p}$$

Define

$$\psi'(x) := \psi_*(\tau x)$$

Define the Gauss sums

$$\gamma'\left(\frac{u}{\pi^a}\right) := \int_{O_K} \psi'\left(\frac{ux^2}{\pi^a}\right)\, dx = \int_{O_K} e^{2\pi i \mathrm{tr}\left(\frac{\tau u x^2}{\pi^{r+a}}\right)}\, dx$$

$$\gamma'\left(\frac{u}{\pi^a}, \frac{u'}{\pi^{a'}}\right) := \int_{O_K} \psi'\left(\frac{ux^2}{\pi^a} + \frac{u'x}{\pi^{a'}}\right)\, dx = \int_{O_K} e^{2\pi i \mathrm{tr}\left(\frac{\tau u x^2}{\pi^{r+a}} + \frac{u'x}{\pi^{r+a'}}\right)}\, dx$$

We will be focusing on these “quadratic form" (referring to the x^2 in the integral) Gauss sums for the bulk of the following subsections. However, there are also “quadratic character" Gauss sums, which will be considered afterwards. This choice of ordering is chosen because at even places, the calculation of quadratic form Gauss sums will be useful in the calculations of quadratic character Gauss sums. This is interestingly in contrast with odd places, where the calculations are more easily done the other way around. Also worth note is that quadratic character Gauss sums are less diverse at an odd place, since there will only be 1 quadratic character, as opposed to the 2^{n+1} of them found at even places.

There is a unique quadratic character $\left(\frac{\cdot}{\mathfrak{p}}\right)$ on $O_{K_\mathfrak{p}}^\times$, which we extend trivially to

$O_{K_{\mathfrak{p}}}$, given by

$$\left(\frac{x}{\mathfrak{p}}\right) = \begin{cases} 1 & x \text{ is a square unit} \\ -1 & x \text{ is a nonsquare unit} \\ 0 & x \in \pi O_K \end{cases}$$

From this, we define the quadratic character Gauss sum for a unit u as

$$\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{u}{\pi^a}\right) := \int_{O_K} \left(\frac{x}{\mathfrak{p}}\right) \psi'\left(\frac{ux}{\pi^a}\right)\, dx$$

6.3.1 Integration Lemmas

We will need three integration lemmas, which will be used here as well as later on.

Lemma 6.9. *Let G be a subset of nonzero measure of O_K and let $\#G$ denote the measure of G. Then, for any integrable function f such that the double integral converges absolutely, we have*

$$\int_{x\in O_K} f(x)dx = \frac{1}{\#G}\int_{x\in O_K}\int_{y\in G} f(x+y)dydx$$

Proof. Start with the right hand side of the above equation. Switch the order of integration (by absolute convergence) to get

$$\frac{1}{\#G}\int_{y\in G}\int_{x\in O_K} f(x+y)dxdy$$

For the inner integral, make the change of variables $x_{\text{new}} = x_{\text{old}} + y$. This has trivial Jacobian and we get

$$\frac{1}{\#G}\int_{y\in G}\int_{x\in O_K} f(x)dxdy$$

We may now switch the order of integration back and take the trivial integral over y to get our result. □

Lemma 6.10. *Let G be a subset of nonzero measure of $O_K^\times$ and let $\#G$ denote the measure of G. Then, for any integrable function f such that the double integral converges absolutely, we have*

$$\int_{x\in O_K^\times} f(x)dx = \frac{1}{\#G}\int_{x\in O_K^\times}\int_{y\in G} f(xy)dydx$$

The result still holds (with the same proof) if x is being integrated over all of O_K.

Proof. Start with the right hand side of the above equation. Switch the order of integration (by absolute convergence) to get

$$\frac{1}{\#G}\int_{y\in G}\int_{x\in O_K^\times} f(xy)dxdy$$

For the inner integral, make the change of variables $x_{\text{new}} = x_{\text{old}}y$. Since y is a unit, $|y| = 1$ and the Jacobian is trivial. We get

$$\frac{1}{\#G}\int_{y\in G}\int_{x\in O_K^\times} f(x)dxdy$$

We may now switch the order of integration back and take the trivial integral over y to get our result. $\square$

Lemma 6.11. *Still letting # denote taking the measure of a set, we have*

$$\frac{1}{\#O_K^{\times 2}}\int_{x\in O_K^{\times 2}} f(x)dx = \frac{1}{\#O_K^\times}\int_{x\in O_K^\times} f(x^2)dx$$

Proof. We start with the right hand side and apply the change of variables $y = x^2$, which has Jacobian $dy = |2x|dx$. Since x is a unit, the Jacobian is just $dy = |2|dx = |\pi^e|dx = q^{-n}dx$. However, the map $y = x^2$ is not a bijection from $O_K^\times$ to $O_K^{\times 2}$ but is rather a double covering. As such, we will need to introduce an additional factor of 1/2. We get

$$\frac{1}{\#O_K^\times}\int_{x\in O_K^\times} f(x^2)dx = \frac{1}{2}\frac{1}{\#O_K^\times}\int_{y\in O_K^{\times 2}} f(y)q^{-n}dy = \frac{1}{\#O_K^{\times 2}}\int_{x\in O_K^{\times 2}} f(x)dx$$

Note that in the last step, we used that the square units form an index 2^{n+1} subgroup in $O_K^\times$.

Finally, I would like to remark that there is an easier way to get the normalization constants. Since we know that the change of variables $y = x^2$ is a double covering with constant Jacobian, it follows that the two integrals just differ up to a constant factor, regardless of the function f. Taking $f = 1$ then makes this constant factor clear. $\square$

6.3.2 Evaluating the Odd Gauss Sums

We start by evaluating quadratic character Gauss sums.

Proposition 6.12. *One has*

$$\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{u}{\pi^a}\right) = \left(\frac{u}{\mathfrak{p}}\right)\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{1}{\pi^a}\right)$$

and

$$\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{u}{\pi^a}\right) = \begin{cases} 0 & a \neq 1 \\ -(-1)^f q^{-1/2}\left(\frac{u}{\mathfrak{p}}\right) & a = 1,\ p \equiv 1 \ mod\ 4 \\ -(-i)^f q^{-1/2}\left(\frac{u}{\mathfrak{p}}\right) & a = 1,\ p \equiv 3 \ mod\ 4 \end{cases}$$

Proof. The first claim is well known and is just the change of variables $x_{\text{new}} = ux_{\text{old}}$, from which we immediately get

$$\left(\frac{u}{\mathfrak{p}}\right)\int_{O_K}\left(\frac{x}{\mathfrak{p}}\right)\psi'\left(\frac{x}{\pi}\right)\, dx$$

From this, we see that it suffices to prove the second claim only in the case $u = 1$.

For the second claim, it is a well known fact that when integrating a multiplicative character against an additive character, the integral will be 0 unless their conductors are equal. $\left(\frac{\cdot}{\mathfrak{p}}\right)$ has conductor 1, hence the need for $a = 1$. In the case $a = 1$, we have the integral

$$\int_{O_K}\left(\frac{x}{\mathfrak{p}}\right)\psi'\left(\frac{x}{\pi}\right)\, dx$$

To do this, we start by noting that the value of x only matters mod π. Hence, we may convert the integral to a sum to get

$$q^{-1}\sum_{x\in\mathbb{F}_q}\left(\frac{x}{\mathfrak{p}}\right)\psi'\left(\frac{x}{\pi}\right)$$

where we interpret elements of $\mathbb{F}_q$ as lying in O_K via the standard embedding as roots of unity.

By the definition of ψ', this becomes

$$q^{-1}\sum_{x\in\mathbb{F}_q}\left(\frac{x}{\mathfrak{p}}\right)e^{2\pi i \mathrm{tr}_{\mathbb{Q}_p}^{K_0}\mathrm{tr}_{K_0}^{K}\left(\frac{\tau x}{\pi^{r+1}}\right)}$$

Since $x \in K_0$, we may pull it out and use the definition of τ to get

$$q^{-1} \sum_{x \in \mathbb{F}_q} \left(\frac{x}{\mathfrak{p}}\right) e^{2\pi i \mathrm{tr}_{\mathbb{Q}_p}^{K_0}\left(\frac{x}{p}\right)} = q^{-1} \sum_{x \in \mathbb{F}_q} \left(\frac{x}{\mathfrak{p}}\right) e^{2\pi i \frac{1}{p} \mathrm{tr}_{\mathbb{Q}_p}^{K_0}(x)}$$

We may now switch to a sum defined entirely within the residue field, yielding

$$q^{-1} \sum_{x \in \mathbb{F}_q} \left(\frac{N_{\mathbb{F}_p}^{\mathbb{F}_q}(x)}{p}\right) e^{2\pi i \frac{1}{p} \mathrm{tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(x)}$$

Evaluating this Gauss sum is now easy and is given by the Hasse-Davenport lifting relation. This relation says that the Gauss sum under consideration is simply

$$(-1)^{f-1} \gamma_0^f$$

where

$$\gamma_0 = p^{-1} \sum_{x \in \mathbb{F}_p} \left(\frac{x}{p}\right) e^{2\pi i \frac{1}{p} x} = p^{-1/2} \begin{cases} 1 & p \equiv 1 \bmod 4 \\ i & p \equiv 3 \bmod 4 \end{cases}$$

is just the classical Gauss sum. As such, we have

$$q^{-1} \sum_{x \in \mathbb{F}_q} \left(\frac{N_{\mathbb{F}_p}^{\mathbb{F}_q}(x)}{p}\right) e^{2\pi i \frac{1}{p} \mathrm{tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(x)} = -q^{-1/2} \begin{cases} (-1)^f & p \equiv 1 \bmod 4 \\ (-i)^f & p \equiv 3 \bmod 4 \end{cases}$$

□

Proposition 6.13. *Let u be a unit, $t \in K$, and without loss of generality, assume $a \geq 0$. (We may do this since if $a < 0$, we get the same result as if $a = 0$.) Then,*

$$\gamma'\left(\frac{u}{\pi^a}, t\right) = \begin{cases} 0 & v_\pi(t) < -a \\ \psi'\left(\frac{-t^2\pi^{2a}}{4u^2}\right) q^{-a/2} & v_\pi(t) \geq -a,\ a \text{ even} \\ -(-1)^f \psi'\left(\frac{-t^2\pi^{2a}}{4u^2}\right) q^{-a/2} \left(\frac{u}{\mathfrak{p}}\right) & v_\pi(t) \geq -a,\ a \text{ odd},\ p \equiv 1 \text{ mod } 4 \\ -(-i)^f \psi'\left(\frac{-t^2\pi^{2a}}{4u^2}\right) q^{-a/2} \left(\frac{u}{\mathfrak{p}}\right) & v_\pi(t) \geq -a,\ a \text{ odd},\ p \equiv 3 \text{ mod } 4 \end{cases}$$

Proof. For the case that $v_\pi(t) < -a$, we invoke integration lemma 6.9 with $G = \pi^a O_K$. We get

$$\int_{O_K} \psi'\left(\frac{ux^2}{\pi^a} + tx\right) dx = \frac{1}{\#G} \int_{x \in O_K} \int_{y \in \pi^a O_K} \psi'\left(\frac{u(x+y)^2}{\pi^a} + t(x+y)\right) dydx$$

Substituting $y_{\text{old}} = \pi^a y_{\text{new}}$, we get

$$\frac{q^{-a}}{\#G} \int_{x \in O_K} \int_{y \in O_K} \psi'\left(\frac{u(x+\pi^a y)^2}{\pi^a} + t(x+\pi^a y)\right) dydx$$

Since ψ' doesn't care about integral inputs, some of the terms go away and we may rearrange the integral to

$$\frac{q^{-a}}{\#G}\int_{x\in O_K}\psi'\left(\frac{ux^2}{\pi^a}\right)\psi'(tx)\int_{y\in O_K}\psi'(t\pi^a y)\,dydx$$

By assumption $t\pi^a\notin O_K$, and so the inner integral vanishes, concluding this case.

As such, we may now take $v_\pi(t)\geq -a$. In this case, we make the substitution $x_{\text{new}}=x_{\text{old}}+\frac{t\pi^a}{2u}$. Notice that crucially we have $\frac{t\pi^a}{2u}\in O_K$. We get

$$\psi'\left(\frac{-t^2\pi^{2a}}{4u^2}\right)\int_{O_K}\psi'\left(\frac{ux^2}{\pi^a}\right)\,dx$$

We will now pull out the leading constant and shift focus entirely to computing the integral

$$\int_{O_K}\psi'\left(\frac{ux^2}{\pi^a}\right)\,dx$$

First of all, if $a\geq 2$, I claim that

$$\int_{O_K^\times}\psi'\left(\frac{ux^2}{\pi^a}\right)\,dx=0$$

and hence

$$\int_{O_K}\psi'\left(\frac{ux^2}{\pi^a}\right)\,dx=\int_{\pi O_K}\psi'\left(\frac{ux^2}{\pi^a}\right)\,dx=q^{-1}\int_{O_K}\psi'\left(\frac{ux^2}{\pi^{a-2}}\right)\,dx$$

Proving the claim is a fairly straightforward application of our integration lemmas. First, by lemma 6.11 we have

$$\int_{O_K^\times}\psi'\left(\frac{ux^2}{\pi^a}\right)\,dx=2\int_{O_K^{\times 2}}\psi'\left(\frac{ux}{\pi^a}\right)\,dx$$

Then, by lemma 6.10 with $G=1+\pi O_K$, this equals

$$2q\int_{x\in O_K^{\times 2}}\int_{y\in 1+\pi O_K}\psi'\left(\frac{uxy}{\pi^a}\right)\,dydx$$

Setting $y_{\text{old}}=1+\pi y_{\text{new}}$, we have

$$2\int_{x\in O_K^{\times 2}}\int_{y\in O_K}\psi'\left(\frac{ux(1+\pi y)}{\pi^a}\right)\,dydx=2\int_{x\in O_K^{\times 2}}\psi'\left(\frac{ux}{\pi^a}\right)\int_{y\in O_K}\psi'\left(\frac{uxy}{\pi^{a-1}}\right)\,dydx$$

Since $a \geq 2$ and ux is a unit, it follows that the inner integral is 0 and the desired result follows.

Letting $\bar{a} \in \{0, 1\}$ denote the value of a mod 2, the claim we just proved implies

$$\int_{O_K} \psi' \left(\frac{ux^2}{\pi^a} \right) \, dx = q^{-(a-\bar{a})/2} \int_{O_K} \psi' \left(\frac{ux^2}{\pi^{\bar{a}}} \right) \, dx$$

It is not hard to see that we have reduced the proposition to just the cases $a = 0$ and $a = 1$. Since the $a = 0$ case is trivial, we now restrict attention to the $a = 1$ case. We must now compute

$$\int_{O_K} \psi' \left(\frac{ux^2}{\pi} \right) \, dx$$

There is a very common argument that shows this quadratic Gauss sum is the same a character Gauss sum. By lemma 6.11, we may rewrite the integral as

$$\int_{O_K} \psi' \left(\frac{ux}{\pi} \right) \begin{cases} 2 & x \text{ is a square unit} \\ 0 & x \text{ is a nonsquare unit} \\ 1 & x \in \pi O_K \end{cases} \, dx$$

Since the integral of $\psi' \left(\frac{ux}{\pi} \right)$ over all of O_K is 0, the above integral is the same as

$$\int_{O_K} \psi' \left(\frac{ux}{\pi} \right) \begin{cases} 1 & x \text{ is a square unit} \\ -1 & x \text{ is a nonsquare unit} \\ 0 & x \in \pi O_K \end{cases} \, dx$$

However, this is just the character Gauss sum $\gamma' \left(\left(\frac{\cdot}{\mathfrak{p}} \right), \frac{u}{\pi} \right)$. $\square$

Remark 6.14. *It is worth special note that when $t = 0$, quadratic Gauss sums are essentially the same as character Gauss sums. There is different dependence on a, but up to some correction factors of the form $q^{something}$ they are basically identical. Namely, one could write the correspondence as*

$$\gamma' \left(\frac{u}{\pi^a}, 0 \right) = \begin{cases} q^{-a/2} \gamma' \left(1, \frac{u}{\pi^0} \right) & a \text{ is even} \\ q^{-(a-1)/2} \gamma' \left(\left(\frac{\cdot}{\mathfrak{p}} \right), \frac{u}{\pi} \right) & a \text{ is odd} \end{cases}$$

(Here, we are letting $\gamma' \left(1, \frac{u}{\pi^0} \right)$ denote a Gauss sum build from the trivial character which is 1 on all of $O_{K_\mathfrak{p}}$.)

This is not surprising, because of the "common argument" we used at the end of the above proof. In the case of even primes, however, the "common argument" fails completely. Despite this, we will still end up seeing a reincarnation of this correspondence between the values of quadratic and character Gauss sums. The differences will involve shuffling the units around and a constant factor (which only depends on $K_{\mathfrak{p}}$) that shows up for a odd. (This is found in proposition 6.83 and the remark following it.)

6.4 $\square_0$, $\square_1$, and δ

The goal of this section is to define and study three functions that show up in the computation of the Gauss sum. These functions will essentially tell us if and how we may write $x \in O_K$ as a norm from the extension $K[\sqrt{\pi}]$.

By local class field theory we know $N_K^{K[\sqrt{\pi}]}(K[\sqrt{\pi}]^\times)$ is a subgroup of $K^\times$ of index 2. Since this subgroup evidently contains $-\pi$, it is of the form $(-\pi)^{\mathbb{Z}} U_N$, for some subgroup $U_N \in O_K^\times$ of index 2. Note that if $N_K^{K[\sqrt{\pi}]}(x + y\sqrt{\pi}) = u \in U_N$, then we have $x \in O_K^\times$, $y \in O_K$ by valuation considerations.

Lemma 6.15. *If $u \in O_K^\times$ and we have x_0 such that $x_0^2 \equiv v \bmod \pi^{2e+1}$, then there is a lift x of x_0 such that $x^2 = v$ in $O_K^\times$.*

Proof. We apply Hensel's lemma with $f(x) = x^2 - v$ and $f'(x) = 2x$. Hensel's lemma states that if $|f'(x_0)|^2 > |f(x_0)|$, then the sequence given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ has a limit x, which is a root of of f.

In this case, x_0 is a unit so $|f'(x_0)|^2 = |2x_0|^2 = \frac{1}{4}$. Thus, the requirement in Hensel's lemma is that $|f(x_0)| < \frac{1}{4}$. This is exactly the same as requiring $x_0^2 \equiv v \bmod \pi^{2e+1}$. $\square$

In particular, if we let $U_i = \{u \in O_K^\times : u \equiv 1 \bmod \pi^i\}$, then every element of U_{2e+1} is square and hence $U_N \supset U_{2e+1}$.

Lemma 6.16. *For all π-adic coefficients c and all $i < 2e$, $\exists u \in U_N$ such that $u \equiv 1 + c\pi^i$ mod π^{i+1}.*

Proof. Choose b so that $b^2 = c$. There are two cases to consider. If i is odd, we have

$$N(1 + b\pi^{i/2}) = 1 - c\pi^i \equiv 1 + c\pi^i \bmod \pi^{i+1}$$

On the other hand, if i is even we have

$$N(1 + b\pi^{i/2}) = (1 + b\pi^{i/2})^2 \equiv 1 + c\pi^i \bmod \pi^{i+1},$$

where the step $2\pi^{i/2} \equiv 0 \bmod \pi^{i+1}$ follows from the inequality $\frac{i}{2} + e > i$. □

Note also that since $q - 1$ is odd, U_N contains all $q - 1$st roots of unity.

Consider the finite field F_q with $q = 2^f$, along with the function $f(x) = \mathrm{tr}_{F_2}^{F_q}(ax^2 + bx)$, where $a, b \in F_q$. We have the following nice lemma which will be used repeatedly and is what the calculation of the Gauss sum boils down to at the end.

Lemma 6.17. *As x varies over F_q, $f(x)$ is identically 0 if $a = b^2$ and takes on the values 0 and 1 equally often otherwise.*

Proof. Treating F_q as an F_2 vector space, $f(x)$ is a linear transform. Hence, one of the two possibilities listed above must occur, depending on whether f is identically 0.

The Frobenius element is $x \mapsto x^2$, so bx and b^2x^2 are conjugate, and hence have the same trace. Therefore, $f(x) = \mathrm{tr}_{F_2}^{F_q}((a + b^2)x^2)$. It is now obvious that this function is identically 0 iff $a + b^2 = 0$. □

Proposition 6.18. $1 + \tilde{\delta}\pi^{2e} \in U_N$ *for exactly half of all $\delta \in F_q$, independently of the lift $\tilde{\delta}$ chosen. Such δ are exactly those satisfying $\delta \in \alpha^{-2} \cdot ker(tr_{F_2}^{F_q})$.*

Proof. Let $U_{N,i} = \{u \in U_N : u \equiv 1 \bmod \pi^i\}$, so that $U_{N,0} = U_N$. We inductively show that for all $0 \leq i < 2e$, $U_{N,i}$ has index 2 in U_i.

The base case is the class field theory result. For the inductive step, we consider the short exact sequences $U_{i+1} \to U_i \to U_i/U_{i+1}$ and $U_{N,i+1} \to U_{N,i} \to U_{N,i}/U_{N,i+1}$. The natural inclusion of the second sequence into the first induces an isomorphism on the quotient, by the previous lemma. The inclusion of the middle term has index 2 by the inductive assumption, so the inclusion on the first terms also has index 2.

It follows that $U_{N,2e}$ has index 2 in U_{2e}. Now using $i = 2e$ in the short exact sequences, we have $U_{2e+1} \to U_{2e} \to U_{2e}/U_{2e+1}$ and $U_{N,2e+1} \to U_{N,2e} \to U_{N,2e}/U_{N,2e+1}$. We just observed the inclusion on the middle terms has index 2. The inclusion on the left terms is an isomorphism since we know $U_N \supset U_{2e+1}$. Hence, the inclusion on the quotients has index 2 and the first part of the result follows.

For the second part, consider

$$N(1 + b\pi^e) = 1 + (b^2 + \alpha^{-1}b)\pi^{2e} = 1 + \alpha^{-2}((\alpha b)^2 + \alpha b)\pi^{2e}$$

As observed in proving lemma 6.17, the map $x^2 + x : F_q \to F_q$ has image exactly the kernel of the trace map. This realizes $q/2$ possible values for δ. Since we can only get half of them, these must be exactly the possible values. $\square$

Note that our construction that realizes the values of δ shows the stronger statement that they can be gotten as perfect squares, not merely norms.

Lemma 6.19. *Suppose $x^2 - \pi y^2 = z^2 - \pi w^2$ for $x, y, z, w \in O_K$. Then $x^2 \equiv z^2 \bmod \pi^{2e+1}$ and $y^2 \equiv w^2 \bmod \pi^{2e}$.*

Proof. Since $x + z = x - z + 2z$, we have

$$v_\pi(x - z) < e \implies v_\pi(x + z) = v_\pi(x - z) \implies v_\pi(x^2 - z^2) = 2v_\pi(x + z) < 2e$$

So, in this case the valuation is smaller than $2e$ and even. On the other hand,

$$v_\pi(x - z) \geq e \implies v_\pi(x + z) \geq e \implies v_\pi(x^2 - z^2) \geq 2e$$

so in this case the valuation is at least $2e$. Together, this implies that if $v_\pi(a^2 - b^2)$ is odd for any $a, b \in O_K$, then $v_\pi(a^2 - b^2) \geq 2e + 1$. From the initial assumption, we have that $v_\pi(x^2 - z^2)$ and $v_\pi(y^2 - w^2)$ are consecutive integers, and so one of them is odd. The result follows. $\square$

Definition 6.20. *Once and for all, fix δ_0 so that $1 + \delta_0 \pi^{2e} \notin U_N$. If $u \in O_K^\times$, define*

$$\delta(u) = \begin{cases} 0 & u \in U_N \\ \delta_0 & u \notin U_N \end{cases}$$

From this, we may write

$$u(1 + \delta(u)\pi^{2e}) = x^2 - \pi y^2$$

Then by the previous lemma, it makes sense to define

$$\square_0(u) := x^2 \bmod \pi^{2e+1} \in O_K/\pi^{2e+1}, \quad \square_1(u) := y^2 \bmod \pi^{2e} \in O_K/\pi^{2e}$$

As stated earlier, these functions describe whether a unit is a norm as well as how to write it as such.

Remark 6.21. *If $u' = uc^2$ with $c \in O_K^\times$, then we clearly have $\square_0(u') = \square_0(u)c^2$. Hence,*

$$\frac{\square_0(u')}{u'} = \frac{\square_0(u)}{u}$$

In other words, the function $u \mapsto \frac{\square_0(u)}{u}$ only cares about the value of u in $O_K^\times / O_K^{\times 2}$.

Furthermore, if we had chosen a different value for δ_0, say δ_0', consider

$$\rho_\delta = \frac{1 + \delta_0' \pi^{2e}}{1 + \delta_0 \pi^{2e}}$$

We have $\rho_\delta \in U_{N,2e}$. From the remarks after (lemma 7) and (proposition 9), this implies that ρ_δ is a perfect square. It follows that $\frac{\square_0(u)}{u}$ is invariant of the choice of δ, and similarly for $\frac{\square_1(u)}{u}$.

Remark 6.22. *The function $u \mapsto \frac{\square_1(u)}{u}$ only cares about the value of u in O_K/π^{2e} and so with the previous remark only depends on $u \in \frac{(O_K/\pi^{2e})^\times}{(O_K/\pi^{2e})^{\times 2}}$.*

Proof. It is clear that u only matters mod π^{2e+1}, since U_{2e+1} is comprised of squares. So, it suffices to prove

$$\frac{\Box_1(u\cdot(1+a\pi^{2e}))}{u\cdot(1+a\pi^{2e})} = \frac{\Box_1(u)}{u} \text{ for } a \in O_K$$

If $1+a\pi^{2e} \in U_{N,2e}$, then it is a perfect square and we have

$$\frac{\Box_1(u\cdot(1+a\pi^{2e}))}{u\cdot(1+a\pi^{2e})} = \frac{\Box_1(u)(1+a\pi^{2e})}{u\cdot(1+a\pi^{2e})} = \frac{\Box_1(u)}{u}$$

Otherwise, write $1+a\pi^{2e} = (1+a'\pi^{2e})(1+\delta_0)$ for $1+a'\pi^{2e} \in U_{N,2e}$. Then we have

$$\frac{\Box_1(u\cdot(1+a\pi^{2e}))}{u\cdot(1+a\pi^{2e})} = \frac{\Box_1(u\cdot(1+\delta_0\pi^{2e}))(1+a'\pi^{2e})}{u\cdot(1+a\pi^{2e})} = \frac{\Box_1(u\cdot(1+\delta_0\pi^{2e}))}{u\cdot(1+\delta_0\pi^{2e})}$$

The key is that $\Box_1(u\cdot(1+\delta_0\pi^{2e})) = \Box_1(u)$ due to how $\Box_1$ is defined. Hence, the numerator is just $\Box_1(u)$. On the other hand, the denominator is just u, since $\Box_1$ is only defined mod π^{2e}. Note that because of this last step, this particular result only holds for $\Box_1$, and not $\Box_0$. $\Box$

Starting from $u(1+\delta(u)\pi^{2e}) = x^2 - \pi y^2$ mod π^{e+1}, we see $u \equiv x^2 + \pi y^2$ mod π^{e+1}. Further, note that when working mod π^{e+1}, squaring is a homomorphism over addition. (Since if we consider $x^2 = (x_0 + x_1\pi + x_2\pi^2 + \ldots)^2$, the lowest valuation crossterm will be $2x_0x_1\pi$, of valuation $e+1$.) It follows that x^2 must be exactly the even valuation terms of u, and πy^2 the odd valuation terms.

In summary, $\Box_0(u)$ is a perfect square satisfying

$$\Box_0(u) \equiv \sum_{i\geq 0} u_{2i}\pi^{2i} \bmod \pi^{e+1}$$

and $\Box_1$ satisfies

$$\pi\Box_1(u) \equiv \sum_{i\geq 0} u_{2i+1}\pi^{2i+1} \bmod \pi^{e+1}$$

This explains the choice of notation. $\Box_i$ is a square number that comes from the coefficients of index congruent to i mod 2. Of course, this only true mod π^{e+1}.

6.5 Computing the Gauss Sum

We will build a factor of τ into the Gauss sum to simplify computation as well as the final formula. It should be clear that computing this Gauss sum is the same as computing the original Gauss sum. Define

$$\psi'(x) = \psi_*(\tau x), \quad \gamma'\left(\frac{u}{\pi^a}\right) = \gamma\left(\frac{\tau u}{\pi^a}\right) = \int_{O_K} \psi'\left(\frac{ux^2}{\pi^a}\right)\, dx$$

Similarly, define

$$\gamma'\left(\frac{u}{\pi^a}, \frac{u'}{\pi^{a'}}\right) = \gamma\left(\frac{\tau u}{\pi^a}, \frac{\tau u'}{\pi^{a'}}\right) = \int_{O_K} \psi_*\left(\frac{\tau u x^2}{\pi^a} + \frac{\tau u' x}{\pi^{a'}}\right)\, dx$$

There is still a slightly more convenient form this sum can take on. Let $t = \frac{u'}{\pi^{a'}} \in K$. For a certain function f given in the statement of proposition (number), we will actually be evaluating sums of the form

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{f(a)}}\right) = \int_{O_K} \psi_*\left(\frac{\tau u x^2}{\pi^a} + \frac{\tau u' x}{\pi^{f(a)+a'}}\right)\, dx$$

It should be clear that evaluating this sum is equivalent to evaluating the original Gauss sum (regardless of f !). Finally, we may assume $a \geq 0$, since if it is negative we may increase it to 0 without affecting the value of the Gauss sum. It is with this setup that we can begin our main computation.

6.5.1 The Main Computation

Proposition 6.23. *Throughout, assume $a \geq 0$ and take $u \in O_K^\times$. Let $t = \frac{u'}{\pi^{a'}}$ with $a' \in \mathbb{Z}$ and $u' \in O_K$. Let $\bar{a} \in \{0,1\}$ denote the value of a mod 2. Similarly let $\overline{a+e} \in \{0,1\}$ be the value of $a+e$ mod 2.*

If $a \geq 2e+2$, then

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{a-e}}\right) = \begin{cases} 0 & t \notin O_K \\ q^{-(a-\bar{a}-2e)/2}\psi'\left(\frac{-\alpha^2 t^2}{\pi^a u}\right)\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right) & t \in O_K \end{cases}$$

If $0 \leq a \leq e$, then

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} 1 & \square_{1-\bar{a}}(u) \equiv t^2 \ \mathit{mod}\ \pi^a \\ 0 & \mathit{else} \end{cases}$$

If $e \le a \le 2e+1$, then

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)\beta(\overline{a+e}) & \square_{1-\bar{a}}(u) \equiv t^2 \ mod\ \pi^{2e-a} \\ 0 & else \end{cases} \tag{6.24}$$

where

$$\beta(\overline{a+e}) = \begin{cases} 1 & \overline{a+e} = 0 \\ \psi_0\left(\frac{1}{4}\left(\frac{\square_1(\alpha_0)}{\alpha_0}\right)\right)\left(-\frac{1}{\sqrt{q}}e^{2\pi i\frac{5}{8}f}\right) & \overline{a+e} = 1 \end{cases}$$

Here, ψ_0 is the standard exponential of K_0. Also recall $\alpha_0 \in O_{K_0}^\times$ is given by $\alpha = \sum_{i=0}^{e-1} \alpha_i \pi^i$.

We will rearrange these formulas into a cleaner form after completing the proof. The forms above are chosen because it is easier to see how they come out of the computations. Note that we end up with two nontrivially equal formulas for the case $a = e$!

Proof. We will have to chip away at this computation one case at a time. We will proceed roughly in the order of the formulas given above. Case 1 will deal with $a \ge 2e+2$.

Case 1a: This subcase deals with the further assumption $t \notin O_K$, or equivalently $a' > 0$.

We are starting with the expression

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{a-e}}\right) = \int_{O_K} \psi'\left(\frac{ux^2}{\pi^a} + \frac{u'x}{\pi^{a-e+a'}}\right)\, dx$$

Let $y_1 = \sum_{i=0}^{a-e+a'-2} x_i\pi^i$ and $\pi^{a-e+a'-1}y_2 = \pi^{a-e+a'-1}\sum_{i=0}^{\infty} x_{i+a-e+a'-1}\pi^i$ be a partitioning of the π-adic representation of x so that $x = y_1 + \pi^{a-e+a'-1}y_2$. We write

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{a-e}}\right) = \int_{O_K} \psi'\left(\frac{u(y_1+\pi^{a-e+a'-1}y_2)^2}{\pi^a} + \frac{u'(y_1+\pi^{a-e+a'-1}y_2)}{\pi^{a-e+a'}}\right)\, dx$$

$$= \int_{O_K} \psi'\left(\frac{uy_1^2}{\pi^a} + \frac{u2y_1y_2\pi^{a-e+a'-1}}{\pi^a} + \frac{uy_2^2\pi^{2a-2e+2a'-2}}{\pi^a} + \frac{u'y_1}{\pi^{a-e+a'}} + \frac{u'y_2\pi^{a-e+a'-1}}{\pi^{a-e+a'}}\right)\, dx$$

In the exponential above are five summands. The second summand is an integer since $2 = \pi^e/\alpha$ and $a' > 0$. The third summand is an integer since $2a - 2e - 2 > a \leftrightarrow a \geq 2e + 2$. As such, we get

$$= \int_{O_K} \psi' \left(\frac{uy_1^2}{\pi^a} + \frac{u'y_1}{\pi^{a-e+a'}} + \frac{u'y_2}{\pi} \right) \, dx$$

It immediately follows that the integral only depends on the π-adic coefficients up to $x_{a-e+a'-1}$ and that furthermore for some function f, we may rewrite the integral as

$$= \int_{O_K} \psi' \left(f(x_0, \ldots, x_{a-e+a'-2}) + \frac{u'x_{a-e+a'-1}}{\pi} \right) \, dx$$

Since only finitely many of the π-adic coefficients matter, we may rewrite the integral as a finite sum over these coefficients and find

$$= q^{-(a-e+a'-1)} \sum_{x_0, \ldots, x_{a-e+a'-2}} \psi' \left(f(x_0, \ldots, x_{a-e+a'-2}) \right) \sum_{x_{a-e+a'-1}} \psi' \left(\frac{u'x_{a-e+a'-1}}{\pi} \right)$$

Of course, the sum over $x_{a-e+a'-1}$ is just 0 and we are done. Though, we could make this last step a bit more rigorous by observing

$$\sum_{x_{a-e+a'-1}} \psi' \left(\frac{u'x_{a-e+a'-1}}{\pi} \right) = \int_{O_K} \psi' \left(\frac{u'x}{\pi} \right) \, dx$$

and then using the fact that the integral of a nonconstant exponential is 0.

Case 1b: Now we tackle the case $a \geq 2e + 2$, $t \in O_K$.

Rewrite

$$\int_{O_K} \psi' \left(\frac{ux^2}{\pi^a} + \frac{tx}{\pi^{a-e}} \right) \, dx = \int_{O_K} \psi' \left(\frac{ux^2}{\pi^a} + \frac{2\alpha tx}{\pi^a} \right) \, dx$$

We may complete the square to get

$$= \int_{O_K} \psi' \left(\frac{u}{\pi^a} \left(\left(x + \frac{\alpha t}{u} \right)^2 - \frac{\alpha^2 t^2}{u^2} \right) \right) \, dx$$

Since t is integral, this is the same as

$$\psi' \left(\frac{-\alpha^2 t^2}{\pi^a u} \right) \int_{O_K} \psi' \left(\frac{u}{\pi^a} x^2 \right) \, dx$$

which is the desired result.

Case 1c: Finally, we will handle the case $a \geq 2e+2$ and $u' = 0$. In this case we are concerned with the sum

$$\int_{O_K} \psi'\left(\frac{ux^2}{\pi^a}\right)\, dx$$

We will proceed similarly to case 1a. To do this, we let $y_1 = \sum_{i=0}^{a-e-2} x_i \pi^i$ and $\pi^{a-e-1} y_2 = \pi^{a-e-1} \sum_{i=0}^{\infty} x_{i+a-e-1} \pi^i$ so that $x = y_1 + \pi^{a-e-1} y_2$. We get

$$\begin{aligned} \int_{O_K} \psi'\left(\frac{ux^2}{\pi^a}\right)\, dx &= \int_{O_K} \psi'\left(\frac{u(y_1 + \pi^{a-e-1} y_2)^2}{\pi^a}\right)\, dx \\ &= \int_{O_K} \psi'\left(\frac{uy_1^2}{\pi^a} + \frac{2uy_1y_2\pi^{a-e-1}}{\pi^a} + \frac{u\pi^{2a-2e-2}y_2^2}{\pi^a}\right)\, dx \quad (6.25) \end{aligned}$$

In the exponential, there are three summands. The second summand can be simplified using $2 = \pi^e/\alpha$. The third summand is an integer and can be removed, since $2a - 2e - 2 \geq a \leftrightarrow a \geq 2e + 2$. We get

$$= \int_{O_K} \psi'\left(\frac{uy_1^2}{\pi^a} + \frac{uy_1y_2}{\alpha\pi}\right)\, dx$$

We see that for some function f, we have

$$= \int_{O_K} \psi'\left(f(x_0, \ldots, x_{a-e-2}) + \frac{ux_0x_{a-e-1}}{\alpha\pi}\right)\, dx$$

Just as before, we write this as a finite sum and get

$$= q^{-(a-e-1)} \sum_{x_0,\ldots,x_{a-e-2}} \psi'\left(f(x_0, \ldots, x_{a-e-2})\right) \sum_{x_{a-e-1}} \psi'\left(\frac{ux_0x_{a-e-1}}{\alpha\pi}\right)$$

This time, we see that the sum over x_{a-e-1} vanishes iff $x_0 \neq 0$. It follows from this that

$$\int_{O_K} \psi'\left(\frac{ux^2}{\pi^a}\right)\, dx = \int_{\pi O_K} \psi'\left(\frac{ux^2}{\pi^a}\right)\, dx = q^{-1} \int_{O_K} \psi'\left(\frac{ux^2}{\pi^{a-2}}\right)\, dx$$

We conclude that for $a \geq 2e + 2$

$$\gamma'\left(\frac{u}{\pi^a}\right) = q^{-1}\gamma'\left(\frac{u}{\pi^{a-2}}\right)$$

This reduces all cases where $a \geq 2e+2$ to just cases where $a \in \{2e, 2e+1\}$. In particular, we have

$$\gamma'\left(\frac{u}{\pi^a}\right) = q^{-(a-\bar{a}-2e)/2}\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)$$

It is left to the reader to check that this fits with the formulas claimed above.

Case 2 will deal with $0 \leq a \leq 2e+1$.

Case 2a: This subcase will further assume $t \notin O_K$, or equivalently $a' > 0$.

We are computing the sum

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \int_{O_K} \psi'\left(\frac{ux^2}{\pi^a} + \frac{u'x}{\pi^{(a+\bar{a})/2+a'}}\right)\, dx$$

Let $y_1 = \sum_{i=0}^{(a+\bar{a})/2+a'-2} x_i\pi^i$ and $\pi^{(a+\bar{a})/2+a'-1}y_2 = \pi^{(a+\bar{a})/2+a'-1}\sum_{i=0}^{\infty} x_{i+(a+\bar{a})/2+a'-1}\pi^i$ so that $x = y_1 + \pi^{(a+\bar{a})/2+a'-1}y_2$. We get

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \int_{O_K} \psi'\left(\frac{u(y_1+\pi^{(a+\bar{a})/2+a'-1}y_2)^2}{\pi^a} + \frac{u'(y_1+\pi^{(a+\bar{a})/2+a'-1}y_2)}{\pi^{(a+\bar{a})/2+a'}}\right)\, dx$$

$$= \int_{O_K} \psi'\left(\frac{uy_1^2}{\pi^a} + \frac{2uy_1y_2\pi^{(a+\bar{a})/2+a'-1}}{\pi^a} + \frac{u\pi^{a+\bar{a}+2a'-2}y_2^2}{\pi^a} + \frac{u'y_1}{\pi^{(a+\bar{a})/2+a'}} + \frac{u'y_2}{\pi}\right)\, dx$$

There are five summands in the exponential. The second summand may be removed since it is an integer. This is slightly nontrivial, but the fact that $2 = \pi^e/\alpha$ reduces this claim to showing the inequality

$(a+\bar{a})/2 + a' - 1 + e \geq a$. This is equivalent to the inequality $2e - 2 + 2a' + \bar{a} \geq a$, which is a combination of $2e - 2 + 2a' + \bar{a} \geq 2e + \bar{a}$, which is obvious and $2e + \bar{a} \geq a$, which follows by casework. The third summand is also an integer, though this time it is obvious. It follows that for some function f, the integral becomes

$$= \int_{O_K} \psi'\left(f(x_0, \ldots, x_{(a+\bar{a})/2+a'-2}) + \frac{u'x_{(a+\bar{a})/2+a'-1}}{\pi}\right)\, dx$$

Just as before, we may write this integral as a finite sum and it is clear that the sum over $x_{(a+\bar{a})/2+a'-1}$ will vanish.

Case 2b: We now consider $0 \leq a \leq e$ and $t \in O_K$.

We are computing the sum

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \int_{O_K} \psi_*\left(\frac{\tau u x^2}{\pi^a} + \frac{\tau t x}{\pi^{(a+\bar{a})/2}}\right)\, dx$$

If $a = 0, t \in O_K$, the integrand is identically 1, so we may as well restrict to $a \geq 1$.

Then from the definition of ψ_*, we have

$$= \int_{O_K} \exp\left(2\pi i \mathrm{tr}^K_{Q_2}\left(\frac{\tau u x^2}{\pi^{r+a}} + \frac{\tau t x}{\pi^{r+(a+\bar{a})/2}}\right)\right)\, dx$$

By the tracechanging formula, this becomes

$$= \int_{O_K} \exp\left(\pi i \mathrm{tr}^{K_0}_{Q_2}\left((ux^2)_{a-1} + (tx)_{(a+\bar{a})/2-1}\right)\right)\, dx$$

To extract the given π-adic coefficients of these products, we appeal to (lemma). Once we additionally note that

$$x^2 \equiv \sum_{i=0}^{\infty} x_i^2 \pi^{2i} \bmod 2$$

we can calculate that the integral becomes

$$= \int_{O_K} \exp\left(\pi i \mathrm{tr}^{K_0}_{Q_2}\left(\sum_{i=0}^{(a+\bar{a})/2-1} u_{a-1-2i} x_i^2 + \sum_{i=0}^{(a+\bar{a})/2-1} t_{(a+\bar{a})/2-1-i} x_i\right)\right)\, dx$$

We can again use the trick where we write the integral as a finite sum to get

$$= \prod_{i=0}^{(a+\bar{a})/2-1} q^{-1} \sum_{x_i} \exp\left(\pi i \mathrm{tr}^{K_0}_{Q_2}\left(u_{a-1-2i} x_i^2 + t_{(a+\bar{a})/2-1-i} x_i\right)\right)$$

Lemma 6.17 now kicks in and tells us that the above quantity is either 0 or 1, and is 0 unless we have

$$u_{a-1-2i} = t^2_{(a+\bar{a})/2-1-i} \qquad 0 \leq i \leq (a+\bar{a})/2 - 1$$

Reindex the conditions to get

$$u_{1-\bar{a}+2i} = t_i^2 \qquad 0 \leq i \leq (a+\bar{a})/2 - 1$$

Note that $1 - \bar{a}$ is either 0 or 1. We now sum the above conditions to get

$$\sum_{i=0}^{(a+\bar{a})/2-1} u_{1-\bar{a}+2i} \pi^{2i} = \sum_{i=0}^{(a+\bar{a})/2-1} t_i^2 \pi^{2i}$$

The left and right sides are just the π-adic expansions of $\square_{1-\bar{a}}(u)$ and t^2, stopping at valuation $a+\bar{a}-2 \in \{a-1, a-2\}$. This implies that all the conditions are equivalent to the single condition

$$\square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^a$$

This concludes the proof of this case.

Case 2c: We now consider $e \leq a \leq 2e+1$ and $t \in O_K$.

As in the previous case, we are computing

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \int_{O_K} \psi_*\left(\frac{\tau u x^2}{\pi^a} + \frac{\tau t x}{\pi^{(a+\bar{a})/2}}\right) dx$$

For $b \geq 0$ (to be chosen soon) let

$$I_y := \int_{y+\pi^b O_K} \psi_*\left(\frac{\tau u x^2}{\pi^a} + \frac{\tau t x}{\pi^{(a+\bar{a})/2}}\right) dx$$

so that

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \sum_{y \in O_K/\pi^b} I_y(u)$$

By the change of variables $x_{\text{old}} = y + \pi^b x_{\text{new}}$, we have

$$I_y = q^{-b} \int_{y+\pi^b O_K} \psi_*\left(\frac{\tau u (y+\pi^b x)^2}{\pi^a} + \frac{\tau t (y+\pi^b x)}{\pi^{(a+\bar{a})/2}}\right) dx$$

$$= q^{-b}\psi'\left(\frac{uy^2}{\pi^a} + \frac{ty}{\pi^{(a+\bar{a})/2}}\right) \int_{O_K} \psi_*\left(\frac{\tau \cdot 2uyx\pi^b}{\pi^a} + \frac{\tau u x^2 \pi^{2b}}{\pi^a} + \frac{\tau t x \pi^b}{\pi^{(a+\bar{a})/2}}\right) dx$$

Set $2 = \pi^e/\alpha$ and use the definition of ψ_* to get

$$= q^{-b}\psi'\left(\frac{uy^2}{\pi^a} + \frac{ty}{\pi^{(a+\bar{a})/2}}\right) \int_{O_K} \exp\left(2\pi i \operatorname{tr}^K_{Q_2}\left(\frac{\tau u y x \pi^{b+e}}{\alpha \pi^{r+a}} + \frac{\tau u x^2 \pi^{2b}}{\pi^{r+a}} + \frac{\tau t x \pi^b}{\pi^{r+(a+\bar{a})/2}}\right)\right) dx$$

We now make the choice of b. Choose $b = \lfloor \frac{a-e}{2} \rfloor = \frac{a-e-\overline{a+e}}{2}$. Also set $y' = (uy)/\alpha$. The integral simplifies to

$$= f(y) \int_{O_K} \exp\left(2\pi i \operatorname{tr}^K_{Q_2}\left(\frac{\tau y' x}{\pi^{r+(a-e+\overline{a+e})/2}} + \frac{\tau u x^2}{\pi^{r+e+\overline{a+e}}} + \frac{\tau t x}{\pi^{r+(e+\bar{a}+\overline{a+e})/2}}\right)\right) dx$$

where to save space we have temporarily set

$$f(y) := q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{uy^2}{\pi^a} + \frac{ty}{\pi^{(a+\bar{a})/2}}\right)$$

We may further compress this down by setting $t' = t + \pi^{(2e+\bar{a}-a)/2}y'$ to get

$$= f(y)\int_{O_K} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K}\left(\frac{\tau t' x}{\pi^{r+(e+\bar{a}+\overline{a+e})/2}} + \frac{\tau u x^2}{\pi^{r+e+\overline{a+e}}}\right)\right)\, dx$$

Note that $2e + \bar{a} - a \geq 0$, so that t' is integral.

Now turn your attention to the integrand and more specifically the two exponents in the denominators there. They are each at least as large as r. Our next goal is to figure out just how much bigger than r they are.

The second exponent is either e or $e+1$ larger than r, depending on $\overline{a+e}$.

For the first exponent, we have

$$1 \leq \frac{e+\bar{a}+\overline{a+e}}{2} = \left\lceil \frac{e+\bar{a}}{2} \right\rceil \leq \left\lceil \frac{e+1}{2} \right\rceil \leq \lceil e \rceil = e$$

It follows we can use the tracechanging formula (proposition 6.6) to get

$$= f(y)\int_{O_K} \exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left((t'x)_{(e+\bar{a}+\overline{a+e})/2-1} + (ux^2)_{e+\overline{a+e}-1} + (\overline{a+e})\frac{(ux^2)_0}{2\alpha_0}\right)\right)\, dx$$

By an application of lemma 6.8, we get

$$= f(y)\int_{O_K} \exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\sum_{i=0}^{(e+\bar{a}+\overline{a+e})/2-1}\left(t'_{(e+\bar{a}+\overline{a+e})/2-1-i}x_i + u_{e+\overline{a+e}-1-2i}x_i^2\right)\right.\right.$$
$$\left.\left. + (\overline{a+e})\frac{u_0 x_0^2}{2\alpha_0}\right)\right)\, dx \quad (6.26)$$

This expression is rather messy, so it can be helpful to write down the first and last terms inside the sigma notation. The sum goes

$$\left(t'_{(e+\bar{a}+\overline{a+e})/2-1}x_0 + u_{e+\overline{a+e}-1}x_0^2\right) + \ldots + \left(t'_0 x_{(e+\bar{a}+\overline{a+e})/2-1} + u_{1-\bar{a}}x^2_{(e+\bar{a}+\overline{a+e})/2-1}\right)$$

Moving on, just as in case 2b we may replace the integral with a finite sum and factor over the various x_i. The integral becomes

$$= f(y) \prod_{i=0}^{(e+\bar{a}+\overline{a+e})/2-1} q^{-1}\sum_{x_i} \exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left(t'_{(e+\bar{a}+\overline{a+e})/2-1-i}x_i + u_{e+\overline{a+e}-1-2i}x_i^2 + \theta\right)\right)$$

Here, we have temporarily let θ denote the quantity $(\overline{a+e})\frac{u_0 x_0^2}{2\alpha_0}$. Notably, this quantity is only nonzero when $i = 0$ and in the case $\overline{a+e} = 1$.

We will now evaluate I_y using case work on $\overline{a+e}$. Once we have evaluated each case separately, things will come back together and we will continue Case 2c as a whole.

Case 2ci: Assume $\overline{a+e} = 0$, so that a and e have the same parity.

Lemma 6.17 tells us that every term in the product is either 0 or 1, depending on i. Hence, I_y is either 0 or $f(y)$. For it to be nonzero, we must have

$$u_{e+\overline{a+e}-1-2i} = {t'}^2_{(e+\bar{a}+\overline{a+e})/2-1-i} \qquad 0 \leq i \leq (e+\bar{a}+\overline{a+e})/2-1$$

Reindexing and setting $\overline{a+e} = 0$ gives the equivalent conditions

$$u_{1-\bar{a}+2i} = {t'}^2_i \qquad 0 \leq i \leq (e+\bar{a})/2-1$$

Just like before, we may add these conditions to get

$$\sum_{i=0}^{(e+\bar{a})/2-1} u_{1-\bar{a}+2i}\pi^{2i} = \sum_{i=0}^{(e+\bar{a})/2-1} {t'}^2_i \pi^{2i}$$

The left and right sides are the π-adic expansions of $\square_{1-\bar{a}}(u)$ and ${t'}^2$, stopping at valuation $e+\bar{a}-2$, which is an even number. It follows that all of the conditions are equivalent to the single condition

$$\square_{1-\bar{a}}(u) \equiv {t'}^2 \bmod \pi^{e+\bar{a}}$$

In conclusion,

$$I_y = \begin{cases} f(y) & \square_{1-\bar{a}}(u) \equiv {t'}^2 \bmod \pi^{e+\bar{a}} \\ 0 & \text{else} \end{cases}$$

Case 2cii: Next, we consider $\overline{a+e} = 1$.

In this case, we have

$$I_y = f(y) \prod_{i=0}^{(e+\bar{a}-1)/2} q^{-1} \sum_{x_i} \exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left(t'_{(e+\bar{a}-1)/2-i}x_i + u_{e-2i}x_i^2 + \theta\right)\right)$$

Since $\overline{a+e} = 1$, the θ term will show up for the $i = 0$ term in the product, and will there take on the value of $\frac{u_0 x_0^2}{2\alpha_0}$. We may now apply lemma 6.17 to see that for $i \geq 1$, every term in the product is 0 or 1. Note that lemma 6.17 tells us nothing when $i = 0$, since the 2 in the denominator will cause us to care about our quantities mod 4.

From here we see I_y is either 0 or $f(y)\beta(1)$,[2] where

$$\beta(1) := q^{-1} \sum_{x_0} \exp\left(\pi i \operatorname{tr}_{Q_2}^{K_0}\left(t'_{(e+\bar{a}-1)/2} x_0 + u_e x_0^2 + \frac{u_0 x_0^2}{2\alpha_0}\right)\right)$$

Furthermore, I_y is nonzero[3] exactly when we have the conditions

$$u_{e-2i} = t'^2_{(e+\bar{a}-1)/2-i} \qquad 1 \leq i \leq (e+\bar{a}-1)/2$$

Or, upon reindexing

$$u_{1-\bar{a}+2i} = t'^2_i \qquad 0 \leq i \leq (e+\bar{a}-3)/2$$

As before, sum these conditions to get

$$\sum_{i=0}^{(e+\bar{a}-3)/2} u_{1-\bar{a}+2i}\pi^{2i} = \sum_{i=0}^{(e+\bar{a}-3)/2} t'^2_i \pi^{2i}$$

These are the starts of the π-adic expansions of $\square_{1-\bar{a}}(u)$ and t'^2_i, stopping at the even valuation $e + \bar{a} - 3$. It follows that the many conditions are equivalent to the single condition

$$\square_{1-\bar{a}}(u) \equiv t'^2 \bmod \pi^{e+\bar{a}-1}$$

In conclusion,

$$I_y = \begin{cases} f(y)\beta(1) & \square_{1-\bar{a}}(u) \equiv t'^2 \bmod \pi^{e+\bar{a}-1} \\ 0 & \text{else} \end{cases}$$

Case 2c, continued: The two branches 2ci and 2cii now come back together.

Letting $\beta(0) := 1$, we have

$$I_y = \begin{cases} f(y)\beta(\overline{a+e}) & \square_{1-\bar{a}}(u) \equiv t'^2 \bmod \pi^{e+\bar{a}-\overline{a+e}} \\ 0 & \text{else} \end{cases}$$

[2]We will deal with evaluating $\beta(1)$ later. Though for now it is worth noting that $\beta(1)$ itself appears to be a Gauss sum, taken over K_0.

[3]Later, we will show $f(y)\beta$ is necessarily nonzero.

Recalling $t' = t + \pi^{(2e+\bar{a}-a)/2}y'$ and $y' = (uy)/\alpha$, we see that I_y is nonzero when

$$\Box_{1-\bar{a}}(u) \equiv (t + \pi^{(2e+\bar{a}-a)/2}y')^2 \equiv t^2 + \pi^{2e+\bar{a}-a}y'^2 \equiv t^2 + \pi^{2e+\bar{a}-a}\frac{u^2y^2}{\alpha^2} \bmod \pi^{e+\bar{a}-\overline{a+e}}$$

This may be split into the pair of conditions

$$\Box_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e+\bar{a}-a}, \qquad \frac{\alpha^2}{u^2}\frac{\Box_{1-\bar{a}}(u) - t^2}{\pi^{2e+\bar{a}-a}} \equiv y^2 \bmod \pi^{a-e-\overline{a+e}}$$

The second condition simplifies to

$$\frac{\alpha}{u}\frac{\sqrt{\Box_{1-\bar{a}}(u)} - t}{\pi^{(2e+\bar{a}-a)/2}} \equiv y \bmod \pi^{(a-e-\overline{a+e})/2}$$

Recall that $\frac{a-e-\overline{a+e}}{2}$ was exactly the choice of b, so this condition uniquely specifies a value of y mod π^b. Let

$$y_0 = \frac{\alpha}{u}\frac{\sqrt{\Box_{1-\bar{a}}(u)} - t}{\pi^{(2e+\bar{a}-a)/2}}$$

From the definition

$$I_y := \int_{y+\pi^b O_K} \psi_*\left(\frac{\tau u x^2}{\pi^a} + \frac{\tau t x}{\pi^{(a+\bar{a})/2}}\right) dx$$

we know that the value of y only matters mod π^b. It follows that

$$I_y = \begin{cases} f(y_0)\beta(\overline{a+e}) & \Box_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e+\bar{a}-a} \text{ and } y \equiv y_0 \bmod \pi^b \\ 0 & \text{else} \end{cases}$$

Also recall

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \sum_{y \in O_K/\pi^b} I_y(u)$$

We can now see that depending on t, this sum either consists entirely of 0s or contains a unique nonzero term. It follows

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{uy_0^2}{\pi^a} + \frac{ty_0}{\pi^{(a+\bar{a})/2}}\right)\beta(\overline{a+e}) & \Box_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e+\bar{a}-a} \\ 0 & \text{else} \end{cases} \tag{6.27}$$

Note 1: A necessary consequence of our proof thus far is that the above piecewise formula cannot depend on the choice of representative y_0. Inspection should reveal that this is a nontrivial fact, although it can also be proven via direct computation.

Note 2: Since $2e+\bar{a}+a$ is an even number, the condition $\square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e+\bar{a}-a}$ is equivalent to $\square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e-a}$.

We may simplify the expression slightly by plugging in the definition of y_0 to get

$$= q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{u\left(\frac{\alpha}{u}\frac{\sqrt{\square_{1-\bar{a}}(u)}-t}{\pi^{(2e+\bar{a}-a)/2}}\right)^2}{\pi^a} + \frac{t\left(\frac{\alpha}{u}\frac{\sqrt{\square_{1-\bar{a}}(u)}-t}{\pi^{(2e+\bar{a}-a)/2}}\right)}{\pi^{(a+\bar{a})/2}}\right)\beta(\overline{a+e})$$

Light simplification yields

$$= q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{\alpha^2}{u}\frac{\left(\sqrt{\square_{1-\bar{a}}(u)}-t\right)^2}{\pi^{2e+\bar{a}}} + \frac{t\alpha}{u}\frac{\sqrt{\square_{1-\bar{a}}(u)}-t}{\pi^{e+\bar{a}}}\right)\beta(\overline{a+e})$$

Expanding the square,

$$\begin{aligned} &= q^{-(a-e-\overline{a+e})/2}\beta(\overline{a+e})\times \\ &\psi'\left(\frac{\alpha^2}{u}\frac{\square_{1-\bar{a}}(u)}{\pi^{2e+\bar{a}}} - \frac{\alpha^2}{u}\frac{2t\sqrt{\square_{1-\bar{a}}(u)}}{\pi^{2e+\bar{a}}} + \frac{\alpha^2}{u}\frac{t^2}{\pi^{2e+\bar{a}}} + \frac{t\alpha}{u}\frac{\sqrt{\square_{1-\bar{a}}(u)}}{\pi^{e+\bar{a}}} - \frac{t^2\alpha}{u}\frac{1}{\pi^{e+\bar{a}}}\right) \end{aligned} \tag{6.28}$$

The fourth and fifth terms can each be multiplied by $1 = 2\alpha/\pi^e$ to get

$$\begin{aligned} &= q^{-(a-e-\overline{a+e})/2}\beta(\overline{a+e})\times \\ &\psi'\left(\frac{\alpha^2}{u}\frac{\square_{1-\bar{a}}(u)}{\pi^{2e+\bar{a}}} - \frac{\alpha^2}{u}\frac{2t\sqrt{\square_{1-\bar{a}}(u)}}{\pi^{2e+\bar{a}}} + \frac{\alpha^2}{u}\frac{t^2}{\pi^{2e+\bar{a}}} + \frac{2t\alpha^2}{u}\frac{\sqrt{\square_{1-\bar{a}}(u)}}{\pi^{2e+\bar{a}}} - \frac{2t^2\alpha^2}{u}\frac{1}{\pi^{2e+\bar{a}}}\right) \end{aligned} \tag{6.29}$$

After canceling, we get the final result

$$= q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)\beta(\overline{a+e})$$

Therefore, we have proved

$$\begin{aligned} &\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \\ &\begin{cases} q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)\beta(\overline{a+e}) & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e+\bar{a}-a} \\ 0 & \text{else} \end{cases} \end{aligned} \tag{6.30}$$

This concludes the main body of the casework.

Case 2c, $\beta(1)$: This is a follow up to 2cii where we compute $\beta(1)$.

Recall that

$$\beta(1) := q^{-1}\sum_{x_0}\exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left(t'_{(e+\bar{a}-1)/2}x_0 + u_e x_0^2 + \frac{u_0 x_0^2}{2\alpha_0}\right)\right)$$

where the sum is taken over all q possibilities for the π-adic coefficient x_0. There is an implicit assumption $\overline{a+e}=1$, since that is when $\beta(1)$ shows up in our formula. Also recall

$$t' = t + \pi^{(2e+\bar{a}-a)/2}y' \quad , \quad y' = (uy_0)/\alpha = \frac{\sqrt{\Box_{1-\bar{a}}(u)} - t}{\pi^{(2e+\bar{a}-a)/2}} \implies t' = \sqrt{\Box_{1-\bar{a}}(u)} \text{ [4]}$$

So, we get

$$= q^{-1}\sum_{x_0}\exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\left(\sqrt{\Box_{1-\bar{a}}(u)}\right)_{(e+\bar{a}-1)/2} x_0 + u_e x_0^2 + \frac{u_0 x_0^2}{2\alpha_0}\right)\right)$$

$$= q^{-1}\sum_{x_0}\exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\sqrt{(\Box_{1-\bar{a}}(u))_{e+\bar{a}-1}}x_0 + u_e x_0^2 + \frac{u_0 x_0^2}{2\alpha_0}\right)\right)$$

$$= q^{-1}\sum_{x_0}\exp\left(\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\sqrt{u_e}x_0 + u_e x_0^2 + \frac{u_0 x_0^2}{2\alpha_0}\right)\right)$$

By lemma 6.17, we get

$$= q^{-1}\sum_{x_0}\exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{u_0 x_0^2}{4\alpha_0}\right)\right)$$

Next, notice that x_0 mod 2 determines x_0^2 mod 4. It follows that we may rewrite

$$= \int_{O_{K_0}} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{u_0 x^2}{4\alpha_0}\right)\right)\, dx$$

Since u_0 is a π-adic coefficient of u, it is a root of unity and in particular a perfect square. We may absorb it into the variable x and get

$$= \int_{O_{K_0}} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4\alpha_0}\right)\right)\, dx$$

[4] Note we took $y = y_0$ in the definition of y' since although technically β can be evaluated for any choice of y, it was only when $y = y_0$ that we got a contribution to γ. So, this is the only case we care about.

This leads us to observe that the result of this calculation only depends on α_0 and the residue field extension degree f. We can factor out the dependence on α_0 by noting that the quantity we are computing is in fact a Gauss sum for the field $K = K_0$.

For the moment, take $K = K_0$ so that $e = 1$. Choose $\pi = 2$, $\tau = 1$. α and α_0 for this field will then be 1, so any further usage of the variable α_0 will to refer to the one coming from the older, original K. Apologies for any confusion due to this. The Gauss sum we are trying to compute is then $\gamma'_{K_0}\left(\frac{\alpha_0^{-1}}{2^2}\right)$, where the subscript is a reminder of the field we are taking as K. For this Gauss sum, $a = 2$, so $\bar{a} = 0$, $\overline{a+e} = 1$.

Recall

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)\beta(\overline{a+e}) & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e+\bar{a}-a} \\ 0 & \text{else} \end{cases} \tag{6.31}$$

so that we have

$$\gamma'_{K_0}\left(\frac{\alpha_0^{-1}}{2^2}, 0\right) = \begin{cases} \psi_0\left(\frac{1}{4}\left(\frac{\square_1(\alpha_0^{-1})}{\alpha_0^{-1}}\right)\right)\beta(1) & \square_1(\alpha_0^{-1}) \equiv 0 \bmod 1 \\ 0 & \text{else} \end{cases}$$

where ψ_0 is the standard exponential for K_0.

It is clear that the conditional $\square_1(\alpha_0^{-1}) \equiv 0 \bmod 1$ trivially holds. We get

$$\int_{O_{K_0}} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4\alpha_0}\right)\right) dx = \psi_0\left(\frac{1}{4}\left(\frac{\square_1(\alpha_0^{-1})}{\alpha_0^{-1}}\right)\right)\int_{O_{K_0}} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right) dx$$

We can clean this up slightly using

$$\frac{\square_1(\alpha_0^{-1})}{\alpha_0^{-1}} = \frac{\alpha_0^2\square_1(\alpha_0^{-1})}{\alpha_0} = \frac{\square_1(\alpha_0)}{\alpha_0}$$

Regardless, the problem has been reduced to computing an integral that only depends on f. I claim that

$$\int_{O_{K_0}} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right) dx = -\frac{1}{\sqrt{q}}e^{2\pi i \frac{5}{8} f}$$

proof of which will be saved for the next proposition. Besides this one step, this concludes the evaluation of the Gauss sum. $\square$

Remark 6.32. *Although the proof of the proposition does not make this obvious, the formula*

$$\gamma_a(u) = \frac{1}{q}\gamma_{a-2}(u)$$

continues to hold as long as $a - 2 \geq e$ and both Gauss sums are nonzero. This can have some interesting implications, since if $\gamma_e(u)$ is nonzero, it is necessarily 1. This would imply that γ_{e+2i} is positive real. This claim is proven in corollary 6.40.

Proposition 6.33.

$$\sigma_f := \int_{O_{K_0}} \exp\left(2\pi i tr_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right)\, dx = -\frac{1}{\sqrt{q}}e^{2\pi i \frac{5}{8}f}$$

Proof. x^2 mod 4 depends only on x mod 2, so we rewrite the integral as a sum over the leading π-adic coefficient of x. Here, we identify the values of the π-adic coefficient with F_q in the obvious way. We get

$$\sigma_f = \frac{1}{q}\sum_{x\in F_q} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right),$$

We will often abuse notation like this by indexing our sums over elements of the residue field, and let it be understood that the terms in the sum will be the multiplicative lifts of these elements.

We now proceed using casework on whether f is even or odd. The proofs used in each case will be mostly unrelated.

Case 1: Let us first consider what happens when f is even. Let $f = 2n$, so $q = 2^{2n}$. The extension $F_{2^n} \subset F_{2^{2n}}$ determines a subextension $L_0 \subset K_0$. Let

$$\sigma_{f,a} = \sum_{\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(x)=a} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right)$$

so that we have

$$\sigma_f = \frac{1}{q}\sum_{a\in F_{2^n}} \sigma_{f,a}$$

For any $b \in F_{2^n}$, we have $\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(b) = 0$, which implies

$$\sigma_{f,a} = \sum_{\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(x)=a} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{(x+b)^2}{4}\right)\right)$$

Expanding out the square, we have

$$\sigma_{f,a} = \sum_{\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(x)=a} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right) \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{bx}{2}\right)\right) \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{b^2}{4}\right)\right)$$

Since the lift of b lies in L_0, we get

$$\sigma_{f,a} = \sum_{\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(x)=a} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right) \exp\left(2\pi i \mathrm{tr}_{Q_2}^{L_0}\left(\frac{ab}{2}\right)\right) \exp\left(2\pi i \mathrm{tr}_{Q_2}^{L_0}\left(\frac{b^2}{2}\right)\right)$$

or equivalently,

$$\sigma_{f,a} = \sigma_{f,a} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{L_0}\left(\frac{ab+b^2}{2}\right)\right)$$

Lemma 6.17 tells us that if $a = 1$, then $\exp\left(2\pi i \mathrm{tr}_{Q_2}^{L_0}\left(\frac{ab+b^2}{2}\right)\right) = 1$ for all choices of b. On the other hand, if $a \neq 1$, then we may pick b so that $\exp\left(2\pi i \mathrm{tr}_{Q_2}^{L_0}\left(\frac{ab+b^2}{2}\right)\right) \neq 1$, in which case it follows $\sigma_{f,a} = 0$.

We can conclude then, that

$$\sigma_f = \frac{1}{q} \sum_{\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(x)=1} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right)$$

To finish the computation, we will show that the terms in this sum all have the same value. To do this, let $x \in O_{K_0}$ be the lift of some index in the sum. That is, x is a root of unity obeying

$$x + x^{2^n} \equiv 1 \bmod 2$$

Raise both sides to the 2^i to get for all $i \geq 0$,

$$x^{2^i} + x^{2^{n+i}} \equiv 1 \bmod 2$$

Squaring this relation yields for all $i \geq 0$,

$$x^{2^{i+1}} + x^{2^{n+i+1}} \equiv 1 + 2x^{2^{n+i}+2^i} \bmod 4$$

We will now use this last identity to compute the value of the exponential in the sum for σ_f. To do this, we need to find the value mod 4 of

$$\mathrm{tr}_{Q_2}^{K_0}\left(x^2\right) = \sum_{i=0}^{2n-1} x^{2^{i+1}}$$

We can regroup these terms and apply the identity to get

$$\sum_{i=0}^{2n-1} x^{2^{i+1}} = \sum_{i=0}^{n-1} x^{2^{i+1}} + x^{2^{i+n+1}} \equiv \sum_{i=0}^{n-1} 1 + 2x^{2^{n+i}+2^i} \bmod 4$$

We slightly rewrite this sum as

$$n + 2\sum_{i=0}^{n-1}\left(x \cdot x^{2^n}\right)^{2^i}$$

From the fact $x^{2^n} \equiv 1 + x \bmod 2$, we get

$$n + 2\sum_{i=0}^{n-1}\left(x \cdot x^{2^n}\right)^{2^i} \equiv n + 2\sum_{i=0}^{n-1}\left(x + x^2\right)^{2^i} \bmod 4$$

Writing out the sum, we have

$$n + 2\left((x + x^2) + (x^2 + x^4) + \ldots + (x^{2^{n-1}} + x^{2^n})\right) \equiv n + 2 \bmod 4$$

Plugging this back into the sum for σ_f, we have

$$\sigma_f = \frac{1}{q}\sum_{\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(x)=1} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right) = \frac{1}{q}\sum_{\mathrm{tr}_{F_{2^n}}^{F_{2^{2n}}}(x)=1} \exp\left(2\pi i \cdot \frac{n+2}{4}\right)$$

Keeping in mind $q = 2^f = 2^{2n}$ and that there are 2^n terms in the sum, we get

$$\sigma_f = -\frac{1}{\sqrt{q}} e^{2\pi i \frac{n}{4}} = -\frac{1}{\sqrt{q}} e^{2\pi i \frac{5n}{4}} = -\frac{1}{\sqrt{q}} e^{2\pi i \frac{5f}{8}}$$

Case 2: Now we must consider the case when f is odd. The idea is to first compute the magnitude of the Gauss sum, and then do a counting argument to show there is only one way to add up the terms to get something of that magnitude.

To compute the magnitude of σ_f, we write let H denote the quaternions with coefficients in O_{K_0}. That is, for $h \in H$, we may write $h = a+bi+cj+dk,\ a, b, c, d \in O_{K_0}$. Letting N be the standard norm down to O_{K_0}, we may write

$$\sigma_f^4 = \int_{h\in H} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{N(h)}{4}\right)\right)\, dh$$

In order to compute the integral, we start with just the integral over $H^\times$:

$$\int_{h\in H^\times} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{N(h)}{4}\right)\right)\, dh$$

Note that by the standard inverse formula $h^{-1} = (a - bi - cj - dk)/N(h)$, $H^\times$ consists of exactly those elements so that $N(h) \in O_{K_0}^\times$. Also note that the condition to be in $H^\times$ may be worded as $a + b + c + d \not\equiv 0 \bmod 2$. Hence, $H^\times$ has a measure of $1 - \frac{1}{q}$.

Let $\mu(x) = \mathrm{meas}(\{h \in H | N(h) \equiv x \bmod 4\})$. For any function $f : O_{K_0}/4 \to \mathbb{C}$, we have

$$\int_{h\in H^\times} f(N(h))\, dh = \frac{1}{q^2} \sum_{x\in(O_{K_0}/4)^\times} \mu(x) f(x)$$

I claim that all the measures above are nonzero - that is, every class in $(O_{K_0}/4)^\times$ is a norm coming from $H^\times$. This is easily verified directly by considering a norm of the form $0 + x^2 + y^2 + y^2 = x^2 + 2y^2$. Furthermore, all of the measures above are equal. This follows because given x, y, there is $z \in H^\times$ so that $N(z) = y/x$. Then, multiplication by z is a measure preserving bijection on H that maps $\mu(x)$ onto $\mu(y)$. Call this common measure μ. Then, we have

$$\int_{h\in H^\times} f(N(h))\, dh = \frac{\mu}{q^2} \sum_{x\in(O_{K_0}/4)^\times} f(x)$$

Setting $f = 1$, we get

$$1 - \frac{1}{q} = \frac{\mu}{q^2}(q^2 - q),$$

so that $\mu = 1$. Applying this to the desired integral and converting the sum back to an integral, we have

$$\int_{h\in H^\times} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{N(h)}{4}\right)\right)\, dh = \int_{x\in O_{K_0}^\times} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x}{4}\right)\right)\, dx$$

Applying the change of variables $x_{\text{old}} = x_{\text{new}} + 2s$, we see

$$\int_{x\in O_{K_0}^\times} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x}{4}\right)\right)\, dx = \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{s}{2}\right)\right) \int_{x\in O_{K_0}^\times} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x}{4}\right)\right)\, dx$$

Choosing $s \in O_{K_0}^\times$ so that $\mathrm{tr}_{Q_2}^{K_0}(s) = 1$, we conclude that the integral of interest is 0. That is, we have shown

$$\int_{h \in H^\times} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{N(h)}{4}\right)\right) \, dh = 0$$

Hence, it suffices to compute

$$\sigma_f^4 = \int_{h \in H - H^\times} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{N(h)}{4}\right)\right) \, dh$$

Such an h in the domain necessarily satisfies $h \equiv a + bi + cj + (a+b+c)k \bmod 2$. We get

$$\sigma_f^4 = \frac{1}{q} \int_{(a,b,c) \in O_{K_0}^3} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{a^2+b^2+c^2+(a+b+c)^2}{4}\right)\right) \, dadbdc$$

Expanding, we get

$$\sigma_f^4 = \frac{1}{q} \int_{(a,b,c) \in O_{K_0}^3} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{a^2+b^2+c^2+ab+bc+ac}{2}\right)\right) \, dadbdc$$

Since we only care about $a^2+b^2+c^2+ab+bc+ac = a^2+a(b+c)+b^2+c^2+bc$ mod 2, lemma 6.17 tells us that the integral over a will vanish unless $b+c \equiv 1 \bmod 2$. Thus, we assume $c \equiv b+1 \bmod 2$, which occurs in $1/q$ cases. This gives

$$\sigma_f^4 = \frac{1}{q^2} \int_{(a,b) \in O_{K_0}^2} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{a^2+a+b^2+(1+b)^2+b(1+b)}{2}\right)\right) \, dadb$$

Simplifying, we get

$$\sigma_f^4 = \frac{1}{q^2} \int_{(a,b) \in O_{K_0}^2} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{a^2+a+b^2+b+1}{2}\right)\right) \, dadb$$

By lemma 6.17, $\mathrm{tr}(x^2+x) \equiv 0 \bmod 2$, so we get

$$\begin{aligned}\sigma_f^4 &= \frac{1}{q^2} \int_{(a,b) \in O_{K_0}^2} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{1}{2}\right)\right) \, dadb \\ &= \frac{1}{q^2} \int_{(a,b) \in O_{K_0}^2} \exp\left(2\pi i \frac{f}{2}\right) \, dadb = \frac{1}{q^2}(-1)^f\end{aligned}$$

We conclude that $|\sigma_f| = \frac{1}{\sqrt{q}}$.

Now we can start the counting argument. Note that

$$\psi_*\left(\frac{(x+1)^2}{4}\right) = \psi_*\left(\frac{x^2}{4}\right)\psi_*\left(\frac{x}{2}\right)\psi_*\left(\frac{1}{4}\right) = i^f\psi_*\left(\frac{x^2}{4}\right)\psi_*\left(\frac{x}{2}\right)$$

If $\mathrm{tr}_{F_2}^{F_q}(x) = 0$, then it further reduces to $i^f\psi_*\left(\frac{x^2}{4}\right)$. Using this, we write

$$\begin{aligned}\sigma_f &= \int_{\mathrm{tr}_{F_2}^{F_q}(x)=0} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right)\, dx + \int_{\mathrm{tr}_{F_2}^{F_q}(x)=1} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right)\, dx \\ &= \int_{\mathrm{tr}_{F_2}^{F_q}(x)=0} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right)\, dx + \int_{\mathrm{tr}_{F_2}^{F_q}(x)=0} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{(x+1)^2}{4}\right)\right)\, dx \\ &= (1+i^f)\int_{\mathrm{tr}_{F_2}^{F_q}(x)=0} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2}{4}\right)\right)\, dx\end{aligned}$$

$x + x^2 : F_q \to F_q$ is a linear function with kernel of size 2. Lemma 6.17 tells us its image consists of elements of trace 0. It follows that the image consists of every element of trace 0, twice. We apply the change of variables $x_{\text{old}} = x_{\text{new}}^2 + x_{\text{new}}$, yielding

$$\begin{aligned}\sigma_f &= \frac{1+i^f}{2}\int_{x\in F_q} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{(x+x^2)^2}{4}\right)\right)\, dx \\ &= \frac{1+i^f}{2}\int_{x\in F_q} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2+2x^3+x^4}{4}\right)\right)\, dx\end{aligned}$$

Since $x \in O_{K_0}$ is a root of unity, x^2 and x^4 are conjugates and have the same trace.[5] We get

$$\sigma_f = \frac{1+i^f}{2}\int_{x\in F_q} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2+x^3}{2}\right)\right)\, dx$$

Let

$$I = \int_{x\in F_q} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2+x^3}{2}\right)\right)\, dx$$

I is clearly real. An elementary check shows that our desired result holds if I is the rational number $\frac{1}{2^{(f-1)/2}}\left(\frac{2}{f}\right)$, where the second factor is the quadratic symbol. Note that since we know $|\sigma_f|$, we at least know that $I = \pm\frac{1}{2^{(f-1)/2}}\left(\frac{2}{f}\right)$. We will resolve the sign by induction.

For the base case, take $f = 1$. Then, we check

$$I = \int_{x\in F_2} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2+x^3}{2}\right)\right)\, dx = \frac{1}{2}(1+1) = 1$$

[5] This replacement still works even if we use an arbitrary lift.

For the inductive step, assume we have shown the claim for all values $\leq f$ and we wish to prove it for f. Since f is odd, let p denote any prime factor of f, and let $f' = f/p$. Since the quantity $\mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2+x^3}{2}\right)$ is constant on conjugacy classes in F_{2^f}, we may write

$$2^f I = 2^{f'} \int_{x \in F_{2^{f'}}} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2+x^3}{2}\right)\right) dx + \sum_{c \in \text{conj classes in } F_{2^f} - F_{2^{f'}}} \#c \cdot \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{c^2+c^3}{2}\right)\right), \quad (6.34)$$

where to evaluate the function on a conjugacy class just means to evaluate it on any element.

In the sum above, $\#c$ is always divisible by p. Hence, we get

$$2^f I \equiv 2^{f'} \int_{x \in F_{2^{f'}}} \exp\left(2\pi i \mathrm{tr}_{Q_2}^{K_0}\left(\frac{x^2+x^3}{2}\right)\right) dx \bmod p$$

If we let $L_0 \subset K_0$ be the subset corresponding to the residue field $F_{2^{f'}}$, we may reduce this to

$$2^f I \equiv 2^{f'} \int_{x \in F_{2^{f'}}} \exp\left(2\pi i \cdot p \cdot \mathrm{tr}_{Q_2}^{L_0}\left(\frac{x^2+x^3}{2}\right)\right) dx \bmod p$$

However, since p is odd, it won't affect the value of the exponential. Hence, the integral is just $\frac{1}{2^{(f'-1)/2}}\left(\frac{2}{f'}\right)$, by the inductive assumption. We also choose to bring in the fact that $I = \pm\frac{1}{2^{(f-1)/2}}\left(\frac{2}{f}\right)$ to get the equation

$$\pm 2^{(f+1)/2}\left(\frac{2}{f}\right) \equiv 2^{(f'+1)/2}\left(\frac{2}{f'}\right) \bmod p$$

Rewrite this as

$$\pm 2^{(pf'+1)/2}\left(\frac{2}{pf'}\right) \equiv 2^{(f'+1)/2}\left(\frac{2}{f'}\right) \bmod p$$

Rearranging again, we get

$$\pm 2^{(f'(p-1))/2} \equiv \left(\frac{2}{p}\right) \bmod p$$

However, we know $\left(\frac{2}{p}\right) \equiv 2^{(p-1)/2} \bmod p$. Since, f' is odd, its presence doesn't change the congruence. Hence the sign must be $+1$, as desired. $\square$

6.5.2 Cleaning up the Formula

Corollary 6.35. *Let a be any integer and take $u \in O_K^\times$. Let $t \in K$. Let $\bar{a} \in \{0,1\}$ denote the value of a mod 2. Similarly let $\overline{a+e} \in \{0,1\}$ be the value of $a+e$ mod 2. Then*

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = g_0(a)g_1(a,u,t)g_2(a,u,t)g_3(a)$$

Each g_i ($i \geq 1$) is given by a piecewise formula where you check each condition in turn and only use the formula for the first one that holds.

g_0 determines what the magnitude of the Gauss sum should be when the Gauss sum is non-zero.

$$g_0(a) = q^{-max(0,a-e)/2}$$

$g_1(a,u,t)$ is always 0 or 1 and its job is to control whether γ' is zero or not.

$$g_1(a,u,t) = \begin{cases} 1 & \square_{1-\bar{a}}(u) \equiv t^2 \ mod\ \pi^{min(a+\bar{a}, 2e-a+\bar{a})} \\ 0 & else \end{cases}$$

$g_2(a,u,t)$ is the main term in the formula and shows how the complex argument of γ' depends on u, t. It is given by

$$g_2(a,u,t) = \begin{cases} 1 & a \leq e \\ \psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right) & a > e \end{cases}$$

$g_3(a)$ is an extra constant factor that shows up when $a \not\equiv e \bmod 2$.

$$g_3(a) = \begin{cases} 1 & a \leq e \\ 1 & \overline{a+e} = 0 \\ \psi_0\left(\frac{1}{4}\frac{\square_1(\alpha_0)}{\alpha_0}\right)\left(-e^{2\pi i \frac{5}{8} f}\right) & \overline{a+e} = 1 \end{cases}$$

Here, ψ_0 is the standard exponential of K_0. Also recall $\alpha_0 \in O_{K_0}^\times$ is given by $\alpha = \sum_{i=0}^{e-1} \alpha_i \pi^i$.

Remark 6.36. *In the proof below, we will show that the function*

$$g_2^{alt}(a,u,t) = \begin{cases} 1 & a < e \\ \psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right) & a \geq e \end{cases}$$

also would work in our formula for the Gauss sum! Surprisingly, in the case $a = e$, our two different formulas for $g_2(e, u, t)$ are not necessarily equal to each other! This is not a contradiction because if the two possibilities are not the same then we will have $g_1(a, u, t) = 0$ and the difference does not matter. This does mean that our definition would be more natural if we combined g_1 and g_2 into a single function (since $g_1(a, u, t)g_2(a, u, t) = g_1(a, u, t)g_2^{alt}(a, u, t)$). However, we have opted to keep them separate to make the algebra a bit easier to handle.

Proof. We use the formulas we just proved in proposition 6.23. For convenience we recall them.

If $a \geq 2e + 2$, then

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{a-e}}\right) = \begin{cases} 0 & t \notin O_K \\ q^{-(a-\bar{a}-2e)/2}\psi'\left(\frac{-\alpha^2 t^2}{\pi^a u}\right)\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right) & t \in O_K \end{cases}$$

If $0 \leq a \leq e$, then

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} 1 & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^a \\ 0 & \text{else} \end{cases}$$

If $e \leq a \leq 2e + 1$, then

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} q^{-(a-e-\overline{a+e})/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)\beta(\overline{a+e}) & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e-a} \\ 0 & \text{else} \end{cases} \tag{6.37}$$

where

$$\beta(\overline{a+e}) = \begin{cases} 1 & \overline{a+e} = 0 \\ \psi_0\left(\frac{1}{4}\left(\frac{\square_1(\alpha_0)}{\alpha_0}\right)\right)\left(-\frac{1}{\sqrt{q}}e^{2\pi i \frac{5}{8} f}\right) & \overline{a+e} = 1 \end{cases}$$

Note that this implies if $e \leq a \leq 2e + 1$, then

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} q^{-(a-e)/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)g_3(a) & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e-a} \\ 0 & \text{else} \end{cases}$$

We now proceed by mild casework. The case of $a < 0$ is easily done by hand and does not require our formulas. In this case

$$\begin{aligned} \gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) &= \int_{O_K} \psi'\left(\frac{ux^2}{\pi^a} + \frac{tx}{\pi^{(a+\bar{a})/2}}\right) dx \\ &= \int_{O_K} \psi'\left(\frac{tx}{\pi^{(a+\bar{a})/2}}\right) dx = \mathbb{1}_{\pi^{(a+\bar{a})/2}O_K}(t) \end{aligned}$$

This last condition is the same thing as

$$t^2 \equiv 0 \bmod \pi^{min(a+\bar{a},2e-a+\bar{a})}$$

(Note that since $a < 0$, we used $\pi^{min(a+\bar{a},2e-a+\bar{a})} = \pi^{a+\bar{a}}$.) Furthermore, since $a+\bar{a} \leq 0$, we always have $\square_{1-\bar{a}}(u) \equiv 0 \bmod \pi^{min(a+\bar{a},2e-a+\bar{a})}$. Hence the above condition is the same as

$$\square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{min(a+\bar{a},2e-a+\bar{a})}$$

which proves the $a < 0$ case.

If $0 \leq a \leq e$, then the formulas we proved tell us

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} 1 & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^a \\ 0 & \text{else} \end{cases}$$

Since $a \leq 2e-1$, two perfect squares are congruent mod π^a iff they are congruent mod $\pi^{a+\bar{a}}$. Furthermore, for $a \leq e$, $min(a+\bar{a}, 2e-a+\bar{a}) = a+\bar{a}$, so we get

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} 1 & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{min(a+\bar{a},2e-a+\bar{a})} \\ 0 & \text{else} \end{cases} = g_1(a,u,t)$$

It is now straightforward to check this has the desired form.

If $e \leq a \leq 2e+1$, then the formulas we proved tell us

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} q^{-(a-e)/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)g_3(a) & \square_{1-\bar{a}}(u) \equiv t^2 \bmod \pi^{2e-a} \\ 0 & \text{else} \end{cases}$$

We apply similar reasoning to the previous case but this time using $min(a+\bar{a}, 2e-a+\bar{a}) = 2e-a+\bar{a}$ to conclude that

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = q^{-(a-e)/2}\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)-t^2}{u}\right)\right)g_1(a,u,t)g_3(a)$$

It is now straightforward to check this has the desired form. Note that in the case $a = e$ we have indeed gotten a different result for $g_2(e,u,t)$. There is also the presence of a g_3 term, although $g_3(e) = 1$ so we may ignore it in the case $a = e$. Since the value of the Gauss sum cannot depend on how we calculate it, it follows that if the two possible forms of $g_2(e,u,t)$ disagree, we must have $g_1(a,u,t) = 0$ to avoid a contradiction.

Finally, if $a \geq 2e+2$, let t' be such that $t'/\pi^{a-e} = t/\pi^{(a+\bar{a})/2}$. Then we have

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} 0 & t' \notin O_K \\ q^{-(a-\bar{a}-2e)/2}\psi'\left(\frac{-\alpha^2 t'^2}{\pi^a u}\right)\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right) & t' \in O_K \end{cases}$$

We now substitute in the value of $\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)$, which gives

$$\gamma'\left(\frac{u}{\pi^a}, \frac{t}{\pi^{(a+\bar{a})/2}}\right) = \begin{cases} 0 & t' \notin O_K \\ q^{-(a-e)/2}\psi'\left(\frac{-\alpha^2 t'^2}{\pi^a u}\right)\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)}{u}\right)\right) g_3(a) & t' \in O_K \end{cases}$$

Since $t' = t/\pi^{e+(-a+\bar{a})/2}$, it is now easy to check that

$$\psi'\left(\frac{-\alpha^2 t'^2}{\pi^a u}\right)\psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\left(\frac{\square_{1-\bar{a}}(u)}{u}\right)\right) = \psi'\left(\frac{\alpha^2}{\pi^{2e+\bar{a}}}\frac{\square_{1-\bar{a}}(u) - t^2}{u}\right)$$

which gives the correct g_2 term. Finally, the condition for γ' to be nonzero can be rewritten as $t'^2 \in O_K$, or that $t^2 \in \pi^{2e-a+\bar{a}}O_K$. Since $a \geq 2e+2$, $min(a+\bar{a}, 2e-a+\bar{a}) = 2e-a+\bar{a}$ and this is the same as

$$t^2 \equiv 0 \bmod \pi^{min(a+\bar{a},2e-a+\bar{a})}$$

Furthermore, we have $2e-a+\bar{a} < 0$ and so we always have

$$\square_{1-\bar{a}}(u) \equiv 0 \bmod \pi^{min(a+\bar{a},2e-a+\bar{a})}$$

which gives us the desired g_1.

$\square$

Remark 6.38. *It takes a fair bit of extra work to determine the exact constant value that g_3 can take on, and said work feels rather orthogonal to the rest of the computation. Also of note is that for many of the calculations I have done using this formula, the value of g_3 always finds a way to cancel out in the end. Perhaps there is some sort of symmetry to the computations that is lacked by g_3 that causes this to happen.*

6.5.3 Gauss Sum Identities

We now prove two very useful identities - the Gauss reflection formula and the blurring lemma.

Looking at the above formulas, one may notice some interesting symmetries if one reflects a across the value $a = e$. This is described more explicitly in the following “reflection formula". Here, we will take $t = 0$, although it is not difficult to describe a formula for nonzero t.

Proposition 6.39. *For ANY integer a and unit u we have*

$$\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)\overline{\gamma'\left(\frac{u}{\pi^{a}}\right)} = q^{-(a+\bar{a})/2}\gamma'\left(\frac{u}{\pi^{2e-a}}\right)$$

In particular, since negating the input to a Gauss sum conjugates the output value, this is equivalent to

$$\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)\gamma'\left(\frac{-u}{\pi^{a}}\right) = q^{-(a+\bar{a})/2}\gamma'\left(\frac{u}{\pi^{2e-a}}\right)$$

Proof. By the above corollary, we must verify the equality

$$q^{-(e+\bar{a})/2}g_2(2e+\bar{a},u,0)g_3(2e+\bar{a})\cdot q^{-max(0,a-e)/2}g_1(a,u,0)\overline{g_2(a,u,0)g_3(a)} =$$

$$q^{-(a+\bar{a})/2}\cdot q^{-max(0,e-a)/2}g_1(2e-a,u,0)g_2(2e-a,u,0)g_3(2e-a)$$

First, we check that the powers of q cancel out. That is, we want to check

$$q^{-(e+\bar{a})/2}q^{-max(0,a-e)/2} = q^{-(a+\bar{a})/2}q^{-max(0,e-a)/2}$$

This is equivalent to verifying

$$(e+\bar{a}) + max(0,a-e) = (a+\bar{a}) + max(0,e-a)$$

which follows from the general identity (for all $x \in \mathbb{R}$)

$$x + max(-x,0) = max(0,x)$$

It follows that the desired statement reduces to checking that

$$g_2(2e+\bar{a},u,0)g_3(2e+\bar{a})\cdot g_1(a,u,0)\overline{g_2(a,u,0)g_3(a)} = g_1(2e-a,u,0)g_2(2e-a,u,0)g_3(2e-a)$$

We now check that the g_3 terms on each side are equal. That is, we check that

$$g_3(2e+\bar{a})\overline{g_3(a)} = g_3(2e-a) \leftrightarrow g_3(2e+\bar{a}) = g_3(a)g_3(2e-a)$$

If $a = e$ then all three g_3 terms are 1 and the equation is true. If $a \neq e$ then note that the equation is symmetric under $a \mapsto 2e - a$ and so wlog assume $a > e$. In this case, the desired equation becomes $g_3(2e + \bar{a}) = g_3(a)$ but this is now obvious from the definition of g_3. Thus the proof will be complete if we can verify the equation

$$g_2(2e+\bar{a},u,0)g_1(a,u,0)\overline{g_2(a,u,0)} = g_1(2e-a,u,0)g_2(2e-a,u,0)$$

Note that the g_1 terms on each side are clearly equal since $g_1(a, u, 0) = g_1(2e - a, u, 0)$. However, we cannot cancel them since g_1 may be zero. Regardless, rearrange the equation to get

$$g_2(2e+\bar{a},u,0)g_1(a,u,0) = g_1(a,u,0)g_2(a,u,0)g_2(2e-a,u,0)$$

Without loss of generality we will only prove this equation in the case $a \geq e$ since the equation is unchanged if we switch a for $2e - a$. By remark 6.36, this equation may be rewritten as

$$g_2(2e+\bar{a},u,0)g_1(a,u,0) = g_1(a,u,0)g_2^{alt}(a,u,0)g_2(2e-a,u,0)$$

Since $a \geq e$, we have $g_2(2e - a, u, 0) = 1$. Furthermore, it is straightfoward to check from the definitions that for $a \geq e$ we have $g_2(2e + \bar{a}, u, 0) = g_2^{alt}(a, u, 0)$. Hence the equation has been verified. $\square$

Corollary 6.40. *(i) For any $a \geq e$ and unit u, $\gamma'\left(\frac{u}{\pi^{a+2}}\right) = q^{-1}\gamma'\left(\frac{u}{\pi^{a}}\right)$, as long as both Gauss sums are nonzero. In particular, both Gauss sums are nonzero as long as $a \geq 2e$.*

(ii) If $\gamma'\left(\frac{u}{\pi^e}\right) \neq 0$ *and* $a \equiv e$ *mod 2, then*

$$\gamma'\left(\frac{u}{\pi^a}\right) = \begin{cases} 1 & a \leq e \\ q^{-(a-e)/2} & a \geq e \end{cases}$$

Proof. To prove (i), it suffices to show that the two Gauss sums differ by a positive real factor. That the factor is q^{-1} then would follow easily, since corollary 6.35 tells us the magnitudes of the Gauss sums. In order to actually prove they differ by a positive real factor, we instead show that $\gamma'\left(\frac{u}{\pi^a}\right)$ and $\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)$ differ by a positive real factor. The same argument will also show that $\gamma'\left(\frac{u}{\pi^{a+2}}\right)$ and $\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)$ differ by a positive real factor. Let us do this now.

The Gauss reflection formula tells us that

$$\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)\overline{\gamma'\left(\frac{u}{\pi^a}\right)} = q^{-(a+\bar{a})/2}\gamma'\left(\frac{u}{\pi^{2e-a}}\right)$$

Swapping a for $2e-a$ gives us

$$\gamma'\left(\frac{u}{\pi^{2e+\bar{a}}}\right)\overline{\gamma'\left(\frac{u}{\pi^{2e-a}}\right)} = q^{-(2e-a+\bar{a})/2}\gamma'\left(\frac{u}{\pi^a}\right)$$

In the case $a \geq e$, we can use corollary 6.35 to see that $\gamma'\left(\frac{u}{\pi^{2e-a}}\right)$ is a positive real number (it cannot be 0 since we assume $\gamma'\left(\frac{u}{\pi^a}\right) \neq 0$) and the result follows. Finally, the last comment about the Gauss sums being nonzero when $a \geq 2e$ is just a quick application of 6.35.

To prove (ii), if $\gamma'\left(\frac{u}{\pi^e}\right) \neq 0$, then corollary 6.35 tells us that $\gamma'\left(\frac{u}{\pi^e}\right) = 1$. The result now follows by repeated application of (i). □

Part (i) of the above corollary does not work for all choices of $a \in \mathbb{Z}$. The following "blurring lemma" gives a generalization that holds for all choices of a.

Lemma 6.41. *For a unit* u *and integer* $a \geq 3$*,*

$$\int_{y \in O_K} \gamma'\left(\frac{u(1+\pi^{a-2}y)}{\pi^a}\right) = q^{-1}\gamma'\left(\frac{u}{\pi^{a-2}}\right)$$

We may think of this lemma as follows. If we take a Gauss sum with exponent a, then obviously the Gauss sum can only see the value of u mod π^a. If we then take the Gauss sum and blur the value of u by averaging its value over $u + \pi^{a-2}O_K$, we end up with something that can only see the value of u mod π^{a-2}. In fact, we will get precisely another Gauss sum with exponent $a - 2$.

The primary use of the blurring lemma will be to actually raise the exponent in the denominator of our Gauss sum. When doing explicit calculations, the g_1 term in the Gauss sum causes lots of problems. Since it is effectively a characteristic function, it ends up changing our nice regions of integration (such as O_K) into very unwieldy sets. By raising the exponent a sufficiently high, the g_1 term will become identically 1 and this will no longer be a problem.

The proof of this lemma is very easy, requiring only writing out the Gauss sum as an integral and then interchanging the two integrals. We may stretch the statement quite a bit while maintaining the same simple proof. First of all, we may consider averaging u over different sets (for example, consider replacing the $a - 2$ term with $a - 4$ in the statement above). Second, we may consider Gauss sums that contain more than just a quadratic term. The resulting more general blurring lemma is not quite as pretty, but will be useful later on.

Lemma 6.42. *Let $c_2, c_1, c_0 \in K$. Let $k \geq 1$ be an integer such that all three of $\pi^k c_2, \pi^k c_1, \pi^k c_0$ lie in O_K. Then, we have (rather trivially)*

$$q^k \int_{y \in 1+\pi^k O_K} \int_{x \in O_K} \psi'\left(yc_2x^2 + yc_1x + yc_0\right)\, dxdy = \int_{x \in O_K} \psi'\left(c_2x^2 + c_1x + c_0\right)\, dx$$

On the other hand, if exactly two of $\pi^k c_2, \pi^k c_1, \pi^k c_0$ lie in O_K, let $j \in \{0, 1, 2\}$ denote which of the three is non-integral. Further, let $v = -v_\pi(\pi^k c_j)$ be a positive integer *telling us how non-integral this term is. Then we have*

$$q^k \int_{y \in 1+\pi^k O_K} \int_{x \in O_K} \psi'\left(yc_2x^2 + yc_1x + yc_0\right)\, dxdy =$$
$$\begin{cases} 0 & j = 0 \\ q^{-v} \int_{x \in O_K} \psi'\left(\pi^{2v}c_2x^2 + \pi^v c_1 x + c_0\right) dx & j = 1 \\ q^{-\lceil v/2 \rceil} \int_{x \in O_K} \psi'\left(\pi^{2\lceil v/2\rceil}c_2x^2 + \pi^{\lceil v/2 \rceil} c_1 x + c_0\right) dx & j = 2 \end{cases} \tag{6.43}$$

Proof. In any case, substitute $y_{\text{old}} = 1 + \pi^k y_{\text{new}}$ to get

$$\int_{y \in O_K} \int_{x \in O_K} \psi'\left(c_2x^2 + c_1x + c_0\right) \psi'\left(y\pi^k c_2x^2 + y\pi^k c_1 x + y\pi^k c_0\right)\, dxdy$$

Swapping order of integration, this becomes

$$\int_{x \in O_K} \psi'\left(c_2x^2 + c_1x + c_0\right) \int_{y \in O_K} \psi'\left(y\pi^k c_2x^2 + y\pi^k c_1 x + y\pi^k c_0\right)\, dydx$$

If all three of $\pi^k c_0, \pi^k c_1, \pi^k c_2$ lie in O_K, then the inner integral is identically 1 and the result follows trivially. Now, we proceed with the remaining cases, performing casework on j.

Case 0: $j = 0$

In this case, we have

$$\int_{x \in O_K} \psi'\left(c_2x^2 + c_1x + c_0\right) \int_{y \in O_K} \psi'\left(y\pi^k c_0\right)\, dydx$$

By assumption, $pi^k c_0$ is non-integral and so the inner integral vanishes.

Case 1: $j = 1$

In this case, we have

$$\int_{x \in O_K} \psi'\left(c_2x^2 + c_1x + c_0\right) \int_{y \in O_K} \psi'\left(y\pi^k c_1 x\right)\, dydx$$

The inner integral evaluates to the indicator function $\mathbb{1}_{O_K}(\pi^k c_1 x)$, and hence asserts that $x \in \pi^v O_K$. We get

$$\int_{x \in \pi^v O_K} \psi'\left(c_2x^2 + c_1x + c_0\right) dx = q^{-v} \int_{x \in O_K} \psi'\left(\pi^{2v}c_2x^2 + \pi^v c_1 x + c_0\right) dx$$

Case 2: $j = 2$

In this case, we have

$$\int_{x\in O_K} \psi'\left(c_2x^2 + c_1x + c_0\right) \int_{y\in O_K} \psi'\left(y\pi^k c_2x^2\right)\, dydx$$

The inner integral evaluates to the indicator function $\mathbb{1}_{O_K}(\pi^k c_2 x^2)$, and hence asserts that $x^2 \in \pi^v O_K$. This is equivalent to $x \in \pi^{\lceil v/2\rceil} O_K$. We get

$$\int_{x\in\pi^{\lceil v/2\rceil} O_K} \psi'\left(c_2x^2 + c_1x + c_0\right) dx = q^{-\lceil v/2\rceil} \int_{x\in O_K} \psi'\left(\pi^{2\lceil v/2\rceil} c_2x^2 + \pi^{\lceil v/2\rceil} c_1x + c_0\right) dx$$

$\square$

Remark 6.44. *We may obtain the earlier blurring lemma from this more powerful version by setting $c_2 = u/\pi^a, c_1 = c_0 = 0$, and taking $k = a - 2$.*

This choice of parameters gives $j = 2$ and $v = 2$ and we get exactly the desired result. Note that we need to take $a \geq 3$ so that we have the $k \geq 1$ hypothesis.

6.6 Classification of γ

The goal of this section is to determine some character-like properties possessed by $\gamma_a(u)$, and eventually give a classification of said function.

6.6.1 Computational Lemmas

The main term in the formula for $\gamma_u(a)$ was given as

$$g_2(a,u) = \begin{cases} 1 & a \leq e \\ \psi'\left(\frac{1}{4\pi}\frac{\square_0(u)}{u}\right) & a \text{ odd} \\ \psi'\left(\frac{1}{4}\frac{\square_1(u)}{u}\right) & a \text{ even} \end{cases}$$

Excluding the $a \leq e$ case where $g_2 = 1$, there is a close relation between the possibilities when a is even or odd. In fact, the two functions differ by a quadratic character of u. We will refer to this as the “parity shifting lemma".

Lemma 6.45.

$$\psi'\left(\frac{1}{4\pi}\frac{\Box_0(u)}{u}\right) = \psi'\left(\frac{1}{4}\frac{\Box_1(u)}{u}\right)\zeta_8^f \chi_{K[\sqrt{\pi}]}(u),$$

where $\zeta_8 = e^{(2\pi i)/8}$ *and* $\chi_{K[\sqrt{\pi}]}(u)$ *is the quadratic character returning* ± 1 *depending on whether* u *is a norm from* $K[\sqrt{\pi}]$*.*

Proof. We immediately get

$$\psi'\left(\frac{1}{4\pi}\frac{\Box_0(u) - \pi\Box_1(u)}{u}\right) = \psi'\left(\frac{1}{4\pi}\frac{u(1+\pi^{2e}\delta(u))}{u}\right) = \psi'\left(\frac{1}{4\pi}\right)\psi'\left(\frac{\alpha^2\delta(u)}{\pi}\right)$$

The first multiplicand is

$$\psi'\left(\frac{1}{4\pi}\right) = \psi_*\left(\frac{\tau}{4\pi}\right) = \exp\left(2\pi i \mathrm{tr}^K_{Q_2}\left(\frac{\tau}{4\pi^{r+1}}\right)\right) = \exp\left(2\pi i \mathrm{tr}^{K_0}_{Q_2}\left(\frac{1}{8}\right)\right) = \zeta_8^f$$

Recalling $\delta_0 \in K_0$, the second term satisfies

$$\psi'\left(\frac{\alpha^2\delta(u)}{\pi}\right) = \psi_*\left(\frac{\tau\alpha^2\delta(u)}{\pi}\right) = \exp\left(2\pi i \mathrm{tr}^K_{Q_2}\left(\frac{\tau\alpha^2\delta(u)}{\pi^{r+1}}\right)\right) = \exp\left(2\pi i \mathrm{tr}^{K_0}_{Q_2}\left(\frac{\alpha^2\delta(u)}{2}\right)\right)$$

By proposition 9, we know δ_0 is characterized by

$$\mathrm{tr}^{K_0}_{Q_2}\left(\frac{\alpha^2\delta(u)}{2}\right) \equiv 1 \bmod 2$$

Hence, we have

$$\exp\left(2\pi i \mathrm{tr}^{K_0}_{Q_2}\left(\frac{\alpha^2\delta(u)}{2}\right)\right) = \chi_{K[\sqrt{\pi}]}(u),$$

□

Remark 6.46. *During the computation above we calculated*

$$\psi'\left(\frac{1}{4\pi}\right) = \zeta_8^f,$$

which will be useful to note for later.

The parity shifting lemma can also apply directly to Gauss sums. We choose to leave the formula in terms of the function g_3 from the explicit formula for the Gauss sum (corollary 6.35).

Corollary 6.47.

$$\gamma'\left(\frac{u}{\pi^{2e+1}}\right) = \gamma'\left(\frac{u}{\pi^{2e}}\right) q^{-1/2} g_3(2e+1)\zeta_8^f \chi_{K[\sqrt{\pi}]}(u)$$

Proof. Follows immediately from the explicit formula for the Gauss sum. □

Corollary 6.48. *Let $a > e$ and assume $\gamma_a(u) \neq 0$. Then*

$$\left(\frac{\gamma_a(u)}{|\gamma_a(u)|}\right)^4 = -1^n,$$

where $n = ef$ is just the degree of the field extension.

Proof. This can likely be proven using a quaternion argument, but we will instead do casework using the formulas we have built up. We know $\frac{\gamma_a(u)}{|\gamma_a(u)|} = g_1(a,u)g_2(a,u)g_3(a)$, so we can consider each factor in turn. $g_1(a,u)$ is always 0 or 1, so we may ignore it. It is clear $\psi'\left(\frac{1}{4}\frac{\square_1(u)}{u}\right)^4 = 1$, so the above lemma tells us

$$g_2(a,u)^4 = \begin{cases} 1 & a \leq e \\ (-1)^f & a \text{ odd} \\ 1 & a \text{ even} \end{cases}$$

It is also clear that

$$g_3(a)^4 = \begin{cases} 1 & a \leq e \\ 1 & a \equiv e \bmod 2 \\ (-1)^f & a \not\equiv e \bmod 2 \end{cases}$$

Since we assume $a > e$, we may write $g_2(a,u)^4 = (-1)^{af}$ and $g_3(a)^4 = (-1)^{(a-e)f}$. Hence, under our assumptions,

$$\left(\frac{\gamma_a(u)}{|\gamma_a(u)|}\right)^4 = (-1)^{af}(-1)^{(a-e)f} = (-1)^n$$

□

In fact, the parity shifting lemma isn't the only interesting character-related identity satisfied by $\gamma_a(u)$. We will build up to another identity through a series of lemmas.

Lemma 6.49. *For any two units u, u', we have*

$$\delta(u)+\delta(u') \equiv \delta(uu') \ mod\ \pi, \quad (1+\delta(u)\pi^{2e})(1+\delta(u')\pi^{2e}) \equiv (1+\delta(uu')\pi^{2e}) \ mod\ \pi^{2e+1}$$

Proof. Immediate from the definition of δ. □

Lemma 6.50. *Recall that $\Box_0$ and $\pi\Box_1$ are functions defined mod π^{2e+1}. For arbitrarily chosen squareroots, we have*

$$\Box_0(uu') = \Box_0(u)\Box_0(u') + 2\pi\sqrt{\Box_0(u)\Box_0(u')\Box_1(u)\Box_1(u')} + \pi^2\Box_1(u)\Box_1(u')$$

$$\Box_1(uu') = \Box_0(u)\Box_1(u') + 2\sqrt{\Box_0(u)\Box_0(u')\Box_1(u)\Box_1(u')} + \Box_0(u')\Box_1(u)$$

Proof. From the definitions, we have

$$uu'(1+\delta(u)\pi^{2e})(1+\delta(u')\pi^{2e}) \equiv (\Box_0(u) - \pi\Box_1(u))(\Box_0(u') - \pi\Box_1(u')) \text{ mod } \pi^{2e+1}$$

We simultaneously rewrite both sides of the equation. We use the previous lemma on the left side and recognize the terms on the right as norms from $K[\sqrt{\pi}]$. Note that we arbitrarily choose the squareroots when doing this. We get

$$\begin{aligned} &uu'(1+\delta(uu')\pi^{2e}) \equiv \\ &\qquad N\left(\sqrt{\Box_0(u)} + \sqrt{\pi}\sqrt{\Box_1(u)}\right) N\left(\sqrt{\Box_0(u')} + \sqrt{\pi}\sqrt{\Box_1(u')}\right) \text{ mod } \pi^{2e+1} \end{aligned} \tag{6.51}$$

Combining the norms on the right yields

$$\begin{aligned} &uu'(1+\delta(uu')\pi^{2e}) \equiv \\ &N\left(\left(\sqrt{\Box_0(u)\Box_0(u')} + \pi\sqrt{\Box_1(u)\Box_1(u')}\right) + \pi\left(\sqrt{\Box_0(u)\Box_1(u')} + \sqrt{\Box_0(u')\Box_1(u)}\right)\right) \text{ mod } \pi^{2e+1} \end{aligned} \tag{6.52}$$

The result follows from the definition of $\Box_0$ and $\Box_1$. □

Lemma 6.53. *For any units u, u', we have*

$$\frac{\Box_0(uu')}{uu'} \equiv -1 + \frac{\Box_0(u)}{u} + \frac{\Box_0(u')}{u'} \ mod\ \pi^{e+1}$$

$$\frac{\square_1(uu')}{uu'} \equiv \frac{\square_1(u)}{u} + \frac{\square_1(u')}{u'} \ \mathit{mod}\ \pi^e$$

$$\frac{\sqrt{\square_0(uu')\square_1(uu')}}{uu'} \equiv \frac{\sqrt{\square_0(u)\square_1(u)}}{u} + \frac{\sqrt{\square_0(u')\square_1(u')}}{u'} \ \mathit{mod}\ \pi^e$$

Proof. By lemma 6.50, we have

$$\frac{\square_0(uu')}{uu'} \equiv \frac{\square_0(u)\square_0(u') + \pi^2\square_1(u)\square_1(u')}{uu'} \bmod \pi^{e+1}$$

Using $u \equiv \square_0(u) - \pi\square_1(u) - \pi^{2e}u\delta(u) \bmod \pi^{2e+1}$, we get

$$\frac{\square_0(uu')}{uu'} \equiv \frac{(u + \pi\square_1(u))(u' + \pi\square_1(u')) + \pi^2\square_1(u)\square_1(u')}{uu'} \bmod \pi^e$$

We may expand to get

$$\frac{\square_0(uu')}{uu'} \equiv 1 + \frac{\pi\square_1(u)}{u} + \frac{\pi\square_1(u')}{u'} \bmod \pi^{e+1}$$

Finally, another application of $u \equiv \square_0(u) - \pi\square_1(u) - \pi^{2e}u\delta(u) \bmod \pi^{2e+1}$ gives

$$\frac{\square_0(uu')}{uu'} \equiv -1 + \frac{\square_0(u)}{u} + \frac{\square_0(u')}{u'} \bmod \pi^{e+1}$$

The proof of the second identity is extremely similar.

$$\frac{\square_1(uu')}{uu'} \equiv \frac{\square_0(u)\square_1(u') + \square_0(u')\square_1(u)}{uu'} \equiv \frac{(u + \pi\square_1(u))\square_1(u') + (u' + \pi\square_1(u'))\square_1(u)}{uu'} \bmod \pi^e \quad (6.54)$$

The result now follows by expanding out the numerator.

For the final identity, we again start with the previous lemma to get

$$\frac{\sqrt{\square_0(uu')\square_1(uu')}}{uu'} \equiv \frac{\left(\sqrt{\square_0(u)\square_0(u')} + \pi\sqrt{\square_1(u)\square_1(u')}\right)\left(\sqrt{\square_0(u)\square_1(u')} + \sqrt{\square_0(u')\square_1(u)}\right)}{uu'} \bmod \pi^e \quad (6.55)$$

Note because we are taking squareroots of $\square_1$, which is only defined mod π^{2e}, these equations are only defined mod π^e. Expanding and recollecting the terms yields

$$\frac{\sqrt{\square_0(uu')\square_1(uu')}}{uu'} \equiv \frac{\sqrt{\square_0(u)\square_1(u)}(\square_0(u') + \pi\square_1(u')) + \sqrt{\square_0(u')\square_1(u')}(\square_0(u) + \pi\square_1(u))}{uu'} \bmod \pi^e \quad (6.56)$$

The result now follows immediately. (Note $\pi\square_1(u') \equiv -\pi\square_1(u') \bmod \pi^{e+1}$.) $\square$

The next proposition is an improvement of the previous lemma. It cannot be further improved since $\square_0$ and $\square_1$ are only defined mod π^{2e+1} and π^{2e}.

Proposition 6.57. *For any units u, u', we have*

$$\frac{\square_0(uu')}{uu'} \equiv$$
$$-1+\frac{\square_0(u)}{u}+\frac{\square_0(u')}{u'}+2\pi\frac{\sqrt{\square_0(u)\square_1(u)}}{u}\frac{\sqrt{\square_0(u')\square_1(u')}}{u'}+2\pi^2\frac{\square_1(u)}{u}\frac{\square_1(u')}{u'} \mod \pi^{2e+1} \tag{6.58}$$

$$\frac{\square_1(uu')}{uu'} \equiv \frac{\square_1(u)}{u}+\frac{\square_1(u')}{u'}+2\frac{\sqrt{\square_0(u)\square_1(u)}}{u}\frac{\sqrt{\square_0(u')\square_1(u')}}{u'}+2\pi\frac{\square_1(u)}{u}\frac{\square_1(u')}{u'} \mod \pi^{2e}$$

Proof. By lemma 6.50, we have

$$\frac{\square_1(uu')}{uu'} \equiv \frac{\square_0(u)\square_1(u') + 2\sqrt{\square_0(u)\square_0(u')\square_1(u)\square_1(u')} + \square_0(u')\square_1(u)}{uu'} \mod \pi^{2e}$$

Using $u \equiv \square_0(u) - \pi\square_1(u) - \pi^{2e}u\delta(u) \mod \pi^{2e+1}$, rewrite this as

$$\frac{\square_1(uu')}{uu'} \equiv$$
$$\frac{(u+\pi\square_1(u))\square_1(u') + 2\sqrt{\square_0(u)\square_0(u')\square_1(u)\square_1(u')} + (u'+\pi\square_1(u'))\square_1(u)}{uu'} \mod \pi^{2e} \tag{6.59}$$

The formula for $\square_1$ then follows immediately.

To get the corresponding formula for $\square_0$, we use three applications of $\pi\square_1(u) \equiv -u + \square_0(u) - \pi^{2e}u\delta(u) \mod \pi^{2e+1}$ to get

$$\frac{\pi\square_1(uu')}{uu'} - \frac{\pi\square_1(u)}{u} - \frac{\pi\square_1(u')}{u'} \equiv$$
$$1 + \frac{\square_0(uu')}{uu'} - \frac{\square_0(u)}{u} - \frac{\square_0(u')}{u'} - \pi^{2e}(\delta(uu') - \delta(u) - \delta(u')) \mod \pi^{2e+1} \tag{6.60}$$

By lemma 6.49, the δ terms cancel, yielding

$$\frac{\pi\square_1(uu')}{uu'} - \frac{\pi\square_1(u)}{u} - \frac{\pi\square_1(u')}{u'} \equiv 1 + \frac{\square_0(uu')}{uu'} - \frac{\square_0(u)}{u} - \frac{\square_0(u')}{u'} \mod \pi^{2e+1}$$

Taking the half of the proposition we already proved and multiplying by π, we see

$$\frac{\pi\Box_1(uu')}{uu'} \equiv$$
$$\frac{\pi\Box_1(u)}{u} + \frac{\pi\Box_1(u')}{u'} + 2\pi\frac{\sqrt{\Box_0(u)\Box_1(u)}}{u}\frac{\sqrt{\Box_0(u')\Box_1(u')}}{u'} + 2\pi^2\frac{\Box_1(u)}{u}\frac{\Box_1(u')}{u'} \bmod \pi^{2e+1} \tag{6.61}$$

The result now follows immediately. $\Box$

6.6.2 The Gauss Sum as a Quadratic Form

As stated previously, when $a > e$ is even, the main term g_2 is $\psi'\left(\frac{1}{4}\frac{\Box_1(u)}{u}\right)$. We may rewrite it as

$$\psi'\left(\frac{1}{4}\frac{\Box_1(u)}{u}\right) = \exp\left(2\pi i\ \mathrm{tr}^K_{Q_2}\left(\frac{\tau}{\pi^r}\frac{1}{4}\frac{\Box_1(u)}{u}\right)\right) = \exp\left(\frac{\pi i}{2}\mathrm{tr}^K_{Q_2}\left(\frac{\tau}{\pi^r}\frac{\Box_1(u)}{u}\right)\right) = i^{q(u)},$$

where

$$q : O_K \to \mathbb{Z}/4, \quad q(u) \equiv \mathrm{tr}^K_{Q_2}\left(\frac{\tau}{\pi^r}\frac{\Box_1(u)}{u}\right) \bmod 4$$

From remark 6.22, we know that q only depends on $u \in \frac{(O_K/\pi^{2e})^\times}{(O_K/\pi^{2e})^{\times 2}}$. There is a group isomorphism ϕ from the additive group $(\mathbb{Z}/2)^n$ to the multiplicative group $\frac{(O_K/\pi^{2e})^\times}{(O_K/\pi^{2e})^{\times 2}}$. Hence, we have a function

$$\bar{q} = q \circ \phi : (\mathbb{Z}/2)^n \to \mathbb{Z}/4$$

Letting $u = \phi(v),\ u' = \phi(v')$, we have

$$\bar{q}(v + v') \equiv \mathrm{tr}^K_{Q_2}\left(\frac{\tau}{\pi^r}\frac{\Box_1(uu')}{uu'}\right) \bmod 4$$

By the above proposition 6.57, we get

$$\bar{q}(v + v') \equiv \mathrm{tr}^K_{Q_2}\left(\frac{\tau}{\pi^r}\left(\frac{\Box_1(u)}{u} + \frac{\Box_1(u')}{u'} + 2\frac{\sqrt{\Box_0(u)\Box_1(u)}}{u}\frac{\sqrt{\Box_0(u')\Box_1(u')}}{u'} + 2\pi\frac{\Box_1(u)}{u}\frac{\Box_1(u')}{u'}\right)\right) \bmod 4 \tag{6.62}$$

So,

$$\bar{q}(v+v') = \bar{q}(v) + \bar{q}(v') + 2B_q(v,v'), \tag{6.63}$$

where

$$B_q(v,v') \equiv \operatorname{tr}_{Q_2}^{K}\left(\frac{\tau}{\pi^r}\left(\frac{\sqrt{\square_0(u)\square_1(u)}}{u}\frac{\sqrt{\square_0(u')\square_1(u')}}{u'} + \pi\frac{\square_1(u)}{u}\frac{\square_1(u')}{u'}\right)\right) \bmod 2$$

We will often abuse notation and allow B_q to directly act on elements of O_K, simply via the above formula.

Lemma 6.57 states that $B_q(v,v')$ is a bilinear function.[6] This is precisely the condition for $\bar{q}$ to be a quadratic form mod 4 in the sense of Brown (and hence also $q(v)$ by abuse of notation). See [Woo93] for more details and [Woo93] section 2 for the definition. In particular, we may take $v = v' = 0$ to see that $\bar{q}(0) = 0$, although this can also been seen via direct computation. Before moving to classify the quadratic form q, we quickly note down the defining property of B_q in terms of Gauss sums, something which will be useful later.

Proposition 6.64.

$$\gamma'\left(\frac{uu'}{\pi^{2e}}\right) = q^{e/2}\gamma'\left(\frac{u}{\pi^{2e}}\right)\gamma'\left(\frac{u'}{\pi^{2e}}\right)e^{\pi i B_q(u,u')}$$

Now we classify q. By the section 3 theorem of [Woo93], a mod 4 quadratic form with nondegenerate B_q is completely classified by the three things: the dimension of its domain, whether or not B_q is alternating, and the Brown σ invariant which takes a value in $\mathbb{Z}/8$. We will look at each of these properties of q in turn.

Lemma 6.65. *For $z \in O_K$, The set of solutions to $x^2 \equiv z \bmod \pi^{2e}$ is either empty or is a unique equivalence class mod π^e. The solutions to $x^2 \equiv z \bmod \pi^{2e-1}$ is either empty or is again a unique equivalence class mod π^e.*

Proof. It suffices to prove two statements:

$$x_1 \equiv x_2 \bmod \pi^e \implies x_1^2 \equiv x_2^2 \bmod \pi^{2e} \quad\text{and}\quad x_1 \not\equiv x_2 \bmod \pi^e \implies x_1^2 \not\equiv x_2^2 \bmod \pi^{2e-1}$$

[6] This function is closely related to the local Hilbert symbol, possibly equal to it.

The first statement shows that it suffices to consider congruence classes mod π^e. The second shows that each such class goes to a unique value mod π^{2e-1} (and hence also a unique value mod π^{2e}), and so the result would follow.

To prove the first statement, if $x_1 \equiv x_2 \bmod \pi^e$ then we may write $x_2 = x_1 + u\pi^a$ for $u \in O_K^\times$ and $a \geq e$. Then we have

$$(x_1 + u\pi^a)^2 = x_1^2 + 2u\pi^a + u^2\pi^{2a} \equiv x_1^2 \bmod \pi^{2e}$$

For the second statement, if $x_1 \not\equiv x_2 \bmod \pi^e$ then we may write $x_2 = x_1 + u\pi^a$ for $u \in O_K^\times$ and $0 \leq a < e$. Then we have

$$(x_1 + u\pi^a)^2 = x_1^2 + 2u\pi^a + u^2\pi^{2a}$$

So, we must show $2u\pi^a + u^2\pi^{2a} \not\equiv 0 \bmod \pi^{2e-1}$. To do this, note $v_\pi(2u\pi^a) = e + a$ and $v_\pi(u^2\pi^{2a}) = 2a < e + a$. Hence, we have $v_\pi(2u\pi^a + u^2\pi^{2a}) = 2a \leq 2e - 2$ and so $2u\pi^a + u^2\pi^{2a} \not\equiv 0 \bmod \pi^{2e-1}$. $\square$

Proposition 6.66. *B_q is nonsingular.*

Proof. To show nonsingularity, take $a = 2e$, so that

$$\gamma_{2e}(u) = q^{-e/2} g_3(a) g_2(a, u) = q^{-e/2} g_3(a) i^{\tilde{q}(\phi^{-1}(u))}$$

If v_0 were an element in the kernel of B_q, then we would have $q(v_0 + v) = q(v_0) + q(v)$ for all v. The above formula then would imply that for $u_0 = \phi(v_0)$, we would have

$$\gamma_{2e}(u_0 u) = i^{\tilde{q}(v_0)} \gamma_{2e}(u)$$

for all units u. For convenience, we will let $c = i^{\tilde{q}(v_0)}$, since it is just some constant fourth root of unity.

We may rewrite the above equation as (for all $u \in O_K^\times$)

$$c^{-1} = \gamma_{2e}(u)\gamma_{2e}(u_0 u)^{-1} = \int_{O_K} \psi'\left(\frac{ux^2}{\pi^{2e}}\right) dx \left(\int_{O_K} \psi'\left(\frac{u_0 u y^2}{\pi^{2e}}\right) dy\right)^{-1}$$

The key to proceeding is to note that the Gauss sum we are taking the reciprocal of has complex norm $q^{-e/2}$. Hence, we may write

$$c^{-1} = q^e \int_{O_K} \psi'\left(\frac{ux^2}{\pi^{2e}}\right) dx\, \overline{\int_{O_K} \psi'\left(\frac{u_0uy^2}{\pi^{2e}}\right) dy}$$

We may distribute the conjugation across the integral and combine the integrals to get

$$c^{-1} = q^e \int_{O_K} \psi'\left(\frac{ux^2}{\pi^{2e}}\right) dx \int_{O_K} \psi'\left(\frac{-u_0uy^2}{\pi^{2e}}\right) dy = q^e \int_{O_K}\int_{O_K} \psi'\left(\frac{u(x^2-u_0y^2)}{\pi^{2e}}\right) dxdy$$

Now, integrate both sides over $u \in O_K^\times$. We get

$$c^{-1}\left(1-\frac{1}{q}\right) = q^e \int_{O_K}\int_{O_K}\int_{O_K^\times} \psi'\left(\frac{u(x^2-u_0y^2)}{\pi^{2e}}\right) dudxdy$$

Let's look at the innermost integral more closely. We have

$$\int_{O_K^\times} \psi'\left(\frac{u(x^2-u_0y^2)}{\pi^{2e}}\right) du = \int_{O_K} \psi'\left(\frac{u(x^2-u_0y^2)}{\pi^{2e}}\right) du - \int_{\pi O_K} \psi'\left(\frac{u(x^2-u_0y^2)}{\pi^{2e}}\right) du$$

$$= \int_{O_K} \psi'\left(\frac{u(x^2-u_0y^2)}{\pi^{2e}}\right) du - \frac{1}{q}\int_{O_K} \psi'\left(\frac{u(x^2-u_0y^2)}{\pi^{2e-1}}\right) du$$

The first integral is a characteristic function for $u_0 \equiv (x/y)^2 \bmod \pi^{2e}$, whereas the second is a characteristic function for $u_0 \equiv (x/y)^2 \bmod \pi^{2e-1}$. Plugging this back into the triple integral, we can write out the result using measures of certain sets:

$$c^{-1}\left(1-\frac{1}{q}\right) =$$
$$q^e\left(\text{meas}(\{(x,y)|x^2 \equiv u_0y^2 \bmod \pi^{2e}\}) - \frac{1}{q}\text{meas}(\{(x,y)|x^2 \equiv u_0y^2 \bmod \pi^{2e-1}\})\right) \tag{6.67}$$

It is clear that since c is a fourth root of unity, and all other terms above are real, $c = \pm 1$. First we show $c = -1$ is impossible. By the lemma, for each choice of y, there are at most measure q^{-e} choices of x so that $x^2 \equiv u_0y^2 \bmod \pi^{2e-1}$. Hence,

$$\frac{1}{q}\text{meas}(\{(x,y)|x^2 \equiv u_0y^2 \bmod \pi^{2e-1}\}) \le \frac{1}{q}q^{-e} \le \left(1-\frac{1}{q}\right)q^{-e}$$

However,

$$\text{meas}(\{(x,y)|x^2 \equiv u_0y^2 \bmod \pi^{2e}\}) > 0,$$

since there are solutions whenever $x \equiv y \equiv 0 \bmod \pi^{2e}$. Therefore, we have

$$\begin{aligned}\text{meas}(\{(x,y)|x^2 \equiv u_0 y^2 \bmod \pi^{2e}\}) - \frac{1}{q}\text{meas}(\{(x,y)|x^2 \equiv u_0 y^2 \bmod \pi^{2e-1}\}) \\ > 0 - \left(1 - \frac{1}{q}\right) q^{-e}, \quad (6.68)\end{aligned}$$

so this quantity can never be negative enough for $c = -1$ to work. Therefore, any solution must use $c = 1$. We have

$$\begin{aligned}\text{meas}(\{(x,y)|x^2 \equiv u_0 y^2 \bmod \pi^{2e}\}) - \frac{1}{q}\text{meas}(\{(x,y)|x^2 \equiv u_0 y^2 \bmod \pi^{2e-1}\}) \\ \leq \left(1 - \frac{1}{q}\right) \text{meas}(\{(x,y)|x^2 \equiv u_0 y^2 \bmod \pi^{2e}\}) \leq \left(1 - \frac{1}{q}\right) q^{-e},\end{aligned}$$

where the last step is our (lemma).

Therefore, for $c = 1$ to occur, we must have the equality case in the above inequalities. In particular, $x^2 \equiv u_0 y^2 \bmod \pi^{2e}$ must have a solution for some unit value of y. Hence, u_0 must be a square mod π^{2e}. This is equivalent to saying $v_0 = \phi^{-1}(u_0) = 0$, which concludes the proof that B_q is nonsingular. $\square$

Proposition 6.69. *B_q is alternating (that is, $B_q(v, v)$ is identically 0) iff $x^2 \equiv -1$ mod π^{2e} has any solutions.*

Proof. Start from the equation

$$\bar{q}(v + v) = \bar{q}(v) + \bar{q}(v) + 2B_q(v, v)$$

We know $\bar{q}(2v) = \bar{q}(0) = 0$, so this implies $\bar{q}(v) \equiv B_q(v, v) \bmod 2$. Thus, it suffices to ask whether $\bar{q}(v)$ is identically 0 mod 2 instead.

To continue, we again use the equation

$$\gamma_{2e}(u) = q^{-e/2} g_3(a) i^{\tilde{q}(\phi^{-1}(u))}$$

In particular, if we square this equation, we get

$$\gamma_{2e}(u)^2 = q^{-e} g_3(a)^2 (-1)^{\tilde{q}(\phi^{-1}(u))}$$

We may rewrite the left side to get

$$\int_{O_K}\int_{O_K}\psi'\left(\frac{u(x^2-(-1)y^2)}{\pi^{2e}}\right)\,dxdy = q^{-e}g_3(a)^2(-1)^{\tilde{q}(\phi^{-1}(u))}$$

Now the left side is exactly the function we studied in the proof of the previous proposition, with $u_0 = -1$. In particular, if $x^2 \equiv -1 \bmod \pi^{2e}$ has no solutions, then we know that said function is non-constant. This implies that the right hand side is non-constant, and so $\bar{q}(v)$ is not identically 0 mod 2.

On the other hand, if $x^2 \equiv -1 \bmod \pi^{2e}$ has a solution x_0, then write

$$\gamma_{2e}(u)^2 = \int_{O_K}\psi'\left(\frac{ux^2}{\pi^{2e}}\right)dx\,\int_{O_K}\psi'\left(\frac{uy^2}{\pi^{2e}}\right)dy$$

We may change variables $y_{\text{new}} = x_0 y_{\text{old}}$ to get

$$\gamma_{2e}(u)^2 = \int_{O_K}\psi'\left(\frac{ux^2}{\pi^{2e}}\right)dx\,\int_{O_K}\psi'\left(\frac{-uy^2}{\pi^{2e}}\right)dy = |\gamma_{2e}(u)|^2 = q^{-e}$$

Hence, $\gamma_{2e}(u)^2 = q^{-e}g_3(a)^2(-1)^{\tilde{q}(\phi^{-1}(u))}$ is a constant function, so $\tilde{q}(v)$ is constant mod 2. We know $\tilde{q}(0) = 0$, so it must be identically 0 mod 2. $\square$

Corollary 6.70. *If B_q is alternating, then e is even.*

Proof. We know $x^2 \equiv -1 \bmod \pi^{2e}$ has a solution. Write

$$x^2 = -1 + 4a + \pi^{2e+1}b$$

where a is a root of unity or 0, and $b \in O_K$.

Since every member of U_{2e+1} is a perfect square, we may assume without loss of generality that $b = 0$. We get

$$x^2 + 1 - 4a = 0$$

Now, letting $x = y + 1$, so we get

$$y^2 + 2y + 2 - 4a = 0$$

Hence, y is a root of an Eisenstein polynomial over K_0 and so $K_0[y]$ has even ramification index. It follows that $K \supset K_0[y]$ also has even ramification index. $\square$

Finally, we look into the σ invariant. Per Brown's theorem (as quoted in [Woo93] section 2), σ is given by the formula

$$\sum_{v \in (\mathbb{Z}/2)^n} i^{\bar{q}(v)} = q^{e/2} \zeta_8^{\sigma}$$

We may rewrite this equation as

$$g_3(2e)^{-1} q^{-e/2} \sum_{v \in (\mathbb{Z}/2)^n} \gamma_{2e}(\phi(v)) = \zeta_8^{\sigma}$$

As u varies in $O_K^{\times}$, it represents each possible value of v equally. Hence, we see that for some positive real constant c,

$$\int_{u \in O_K^{\times}} \gamma_{2e}(u)\, du = c \sum_{v \in (\mathbb{Z}/2)^n} \gamma_{2e}(\phi(v))$$

Therefore, for some different positive constant c, we have

$$g_3(2e)^{-1} \int_{O_K^{\times}} \gamma_{2e}(u)\, du = c\zeta_8^{\sigma}$$

Expanding out the definition of γ, we get

$$g_3(2e)^{-1} \int_{O_K^{\times}} \int_{O_K} \psi' \left(\frac{ux^2}{\pi^{2e}} \right) \, dx\, du = c\zeta_8^{\sigma}$$

We change the order of integration to get

$$g_3(2e)^{-1} \int_{O_K} \int_{O_K^{\times}} \psi' \left(\frac{ux^2}{\pi^{2e}} \right) \, du\, dx = c\zeta_8^{\sigma}$$

As we have calculated many times so far, we may write

$$\int_{O_K^{\times}} \psi' \left(\frac{ux^2}{\pi^{2e}} \right) \, du = \int_{O_K} \psi' \left(\frac{ux^2}{\pi^{2e}} \right) \, du - \frac{1}{q} \int_{O_K} \psi' \left(\frac{ux^2}{\pi^{2e-1}} \right) \, du$$

so that we get

$$g_3(2e)^{-1} \left(\mathrm{meas}(\{x | x^2 \equiv 0 \bmod \pi^{2e}\}) - \frac{1}{q} \mathrm{meas}(\{x | x^2 \equiv 0 \bmod \pi^{2e-1}\}) \right) = c\zeta_8^{\sigma}$$

These conditions are both equivalent to $x \equiv 0 \bmod \pi^e$, so we get

$$g_3(2e)^{-1} \left(1 - \frac{1}{q} \right) q^{-e} = c\zeta_8^{\sigma}$$

from which we conclude

$$g_3(2e)^{-1} = \zeta_8^\sigma$$

Writing this out more explicitly (and since $a = 2e$),

$$\zeta_8^\sigma = \begin{cases} 1 & e \equiv 0 \bmod 2 \\ \psi_0\left(\frac{1}{4}\frac{\square_1(\alpha_0)}{\alpha_0}\right)\left(-e^{2\pi i \frac{5}{8} f}\right) & e \equiv 1 \bmod 2 \end{cases}$$

Therefore,

$$\sigma = \begin{cases} 0 & e \equiv 0 \bmod 2 \\ 2\mathrm{tr}_{Q_2}^{K_0}\left(\frac{\square_1(\alpha_0)}{\alpha_0}\right) + 4 + 5f \bmod 8 & e \equiv 1 \bmod 2 \end{cases}$$

We now put everything together into a single proposition.

Proposition 6.71. *Let v be the element of $(\mathbb{Z}/2)^n$ corresponding to $u \in O_K$. Then, we have*

$$g_2(a,u) = \begin{cases} 1 & a \leq e \\ i^{\bar{q}(v)}\zeta_8^f \chi_{K[\sqrt{\pi}]}(u) & a \text{ odd} \\ i^{\bar{q}(v)} & a \text{ even} \end{cases}$$

where $\bar{q} : (\mathbb{Z}/2)^n \to \mathbb{Z}/4$ is a quadratic form.

If $x^2 \equiv -1$ mod π^{2e} has a solution, then $\bar{q}$ is equivalent under basechange to

$$\sum_{i=1}^{n/2} 2x_{2i-1}x_{2i}$$

Otherwise, $\bar{q}$ is equivalent under basechange to

$$\sum_{i=1}^{n} \pm x_i^2,$$

Let p_q be the number of positive coefficients. Then p_q is only well defined mod 4 and given by

$$p_q \equiv \frac{\sigma + n}{2} \equiv \begin{cases} n/2 & e \equiv 0 \textit{ mod } 2 \\ tr_{Q_2}^{K_0}\left(\frac{\square_1(\alpha_0)}{\alpha_0}\right) + 2 + \frac{1}{2}(5+e)f & e \equiv 1 \textit{ mod } 2 \end{cases}$$

Proof. We have already checked everything except for the basechange results. In the first case, B_q is alternating, so by corollary 6.70 e is even. In this case, $\sigma = 0$ and it follows by part 7 of Brown's theorem that the associated Arf invariant is 0. This implies that q has the desired form.

If B_q is not alternating, then [Woo93] tells again tells us $\bar{q}$ has the desired form and further tells us $p_q \equiv \frac{\sigma+n}{2}$ mod 4. It also provides a basechange that turns four +1 coefficients into -1 coefficients or vice-versa, so p_q is only well defined mod 4. $\square$

6.6.3 The Squareness Operator

Concluding our calculation of various properties of B_q, we will give some examples of subspaces that are orthogonal compliments. This particular calculation will be used in the computation of the character Gauss sum.

Definition 6.72. *Given $u \in O_K^\times$, we define the squareness of u, denoted $sq(u)$. $sq(u)$ will be the smallest positive integer k such that u is congruent to a square mod π^k. If u is a perfect square in $O_K^\times$, then we let $sq(u) = \infty$.*

For example, taking $e = 20$, we have $sq(1 + \pi^2 + \pi^4 + \pi^5 + \pi^7) = 5$.

For later applications, it will be convenient to define squareness for all $x \in K$. To do this, let u be a unit, k be any integer and then define $sq(u\pi^{2k}) := sq(u)$. This defines squareness for any even valuation $x \in K$. For x with odd valuation, define $sq(x) = 0$.

Remark 6.73. *For any unit u, $sq(u) \in \{1, 3, \dots, 2e-1, 2e, \infty\}$. That is, it is either an odd number from 1 to $2e-1$, or it is $2e$ or ∞.*

Remark 6.74. *For any $x, y \in K$, we have $sq(xy^2) = sq(x)$. We will not need this fact, but it is a nice property.*

Remark 6.75. *An alternate description of squareness which holds for all $x \in K$ is as the valuation*
$sq(x) = v_\pi(\square_1(x)) - v_\pi(x) + 1.$

For even k satisfying $0 \leq k \leq 2e$, let W_k denote the $f(e - k/2)$ dimensional subspace of $\frac{(O_K/\pi^{2e})^\times}{(O_K/\pi^{2e})^{\times 2}}$ consisting of elements that can be represented by units of the form $1+\sum_{i=k+1}^{\infty} a_i\pi^i$. To use the just introduced terminology, these are all u such that $sq(u) \geq k+1$. By the remark above, this is the same as units such that $v_\pi(\square_1(u)) \geq k$.

Lemma 6.76. $W_k^\perp = W_{2e-k}$

Proof. We will show that $< W_k, W_{2e-k} >= 0$. Since our inner product is non-degenerate and the dimensions of the two spaces add up to $fe = n$, the claim will immediately follow.

It will not hurt to recall the definition of B_q:

$$B_q(v,v') \equiv \ \mathrm{tr}_{Q_2}^K \left(\frac{\tau}{\pi^r}\left(\frac{\sqrt{\square_0(u)\square_1(u)}}{u}\frac{\sqrt{\square_0(u')\square_1(u')}}{u'} + \pi\frac{\square_1(u)}{u}\frac{\square_1(u')}{u'}\right)\right) \mod 2$$

If $v \in W_k$ and $v' \in W_{2e-k}$, we know that $\square_1(u) \in \pi^k O_K$ and $\square_1(u') \in \pi^{2e-k}O_K$. It immediately follows that the out of the two terms being summed in the definition of B_q, the first is a multiple of $\pi^e = 2$ and the second is a multiple of $\pi^{2e} = 4$. It follows that the inner product is 0. $\square$

6.7 Computing Quadratic Character Gauss Sums

We know $\#O_K^\times/O_K^{\times 2} \cong (\mathbb{Z}/2)^{n+1}$, a power of the two element cyclic group. It follows that $O_K^\times$ has precisely 2^{n+1} quadratic characters.

Let $\chi_{u_0}(u) = e^{\pi i B_q(u,u_0)}$ denote a quadratic character. It is clear from the definitions that $\chi_{u_0}\chi_{u_1} = \chi_{u_0u_1}$. Since B_q is non-degenerate, we know that as u_0 varies, we get 2^n distinct quadratic characters of this form.

Lemma 6.77. *The conductor of χ_{u_0} is given by*

$$\mathit{conductor}(\chi_{u_0}) = \begin{cases} 0 & u_0 \in (O_K/\pi^{2e})^{\times 2} \\ 2e+1-sq(u_0) & u_0 \notin (O_K/\pi^{2e})^{\times 2} \end{cases}$$

In particular, the conductor of any such character is always even.

Proof. The first case is easy since $u_0 \in (O_K/\pi^{2e})^{\times 2}$ is exactly the condition for the χ_{u_0} to be the trivial character.

Otherwise, we know that u_0 is not a square mod π^{2e} and hence $sq(u_0)$ must be an odd number from 1 to $2e-1$. Following the notation at the end of the previous section, we know that $u_0 \in W_{sq(u_0)-1}$. Hence, by the proposition there, we know that $B_q(u_0, W_{2e+1-sq(u_0)}) = 0$. It follows that χ_{u_0} is trivial on $1 + \pi^{2e+2-sq(u_0)}O_K$. However, $2e+2-sq(u_0)$ is an odd number. Since our character is trivial on squares it follows that in fact χ_{u_0} is trivial on $1 + \pi^{2e+1-sq(u_0)}O_K$. Hence, the conductor is at most $2e+1-sq(u_0)$.

Reversing the argument shows that the conductor can be no smaller than this. After all, if χ were trivial on the larger set $1 + \pi^{2e-sq(u_0)}$ then we would have $B_q(u_0, W_{2e-1-sq(u_0)}) = 0$. From this, it would follow from the proposition that $u_0 \in W_{sq(u_0)+1}$, and hence has squareness at least $sq(u_0)+2$, a contradiction. □

We know of another quadratic character on $O_K^\times$, namely $\chi_{K[\sqrt{\pi}]}$ from the earlier section on the classification of γ. This character has conductor $2e+1$, and hence isn't any of our 2^n earlier characters. This lets us expand our collection of characters. Namely, for $i \in \{0,1\}$ let $\chi^i_{u_0}(u) := \chi_{u_0}(u) * (\chi_{K[\sqrt{\pi}]}(u))^i$.

Corollary 6.78. *For some choice of $\chi^i_{u_0}$, let c_χ denote its conductor. Then,*

$$c_\chi = \begin{cases} 0 & u_0 \in (O_K/\pi^{2e})^{\times 2},\ i=0 \\ 2e+1-sq(u_0) & u_0 \notin (O_K/\pi^{2e})^{\times 2},\ i=0 \\ 2e+1 & i=1 \end{cases}$$

The next proposition now describes all characters of $O_K^\times$.

Proposition 6.79. *Given any quadratic character χ of $O_K^\times$, there is a unique choice of $u_0 \in \frac{(O_K/\pi^{2e})^\times}{(O_K/\pi^{2e})^{\times 2}}$ and $i \in \{0,1\}$ such that $\chi^i_{u_0} = \chi$.*

Proof. This just comes down to verifying that the 2^{n+1} possibilities for $\chi^i_{u_0}(u)$ are all distinct. However, this is obviously true since this collection of characters is a group and only one of the characters has conductor 0 (and hence is trivial). Since we know $O_K^\times$ has exactly 2^{n+1} quadratic characters, this must be all of them. □

We now begin our evaluation of quadratic character Gauss sums. We start with a lemma in which we calculate an easier integral. We then use the lemma to calculate an important intermediate quantity which we will end up needing again later. Finally, we specify what we mean by “quadratic character Gauss sum" and give our formula in a final proposition. Our convention will be for a nontrivial multiplicative character $\chi_{u_0}^i(u)$ to be zero on πO_K and for the trivial character to be 1 everywhere (including on πO_K).

Lemma 6.80. *Let $k \geq 0$ be an integer. Then*

$$\int_{x \in O_K^\times} \gamma'\left(\frac{x}{\pi^k}\right) dx = \begin{cases} (1-q^{-1})q^{-k/2} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Proof. Although possible without, we opt to change the order of integration because it is much faster in this case. First, write the Gauss sum back out as an integral.

$$\int_{x \in O_K^\times} \int_{y \in O_K} \psi'\left(\frac{xy^2}{\pi^k}\right) dy dx$$

In the case $k = 0$, the integrand is identically 1 and the result follows, so we will now assume $k > 0$. Since the domain of integration is compact, we may interchange the order of integration. Additionally rewriting the domain of x, we get

$$\int_{y \in O_K} \left(\int_{x \in O_K} \psi'\left(\frac{xy^2}{\pi^k}\right) dx - \int_{x \in \pi O_K} \psi'\left(\frac{xy^2}{\pi^k}\right) dx\right) dy$$

A very slight change of variables yields

$$\int_{y \in O_K} \left(\int_{x \in O_K} \psi'\left(\frac{xy^2}{\pi^k}\right) dx - q^{-1} \int_{x \in O_K} \psi'\left(\frac{xy^2}{\pi^{k-1}}\right) dx\right) dy$$

The integrals give us indicator functions.

$$\int_{y \in O_K} \left(\mathbb{1}_{\pi^k O_K}(y^2) - q^{-1}\mathbb{1}_{\pi^{k-1} O_K}(y^2)\right) dy$$

In the case of k even, both indicator functions are equivalent to asking $y \in \pi^{k/2} O_K$ and we get

$\#(\pi^{k/2}O_K) - q^{-1}\#(\pi^{k/2}O_K) = (1-q^{-1})q^{-k/2}$, where $\#$ indicates the measure of the set.

In the case of k odd, the first indicator function asks that $y \in \pi^{(k+1)/2}O_K$, whereas the second asks that $y \in \pi^{(k-1)/2}O_K$. This time, the calculation yields $q^{-(k+1)/2} - q^{-1}q^{-(k-1)/2} = 0$. $\square$

We now calculate the aforementioned intermediate quantity. If multiple conditions are met in the piecewise function, only use the formula for the first condition that holds.

Proposition 6.81. *Let a be an integer and u, u_0 be units. Assume a is non-negative.*[7] *Then,*

$$\int_{x \in O_K^\times} \chi_{u_0}^i(x)\gamma'\left(\frac{ux}{\pi^a}\right)\, dx =$$

$$(1-q^{-1})q^{-(a-e)/2}\chi_{u_0}^i(u)\left(\zeta_8^f g_3(2e+1)\right)^i \overline{\gamma'\left(\frac{u_0}{\pi^{2e}}\right)} \cdot \begin{cases} 0 & a < c_\chi \\ 0 & a \not\equiv c_\chi \ mod\ 2 \\ 1 & a \equiv c_\chi \ mod\ 2 \end{cases} \tag{6.82}$$

In the case the integral is nonzero, our expression may alternately be written

$$(1-q^{-1})q^{-(a-e-i)/2}\chi_{u_0}^i(u)\chi_{K[\sqrt{\pi}]}(u_0)\left(\zeta_8^f g_3(2e+1)\right)^{2i} \overline{\gamma'\left(\frac{u_0}{\pi^{2e+i}}\right)}$$

Proof. A quick change of variables $x_{\text{new}} = ux_{\text{old}}$ allows one to factor a copy of $\chi_{u_0}^i(u)$ out of the integral. This reduces the problem to verifying our formula in the case $u = 1$, which we assume from here on out. We proceed by lots of casework.

Case 1: We start with the case of trivial $\chi_{u_0}^i$ (which happens when $u_0 = 1$, $i = 0$). In this case, our integral becomes exactly the one from the previous lemma and hence equals

$$\begin{cases} (1-q^{-1})q^{-a/2} & a \text{ even} \\ 0 & a \text{ odd} \end{cases}$$

[7] $a < 0$ gives the same result as taking $a = 0$ in the integral, so the formula does handle the case of negative a. The assumption $a \geq 0$ is mainly just for a cleaner right hand side.

On the other hand, since by assumption we can't have $a < c_\chi$ in this case, the right hand side is just

$$(1-q^{-1})q^{-(a-e)/2}\,\overline{\gamma'\left(\frac{1}{\pi^{2e}}\right)} \cdot \begin{cases} 0 & a \not\equiv 0 \bmod 2 \\ 1 & a \equiv 0 \bmod 2 \end{cases}$$

It is clear these two expressions are equal upon observing

$$\gamma'\left(\frac{1}{\pi^{2e}}\right) = q^{-e/2}\psi'\left(\frac{\Box_1(1)}{1}\right) = q^{-e/2}\psi'(0) = q^{-e/2}$$

Case 2: Now we assume $\chi^i_{u_0}$ is nontrivial. Writing out the Gauss sum, we want to calculate

$$\int_{x\in O_K^\times} \chi^i_{u_0}(x) \int_{t\in O_K} \psi'\left(\frac{xt^2}{\pi^a}\right)\, dtdx$$

For some integer $k > 0$ to be chosen, we can use the integration lemma 6.10 to rewrite this as

$$\int_{x\in O_K^\times} q^k \int_{y\in 1+\pi^k O_K} \chi^i_{u_0}(xy) \int_{t\in O_K} \psi'\left(\frac{xyt^2}{\pi^a}\right)\, dtdydx$$

Case 2a: $a < c_\chi$

In this case, take $k = c_\chi - 1$ so the integral becomes

$$\int_{x\in O_K^\times} q^k \int_{y\in 1+\pi^k O_K} \chi^i_{u_0}(xy)dy \int_{t\in O_K} \psi'\left(\frac{xt^2}{\pi^a}\right)\, dtdx$$

The integral over y is clearly 0, and so the entire expression is just 0.

Case 2b: $a > c_\chi$

In this case, we take $k = c_\chi$ and the integral becomes

$$\int_{x\in O_K^\times} \chi^i_{u_0}(x) q^k \int_{y\in 1+\pi^k O_K} \int_{t\in O_K} \psi'\left(\frac{xyt^2}{\pi^a}\right)\, dtdydx$$

Apply the blurring lemma 6.42 with parameters $(c_0, c_1, c_2, j, v) = (0, 0, x/\pi^a, 2, a - c_\chi)$ to get

$$q^{-\lceil (a-c_\chi)/2\rceil} \int_{x\in O_K^\times} \chi^i_{u_0}(x) \int_{t\in O_K} \psi'\left(\frac{xt^2}{\pi^{a-2\lceil (a-c_\chi)/2\rceil}}\right)\, dtdx$$

If $a \not\equiv c_\chi \bmod 2$ then this new integral is handled by case 2a and is equal to 0. If $a \equiv c_\chi \bmod 2$ then the integral becomes

$$q^{-(a-c_\chi)/2} \int_{x \in O_K^\times} \chi_{u_0}^i(x) \int_{t \in O_K} \psi' \left(\frac{xt^2}{\pi^{c_\chi}} \right) \, dt dx$$

It is easy to check that this formula reduces things to the case $a = c_\chi$. So, we just need to handle that case and we are done.

Case 2ci: $a = c_\chi$, $i = 1$

In this case, $c_\chi = 2e+1$. Apply the parity shifting lemma 6.47 to get

$$\int_{x \in O_K^\times} \chi_{u_0}(x) \chi_{K[\sqrt{\pi}]}(x) \gamma' \left(\frac{x}{\pi^{2e+1}} \right) dx = q^{-1/2} \zeta_8^f g_3(2e+1) \int_{x \in O_K^\times} \chi_{u_0}(x) \gamma' \left(\frac{x}{\pi^{2e}} \right) dx$$

Taking a moment to recall $\chi_{u_0}(x) = e^{\pi i B_q(u_0,x)}$, the defining property of B_q, proposition 6.63 tells us

$$\gamma' \left(\frac{u_0 x}{\pi^{2e}} \right) = q^{e/2} \gamma' \left(\frac{u_0}{\pi^{2e}} \right) \gamma' \left(\frac{x}{\pi^{2e}} \right) \chi_{u_0}(x)$$

Since $\left| \gamma' \left(\frac{u_0}{\pi^{2e}} \right) \right|^2 = q^{-e}$, we may multiply both sides by $\overline{\gamma' \left(\frac{u_0}{\pi^{2e}} \right)}$ to see

$$\overline{\gamma' \left(\frac{u_0}{\pi^{2e}} \right)} \gamma' \left(\frac{u_0 x}{\pi^{2e}} \right) = q^{-e/2} \gamma' \left(\frac{x}{\pi^{2e}} \right) \chi_{u_0}(x)$$

Applying this to the integral at hand gives

$$q^{(e-1)/2} \zeta_8^f g_3(2e+1) \overline{\gamma' \left(\frac{u_0}{\pi^{2e}} \right)} \int_{x \in O_K^\times} \gamma' \left(\frac{u_0 x}{\pi^{2e}} \right) dx$$

And after the easy change of variables $x_{\text{new}} = u_0 x_{\text{old}}$, we get

$$q^{(e-1)/2} \zeta_8^f g_3(2e+1) \overline{\gamma' \left(\frac{u_0}{\pi^{2e}} \right)} \int_{x \in O_K^\times} \gamma' \left(\frac{x}{\pi^{2e}} \right) dx$$

This last integral is just $(1-q^{-1})q^{-e}$ by the previous lemma, and we get

$$(1-q^{-1}) q^{-(e+1)/2} \zeta_8^f g_3(2e+1) \overline{\gamma' \left(\frac{u_0}{\pi^{2e}} \right)}$$

Invoking the parity shifting lemma 6.47 one more time, we may choose to rewrite this as

$$(1-q^{-1}) q^{-e/2} \zeta_8^{2f} g_3(2e+1)^2 \chi_{K[\sqrt{\pi}]}(u_0) \overline{\gamma' \left(\frac{u_0}{\pi^{2e+1}} \right)}$$

Case 2cii: $a = c_\chi$, $i = 0$, and $\chi_{u_0}^i$ is nontrivial

In this case, $c_\chi = 2e + 1 - sq(u_0)$ is an even number from 2 to $2e$, inclusive. We have to compute

$$\frac{1}{1-q^{-1}} \int_{x \in O_K^\times} \chi_{u_0}(x)\gamma'\left(\frac{x}{\pi^{2e+1-sq(u_0)}}\right) dx$$

The first step is to apply the blurring lemma 6.42 to go from the even exponent $2e + 1 - sq(u_0)$ to the even exponent $2e$. Recalling what the blurring lemma says, we have

$$q^{2e+1-sq(u_0)} \int_{y \in 1+\pi^{2e+1-sq(u_0)}O_K} \gamma'\left(\frac{xy}{\pi^{2e}}\right) dy = q^{-(sq(u_0)-1)/2}\gamma'\left(\frac{x}{\pi^{2e+1-sq(u_0)}}\right)$$

However, instead of applying this as is, we may take advantage of the floor functions in the statement of the blurring lemma to get the slightly stronger statement

$$q^{2e+2-sq(u_0)} \int_{y \in 1+\pi^{2e+2-sq(u_0)}O_K} \gamma'\left(\frac{xy}{\pi^{2e}}\right) dy = q^{-(sq(u_0)-1)/2}\gamma'\left(\frac{x}{\pi^{2e+1-sq(u_0)}}\right)$$

Using this, we see our original integral equals

$$\frac{1}{1-q^{-1}} \int_{x \in O_K^\times} \chi_{u_0}(x) q^{(sq(u_0)-1)/2} q^{2e+2-sq(u_0)} \int_{y \in 1+\pi^{2e+2-sq(u_0)}O_K} \gamma'\left(\frac{xy}{\pi^{2e}}\right) dydx$$

We may rearrange the terms to get

$$\frac{q^{(sq(u_0)-1)/2} q^{2e+2-sq(u_0)}}{1-q^{-1}} \int_{x \in O_K^\times} \int_{y \in 1+\pi^{2e+2-sq(u_0)}O_K} \chi_{u_0}(x)\gamma'\left(\frac{xy}{\pi^{2e}}\right) dydx$$

I now claim that for any $y \in 1 + \pi^{2e+2-sq(u_0)}O_K$, we have $\chi_{u_0}(y) = 1$. The reason for this is that $u_0 \in W_{sq(u_0)-1}$ and $y \in W_{2e+1-sq(u_0)}$, which are orthogonal with respect to B_q. Hence, we may rewrite the integral as

$$\frac{q^{(sq(u_0)-1)/2} q^{2e+2-sq(u_0)}}{1-q^{-1}} \int_{x \in O_K^\times} \int_{y \in 1+\pi^{2e+2-sq(u_0)}O_K} \chi_{u_0}(xy)\gamma'\left(\frac{xy}{\pi^{2e}}\right) dydx$$

Using the same reasoning as in case 1, we may use the defining property of B_q to rewrite the integral as

$$\frac{q^{e/2} q^{(sq(u_0)-1)/2} q^{2e+2-sq(u_0)}}{1-q^{-1}} \overline{\gamma'\left(\frac{u_0}{\pi^{2e}}\right)} \int_{x \in O_K^\times} \int_{y \in 1+\pi^{2e+2-sq(u_0)}O_K} \gamma'\left(\frac{u_0 xy}{\pi^{2e}}\right) dydx$$

Using the integration lemma 6.10, the integral over x absorbs the integral over y and we get

$$\frac{q^{e/2} q^{(sq(u_0)-1)/2}}{1-q^{-1}} \overline{\gamma'\left(\frac{u_0}{\pi^{2e}}\right) \int_{x \in O_K^\times} \gamma'\left(\frac{u_0 x}{\pi^{2e}}\right) dydx}$$

This is again $(1-q^{-1})q^{-e}$ by the previous lemma. We get a final answer of

$$q^{(sq(u_0)-e-1)/2} \overline{\gamma'\left(\frac{u_0}{\pi^{2e}}\right)}$$

This concludes the casework. ∎

Proposition 6.83. *Let c_χ denote the conductor of $\chi^i_{u_0}$. For units u_0, u and any integer a, the quadratic character Gauss sum is given by*

$$\begin{aligned}\gamma'\left(\chi^i_{u_0}, \frac{u}{\pi^a}\right) &:= \int_{x \in O_K} \chi^i_{u_0}(x) \psi'\left(\frac{ux}{\pi^a}\right)\, dx \\ &= \begin{cases} 0 & max(0,a) \neq c_\chi \\ \chi^i_{u_0}(u) q^{(sq(u_0)-e-1)/2} \overline{\gamma'\left(\frac{u_0}{\pi^{2e}}\right)} & i = 0 \\ \chi^i_{u_0}(u) q^{-(e+1)/2} \zeta_8^f g_3(2e+1) \overline{\gamma'\left(\frac{u_0}{\pi^{2e}}\right)} & i = 1 \end{cases}\end{aligned}$$

Note that in the latter two cases, we have

$$\left|\gamma'\left(\chi^i_{u_0}, \frac{u}{\pi^a}\right)\right| = q^{-conductor(\chi^i_{u_0})/2}$$

A quick remark before the proof:

Remark 6.84. *The $i = 1$ case can also be written in the alternate form*

$$\chi^i_{u_0}(u) q^{-e/2} \zeta_8^{2f} g_3(2e+1)^2 \chi_{K[\sqrt{\pi}]}(u_0) \overline{\gamma'\left(\frac{u_0}{\pi^{2e+1}}\right)}$$

When defining the character $\chi^i_{u_0}$, the value of u_0 only matters mod π^{2e}, hence we may always choose u_0 so that $\chi_{K[\sqrt{\pi}]}(u_0) = 1$. This explains remark 6.14, where the constant we alluded to is $\zeta_8^{2f} g_3(2e+1)^2$.

Proof. Note that $\max(0,a)$ is just the conductor of the additive character $\psi'(ux/\pi^a)$. The fact that the conductors of the additive and multiplicative characters must match in order to get a nonzero result is a well known fact.This handles the first case.

We next tackle the case of a trivial character, which can be easily checked by hand. This case happens when $u_0 = 1, i = 0$ and entirely boils down to the single computation

$$\overline{\gamma'\left(\frac{1}{\pi^{2e}}\right)} = q^{-e/2}\overline{\psi'\left(\frac{\square_1(1)}{4}\right)} = q^{-e/2}\overline{\psi'(0)} = q^{-e/2}$$

From here on out we assume that χ is nontrivial. We could also assume that $a = c_\chi$. However, we will hold off from making this second assumption and continue for the moment with general a. By substituting $x_{\text{new}} = ux_{\text{old}}$, we get

$$\begin{aligned}\int_{x\in O_K^\times} \chi_{u_0}^i(x)\psi'\left(\frac{ux}{\pi^a}\right)\, dx &= \int_{x\in O_K^\times} \chi_{u_0}^i(u^{-1}x)\psi'\left(\frac{x}{\pi^a}\right)\, dx \\ &= \chi_{u_0}^i(u^{-1})\int_{x\in O_K^\times} \chi_{u_0}^i(x)\psi'\left(\frac{x}{\pi^a}\right)\, dx\end{aligned}$$

Since $\chi_{u_0}^i$ is quadratic, we have $\chi_{u_0}^i(u^{-1}) = \chi_{u_0}^i(u)$. We may now pull out this leading factor and focus on the rest of the integral. The first step is to change the domain of the integral using the integration lemma 6.10. We get

$$\int_{x\in O_K^\times} \chi_{u_0}^i(x)\psi'\left(\frac{x}{\pi^a}\right)\, dx = \frac{1}{\#O_K^{\times 2}}\int_{x\in O_K^\times}\int_{y\in O_K^{\times 2}} \chi_{u_0}^i(xy)\psi'\left(\frac{xy}{\pi^a}\right)\, dydx$$

The multiplicative character is invariant to squares and so, we get

$$\int_{x\in O_K^\times} \chi_{u_0}^i(x)\frac{1}{\#O_K^{\times 2}}\int_{y\in O_K^{\times 2}} \psi'\left(\frac{xy}{\pi^a}\right)\, dydx$$

Applying the integration lemma 6.11, we get

$$\int_{x\in O_K^\times} \chi_{u_0}^i(x)\frac{1}{1-q^{-1}}\int_{y\in O_K^\times} \psi'\left(\frac{xy^2}{\pi^a}\right)\, dydx$$

Since this holds for general a, we know by our earlier work that the above integral vanishes whenever $a < c_\chi$.[8] As such, we may write

$$\int_{x\in O_K^\times} \chi_{u_0}^i(x)\frac{1}{1-q^{-1}}\int_{y\in O_K^\times} \psi'\left(\frac{xy^2}{\pi^{c_\chi}}\right)\, dydx$$

$$= \frac{1}{1-q^{-1}}\int_{x\in O_K^\times} \chi_{u_0}^i(x)\int_{y\in O_K^\times} \psi'\left(\frac{xy^2}{\pi^{c_\chi}}\right) + q^{-1}\psi'\left(\frac{xy^2}{\pi^{c_\chi-2}}\right) + q^{-2}\psi'\left(\frac{xy^2}{\pi^{c_\chi-4}}\right) + \dots\, dydx$$

[8] We need χ to be nontrivial for this to be true!

Collecting terms yields an integral of y over all of O_K and not just $O_K^\times$.

$$= \frac{1}{1-q^{-1}} \int_{x \in O_K^\times} \chi_{u_0}^i(x) \int_{y \in O_K} \psi' \left(\frac{xy^2}{\pi^{c_\chi}} \right) \, dydx = \frac{1}{1-q^{-1}} \int_{x \in O_K^\times} \chi_{u_0}^i(x) \gamma' \left(\frac{x}{\pi^{c_\chi}} \right) dx$$

This last integral is just the one given by the previous proposition. Plugging in its value completes the proof. □

Chapter 7

Computing the Local Whittaker Function at Finite Places

We resume letting K denote a totally real number field of degree $n > 1$. Throughout this section, we will take $dim(V) = 1$, where V is the vector space defining $E^{l,\mu}$. We identify $V \cong K_{\mathfrak{p}}$ by selecting some basis $\mathscr{B}$. In this basis, the quadratic form on V will take on the form $Q(x) = \kappa x^2$. The character χ_V will then be precisely $\langle x, 2\kappa\rangle_{\mathbb{A}}$, so we see that this use of κ is consistent with our previous usage. Our goal for this chapter will be to evaluate the local Whittaker functions $W_{m,\mathfrak{p}}(s, \Phi_{\mathfrak{p}})$ at finite primes. We will do this by first reducing the integral $I_W(\mu, m, \Delta s)$ to a related integral I^* that has no dependence on κ. We will then evaluate I^* directly.

7.1 Reducing I_W to I^*

Fix a finite place $\mathfrak{p}$, which may be even or odd. Extend the definition of e to odd primes by in general setting $e = v_{\mathfrak{p}}(2)$. That is, $e = 0$ for all odd primes. Recall that we had the integral

$$I_W(\mu, m, \Delta s) = \int_{y \in K_{\mathfrak{p}}} \int_{\vec{x} \in (\mu + O_{K_{\mathfrak{p}}}) \times O_{K_{\mathfrak{p}}}^{2\Delta s}} \psi'(y(\kappa x_0^2 + \sum_i x_i x_i' - m)) d\vec{x} dy$$

where $\vec{x} = (x_0, x_1, x_1', \dots x_{\Delta s}, x_{\Delta s}')$ and the measures are chosen so that the integral over $O_{K_\mathfrak{p}} \times O_{K_\mathfrak{p}}^{2\Delta s+1}$ is 1. By remark 5.41, we may also write this as

$$I_W(\mu, m, \Delta s) = \lim_{k\to\infty} \int_{y\in(1/\pi^k)O_{K_\mathfrak{p}}} \int_{\vec{x}\in(\mu+O_{K_\mathfrak{p}})\times O_{K_\mathfrak{p}}^{2\Delta s}} \psi'(y(\kappa x_0^2 + \sum_i x_i x_i' - m)) d\vec{x} dy$$

The following lemma will help remove most of the coordinates of integration.

Lemma 7.1.

$$\int_{x,y\in O_{K_\mathfrak{p}}^2} \psi'(txy)\, dydx = q^{min(0,v_\pi(t))}$$

Proof. Integrating over y, we get an indicator function.

$$\int_{x\in O_{K_\mathfrak{p}}} \mathbb{1}_{O_{K_\mathfrak{p}}}(tx)\, dx = measure(\{x \in O_{K_\mathfrak{p}} | tx \in O_{K_\mathfrak{p}}\})$$

It is now easy to directly check that the measure in question is exactly $q^{min(0,v_\pi(t))}$. $\square$

To use this lemma, first break up the outer integral by valuation and get

$$I_W(\mu, m, \Delta s) = \int_{y\in O_{K_\mathfrak{p}}} \int_{\vec{x}\in(\mu+O_{K_\mathfrak{p}})\times O_{K_\mathfrak{p}}^{2\Delta s}} \psi'(y(\kappa x_0^2 + \sum_i x_i x_i' - m)) d\vec{x} dy$$

$$+ \sum_{k=1}^{\infty} \int_{y\in(1/\pi^k)O_{K_\mathfrak{p}}^\times} \int_{\vec{x}\in(\mu+O_{K_\mathfrak{p}})\times O_{K_\mathfrak{p}}^{2\Delta s}} \psi'(y(\kappa x_0^2 + \sum_i x_i x_i' - m)) d\vec{x} dy$$

Applying the lemma to this yields

$$= \int_{y\in O_{K_\mathfrak{p}}} \int_{x_0\in\mu+O_{K_\mathfrak{p}}} \psi'(y(\kappa x_0^2 - m)) dx dy +$$
$$\sum_{k=1}^{\infty} q^{-k\Delta s} \int_{y\in(1/\pi^k)O_{K_\mathfrak{p}}^\times} \int_{x_0\in\mu+O_{K_\mathfrak{p}}} \psi'(y(\kappa x_0^2 - m)) dx dy \quad (7.2)$$

We see that Δs now only shows up in the exponent, and so we will make the substitution $X = q^{-\Delta s}$. This will reframe the computation as computing some power series in X. Specifically, we are computing

$$I_W(\mu, m, \Delta s) = \int_{y\in O_{K_\mathfrak{p}}} \int_{x\in\mu+O_{K_\mathfrak{p}}} \psi'(y(\kappa x^2 - m)) dx dy +$$
$$\sum_{k=1}^{\infty} X^k \int_{y\in(1/\pi^k)O_{K_\mathfrak{p}}^\times} \int_{x\in\mu+O_{K_\mathfrak{p}}} \psi'(y(\kappa x^2 - m)) dx dy \quad (7.3)$$

We will let $I_k(\mu, m, \kappa)$, $(0 \le k)$ denote the coefficient of X^k in the above series. Since we are in the $dim(V) = 1$ case, these integrals only depend on 4 parameters, so we chose to unsuppress the dependence on κ.

We define a related integral that is independent of κ. For $k \in \mathbb{Z}$, let

$$I_k^*(\mu, m) = \int_{y \in (1/\pi^k) O_{K_\mathfrak{p}}^\times} \int_{x \in \mu + O_{K_\mathfrak{p}}} \psi'(y(x^2 - m)) dx dy$$

I^* will be easier to evaluate, since the lack of κ will cut down on the necessary casework. Knowing the value of I^* will also suffice, as the following proposition shows.

Proposition 7.4. *(i) For $k > 0$*

$$I_k(\mu, m, \kappa) = |\kappa|_\mathfrak{p}^{-1} I^*_{k - v_\pi(\kappa)}(\mu, m/\kappa)$$

(ii) For $\kappa \in O_{K_\mathfrak{p}}$

$$I_0(\mu, m, \kappa) = \begin{cases} \mathbb{1}_{O_{K_\mathfrak{p}}}(\kappa\mu^2 - m) & 2\kappa\mu \in O_{K_\mathfrak{p}} \\ |2\kappa\mu|_\mathfrak{p}^{-1} \ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\kappa\mu^2 - m}{2\kappa\mu}\right) & 2\kappa\mu \notin O_{K_\mathfrak{p}} \end{cases}$$

(iii) For $\kappa \notin O_{K_\mathfrak{p}}$

$$I_0(\mu, m, \kappa) = \begin{cases} |\kappa|_\mathfrak{p}^{-1} \mathbb{1}_{O_{K_\mathfrak{p}}}(\mu^2 - m/\kappa) & 2\mu \in O_{K_\mathfrak{p}} \\ |2\kappa\mu|_\mathfrak{p}^{-1} \ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\kappa\mu^2 - m}{2\kappa\mu}\right) & 2\mu \notin O_{K_\mathfrak{p}} \end{cases} + |\kappa|_\mathfrak{p}^{-1} \sum_{k=1}^{-v_\pi(\kappa)} I_k^*(\mu, m/\kappa)$$

Proof. For (i), simply substitute $y_{\text{new}} = \kappa y_{\text{old}}$.

For (ii), apply the substitution $x_{\text{old}} = \mu + x_{\text{new}}$ to get

$$\int_{y \in O_{K_\mathfrak{p}}} \int_{x \in \mu + O_{K_\mathfrak{p}}} \psi'(y(\kappa x^2 - m)) dx dy = \int_{y \in O_{K_\mathfrak{p}}} \int_{x \in O_{K_\mathfrak{p}}} \psi'(y(\kappa x^2 + 2\kappa\mu x + \kappa\mu^2 - m)) dx dy$$

Since $\kappa y x^2 \in O_{K_\mathfrak{p}}$, that summand doesn't affect the value of ψ'. This gives us

$$I_0(\mu, m, \kappa) = \int_{y \in O_{K_\mathfrak{p}}} \int_{x \in O_{K_\mathfrak{p}}} \psi'(2\kappa\mu x y + \kappa\mu^2 y - m y) dx dy$$

Integrating over x yields the indicator function $\mathbb{1}_{O_{K_\mathfrak{p}}}(2\kappa\mu y)$. If $2\kappa\mu \in O_{K_\mathfrak{p}}$, the integral becomes

$$\int_{y \in O_{K_\mathfrak{p}}} \psi'(\kappa\mu^2 y - m y) dy = \mathbb{1}_{O_{K_\mathfrak{p}}}(\kappa\mu^2 - m)$$

Otherwise, if $2\kappa\mu \notin O_{K_\mathfrak{p}}$, the integral becomes

$$\int_{y\in(2\kappa\mu)^{-1}O_{K_\mathfrak{p}}} \psi'(\kappa\mu^2 y - my)dy = |2\kappa\mu|_\mathfrak{p}^{-1}\mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\kappa\mu^2 - m}{2\kappa\mu}\right)$$

as desired.

For (iii), again substitute $y_{\text{new}} = \kappa y_{\text{old}}$ to get

$$I_0(\mu, m, \kappa) = |\kappa|_\mathfrak{p}^{-1}\int_{y\in\kappa O_{K_\mathfrak{p}}}\int_{x\in\mu+O_{K_\mathfrak{p}}} \psi'(y(x^2 - m/\kappa))dxdy$$

Part (ii) applied to the expression $I_0(\mu, m/\kappa, 1)$ tells us

$$\int_{y\in O_{K_\mathfrak{p}}}\int_{x\in\mu+O_{K_\mathfrak{p}}} \psi'(y(x^2 - m/\kappa))dxdy = \begin{cases} \mathbb{1}_{O_{K_\mathfrak{p}}}(\mu^2 - m/\kappa) & 2\mu \in O_{K_\mathfrak{p}} \\ |2\mu|_\mathfrak{p}^{-1}\ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\mu^2 - m/\kappa}{2\mu}\right) & 2\mu \notin O_{K_\mathfrak{p}} \end{cases}$$

from which the result quickly follows. □

Corollary 7.5. *If κ is an integral unit, then we have*

(i) For $k > 0$

$$I_k(\mu, m, \kappa) = I_k^*(\mu, m/\kappa)$$

(ii) For $k = 0$

$$I_0(\mu, m, \kappa) = \begin{cases} \mathbb{1}_{O_{K_\mathfrak{p}}}(\kappa\mu^2 - m) & 2\mu \in O_{K_\mathfrak{p}} \\ |2\mu|_\mathfrak{p}^{-1}\ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\kappa\mu^2 - m}{2\mu}\right) & 2\mu \notin O_{K_\mathfrak{p}} \end{cases}$$

7.2 Computing I^*

For $k \in \mathbb{Z}$, we will compute the value of the integral

$$I_k^*(\mu, m) = \int_{y\in(1/\pi^k)O_{K_\mathfrak{p}}^\times}\int_{x\in\mu+O_{K_\mathfrak{p}}} \psi'(y(x^2 - m))dxdy$$

Remark 7.6. *The nature of this computation changes quite wildly depending on whether one chooses to swap the order of integration. In the current order, this requires an involved Gauss sum computation. However, if one reverses the order, it becomes a measure problem. Namely, one can see that*

$$\begin{aligned} I_k^*(\mu, m) = {} & q^k meas(\{x \in \mu + O_{K_\mathfrak{p}} | x^2 \in m + \pi^k O_{K_\mathfrak{p}}\}) \\ & - q^{k-1} meas(\{x \in \mu + O_{K_\mathfrak{p}} | x^2 \in m + \pi^{k-1} O_{K_\mathfrak{p}}\}) \quad (7.7) \end{aligned}$$

This can be evaluated directly, although even with clever use of symmetry, one still must perform similarly tedious computation. We avoid the measure approach because it seems to generalize poorly. Notably, [KY10] were able to generalize the Gauss sum approach to the case of $\dim(V) > 1$ when $K = \mathbb{Q}$ in theorems 4.3 and 4.4. On the other hand, if one uses the measure approach, one will need to replace the quadratic form x^2 in the measures above with some arbitrary multivariable quadratic form Q.

The easiest case is when $k \leq 0$. This can be done via the measure theory approach, since in this case the calculations reduce to linear equations (and the answer looks just like equation (7.7)). However, we opt to repeat part of the argument from proposition 7.4 instead.

Proposition 7.8. *If $k \leq 0$ then*

$$I_k^*(\mu, m) = \begin{cases} q^k \mathbb{1}_{\pi^k O_{K_\mathfrak{p}}}(\mu^2 - m) - q^{k-1} \mathbb{1}_{\pi^{k-1} O_{K_\mathfrak{p}}}(\mu^2 - m) & k \leq v_\pi(2\mu) \\ 0 & k > v_\pi(2\mu) \end{cases}$$

Proof. Substitute $x_{\text{old}} = \mu + x_{\text{new}}$ to get

$$I_k^*(\mu, m) = \int_{y \in \pi^{-k} O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi'(yx^2 + 2\mu xy + \mu^2 y - my))dxdy$$

yx^2 will always be integral, so we may drop it, yielding

$$I_k^*(\mu, m) = \int_{y \in \pi^{-k} O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi'(2\mu xy + \mu^2 y - my))dxdy$$

Integrating over x yields the indicator function $\mathbb{1}_{O_{K_\mathfrak{p}}}(2\mu y)$. In the case $2\mu\pi^{-k} \notin O_{K_\mathfrak{p}}$, this condition can never hold for any y and we get 0. Otherwise, it holds for all y and we get

$$I_k^*(\mu, m) = \int_{y \in \pi^{-k} O_{K_\mathfrak{p}}^\times} \psi'(\mu^2 y - my))dxdy$$

□

We now turn our attention to I_k^* for $k > 0$. The following relative of the blurring lemma 6.42 will show that our calculation is rather uninteresting when $\mu \notin \frac{1}{2} O_{K_\mathfrak{p}}$.

Proposition 7.9. *Let $c_2, c_1, c_0 \in K$. Assume that at least one of the c_i is non-integral and that at least one of $v_\pi(c_1) < v_\pi(c_2)$ or $v_\pi(c_0) < v_\pi(c_2)$ holds. Then*

$$\int_{u \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi'(uc_2x^2 + uc_1x + uc_0)\, dxdu = \begin{cases} -q^{-1} & c_2, c_1 \in O_{K_\mathfrak{p}}, v_\pi(c_0) = -1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We proceed by casework on j.

Case 1: $v_\pi(c_0) < v_\pi(c_1)$

Case 1a: $v_\pi(c_0) = -1$

Then, our hypotheses tell us we are in the particular case $c_2, c_1 \in O_{K_\mathfrak{p}}, v_\pi(c_0) = -1$. We may calculate

$$\int_{u \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi'(uc_0)\, dxdu = \int_{u \in O_{K_\mathfrak{p}}^\times} \psi'\left(\frac{u}{\pi}\right) du = -\int_{u \in \pi O_{K_\mathfrak{p}}} \psi'\left(\frac{u}{\pi}\right) du = -q^{-1}$$

Case 1b: $v_\pi(c_0) < -1$

Using integration lemma 6.10, rewrite the integral as

$$\int_{u \in O_{K_\mathfrak{p}}^\times} q^{-v_\pi(c_0)-1} \int_{u' \in 1+\pi^{-v_\pi(c_0)-1}O_{K_\mathfrak{p}}} \int_{x \in O_{K_\mathfrak{p}}} \psi'(u'uc_2x^2 + u'uc_1x + u'uc_0)\, dxdu'du$$

We may then apply the blurring lemma 6.42 to the inner two integrals. Specifically, if we set the parameters in the blurring lemma called (c_2, c_1, c_0, j, v) to the values $(uc_2, uc_1, uc_0, 0, 1)$, it tells us that

$$q^{-v_\pi(c_0)-1} \int_{u' \in 1+\pi^{-v_\pi(c_0)-1}O_{K_\mathfrak{p}}} \int_{x \in O_{K_\mathfrak{p}}} \psi'(u'uc_2x^2 + u'uc_1x + u'uc_0)\, dxdu' = 0$$

which concludes this case.

Case 1: $v_\pi(c_1) \le v_\pi(c_0)$

Set $v = -v_\pi(c_1)$. We proceed by induction on v for $v \ge 1$.

For the base case, if $v = 1$, then by hypothesis c_2 must be integral. Thus, we have

$$\int_{u \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi'(uc_1x + uc_0)\, dxdu$$

The inner integral gives us the indicator function $\mathbb{1}_{O_{K_\mathfrak{p}}}(u/\pi)$, which asserts that $u \in \pi O_K$. However, this never happens since u is a unit, and hence the integral is 0.

For the inductive step, let $v > 1$. First we show that we may assume $v_\pi(c_1) < v_\pi(c_0)$ by replacing our integral with an appropriate equivalent integral if necessary. Indeed, if $v_\pi(c_1) = v_\pi(c_0)$, then let u_0 denote the unit $\frac{c_0}{c_1}$. Then make the substitution $x_{\text{old}} = x_{\text{new}} - u_0$. This yields

$$\int_{u \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi'(uc_2x^2 + uc_1x + uc_0)\, dxdu = \int_{u \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi(uc_2x^2 + u(c_1 - 2c_2u_0)x + uc_2u_0^2)\, dxdu \quad (7.10)$$

By our assumptions, $v_\pi(c_1) = v_\pi(c_0) < v_\pi(c_2)$. It follows that $v_\pi(c_1 - 2c_2u_0) = v_\pi(c_1) < v_\pi(c_2)$ and so the new integral suffices.

Now that we may assume $v_\pi(c_1) < v_\pi(c_0)$, we follow a similar argument to case 1b. Apply lemma 6.10 to rewrite our integral as

$$\int_{u \in O_{K_\mathfrak{p}}^\times} q^{v-1} \int_{u' \in 1+\pi^{v-1}O_{K_\mathfrak{p}}} \int_{x \in O_{K_\mathfrak{p}}} \psi'(u'uc_2x^2 + u'uc_1x + u'uc_0)\, dxdu'du$$

The blurring lemma with parameters $(c_2, c_1, c_0, j, v) = (uc_2, uc_1, uc_0, 0, 1)$ tells us that

$$q^{v-1} \int_{u' \in 1+\pi^{v-1}O_{K_\mathfrak{p}}} \int_{x \in O_{K_\mathfrak{p}}} \psi'(u'uc_2x^2 + u'uc_1x + u'uc_0)\, dxdu' = q^{-1} \int_{x \in O_{K_\mathfrak{p}}} \psi(\pi^2uc_2x^2 + \pi uc_1x + uc_0)\, dx \quad (7.11)$$

Plugging this in shows that

$$\int_{u \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi'(uc_2x^2 + uc_1x + uc_0)\, dxdu = q^{-1} \int_{u \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi(\pi^2uc_2x^2 + \pi uc_1x + uc_0)\, dxdu \quad (7.12)$$

Since we had $v_\pi(c_1) < v_\pi(c_0)$, it follows that $v_\pi(\pi c_1) \leq v_\pi(c_0)$ and so this integral is 0 by the inductive hypothesis. $\square$

Corollary 7.13. *Let $k > 0$. If either $\mu \notin \frac{1}{2}O_{K_\mathfrak{p}}$ or $\mu^2 - m \notin O_{K_\mathfrak{p}}$ then*

$$I_k^*(\mu, m) = 0$$

Proof. By definition,

$$I_k^*(\mu, m) = \int_{y\in(1/\pi^k)O_{K_\mathfrak{p}}^\times} \int_{x\in\mu+O_{K_\mathfrak{p}}} \psi'(y(x^2-m))dxdy$$

For fixed y, write $y = u_y/\pi^k$ for a unit u_y. For this particular y, the inner integral then takes on the form

$$\int_{x\in O_{K_\mathfrak{p}}} \psi'\left(\frac{u_y}{\pi^k}x^2 + 2\mu\frac{u_y}{\pi^k}x + \frac{u_y}{\pi^k}(\mu^2-m)\right)dx$$

The result now follows immediately from the previous proposition. □

7.3 Computing I^* part 2

We will now tackle the remaining case where $k > 0$, $\mu \in \frac{1}{2}O_{K_\mathfrak{p}}$, and $\mu^2 - m \in O_{K_\mathfrak{p}}$.

7.3.1 A Reduction Formula

This case will be the most involved. The next proposition notes that I^* can be written as the integral of a Gauss sum against an additive character. Additionally, (as long as μ obeys the assumptions of this section) the value of this integral does not depend on the specific value of μ.

Proposition 7.14.

$$I_k^*(\mu, m) = q^{k+e}\int_{y\in O_{K_\mathfrak{p}}^\times} \gamma'\left(\frac{y}{\pi^{2e+k}}\right)\psi'\left(-\frac{my}{\pi^k}\right)dy \tag{7.15}$$

In the case $\mathfrak{p}$ is even, this is the same as

$$q^{(k+2e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(y))^{\bar{k}}\gamma'\left(\frac{y}{\pi^{2e}}\right)\psi'\left(-\frac{my}{\pi^k}\right)dy$$

Proof. First recall the definition

$$I_k^*(\mu, m) = \int_{y\in(1/\pi^k)O_{K_\mathfrak{p}}^\times} \int_{x\in\mu+O_{K_\mathfrak{p}}} \psi'(y(x^2-m))dxdy$$

Changing variables yields

$$q^k \int_{y \in O_{K_\mathfrak{p}}^\times} \int_{x \in \mu + O_{K_\mathfrak{p}}} \psi' \left(\frac{y}{\pi^k} (x^2 - m) \right) dxdy$$

We perform casework depending on whether $\mathfrak{p}$ is an odd or even prime.

Case 1: $\mathfrak{p}$ is odd.

This is quite straightforward. By assumption, μ will be integral and so $\mu + O_{K_\mathfrak{p}} = O_{K_\mathfrak{p}}$, which immediately yields the desired relation.

Case 2: $\mathfrak{p}$ is even.

This is more difficult. First perform another change of variables to get

$$q^k \int_{y \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi' \left(\frac{y}{\pi^k} (x^2 + 2\mu x + \mu^2 - m) \right) dxdy$$

The key to proceeding will be to use the blurring lemma 6.42 to greatly increase the exponent k in the denominator. The idea here is that for d sufficiently large, the g_1 term in the Gauss sum formula will be trivial, making further work significantly easier. To this end, for any integer $d > 1$, we may apply the blurring lemma with parameters (c_2, c_1, c_0, j, v) equal to $(y/\pi^{k+2d}, 2y\mu/\pi^{k+d}, y(\mu^2 - m)/\pi^k, 2, 2d)$. This yields the integral

$$q^k \int_{y \in O_{K_\mathfrak{p}}^\times} q^d q^k \int_{y' \in 1 + \pi^k O_K} \int_{x \in O_{K_\mathfrak{p}}} \psi' \left(\frac{yy'x^2}{\pi^{k+2d}} + \frac{2yy'\mu x}{\pi^{k+d}} + \frac{yy'(\mu^2 - m)}{\pi^k} \right) dxdy'dy$$

Then apply the integration lemma 6.10 to get

$$q^k q^d \int_{y \in O_{K_\mathfrak{p}}^\times} \int_{x \in O_{K_\mathfrak{p}}} \psi' \left(\frac{yx^2}{\pi^{k+2d}} + \frac{2y\mu x}{\pi^{k+d}} + \frac{y(\mu^2 - m)}{\pi^k} \right) dxdy$$

Rewriting using our definition of γ' gives

$$q^k q^d \int_{y \in O_{K_\mathfrak{p}}^\times} \psi' \left(\frac{y(\mu^2 - m)}{\pi^k} \right) \gamma' \left(\frac{y}{\pi^{k+2d}}, \frac{2y\mu}{\pi^{k+d}} \right) dy$$

The general formula for the Gauss sum then tells us

$$q^k q^d q^{-(k+2d-e)/2} \int_{y \in O_{K_\mathfrak{p}}^\times} \psi' \left(\frac{y(\mu^2 - m)}{\pi^k} \right) g_1(a,u,t) g_2(a,u,t) g_3(a) dy$$

where $a = k + 2d, u = y, t = 2y\mu/\pi^{(k-\bar{k})/2}$. ($\bar{k} \in \{0, 1\}$ is the value of k mod 2.) Let us now go through each g_i to see what it contributes to the integral.

Recalling the formula for g_1, we have

$$g_1 = \begin{cases} 1 & \square_{1-\bar{k}}(y) \equiv \frac{4y^2\mu^2}{\pi^{k-\bar{k}}} \bmod \pi^{min(2d+k+\bar{k}, 2e-2d-k+\bar{k})} \\ 0 & \text{else} \end{cases}$$

As long as d is sufficiently large ($d \geq e/2$ suffices), we can see that the first condition will be taken mod $\pi^{2e-2d-k+\bar{k}}$. However, we clearly have $\square_{1-\bar{k}}(y) \equiv \frac{4y^2\mu^2}{\pi^{k-\bar{k}}} \equiv 0 \bmod \pi^{-k+\bar{k}}$. (Since under current assumptions $k \geq 0$, $y \in O_{K_\mathfrak{p}}^\times \subset O_{K_\mathfrak{p}}$, and $2\mu \in O_{K_\mathfrak{p}}$.) Thus, as long as $d \geq e$, the condition will be met and g_1 will be identically 1, regardless of y.

From now on, we assume $d \geq e$ so that this is the case. Though, we will not need to remember this for long, as all occurrences of d in the formula are about to cancel out. Recalling the formula for g_2, we now get

$$q^k q^d q^{-(k+2d-e)/2} \int_{y \in O_{K_\mathfrak{p}}^\times} \psi'\left(\frac{y(\mu^2 - m)}{\pi^k}\right) \psi'\left(\frac{1}{4\pi^{\bar{k}}} \frac{\square_{1-\bar{k}}(y) - \frac{4y^2\mu^2}{\pi^{k-\bar{k}}}}{y}\right) g_3(k + 2d) dy$$

This simplifies to

$$q^{(k+e)/2} \int_{y \in O_{K_\mathfrak{p}}^\times} \psi'\left(\frac{y(\mu^2 - m)}{\pi^k}\right) \psi'\left(\frac{1}{4\pi^{\bar{k}}} \frac{\square_{1-\bar{k}}(y)}{y}\right) \psi'\left(\frac{-y\mu^2}{\pi^k}\right) g_3(k + 2d) dy$$

Things now further simplify drastically. We see that all occurrences of μ cancel out, and thus that this integral does not depend on μ. We may also remove the final occurrence of d from the formula. In particular, since $d \geq e$, it follows that $k + 2d \geq 2e$. Hence, g_3 only cares about the parity of $k + 2d$ and we have $g_3(k + 2d) = g_3(2e + \bar{k})$. Thus, our formula reduces to

$$q^{(k+e)/2} g_3(2e + \bar{k}) \int_{y \in O_{K_\mathfrak{p}}^\times} \psi'\left(\frac{1}{4\pi^{\bar{k}}} \frac{\square_{1-\bar{k}}(y)}{y}\right) \psi'\left(-\frac{my}{\pi^k}\right) dy$$

There are two ways we can clean up this formula. The first possibility is to apply the formula for the Gauss sum to the quantity $\gamma'\left(\frac{y}{\pi^{2e+k}}\right)$. This yields the quantity

$$q^{k+e} \int_{y \in O_{K_\mathfrak{p}}^\times} \gamma'\left(\frac{y}{\pi^{2e+k}}\right) \psi'\left(-\frac{my}{\pi^k}\right) dy$$

The other option is to apply either parity shifting lemma 6.45 to get

$$q^{(k+e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(y))^{\bar{k}}\psi'\left(\frac{1}{4}\frac{\square_1(y)}{y}-\frac{my}{\pi^k}\right)dy$$

Then apply the formula for the Gauss sum to the quantity $\gamma'\left(\frac{y}{\pi^{2e}}\right)$ to get

$$q^{(k+2e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(y))^{\bar{k}}\gamma'\left(\frac{y}{\pi^{2e}}\right)\psi'\left(-\frac{my}{\pi^k}\right)dy$$

$\square$

7.3.2 Two Important Functions

We will evaluate the integral in proposition 7.14 momentarily. First we must introduce two functions that will show up in the computation and are defined in [Su16] section 2. The first is a character $\left(\frac{m}{\mathfrak{p}}\right)$ and the second is a function $\mathfrak{f}$. We start with the character.

$\left(\frac{m}{\mathfrak{p}}\right)$ is the character giving the behavior of $\mathfrak{p}$ in the extension $K[\sqrt{m}]$. Explicitly,

$$\left(\frac{m}{\mathfrak{p}}\right)=\begin{cases}1 & \mathfrak{p}\text{ is split}\\ -1 & \mathfrak{p}\text{ is inert}\\ 0 & \mathfrak{p}\text{ is ramified}\end{cases}$$

The behavior of the prime $\mathfrak{p}$ in such an extension is given explicitly by the following proposition. We label the case $K_\mathfrak{p}[\sqrt{m}]=K_\mathfrak{p}$ as split in analogy with the global case.

Proposition 7.16. *If $\mathfrak{p}$ is an odd prime*

$$K_\mathfrak{p}[\sqrt{m}]\ \textit{is}\ \begin{cases}\textit{ramified} & v_\pi(m)\ \textit{is odd}\\ \textit{inert} & u_m \bmod \pi\ \textit{is nonsquare}\\ \textit{split} & u_m \bmod \pi\ \textit{is square}\end{cases}$$

If $\mathfrak{p}$ is an even prime

$$K_\mathfrak{p}[\sqrt{m}]\ \textit{is}\ \begin{cases}\textit{ramified} & sq(u_m)<2e\ \textit{or}\ v_\pi(m)\ \textit{is odd}\\ \textit{inert} & sq(u_m)=2e\\ \textit{split} & sq(u_m)=\infty\end{cases}$$

Proof. Case 1: $\mathfrak{p}$ is odd.

This case is kind of trivial, but we go over the motions anyway.

The assertion for odd $v_\pi(m)$ is obvious, so restrict attention to when $v_\pi(m)$ is even. In this case $K_\mathfrak{p}[\sqrt{m}] = K_\mathfrak{p}[\sqrt{u_m}]$.

If u_m mod π is square, then u_m is square and the extension is split.

Finally, if u_m mod π is nonsquare, then u_m is nonsquare. Any two such u_m differ by a square factor and hence all generate the same extension. K has at least one inert extension, so this must be it.

Case 2: $\mathfrak{p}$ is even.

The assertion for odd $v_\pi(m)$ is obvious, so consider the case where $v_\pi(m)$ is even. In this case, $K_\mathfrak{p}[\sqrt{m}] = K_\mathfrak{p}[\sqrt{u_m}]$. It is clear that the extension splits iff u_m is already a perfect square, which is to say $sq(u_m) = \infty$.

In the case that $sq(u_m) < 2e$, the squareness will be odd. We may choose to multiply u_m by a perfect square so that it's π-adic expansion starts $u_m = 1 + a_{sq(u_m)}\pi^{sq(u_m)} + \ldots$, since doing so will affect neither the squareness of u_m nor the type of extension of $K_\mathfrak{p}$ we are generating. We can then consider the pair of conjugates $\sqrt{u_m} + 1$ and $\sqrt{u_m} - 1$. They must have the same valuation in $K_\mathfrak{p}[\sqrt{u_m}]$ and furthermore their product is $(\sqrt{u_m}+1)(\sqrt{u_m}-1) = u_m - 1 = a_{sq(u_m)}\pi^{sq(u_m)} + \ldots$ It follows that (for π a uniformizer of $K_\mathfrak{p}$) $v_\pi((\sqrt{u_m}+1)(\sqrt{u_m}-1)) = sq(u_m)$ is odd. Hence, our extension is ramified.

Finally, any two units u_m of squareness $2e$ differ by a square unit factor and hence generate the same extension. Thus, there is exactly one extension unaccounted for above. Since we know $K_\mathfrak{p}$ has at least one inert extension, this must be it. $\square$

Definition 7.17. *Let $\mathfrak{f}(m)$ be given by the equation*

$$2\mathfrak{f}(m) = v_\pi(m) + 2e - \eta(m)$$

where $\eta(m)$ is given in terms of a relative discriminant.

$$\eta(m) = v_\pi \left(Disc(K_\mathfrak{p}[\sqrt{m}]/K_\mathfrak{p}) \right)$$

If $m = 0$, then take $\mathfrak{f}(m) = \infty$. If $K_\mathfrak{p}[\sqrt{m}] = K_\mathfrak{p}$, take the discriminant to be 1.

Lemma 7.18. *The valuation of the relative discriminant is given by*

$$\eta(m) = \begin{cases} 1 & \mathfrak{p} \text{ is odd and } v_\pi(m) \text{ is odd} \\ 0 & \mathfrak{p} \text{ is odd and } v_\pi(m) \text{ is even} \\ 2e + 1 - sq(m) & \mathfrak{p} \text{ is even and } sq(m) < 2e \\ 0 & \mathfrak{p} \text{ is even and } sq(m) \in \{2e, \infty\} \end{cases}$$

The right hand side of the equation defining $\mathfrak{f}$ is always even, so that $\mathfrak{f}(m)$ is an integer. Furthermore, if $\mu \in (1/2)O_{K_\mathfrak{p}}$ and $m \in -\mu^2 + O_{K_\mathfrak{p}}$, then $\mathfrak{f}(-m)$ is non-negative.

Proof. We start by calculating η. Note that for this step we only care about the value of m up to a square factor. Temporarily let $L = K_\mathfrak{p}[\sqrt{m}]$. Let π_L denote a uniformizer (to be chosen) of L. Continue to let π with no subscript denote a uniformizer of $K_\mathfrak{p}$.

Case 1: $\mathfrak{p}$ is odd

If $v_\pi(m)$ is even then by the previous proposition either $K_\mathfrak{p}[\sqrt{m}] = K_\mathfrak{p}$ and $Disc(L/K_\mathfrak{p}) = 1 \implies \eta = 0$ or the extension is inert (and in particular unramified) so $\eta = 0$.

If $v_\pi(m)$ is odd wlog take $v_\pi(m) = 1$. Then $\pi_L = \sqrt{m}$ is a valid choice of uniformizer. Hence O_L has basis $1, \sqrt{m}$ and from the formula for discriminant we have

$$Disc(L/K_\mathfrak{p}) = \begin{vmatrix} 1 & \sqrt{m} \\ 1 & -\sqrt{m} \end{vmatrix}^2 = 4m$$

Since we took $v_\pi(m) = 1$, we have $\eta = 1$.

Case 2: $\mathfrak{p}$ is even

If $sq(m) \in \{2e, \infty\}$, then similarly to the last case we see $L/K_\mathfrak{p}$ is unramified and so $\eta = 0$.

If $sq(m) = 0$, then $v_\pi(m)$ is odd so wlog $v_\pi(m) = 1$. The argument then proceeds in exactly the same manner as the last case except that at the end we get $\eta(m) = v_\pi(4m) = 2e + 1$.

Finally, if $1 \leq sq(m) \leq 2e - 1$, then $sq(m)$ is odd by remark 6.73. In this case the previous proposition tells us the extension is ramified and from its proof we know that (up to choice of an equivalent m) we may choose the uniformizer $\pi_L = (\sqrt{u_m} + 1)/\pi^{(sq(m)-1)/2}$. The discriminant formula then gives

$$Disc(L/K_\mathfrak{p}) = \begin{vmatrix} 1 & (\sqrt{u_m} + 1)/\pi^{(sq(m)-1)/2} \\ 1 & (-\sqrt{u_m} + 1)/\pi^{(sq(m)-1)/2} \end{vmatrix}^2 = 4u_m/\pi^{sq(m)-1}$$

So, $\eta(m) = v_\pi(4u_m/\pi^{sq(m)-1}) = 2e + 1 - sq(m)$.

We next check the evenness claim, although we only do it in the case $\mathfrak{p}$ is even and $sq(m) < 2e$, since the other cases are trivial to check. In this case, if $v_\pi(m)$ is odd, then by definition $sq(m) = 0$ and we are done. If $v_\pi(m)$ is even, then remark 6.73 tells us $sq(m)$ will be odd and we are done.

Finally, for the non-negativity claim, our assumption is equivalent to $v_\pi(m) \geq -2e$. We proceed by casework.

Case 1: $\mathfrak{p}$ is odd

In this case, we have $e = 0$ so $v_\pi(-m) \geq 0$. If $v_\pi(-m)$ is even, then $2\mathfrak{f}(-m) = v_\pi(-m) \geq 0$ and we are done. If $v_\pi(-m)$ is odd, then we must have $v_\pi(-m) \geq 1$. We then see that $2\mathfrak{f}(-m) = v_\pi(-m) - 1 \geq 0$ and we are done.

Case 2: $\mathfrak{p}$ is even, $sq(-m) \in \{2e, \infty\}$

In this case we have $2\mathfrak{f}(-m) = v_\pi(-m) + 2e \geq 0$ and we are done.

Case 3: $\mathfrak{p}$ is even, $sq(-m) = 0$

This is the case of $v_\pi(-m)$ odd, and we must have $\mu \in O_K$ since otherwise we would have $v_\pi(-m) = 2v_\pi(\mu)$ which is even. It follows that $v_\pi(-m) \geq 1$. Then we have $2\mathfrak{f}(-m) = v_\pi(-m) + 2e - (2e + 1) \geq 0$.

Case 4: $\mathfrak{p}$ is even, $0 < sq(-m) < 2e$, $-m \in O_{K_\mathfrak{p}}$

In this case

$$2\mathfrak{f}(-m) = v_\pi(-m) + 2e - (2e + 1 - sq(-m)) = v_\pi(-m) + (sq(-m) - 1)$$

Both summands are nonnegative by assumption and we are done.

Case 5: $\mathfrak{p}$ is even, $0 < sq(-m) < 2e$, $-m \notin O_{K_\mathfrak{p}}$

In this case, we have $-m = \mu^2 + O_{K_\mathfrak{p}}$ for $\mu \notin O_{K_\mathfrak{p}}$. Let $\mu = u_\mu \pi^{v_\pi(\mu)}$. Then we have $sq(-m) = sq(u_\mu^2 + \pi^{-2v_\pi(\mu)}x)$ for some $x \in O_{K_\mathfrak{p}}$. In particular, this immediately implies $sq(-m) \geq -2v_\pi(\mu) = -v\pi(m)$. Since the squareness must be odd, we have the stronger statement $sq(-m) \geq -v_\pi(m) + 1$. We now write

$$2\mathfrak{f}(-m) = v_\pi(-m) + 2e - (2e + 1 - sq(-m)) = v_\pi(-m) + (sq(-m) - 1) \geq 0$$

□

Remark 7.19. *Given some m in the global field K, one may consider the value of $\mathfrak{f}$ at varying finite places $\mathfrak{p}$, which we denote by $\mathfrak{f}_\mathfrak{p}(m)$. Then, one will have $\mathfrak{f}(m) = 0$ at almost all finite places.*

Proof. At almost all places, we will have $\mathfrak{p}$ odd and $v_\pi(m) = 0$. Plugging this into the above formulas gives the desired fact. □

Remark 7.20. *Define an ideal $I_\mathfrak{f} = \prod_{\mathfrak{p}<\infty} \mathfrak{p}^{\mathfrak{f}_\mathfrak{p}(m)}$. Then, this ideal has the property $(4m) = Disc(K[\sqrt{m}]/K)I_\mathfrak{f}^2$. This shows that $I_\mathfrak{f}$ is the ideal showing up in the definition of Hurwitz class number.*

Recall equation (7.15), which says

$$I_k^*(\mu, m) = q^{k+e} \int_{y \in O_{K_\mathfrak{p}}^\times} \gamma' \left(\frac{y}{\pi^{2e+k}} \right) \psi' \left(-\frac{my}{\pi^k} \right) dy$$

7.3.3 Finishing the Calculation

We can now give an explicit formula for $I_k^*(\mu, m)$ by evaluating this integral. The output expressions are slightly unnatural for the sake of unifying all cases into a single formula.

Proposition 7.21. *Continue to assume that $k > 0$, $\mu \in \frac{1}{2}O_{K_\mathfrak{p}}$, and $\mu^2 - m \in O_{K_\mathfrak{p}}$.*

If $k > 0$ is even, then

$$I_k^*(\mu, m) = q^{k/2} \begin{cases} 1 - q^{-1} & k \leq 2\mathfrak{f}(m) \\ -q^{-1}\left(1 - \left|\left(\frac{m}{\mathfrak{p}}\right)\right|\right) & k = 2\mathfrak{f}(m) + 2 \\ 0 & k \geq 2\mathfrak{f}(m) + 4 \end{cases}$$

If k is odd, then

$$I_k^*(\mu, m) = q^{k/2} \begin{cases} q^{\mathfrak{f}(m)}\left(\frac{m}{\mathfrak{p}}\right) & k = 2\mathfrak{f}(m) + 1 \\ 0 & k \neq 2\mathfrak{f}(m) + 1 \end{cases}$$

In the case $m = 0$, we have $\mathfrak{f}(m) = \infty$ and these formulas reduce to

$$I_k^*(\mu, 0) = q^{k/2} \begin{cases} 1 - q^{-1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Proof. We proceed via extensive casework. The case $m = 0$ is handled first via an earlier lemma. All other cases implicitly assume $m \neq 0$.

Case 0: $m = 0$

We have

$$I_k^*(\mu, 0) = q^{k+e} \int_{y \in O_{K_\mathfrak{p}}^\times} \gamma'\left(\frac{y}{\pi^{2e+k}}\right) dy$$

Applying lemma 6.80 yields

$$I_k^*(\mu, 0) = q^{k+e} q^{-(2e+k)/2} \begin{cases} 1 - q^{-1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

which is the desired result.

Case 1a: $\mathfrak{p}$ is odd and k is even.

Since $\mathfrak{p}$ is odd, we know that

$$2\mathfrak{f}(m) = v_\pi(m) - \begin{cases} 1 & \mathfrak{p} \text{ is odd and } v_\pi(m) \text{ is odd} \\ 0 & \mathfrak{p} \text{ is odd and } v_\pi(m) \text{ is even} \end{cases}$$

From the formula for the Gauss sum, we have

$$q^{k+e}\int_{y\in O_{K_\mathfrak{p}}^\times}\gamma'\left(\frac{y}{\pi^{2e+k}}\right)\psi'\left(-\frac{my}{\pi^k}\right)dy = q^{k/2}\int_{y\in O_{K_\mathfrak{p}}^\times}\psi'\left(-\frac{my}{\pi^k}\right)dy$$
$$= q^{k/2}\int_{y\in O_{K_\mathfrak{p}}^\times}\psi'\left(-\frac{u_m y}{\pi^{k-v_\pi(m)}}\right)dy \quad (7.22)$$

We can then do further casework off to evaluate this latest integral.

If $k \le 2\mathfrak{f}(m)$, then in particular we have $k - v_\pi(m) \le -\eta(m) \le 0$, and so the integrand is identically 1. Thus, we get $1 - q^{-1}$.

If $k = 2\mathfrak{f}(m) + 2$, the integral becomes

$$q^{k/2}\int_{y\in O_{K_\mathfrak{p}}^\times}\psi'\left(-\frac{u_m y}{\pi^{2-\eta(m)}}\right)dy = \begin{cases} 0 & \eta(m) = 0 \\ -q^{-1} & \eta(m) = 1 \end{cases}$$

The integral vanishes when $\eta(m) = 0$, which is exactly when $v_\pi(m)$ is even. This is equivalent to asking $\left|\left(\frac{m}{\mathfrak{p}}\right)\right| = 1$.

Finally, if $k \ge 2\mathfrak{f}(m) + 4$, then we have $k - v_\pi(m) \ge 4 - \eta(m) \ge 3$, and so the integral vanishes.

Case 1b: $\mathfrak{p}$ is odd and k is odd.

From the formula for the Gauss sum, we have

$$q^{k+e}\int_{y\in O_{K_\mathfrak{p}}^\times}\gamma'\left(\frac{y}{\pi^{2e+k}}\right)\psi'\left(-\frac{my}{\pi^k}\right)dy = q^{(k+1)/2}\int_{y\in O_{K_\mathfrak{p}}^\times}\left(\frac{y}{\mathfrak{p}}\right)\gamma'\left(\frac{1}{\pi}\right)\psi'\left(-\frac{my}{\pi^k}\right)dy$$

If we pull the $\gamma'\left(\frac{1}{\pi}\right)$ out of the integral, what remains is a character Gauss sum and so we get

$$q^{(k+1)/2}\gamma'\left(\frac{1}{\pi}\right)\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{-m}{\pi^k}\right) = q^{(k+1)/2}\gamma'\left(\frac{1}{\pi}\right)\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{-u_m}{\pi^{k-v_\pi(m)}}\right)$$

By proposition 6.12, the Gauss sum is zero unless $k - v_\pi(m) = 1$. Since k is odd, this can only happen if $v_\pi(m)$ is even. However, this would imply $\eta(m) = 0$, and so $2\mathfrak{f}(m) = v_\pi(m)$. Hence, our condition is $k = v_\pi(m) + 1 = 2\mathfrak{f}(m) + 1$. Now that we know when a nonzero value can occur, a particularly slick way to proceed is to use the fact

$\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{-u_m}{\pi}\right) = \left(\frac{-u_m}{\mathfrak{p}}\right)\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{1}{\pi}\right)$ and the formula in remark 6.14 to write

$$q^{(v_\pi(m)+2)/2}\gamma'\left(\frac{1}{\pi}\right)\gamma'\left(\left(\frac{\cdot}{\mathfrak{p}}\right), \frac{-u_m}{\pi}\right) = q^{(v_\pi(m)+2)/2}\gamma'\left(\frac{1}{\pi}\right)^2\left(\frac{-u_m}{\mathfrak{p}}\right)$$

By proposition 6.13, this becomes

$$q^{v_\pi(m)/2}\begin{cases}1 & p \equiv 1 \bmod 4 \\ (-1)^f & p \equiv 3 \bmod 4\end{cases} \cdot \left(\frac{-u_m}{\mathfrak{p}}\right)$$

Now, we remark that

$$\left(\frac{-1}{\mathfrak{p}}\right) = (-1)^{\frac{p^f-1}{2}} = \begin{cases}1 & p \equiv 1 \bmod 4 \\ (-1)^f & p \equiv 3 \bmod 4\end{cases}$$

where the second equality is just checked manually. Hence, the piecewise function cancels and we are left with

$$q^{v_\pi(m)/2}\left(\frac{u_m}{\mathfrak{p}}\right)$$

To convert this to the desired expression, note that since $v_\pi(m)$ is even, we have $\left(\frac{u_m}{\mathfrak{p}}\right) = \left(\frac{m}{\mathfrak{p}}\right)$.

Case 2: $\mathfrak{p}$ is even.

By proposition 7.14, it suffices to evaluate the formula

$$q^{(k+2e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(y))^{\bar{k}}\gamma'\left(\frac{y}{\pi^{2e}}\right)\psi'\left(-\frac{my}{\pi^k}\right)dy$$

To proceed from here, we will use a procedure very similar to that used to evaluate the character Gauss sums.[1] Write $m = u_m\pi^{v_\pi(m)}$ and substitute $y_{\text{new}} = u_m y_{\text{old}}$ to get

$$q^{(k+2e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(y/u_m))^{\bar{k}}\gamma'\left(\frac{y/u_m}{\pi^{2e}}\right)\psi'\left(-\frac{y}{\pi^{k-v_\pi(m)}}\right)dy$$

We can move the u_m terms from denominator to numerator to make things look prettier. This is possible because χ is quadratic and the fact that from the definition

[1]The reason for the similar procedure is that our integrand is roughly of the form $e^{(1/2)\pi i q(y)}\psi(my/\pi^k)$, whereas the integrand for the character Gauss sum is $e^{\pi i B_q(u_0,y)}\psi(my/\pi^k)$. That is, both are an exponential of something related to the quadratic form q times a standard exponential in y.

of Gauss sum, it is clear $\gamma'(u^2x) = \gamma'(x)$ for any unit u and any $x \in K$. We get

$$q^{(k+2e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(u_m y))^{\bar{k}}\gamma'\left(\frac{u_m y}{\pi^{2e}}\right)\psi'\left(-\frac{y}{\pi^{k-v_\pi(m)}}\right)dy$$

As before, the trick to evaluating this integral is to apply our integration lemmas. By lemma 6.10, we get

$$q^{(k+2e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\cdot \int_{y\in O_{K_\mathfrak{p}}^\times}\frac{1}{\#O_{K_\mathfrak{p}}^{\times 2}}\int_{x\in O_{K_\mathfrak{p}}^{\times 2}}(\chi_{K[\sqrt{\pi}]}(u_m yx))^{\bar{k}}\gamma'\left(\frac{u_m yx}{\pi^{2e}}\right)\psi'\left(-\frac{yx}{\pi^{k-v_\pi(m)}}\right)dxdy \quad (7.23)$$

The first two factors of the integrand are unaffected by square unit x and so we get

$$q^{(k+2e)/2}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\cdot \int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(u_m y))^{\bar{k}}\gamma'\left(\frac{u_m y}{\pi^{2e}}\right)\frac{1}{\#O_{K_\mathfrak{p}}^{\times 2}}\int_{x\in O_{K_\mathfrak{p}}^{\times 2}}\psi'\left(-\frac{yx}{\pi^{k-v_\pi(m)}}\right)dxdy \quad (7.24)$$

By lemma 6.11, this is the same as

$$\frac{q^{(k+2e)/2}}{1-q^{-1}}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(u_m y))^{\bar{k}}\gamma'\left(\frac{u_m y}{\pi^{2e}}\right)\int_{x\in O_{K_\mathfrak{p}}^\times}\psi'\left(-\frac{yx^2}{\pi^{k-v_\pi(m)}}\right)dxdy$$

The inner integral looks very much like a Gauss sum and may easily be realized in terms of such by re-expressing it as an integral over O_K minus an integral over πO_K. This gives us

$$\frac{q^{(k+2e)/2}}{1-q^{-1}}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})\cdot \int_{y\in O_{K_\mathfrak{p}}^\times}(\chi_{K[\sqrt{\pi}]}(u_m y))^{\bar{k}}\gamma'\left(\frac{u_m y}{\pi^{2e}}\right)\left(\gamma'\left(\frac{-y}{\pi^{k-v_\pi(m)}}\right)-\frac{1}{q}\gamma'\left(\frac{-y}{\pi^{k-v_\pi(m)-2}}\right)\right)dy \quad (7.25)$$

We may expand $\gamma'\left(\frac{u_m y}{\pi^{2e}}\right)$ using the definition of B_q. In particular, we know that

$$\gamma'\left(\frac{u_m y}{\pi^{2e}}\right)=\gamma'\left(\frac{u_m}{\pi^{2e}}\right)\gamma'\left(\frac{y}{\pi^{2e}}\right)\chi_{u_m}(y)$$

Using this and rearranging a few terms, our integral becomes

$$\frac{q^{(k+3e)/2}}{1-q^{-1}}\zeta_8^{f\bar{k}}g_3(2e+\bar{k})(\chi_{K[\sqrt{\pi}]}(u_m))^{\bar{k}}\gamma'\left(\frac{u_m}{\pi^{2e}}\right)\cdot \int_{y\in O_{K_\mathfrak{p}}^\times}\chi_{u_m}^{\bar{k}}(y)\gamma'\left(\frac{y}{\pi^{2e}}\right)\left(\gamma'\left(\frac{-y}{\pi^{k-v_\pi(m)}}\right)-\frac{1}{q}\gamma'\left(\frac{-y}{\pi^{k-v_\pi(m)-2}}\right)\right)dy \quad (7.26)$$

By the parity shifting lemma 6.47, this equals

$$q^{(\overline{k-v_\pi(m)})/2}\frac{q^{(k+3e)/2}}{1-q^{-1}}\zeta_8^{f(\bar{k}-\overline{k-v_\pi(m)})}\frac{g_3(2e+\bar{k})}{g_3(2e+\overline{k-v_\pi(m)})}(\chi_{K[\sqrt{\pi}]}(u_m))^{\bar{k}}\gamma'\left(\frac{u_m}{\pi^{2e}}\right)\cdot$$

$$\int_{y\in O_{K_\mathfrak{p}}^\times}\chi_{u_m}^{\overline{v_\pi(m)}}(y)\gamma'\left(\frac{y}{\pi^{2e+\overline{k-v_\pi(m)}}}\right)\left(\gamma'\left(\frac{-y}{\pi^{k-v_\pi(m)}}\right)-\frac{1}{q}\gamma'\left(\frac{-y}{\pi^{k-v_\pi(m)-2}}\right)\right)dy$$

By the reflection formula (proposition 6.39), this equals

$$\frac{q^{(v_\pi(m)+3e)/2}}{1-q^{-1}}\zeta_8^{f(\bar{k}-\overline{k-v_\pi(m)})}\frac{g_3(2e+\bar{k})}{g_3(2e+\overline{k-v_\pi(m)})}(\chi_{K[\sqrt{\pi}]}(u_m))^{\bar{k}}\gamma'\left(\frac{u_m}{\pi^{2e}}\right)\cdot$$

$$\int_{y\in O_{K_\mathfrak{p}}^\times}\chi_{u_m}^{\overline{v_\pi(m)}}(y)\left(\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)}}\right)-\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)+2}}\right)\right)dy$$

We may now evaluate this in subcases. However, instead of doing k even and odd we will perform casework on the value of $\left(\frac{m}{\mathfrak{p}}\right)$. Also note that since $\mathfrak{p}$ is even, we have

$$2\mathfrak{f}(m)=v_\pi(m)+2e-\begin{cases}2e+1-sq(m) & sq(m)<2e\\ 0 & sq(m)\in\{2e,\infty\}\end{cases}$$

Case 2a: $\left(\frac{m}{\mathfrak{p}}\right)=0$.

In this case, either $v_\pi(m)$ is odd or $sq(u_m)<2e$. As such, this case is equivalent to $\chi_{u_m}^{\overline{v_\pi(m)}}$ being a nontrivial character. In either case, we have $c_\chi=2e+1-sq(m)$ and $2\mathfrak{f}(m)=v_\pi(m)+sq(m)-1$.

From proposition 6.81, (since χ is nontrivial) if we want a nonzero result we must have $2e-k+v_\pi(m)\equiv c_\chi \bmod 2$. However, it is easy to chcek that in this case $c_\chi\equiv v_\pi(m) \bmod 2$, so at very minimum we need k to be even. This reduces our integral to

$$\frac{q^{(v_\pi(m)+3e)/2}}{1-q^{-1}}\frac{\zeta_8^{-f(\overline{v_\pi(m)})}}{g_3(2e+\overline{v_\pi(m)})}\gamma'\left(\frac{u_m}{\pi^{2e}}\right)\cdot$$
$$\int_{y\in O_{K_\mathfrak{p}}^\times}\chi_{u_m}^{\overline{v_\pi(m)}}(y)\left(\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)}}\right)-\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)+2}}\right)\right)dy \quad (7.27)$$

Applying proposition 6.81, we see that

$$\int_{y\in O_{K_\mathfrak{p}}^\times} \chi_{u_m}^{\overline{v_\pi(m)}}(y)\left(\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)}}\right)-\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)+2}}\right)\right)dy =$$

$$(1-q^{-1})\overline{\gamma'\left(\frac{u_m}{\pi^{2e}}\right)}\zeta_8^{f(\overline{v_\pi(m)})}g_3(2e+\overline{v_\pi(m)})\Bigg(q^{-(e-k+v_\pi(m))/2}\begin{cases}1 & k\le v_\pi(m)+sq(m)-1\\ 0 & otherwise\end{cases}$$

$$-q^{-(e+2-k+v_\pi(m))/2}\begin{cases}1 & k\le v_\pi(m)+sq(m)+1\\ 0 & otherwise\end{cases}\Bigg) \quad (7.28)$$

Plugging this in, most terms cancel (don't forget $\left|\gamma'\left(\frac{u_m}{\pi^{2e}}\right)\right|^2=q^{-e}$) and we are left with

$$q^{k/2}\left(\begin{cases}1 & k\le v_\pi(m)+sq(m)-1\\ 0 & otherwise\end{cases}-q^{-1}\begin{cases}1 & k\le v_\pi(m)+sq(m)+1\\ 0 & otherwise\end{cases}\right)$$

Light rearrangement yields (for k even)

$$q^{k/2}\begin{cases}1-q^{-1} & k\le v_\pi(m)+sq(m)-1\\ -q^{-1} & k=v_\pi(m)+sq(m)+1\\ 0 & k\ge v_\pi(m)+sq(m)+3\end{cases}$$

which is the desired formula.

Case 2b: $\left(\frac{m}{\mathfrak{p}}\right)\neq 0$.

In this case, we have $2\mathfrak{f}(m)=v_\pi(m)+2e$. We also know $v_\pi(m)$ is even and so a handful of terms cancel right away, leaving

$$\frac{q^{(v_\pi(m)+3e)/2}}{1-q^{-1}}(\chi_{K[\sqrt{\pi}]}(u_m))^{\bar{k}}\gamma'\left(\frac{u_m}{\pi^{2e}}\right)\cdot\int_{y\in O_{K_\mathfrak{p}}^\times}\left(\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)}}\right)-\gamma'\left(\frac{y}{\pi^{2e-k+v_\pi(m)+2}}\right)\right)dy$$

By proposition 6.81, we have the following expression for the above equation, which we have rewritten using $\mathfrak{f}(m)$.

$$q^{(v_\pi(m)+e)/2}(\chi_{K[\sqrt{\pi}]}(u_m))^{\bar{k}}\cdot$$
$$\left(\begin{cases}0 & 2\mathfrak{f}(m)>k\text{ is odd}\\ q^{-(e-k+v_\pi(m))/2} & 2\mathfrak{f}(m)>k\text{ is even}\\ q^{e/2} & 2\mathfrak{f}(m)\le k\end{cases}-\begin{cases}0 & 2\mathfrak{f}(m)+2>k\text{ is odd}\\ q^{-(e+2-k+v_\pi(m))/2} & 2\mathfrak{f}(m)+2>k\text{ is even}\\ q^{e/2} & 2\mathfrak{f}(m)+2\le k\end{cases}\right) \quad (7.29)$$

(Here, the conditions are referring to the parity of k.)

The only way an odd value of k can give a nonzero result here is when $k = 2\mathfrak{f}(m)+1$,[2] in which case we get

$$q^{\mathfrak{f}(m)}\chi_{K[\sqrt{\pi}]}(u_m) = q^{\mathfrak{f}(m)}\left(\frac{m}{\mathfrak{p}}\right)$$

Now take k to be even for the remainder of this case so that we have

$$q^{k/2}\left(\begin{cases}1 & 2\mathfrak{f}(m) > k \\ q^{(2e-k+v_\pi(m))/2} & 2\mathfrak{f}(m) \le k\end{cases} - q^{-1}\begin{cases}1 & 2\mathfrak{f}(m)+2 > k \\ q^{(2e+2-k+v_\pi(m))/2} & 2\mathfrak{f}(m)+2 \le k\end{cases}\right)$$

By light casework, this is the same as

$$q^{k/2}\begin{cases}1 - q^{-1} & k \le 2\mathfrak{f}(m) \\ 0 & k > 2\mathfrak{f}(m)\end{cases}$$

This matches the desired formula as long as $0 = -q^{-1}\left(1 - \left|\left(\frac{m}{\mathfrak{p}}\right)\right|\right)$. However, this is true by the definition of case 2b. $\square$

7.4 Putting it all together

The above calculations give us everything we need to compute the local Whittaker function. First, $I_k^*(\mu, m)$ is given in all cases by propositions 7.8, 7.13, and 7.21. Proposition 7.4 then tells us how to use $I_k^*(\mu, m)$ to compute the value of $I_k(\mu, m, \kappa)$. Finally, we obtain $I_W(\mu, m, \Delta s)$ as a power series whose coefficients are the $I_k(\mu, m, \kappa)$. We collect these results together in one large proposition so that they are all in one place.

Proposition 7.30.

(a)

$$I_W(\mu, m, \Delta s) = \sum_{k=0}^{\infty} I_k(\mu, m, \kappa) X^k$$

[2]If k is larger, the piecewise functions evaluate to $q^{e/2} - q^{e/2} = 0$, whereas if k is smaller they evaluate to $0 - 0 = 0$.

where $X = q^{-\Delta s}$*. Furthermore,*

(bi) For $k > 0$

$$I_k(\mu, m, \kappa) = |\kappa|_{\mathfrak{p}}^{-1} I^*_{k-v_\pi(\kappa)}(\mu, m/\kappa)$$

(bii) For $\kappa \in O_{K_\mathfrak{p}}$

$$I_0(\mu, m, \kappa) = \begin{cases} \mathbb{1}_{O_{K_\mathfrak{p}}}(\kappa\mu^2 - m) & 2\kappa\mu \in O_{K_\mathfrak{p}} \\ |2\kappa\mu|_{\mathfrak{p}}^{-1} \ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\kappa\mu^2 - m}{2\kappa\mu}\right) & 2\kappa\mu \notin O_{K_\mathfrak{p}} \end{cases}$$

(biii) For $\kappa \notin O_{K_\mathfrak{p}}$

$$I_0(\mu, m, \kappa) = \begin{cases} |\kappa|_{\mathfrak{p}}^{-1} \mathbb{1}_{O_{K_\mathfrak{p}}}(\mu^2 - m/\kappa) & 2\mu \in O_{K_\mathfrak{p}} \\ |2\kappa\mu|_{\mathfrak{p}}^{-1} \ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\kappa\mu^2 - m}{2\kappa\mu}\right) & 2\mu \notin O_{K_\mathfrak{p}} \end{cases} + |\kappa|_{\mathfrak{p}}^{-1} \sum_{k=1}^{-v_\pi(\kappa)} I^*_k(\mu, m/\kappa)$$

Finally,

(ci) If $k \leq 0$ *then*

$$I^*_k(\mu, m) = \begin{cases} q^k \mathbb{1}_{\pi^k O_{K_\mathfrak{p}}}(\mu^2 - m) - q^{k-1} \mathbb{1}_{\pi^{k-1} O_{K_\mathfrak{p}}}(\mu^2 - m) & k \leq v_\pi(2\mu) \\ 0 & k > v_\pi(2\mu) \end{cases}$$

(cii) Let $k > 0$*. If either* $\mu \notin \frac{1}{2} O_{K_\mathfrak{p}}$ *or* $\mu^2 - m \notin O_{K_\mathfrak{p}}$ *then*

$$I^*_k(\mu, m) = 0$$

(ciii) If $k > 0$ *is even,* $\mu \in \frac{1}{2} O_{K_\mathfrak{p}}$*, and* $\mu^2 - m \in O_{K_\mathfrak{p}}$*, then*

$$I^*_k(\mu, m) = q^{k/2} \begin{cases} 1 - q^{-1} & k \leq 2\mathfrak{f}(m) \\ -q^{-1}\left(1 - \left|\left(\frac{m}{\mathfrak{p}}\right)\right|\right) & k = 2\mathfrak{f}(m) + 2 \\ 0 & k \geq 2\mathfrak{f}(m) + 4 \end{cases}$$

(civ) If $k > 0$ *is odd,* $\mu \in \frac{1}{2} O_{K_\mathfrak{p}}$*, and* $\mu^2 - m \in O_{K_\mathfrak{p}}$*, then*

$$I^*_k(\mu, m) = q^{k/2} \begin{cases} q^{\mathfrak{f}(m)} \left(\frac{m}{\mathfrak{p}}\right) & k = 2\mathfrak{f}(m) + 1 \\ 0 & k \neq 2\mathfrak{f}(m) + 1 \end{cases}$$

(cv) If $k > 0$*,* $\mu \in \frac{1}{2} O_{K_\mathfrak{p}}$*, and* $\mu^2 - m \in O_{K_\mathfrak{p}}$ *but* $m = 0$*, the above formulas aren't well defined. However, we still have*

$$I^*_k(\mu, 0) = q^{k/2} \begin{cases} 1 - q^{-1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Although this marks the calculation of $I_W(\mu, m, \Delta s)$ complete, we are still not quite able to conclude the value of $W_{m,\mathfrak{p}}(s, \Phi_{\mathfrak{p},\mu})$. This is because Δs must be a non-negative integer in the defining relation for I_W. Plugging in $dim(V) = 1$ into lemma 5.40, we have

$$W_{m,\mathfrak{p}}(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu}) = \gamma(V) \frac{|det_{\mathscr{B}}(V)|_{\mathfrak{p}}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}} q^{r(1/2-\Delta s)} I_W(\mu, m, \Delta s)$$

We will get around this by using the power series $\sum_{k=0}^{\infty} I_k(\mu, m, \kappa) X^k$ to define the function $I_W(\mu, m, \Delta s)$ for all $s \in \mathbb{C}$, which will the give the value of $W_{m,\mathfrak{p}}(s, \Phi_{\mathfrak{p},\mu})$ for all s.

Proposition 7.31. *The power series $\sum_{k=0}^{\infty} I_k(\mu, m, \kappa) X^k$ converges absolutely for all $s \in \mathbb{C}$ with $Re(s) > 1$ and admits a meromorphic extension to all of $\mathbb{C}$. For $Re(s_0 + \Delta s) > 1$ we have the equality*

$$W_{m,\mathfrak{p}}(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu}) = \gamma(V) \frac{|det_{\mathscr{B}}(V)|_{\mathfrak{p}}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}} q^{r(1/2-\Delta s)} I_W(\mu, m, \Delta s)$$

which gives a meromorphic extension of $W_{m,\mathfrak{p}}$ to all s.

Proof. The analytic properties of our power series follow directly from the calculations summarized in proposition 7.30. If $m \neq 0$, then parts (cii),(ciii),(civ) tell us that $I_k^*(\mu, m)$ vanishes for sufficiently large k. Part (bi) then lets us conclude that the power series is in fact polynomial.

On the other hand, if $m = 0$, parts (cii) and (cv) let us conclude that for $k > 0$, $I_k^*(\mu, m)$ is either identically 0 or given by the formula

$$I_k^*(\mu, 0) = q^{k/2} \begin{cases} 1 - q^{-1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

Hence, the tail of our power series is a geometric series of common ratio $qX^2 = q^{1-2\Delta s}$. This clearly has the desired analytic properties (and even a slightly larger region of convergence).

In order to conclude the second claim of this proposition, first recall lemma 5.7, which tells us that (fixing all parameters besides s), there is a meromorphic function f on the open disc of radius $1/q$ such that for $Re(s) > 1$

$$W_{m,\mathfrak{p}}(s, \Phi_{\mathfrak{p},\mu}) = f(q^{-s})$$

It is clear from this that for $Re(s_0 + \Delta s) > 1$ we may write

$$W_{m,\mathfrak{p}}(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu}) = f(q^{-s_0}q^{-\Delta s}) = f_2(q^{-\Delta s})$$

for some meromorphic function f_2 defined on the open disc of radius q^{s_0}/q.

On the other hand, through part 1 of this proposition, we know that there is a meromorphic function g on the disc of radius $1/q$ such that

$$\gamma(V)\frac{|det_{\mathscr{B}}(V)|_{\mathfrak{p}}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}}q^{r(1/2-\Delta s)}I_W(\mu, m, \Delta s) = g(q^{-\Delta s})$$

Finally, by definition of $I_W(\mu, m, \Delta s)$, we know for $\Delta s \in \mathbb{N}$ sufficiently large[3] that

$$W_{m,\mathfrak{p}}(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu}) = \gamma(V)\frac{|det_{\mathscr{B}}(V)|_{\mathfrak{p}}^{1/2}}{Vol_{\mathscr{B}}(L)^{1/2}}q^{r(1/2-\Delta s)}I_W(\mu, m, \Delta s)$$

does hold. Hence, we have

$$f_2(q^{-\Delta s}) = g(q^{-\Delta s})$$

for a sequence of $q^{-\Delta s}$ that approaches 0. We may then apply analytic continuation to conclude that $f_2 = g$. $\square$

For our specific case of interest, we will want to compute the Whittaker function for $\kappa = -1$. The following proposition condenses proposition 7.30 down into a single formula for $I_W(\mu, m, \Delta s)$ in the case κ is a unit. This proposition is based on equation (2.16) of [KY10] and is also closely related to the work on page 14 of [Su16].

Proposition 7.32. *Assume $\kappa \in O_{K_\mathfrak{p}}^\times$. Let $m' = m/\kappa$ and also assume $\mathfrak{f}(m') \geq 0$. Then $I_W(\mu, m, \Delta s)$ only depends on m and κ through their ratio m', and it is given by two cases.*

[3] Just large enough so we don't have to worry when $W_{m,\mathfrak{p}}(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu})$ is well defined.

If $\mu \notin \frac{1}{2}O_{K_\mathfrak{p}}$, then $I(\mu, m, \Delta s)$ is independent of Δs and is given by

$$I_W(\mu, m, \Delta s) = |2\mu|_\mathfrak{p}^{-1}\ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\mu^2 - m'}{2\mu}\right)$$

If $\mu \in \frac{1}{2}O_{K_\mathfrak{p}}$, then

$$I_W(\mu, m, \Delta s) = \mathbb{1}_{O_{K_\mathfrak{p}}}(\mu^2 - m')\left(\frac{1 - X^2}{1 - \left(\frac{m'}{\mathfrak{p}}\right)X}\right)\rho_\mathfrak{p}$$

where

$$\rho_\mathfrak{p} = \frac{1 - \left(\frac{m'}{\mathfrak{p}}\right)X + \left(\frac{m'}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1} - q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2}}{1 - qX^2}$$

Proof. This makes heavy use of the various parts of proposition 7.30. Parts (bi) and (bii) tell us that

$$I_W(\mu, m, \Delta s) = \begin{cases} \mathbb{1}_{O_{K_\mathfrak{p}}}(\mu^2 - m') & 2\mu \in O_{K_\mathfrak{p}} \\ |2\mu|_\mathfrak{p}^{-1}\ \mathbb{1}_{O_{K_\mathfrak{p}}}\left(\frac{\mu^2 - m'}{2\mu}\right) & 2\mu \notin O_{K_\mathfrak{p}} \end{cases} + \sum_{k=1}^{\infty} I_k^*(\mu, m')X^k$$

In the case that $\mu \notin \frac{1}{2}O_{K_\mathfrak{p}}$, part (cii) lets us conclude all the non-constant terms vanish, from which the desired result follows.

We now tackle the case $\mu \in \frac{1}{2}O_{K_\mathfrak{p}}$. Part (cii) makes it clear I_W vanishes as long as $\mu^2 - m' \notin O_{K_\mathfrak{p}}$, so from here out assume $\mu^2 - m' \in O_{K_\mathfrak{p}}$.

By (ciii), the even degree terms of $I(m, \mu, r)$ sum to

$$1 + \sum_{k=1}^{\mathfrak{f}(m')} q^k(1 - q^{-1})X^{2k} - \left(1 - \left|\left(\frac{m'}{\mathfrak{p}}\right)\right|\right) q^{\mathfrak{f}(m')+1}q^{-1}X^{2\mathfrak{f}(m')+2}$$

On the other hand, by (civ) the sum of the odd degree terms contains at most one element and is given by

$$\left(\frac{m}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1}$$

Putting it all together we get

$$I(m, \mu, r) =$$
$$1 + \sum_{k=1}^{\mathfrak{f}(m')} q^k(1 - q^{-1})X^{2k} - q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+2} + \left|\left(\frac{m'}{\mathfrak{p}}\right)\right| q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+2} + \left(\frac{m'}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1} \tag{7.33}$$

Focusing on all but the last two terms, we may regroup them to get

$$1+\sum_{k=1}^{\mathfrak{f}(m')} q^k(1-q^{-1})X^{2k} - q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+2} =$$

$$\left(1+\sum_{k=1}^{\mathfrak{f}(m')} q^k X^{2k}\right) - \left(\sum_{k=1}^{\mathfrak{f}(m')} q^{k-1}X^{2k} + q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+2}\right) = \sum_{k=0}^{\mathfrak{f}(m')} q^k X^{2k} - \sum_{k=0}^{\mathfrak{f}(m')} q^k X^{2k+2}$$

$$= (1-X^2)\sum_{k=0}^{\mathfrak{f}(m')} q^k X^{2k} = (1-X^2)\frac{1-q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2}}{1-qX^2} \quad (7.34)$$

Plugging this in, we have

$$I(m,\mu,r) = (1-X^2)\frac{1-q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2}}{1-qX^2} + \left|\left(\frac{m'}{\mathfrak{p}}\right)\right| q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+2} + \left(\frac{m'}{\mathfrak{p}}\right) q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1}$$

It is easy to check this expression equals the desired one if $\left(\frac{m'}{\mathfrak{p}}\right) = 0$, so now we turn our attention to when $\left(\frac{m'}{\mathfrak{p}}\right) = \pm 1$. We get

$$(1-X^2)\frac{1-q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2}}{1-qX^2} + q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+2} + \left(\frac{m'}{\mathfrak{p}}\right) q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1}$$

Factoring out a copy of $1+\left(\frac{m'}{\mathfrak{p}}\right)X$, we get

$$\left(1+\left(\frac{m'}{\mathfrak{p}}\right)X\right)\left(\left(1-\left(\frac{m'}{\mathfrak{p}}\right)X\right)\frac{1-q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2}}{1-qX^2} + \left(\frac{m'}{\mathfrak{p}}\right) q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1}\right)$$

The second (and larger) of these two factors simplifies if we put everything over a common denominator of $1-qx^2$. It becomes

$$\frac{1-\left(\frac{m'}{\mathfrak{p}}\right)X - q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2} + \left(\frac{m'}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+3}}{1-qX^2}$$
$$+ \frac{\left(\frac{m'}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1} - \left(\frac{m'}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+3}}{1-qX^2} \quad (7.35)$$

These terms then recombine to form

$$\frac{1-\left(\frac{m'}{\mathfrak{p}}\right)X + \left(\frac{m'}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1} - q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2}}{1-qX^2}$$

Plugging this computation in, we get

$$\left(1+\left(\frac{m'}{\mathfrak{p}}\right)X\right)\left(\frac{1-\left(\frac{m'}{\mathfrak{p}}\right)X + \left(\frac{m'}{\mathfrak{p}}\right)q^{\mathfrak{f}(m')}X^{2\mathfrak{f}(m')+1} - q^{\mathfrak{f}(m')+1}X^{2\mathfrak{f}(m')+2}}{1-qX^2}\right)$$

which is equal to the desired quantity. □

7.5 The cases of interest

We will be constructing $E^{l,\mu}$ in the case of half-integral l, $\kappa = (-1)^{\kappa-1/2}$ (and hence $\chi(x) = \langle x, 2\kappa\rangle_{\mathbb{A}}$). Our main restriction will come from setting $V = K$, $Q(x) = \kappa x^2$ (hence $(x,y)_Q = 2\kappa xy$), and $L = O_K$, in which case we must take $\mu \in O_K/2$. Although trivial, we remark that we will take the basis $\{1\}$ of K (and of the completions $K_{\mathfrak{p}}$). This ensures that the measure dy in $I_{W,\mathfrak{p}}(\mu, m, \Delta s)$ is simply the one such that the set of integral elements has measure 1 (and so lines up with what we have been using this entire time). We also have $det_{\mathscr{B}}(V) = 2\kappa$ and $Vol_{\mathscr{B}}(L) = 1$. All of the calculations in this section except for the calculation of level are directly based on those performed in the proofs of [KY10] theorems 6.1, 6.3, 6.5.

7.5.1 General l

By proposition 7.31 together with proposition 7.32, we get for all finite $\mathfrak{p}$

$$W_{m,\mathfrak{p}}\left(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu}\right) = \gamma_{\mathfrak{p}}(V)|2|_{\mathfrak{p}}^{1/2} q_{\mathfrak{p}}^{r_{\mathfrak{p}}(1/2-\Delta s)}\ \mathbb{1}_{O_{K_{\mathfrak{p}}}}(\mu^2 - \kappa m)\left(\frac{1 - X^2}{1 - \left(\frac{\kappa m}{\mathfrak{p}}\right)X}\right)\rho_{\mathfrak{p}}$$

Proposition 7.36. *At almost all finite places we have $\rho_{\mathfrak{p}} = 1$. Let $\rho = \prod_{\mathfrak{p}<\infty} \rho_{\mathfrak{p}}$. Furthermore, let*

$$\kappa_2 = \begin{cases} 4 & \kappa = 1 \\ 1 & \kappa = -1 \end{cases}$$

Then,

$$\prod_{\mathfrak{p}<\infty} W_{m,\mathfrak{p}}\left(s_0 + \Delta s, \Phi_{\mathfrak{p},\mu}\right) = (-i)^{n\kappa_2} 2^{-n/2} |Disc(K)|^{1/2-\Delta s} \mathbb{1}_{O_K}(\mu^2 - \kappa m) \frac{L\left(\left(\frac{\kappa m}{\cdot}\right), \Delta s\right)}{\zeta_K(2\Delta s)}\rho$$

Proof. How one arrives at the indicator function, L function, and ζ should be clear, so we will now handle the remaining terms one at a time.

$\rho_{\mathfrak{p}} = 1$ whenever $\mathfrak{f}(m) = 0$, which happens at almost all places by remark 7.19. This proves the claim about $\rho_{\mathfrak{p}}$.

For the $|2|_{\mathfrak{p}}^{1/2}$ terms, the fact that

$$\prod_{\mathfrak{p}\leq\infty} |2|_{\mathfrak{p}}^{1/2} = 1$$

lets us infer that

$$\prod_{\mathfrak{p}<\infty} |2|_{\mathfrak{p}}^{1/2} = \prod_{\mathfrak{p}|\infty} |2|_{\mathfrak{p}}^{-1/2} = 2^{-n/2}$$

For the q^r terms, we have

$$\prod_{\mathfrak{p}<\infty} q_{\mathfrak{p}}^{r_{\mathfrak{p}}(1/2-\Delta s)} = \prod_{\mathfrak{p}<\infty} |\pi_{\mathfrak{p}}^{r_{\mathfrak{p}}(1/2-\Delta s)}|_{\mathfrak{p}}^{-1} = \left(\prod_{\mathfrak{p}<\infty} |\partial_{\mathfrak{p}}|_{\mathfrak{p}}^{-1}\right)^{1/2-\Delta s}$$

where $\partial_{\mathfrak{p}}$ is the local different and we used the definition of r being the exponent of the different ideal. However, this last product is the product of the local discriminants, which is the absolute value of the global discriminant.

Finally, for the local factors $\gamma_{\mathfrak{p}}(V)$, recall from proposition 5.33 that

$$\gamma_{\mathfrak{p}}(V) = \gamma_{w,\mathfrak{p}}\left(-\frac{1}{2}det(V)\right)\gamma_{w,\mathfrak{p}}\left(\frac{1}{2}\right)^{2-dim(V)} h_{\mathfrak{p}}(V) = \gamma_{w,\mathfrak{p}}\left(-\kappa\right)\gamma_{w,\mathfrak{p}}\left(\frac{1}{2}\right)$$

where the Hasse invariant vanishes because $dim(V) = 1$. The product formula for the Weil constant then implies

$$\prod_{\mathfrak{p}\leq\infty} \gamma_{\mathfrak{p}}(V) = 1$$

so that

$$\prod_{\mathfrak{p}<\infty} \gamma_{\mathfrak{p}}(V) = \prod_{\mathfrak{p}|\infty} \gamma_{\mathfrak{p}}(V)^{-1} = \prod_{\mathfrak{p}|\infty} \gamma_{w,\mathfrak{p}}\left(-\kappa\right)^{-1}\gamma_{w,\mathfrak{p}}\left(\frac{1}{2}\right)^{-1}$$

Quick casework and fact 3.4 then tell us that this product is just $(-i)^{n\kappa_2}$. $\square$

We finally plug back into equation (5.10) to get (for $s_0 + \Delta s = s > 1$)

$$\begin{aligned} E_m(\vec{\tau}, s, \Phi^{3/2,\mu}) = {} & \delta_m v^{-l/2}\Phi^{l,\mu}(g'_{\vec{\tau}}, s) + \\ & (-i)^{n\kappa_2} 2^{-n/2} |Disc(K)|^{-\Delta s} \mathbb{1}_{O_K}(\mu^2 - \kappa m) \frac{L\left(\left(\frac{\kappa m}{\cdot}\right), \Delta s\right)}{\zeta_K(2\Delta s)} \rho \prod_{\mathfrak{p}|\infty} W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^l) \end{aligned} \quad (7.37)$$

Since we calculated the Archimedean local Whittaker functions in lemma 5.29, this marks a complicated but reasonable to evaluate formula for the Fourier series of $E^{l,\mu}$. Although $E^{l,\mu}$ is only defined for $s > 1$, we may still define its Fourier series at other values of s using equation (7.37) and gives an analytic continuation for

$E_m(\vec{\tau}, s, \Phi^{3/2,\mu})$. We will now try to simplify the formula in the case $s = l - 1$ (and so since $s_0 = dim(V)/2 - 1 = -1/2$ then $\Delta s = l - 1/2$).

We need to be careful since some of the terms can sometimes be 0 or ∞. The following lemma recalls some well known details and tells us what to expect.

Lemma 7.38. *For $K \supsetneq \mathbb{Q}$ totally real, $\zeta_K(0)$ is a zero of degree $n - 1 > 0$, $\zeta_K(1)$ is a simple pole, and $\zeta_K(z)$ is finite nonzero for integers $z \geq 2$.*

For $L = K[\sqrt{\alpha}]$ a nontrivial extension of $K \supsetneq \mathbb{Q}$, let n_+ denote the number of Archimedean places of K where α is positive. Then $\zeta_L(0)$ has a zero of degree $n - 1 + n_+ > 0$. $\zeta_L(1)$ is a simple pole, and $\zeta_L(z)$ is finite nonzero for integers $z \geq 2$.

In the case κm is a perfect square, then $L\left(\left(\frac{\kappa m}{\cdot}\right), z\right) = \zeta_K(z)$ and is described as above. Otherwise, $L\left(\left(\frac{\kappa m}{\cdot}\right), 0\right)$ is a zero of order n_+. $L\left(\left(\frac{\kappa m}{\cdot}\right), z\right)$ is finite and nonzero for all integers $z \geq 1$.

$W_{m,\mathfrak{p}}(\tau, l - 1, \Phi^l_{\mathfrak{p}})$ is zero for $m_{\mathfrak{p}} \leq 0$ and is finite/nonzero for $m_{\mathfrak{p}} > 0$.

Proof. The facts about ζ_K are well known. The properties at $z = 0$ come from the fact that the degree of the zero is the rank of the unit group.

The facts about the L function are just from the formula

$$L\left(\left(\frac{\kappa m}{\cdot}\right), z\right) = \frac{\zeta_{K[\sqrt{\kappa m}]}(z)}{\zeta_K(z)}$$

The properties of $W_{m,\mathfrak{p}}$ are from proposition 5.29. $\square$

As one can tell from the above proposition, the cases $l = 1/2, 3/2$ require the most care since one will potentially have to cancel zeros and poles. In particular, for $s = l - 1$, one should define the quantity $E_m(\vec{\tau}, s, \Phi^{3/2,\mu})$ for $l = 1/2, 3/2$ as the limit as l approaches this value.

Definition 7.39. *Define*

$$E_m(\vec{\tau}, 1/2, \Phi^{3/2,\mu}) := \lim_{s \to 1/2} E_m(\tau, s, \Phi^{3/2,\mu})$$

where $E_m(\vec{\tau}, s, \Phi^{3/2,\mu})$ *is defined by equation* (7.37)*. One may alternately write this as*

$$E_m(\vec{\tau}, 1/2, \Phi^{3/2,\mu}) := \lim_{\Delta s \to 1} E_m(\vec{\tau}, -1/2 + \Delta s, \Phi^{3/2,\mu})$$

However, for the most part we will only use limit notation when necessary since it should generally be clear how things would need to be rewritten. Additionally, although it wouldn't be much more work to discuss the general case that generalizes Cohen's modular forms of arbitrary half-integral weight, we will stick to $l = 1/2, 3/2$.

Lemma 7.40.

$$v^{-l/2}\Phi^{l,\mu}(g'_{\vec{\tau}}, l-1) = \mathbb{1}_{O_K}(\mu)$$

Proof. Start by recalling some notation. Write $\vec{\tau} = (\tau_1, \ldots, \tau_n)$ and let $\tau_j = u_j + iv_j$. We let v denote $\prod_j v_j$ and $g'_{\vec{\tau}} = [g_{\vec{\tau}}, 1]_g$, where $g_{\vec{\tau}}$ is the identity matrix at all finite places and $n(u_j)m(\sqrt{v_j})$ at the jth Archimedean place. Plugging this notation into equation (4.11), we arrive at

$$v^{-l/2}\prod_{\mathfrak{p}<\infty}\Phi_{\mathfrak{p},\mu}([I,1]_L, l-1)\prod_{j=1}^{n}\Phi^{3/2}([n(u_j)m(\sqrt{v_j}), 1]_L, l-1)$$

(Note that since $n(u_j)m(\sqrt{v_j})$ is upper triangular, there is no difference between using Leray or normalized coordinates here.) Evaluating the finite places is easy. Since our sections are standard, we have $\Phi_{\mathfrak{p},\mu}([I,1]_L, l-1) = \Phi_{\mathfrak{p},\mu}([I,1]_L, s_0)$. Recalling definition 4.13, the λ mapping then gives

$$\Phi_{\mathfrak{p},\mu}(g', s_0) = \left(g' \mapsto (\omega_V(g')\phi_\mu(t))|_{t=0}\right)$$

Since ϕ_μ is the characteristic function of $\mu + O_{K_\mathfrak{p}}$, we immediately get $\Phi_{\mathfrak{p},\mu}([I,1]_L, s_0) = \mathbb{1}_{\mu+O_{K_\mathfrak{p}}}(0)$. It follows that the term we are evaluating is just

$$\mathbb{1}_{O_K}(\mu)v^{-l/2}\prod_{j=1}^{n}\Phi^{l}([n(u_j)m(\sqrt{v_j}), 1]_L, l-1)$$

For the jth Archimedean place $\mathfrak{p}$, we use the definition of section (equation (4.3)) to get

$$\Phi^l([n(u_j)m(\sqrt{v_j}), 1]_L, l-1) = \langle\sqrt{v_j}, 2\kappa\rangle_\mathfrak{p}|\sqrt{v_j}|_\mathfrak{p}^l\Phi^l([I,1]_L, l-1) = v_j^{l/2}\Phi^{3/2}([I,1]_L, l-1)$$

However, we have by definition $\Phi^l([I,1]_L, l-1) = 1$. Hence, we get

$$\mathbb{1}_{O_K}(\mu) v^{-l/2} \prod_{j=1}^{n} v_j^{l/2} = \mathbb{1}_{O_K}(\mu)$$

$\square$

7.5.2 The Case $l = 1/2$

Proposition 7.41. *In the case $l = 1/2, \kappa = 1, s = -1/2$, we have*

$$E^{1/2,\mu}(4\vec{\tau}, -1/2) = \sum_{m \in 2\mu + 2O_K} e^{2\pi i \vec{m^2} \cdot \vec{\tau}}$$

Therefore,

$$\sum_{\mu \in \frac{1}{2}O_K/O_K} E^{1/2,\mu}(4\vec{\tau}, -1/2) = \theta_K(\vec{\tau})$$

Proof. In the case $l = 1/2, \kappa = 1$, we have $\Delta s = 0$. Because of this, the $1/\zeta_K(2\Delta s)$ term introduces a pole of order $n-1$ which could cause havoc if it doesn't get canceled out. In the case m is not a perfect square, the above lemma tells us that the L function contributes a zero of order n_+. On the other hand, at least $n_- = n - n_+$ of the Whittaker functions will also be zero (although of unspecified orders). Hence, we will have a zero of order at least n and so $E_m = 0$ in such cases.

We now consider the case where m is a perfect square. In the case $m = 0$, every local Whittaker function will have a zero, which will again overpower the $1/\zeta_K(2\Delta s)$ term. We get

$$E_0(\vec{\tau}, -1/2, \Phi^{1/2,\mu}) = v^{-1/4}\Phi^{1/2,\mu}(g'_{\vec{\tau}}, -1/2) = \mathbb{1}_{O_K}(\mu)$$

If $m \neq 0$, write $m = k^2$. In this case, we have

$$\lim_{\Delta s \to 0} \frac{L\left(\left(\frac{\kappa m}{\cdot}\right), \Delta s\right)}{\zeta_K(2\Delta s)} = \lim_{\Delta s \to 0} \frac{\zeta_K(\Delta s)}{\zeta_K(2\Delta s)} = \frac{1}{2^{n-1}}$$

where the last step is because ζ_K has a zero of order $n-1$. We also have from lemma 5.29

$$\prod_{\mathfrak{p}|\infty} W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^l) = \prod_{\mathfrak{p}|\infty} \frac{(2\pi)^l(-i)^{l-1/2}}{\Gamma(l)} m_{\mathfrak{p}}^{l-1} q^{m_{\mathfrak{p}}} = \prod_{\mathfrak{p}|\infty} \sqrt{2}|k_{\mathfrak{p}}^{-1}| q^{k_{\mathfrak{p}}^2} = 2^{n/2}|N(k)|^{-1} e^{2\pi i \vec{k^2} \cdot \vec{\tau}}$$

Finally, since $\Delta s = 0$, we have $X = q^{-\Delta s} = 1$ and so

$$\rho_{\mathfrak{p}} = \frac{1 - \left(\frac{m}{\mathfrak{p}}\right)X + \left(\frac{m}{\mathfrak{p}}\right)q^{\mathfrak{f}(m)}X^{2\mathfrak{f}(m)+1} - q^{\mathfrak{f}(m)+1}X^{2\mathfrak{f}(m)+2}}{1 - qX^2} = \frac{1 - \left(\frac{m}{\mathfrak{p}}\right) + \left(\frac{m}{\mathfrak{p}}\right)q^{\mathfrak{f}(m)} - q^{\mathfrak{f}(m)+1}}{1 - q}$$

Since m is a square, the "quadratic symbols" are 1 and by definition 7.17 we have $\mathfrak{f}(m) = v_\pi(m)/2 + e$. We get $\rho_{\mathfrak{p}} = q^{v_\pi(m)/2 + e_{\mathfrak{p}}}$. So,

$$\rho = \prod_{\mathfrak{p}<\infty} \rho_{\mathfrak{p}} = 2^n \sqrt{|N(m)|} = 2^n |N(k)|$$

From this, we can plug into equation (7.37) (and use $\kappa_2 = 1$) to get

$$E_{k^2}(\vec{\tau}, -1/2, \Phi^{1/2,\mu}) = 2^{-n/2} \mathbb{1}_{O_K}(\mu^2 - k^2) \frac{1}{2^{n-1}} 2^n |N(k)| 2^{n/2} |N(k)|^{-1} e^{2\pi i \vec{k^2} \cdot \vec{\tau}}$$

which simplifies to

$$E_{k^2}(\vec{\tau}, -1/2, \Phi^{1/2,\mu}) = 2\mathbb{1}_{O_K}(\mu^2 - k^2) e^{2\pi i \vec{k^2} \cdot \vec{\tau}}$$

Set $\mu' = 2\mu \in O_K$ and $k' = 2k$ so we get

$$E_{k^2}(4\vec{\tau}, -1/2, \Phi^{1/2,\mu}) = 2\mathbb{1}_{4O_K}({\mu'}^2 - {k'}^2) e^{2\pi i \vec{k'^2} \cdot \vec{\tau}}$$

The indicator function is equivalent to the condition $k' \in \mu' + 2O_K$ (see the next lemma), so we get

$$E_{k^2}(4\vec{\tau}, -1/2, \Phi^{1/2,\mu}) = 2\mathbb{1}_{2O_K}(\mu' - k') e^{2\pi i \vec{k'^2} \cdot \vec{\tau}}$$

which proves the result. $\square$

Lemma 7.42. *For integral x, y, $x \in y + 2O_K$ iff $x^2 \in y^2 + 4O_K$.*

Proof. The forward implication is trivial since $x = y + 2d \implies x^2 = y^2 + 4yd + 4d^2$. For the reverse, if $x^2 - y^2 = 4d$, then $(x+y)(x-y) = 4d$. Hence, at least one of the two factors is divisible by 2. In either case, we get the desired result. $\square$

7.5.3 The Case $l = 3/2$

We now move onto the case $l = 3/2, \kappa = -1$. In this case $\Delta s = 1$. This will be similar to the previous case, but it require a bit more work so we break up the steps into a series of propositions. Whenever κm is a perfect square, we have $L\left(\left(\frac{\kappa m}{\cdot}\right), \Delta s\right) = \zeta_K(1)$, which introduces a pole to the formula. However, this pole will always cancel with a zero in the Archimedean local Whittaker functions.

Proposition 7.43. *$E_m(\vec{\tau}, 1/2, \Phi^{3/2,\mu})$ vanishes unless either $m = 0$ or $m \gg 0$. In the case $m = 0$,*

$$E_0(\vec{\tau}, 1/2, \Phi^{3/2,\mu}) = \mathbb{1}_{O_K}(\mu)$$

In the case $m \gg 0$,

$$E_m(\vec{\tau}, 1/2, \Phi^{3/2,\mu}) = \mathbb{1}_{O_K}(\mu^2+m)\cdot(-1)^n 2^{2n}\pi^n\sqrt{N(m)}|Disc(K)|^{-1}\frac{L\left(\left(\frac{-m}{\cdot}\right), 1\right)}{\zeta_K(2)}\rho e^{2\pi i\vec{m}\cdot\vec{\tau}}$$

Proof. To show that E_m vanishes, it suffices to show that

$$\lim_{\Delta s\to 1} L\left(\left(\frac{-m}{\cdot}\right), \Delta s\right)\prod_{\mathfrak{p}|\infty} W_{m,\mathfrak{p}}(\tau, s, \Phi^l_{\mathfrak{p}}) = 0$$

We will do this by first considering the possible impact of the L function. In comparison to the Whittaker functions, the L function has a limited ability to contribute to this limit - whenever $-m$ is not a perfect square in K, it will take on a finite, nonzero value. This follows by writing

$$\lim_{\Delta s\to 1} L\left(\left(\frac{-m}{\cdot}\right), \Delta s\right) = \lim_{\Delta s\to 1}\frac{\zeta_{K[\sqrt{-m}]}(\Delta s)}{\zeta_K(\Delta s)}$$

and noting that each zeta function has a simple pole at 1. On the other hand, it $-m$ is a perfect square in K, then we will have $L\left(\left(\frac{-m}{\cdot}\right), \Delta s\right) = \zeta_K(\Delta s)$, and so there will be a simple pole at $\Delta s = 1$. Note that in order for $-m$ to be a perfect square, we must have $m \ll 0$.

The vanishing of E_m now follows by casework. First, assume that $m \neq 0$ and m is neither totally positive nor totally negative. Since m is not totally negative, the L

function will take on a finite, nonzero value. On the other hand, since m is not totally positive, there will be some Archimedean place so that $m_{\mathfrak{p}} < 0$. Then, lemma 5.29 tells us that the local Whittaker function at that place vanishes and so

$$\lim_{\Delta s \to 1} L\left(\left(\frac{-m}{\cdot}\right), \Delta s\right) \prod_{\mathfrak{p}|\infty} W_{m,\mathfrak{p}}(\tau, s, \Phi_{\mathfrak{p}}^{l}) = 0$$

On the other hand, if $m \ll 0$, then the L function has a simple pole. However, every single local Whittaker function will have a simple zero. Since we are assuming $K \neq \mathbb{Q}$, there will be more than one Archimedean place and the result is a net zero.[4]

The case $m = 0$ is handled by lemma 7.40.

Finally, we deal with the case $m \gg 0$. In this case, we are evaluating the expression

$$(-i)^n 2^{-n/2} |Disc(K)|^{-1} \mathbb{1}_{O_K}(\mu^2 + m) \frac{L\left(\left(\frac{-m}{\cdot}\right), 1\right)}{\zeta_K(2)} \rho \prod_{\mathfrak{p}|\infty} W_{m,\mathfrak{p}}(\tau, 1/2, \Phi_{\mathfrak{p}}^{3/2})$$

Since $m \gg 0$, the L function will not have a pole, so we have simply plugged in $s = 1/2$. Lemma 5.29 part (iv) tells us that for $l = 3/2, s = l - 1 = 1/2$, we have

$$W_{m,\mathfrak{p}}(\tau, 1/2, \Phi_{\mathfrak{p}}^{3/2}) = \frac{(2\pi)^{3/2}(-i)}{\Gamma(3/2)} m_{\mathfrak{p}}^{1/2} e^{2\pi i m_{\mathfrak{p}} \tau}$$

Since $\Gamma(3/2) = \sqrt{\pi}/2$, we see that

$$\prod_{\mathfrak{p}|\infty} W_{m,\mathfrak{p}}(\tau, 1/2, \Phi_{\mathfrak{p}}^{3/2}) = (-i)^n 2^{5n/2} \pi^n \sqrt{N(m)} e^{2\pi i \vec{m} \cdot \vec{\tau}}$$

Plugging this in, we get

$$\mathbb{1}_{O_K}(\mu^2 + m) \cdot (-1)^n 2^{2n} \pi^n \sqrt{N(m)} |Disc(K)|^{-1} \frac{L\left(\left(\frac{-m}{\cdot}\right), 1\right)}{\zeta_K(2)} \rho e^{2\pi i \vec{m} \cdot \vec{\tau}}$$

which is the desired expression. □

There is still some room to clean up the expression when $m \gg 0$. We also finally get to see the appearance of the Hurwitz class numbers.

[4] In the case $K = \mathbb{Q}$, one has precisely one zero and one pole, which cancel each other out. In this case, one would end up with additional terms for each negative square number. For example, in Zagier's weight 3/2 modular form, these are precisely the non-holomorphic summands.

Lemma 7.44. *Let $\rho_h(K[\sqrt{-m}])$ denote the "Hurwitification" factor defined earlier. In the case $s = 1/2$, we have*

$$\rho_h(K[\sqrt{-m}]) = 2^n \sqrt{|N(m)|} \frac{|disc(K)|}{\sqrt{|disc(K[\sqrt{-m}])|}} \rho$$

Proof. When $s = 1/2$, we have $\Delta s = 1$ and so $X = q^{-\Delta s} = q^{-1}$. In this case, we have

$$\rho = \prod_{\mathfrak{p}<\infty} \frac{1 - \left(\frac{-m}{\mathfrak{p}}\right) q_{\mathfrak{p}}^{-1} + \left(\frac{-m}{\mathfrak{p}}\right) q_{\mathfrak{p}}^{-\mathfrak{f}(-m)-1} - q_{\mathfrak{p}}^{-\mathfrak{f}(-m)-1}}{1 - q_{\mathfrak{p}}^{-1}}$$

Working from the other side, we have the definition

$$\rho_h(K[\sqrt{-m}]) = \sum_{\mathfrak{c}|I_{\mathfrak{f}}} N(\mathfrak{c}) \prod_{\mathfrak{p}|\mathfrak{c}} \left(1 - \left(\frac{-m}{\mathfrak{p}}\right) N(\mathfrak{p})^{-1}\right)$$

where $\mathfrak{c}$ runs over integral divisors of the integral ideal $I_{\mathfrak{f}}$ and $\mathfrak{p}$ runs over prime divisors of $\mathfrak{c}$. We rewrite this formula as

$$\rho_h(K[\sqrt{-m}]) = \sum_{\mathfrak{c}|I_{\mathfrak{f}}} \prod_{\mathfrak{p}|\mathfrak{c}} \left(N(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{c})} - \left(\frac{-m}{\mathfrak{p}}\right) N(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{c})-1}\right)$$

which may again be rewritten as

$$\rho_h(K[\sqrt{-m}]) = \sum_{\mathfrak{c}|I_{\mathfrak{f}}} \prod_{\mathfrak{p}|I_{\mathfrak{f}}} \begin{cases} N(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{c})} - \left(\frac{-m}{\mathfrak{p}}\right) N(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{c})-1} & \mathfrak{p}|\mathfrak{c} \\ 1 & \mathfrak{p} \nmid \mathfrak{c} \end{cases}$$

This factors as

$$\rho_h(K[\sqrt{-m}]) = \prod_{\mathfrak{p}|I_{\mathfrak{f}}} \sum_{j_{\mathfrak{p}}=0}^{\mathfrak{f}_{\mathfrak{p}}(-m)} \begin{cases} 1 & j_{\mathfrak{p}} = 0 \\ N(\mathfrak{p})^{j_{\mathfrak{p}}} - \left(\frac{-m}{\mathfrak{p}}\right) N(\mathfrak{p})^{j_{\mathfrak{p}}-1} & j_{\mathfrak{p}} > 0 \end{cases}$$

This identity is not the easiest to see, so it may help to note that going in the opposite direction is just the distributive law. The inner sums may now be evaluated as geometric series. We also swap out $N(\mathfrak{p}) = q_{\mathfrak{p}}$, yielding

$$\rho_h(K[\sqrt{-m}]) = \prod_{\mathfrak{p}|I_{\mathfrak{f}}} \left(1 + \frac{q_{\mathfrak{p}} - q_{\mathfrak{p}}^{\mathfrak{f}_{\mathfrak{p}}(-m)+1}}{1 - q_{\mathfrak{p}}} - \left(\frac{-m}{\mathfrak{p}}\right) \frac{q_{\mathfrak{p}} - q_{\mathfrak{p}}^{\mathfrak{f}_{\mathfrak{p}}(-m)}}{1 - q_{\mathfrak{p}}}\right)$$

Combining under a common denominator yields

$$\rho_h(K[\sqrt{-m}]) = \prod_{\mathfrak{p}|I_{\mathfrak{f}}} \left(\frac{1 - \left(\frac{-m}{\mathfrak{p}}\right) q_{\mathfrak{p}} + \left(\frac{-m}{\mathfrak{p}}\right) q_{\mathfrak{p}}^{\mathfrak{f}_{\mathfrak{p}}(-m)} - q_{\mathfrak{p}}^{\mathfrak{f}_{\mathfrak{p}}(-m)+1}}{1 - q_{\mathfrak{p}}}\right)$$

It should now be clear that

$$\frac{\rho_h(K[\sqrt{-m}])}{\rho} = \prod_{\mathfrak{p}<\infty} \frac{-q_{\mathfrak{p}}^{\mathfrak{f}_{\mathfrak{p}}(-m)+1}}{-q_{\mathfrak{p}}} = \prod_{\mathfrak{p}<\infty} q_{\mathfrak{p}}^{\mathfrak{f}_{\mathfrak{p}}(-m)} = N(I_{\mathfrak{f}})$$

To convert $N(I_{\mathfrak{f}})$ to the desired expression, we can use the relation $(-4m) = Disc(K[\sqrt{-m}]/K)I_{\mathfrak{f}}^2$, which nets us

$$N(I_{\mathfrak{f}}) = 2^n\sqrt{|N(m)|}\frac{1}{\sqrt{|N(Disc(K[\sqrt{m}]/K))|}}$$

The formula $Disc(K[\sqrt{m}]) = Disc(K)^2N(Disc(K[\sqrt{m}]/K))$ then gives us the desired result. □

Lemma 7.45. *The reflection formula for ζ_K tells us*

$$\frac{1}{\zeta_K(2)} = \frac{|Disc(K)|^{3/2}}{(-2\pi^2)^n}\frac{1}{\zeta_K(-1)}$$

and

$$\frac{L\left(\left(\frac{-m}{\cdot}\right),0\right)}{L\left(\left(\frac{-m}{\cdot}\right),1\right)} = \pi^{-n}\sqrt{|Disc(K[\sqrt{-m}])/Disc(K)|}$$

Proof. This is just an arrangement of the well known reflection formula, so the proof is omitted. However, we mention that the reflection formula for the L function was obtained from the zeta reflection formula via

$$L\left(\left(\frac{-m}{\cdot}\right),s\right) = \frac{\zeta_{K[\sqrt{-m}]}(s)}{\zeta_K(s)}$$

□

Corollary 7.46. *$E_m(\vec{\tau}, 1/2, \Phi^{3/2,\mu})$ vanishes unless either $m = 0$ or $m \gg 0$. In the case $m = 0$,*

$$E_0(\vec{\tau}, 1/2, \Phi^{3/2,\mu}) = \mathbb{1}_{O_K}(\mu)$$

In the case $m \gg 0$,

$$E_m(\vec{\tau}, 1/2, \Phi^{3/2,\mu}) = \mathbb{1}_{O_K}(\mu^2 + m) \cdot \frac{1}{\zeta_K(-1)}\frac{2^{n-1}}{Q_{K[\sqrt{-m}]}}\frac{H(m)}{h_K}e^{2\pi i\vec{m}\cdot\vec{\tau}}$$

for some $Q_{-m} \in \{1,2\}$ that will be determined later. This may alternately be written as

$$E_m(\vec{\tau}, 1/2, \Phi^{3/2,\mu}) = \mathbb{1}_{O_K}(\mu^2+m) \cdot \frac{L\left(\left(\frac{-m}{\cdot}\right), 0\right)}{\zeta_K(-1)} \rho_h(K[\sqrt{-m}]) e^{2\pi i \vec{m}\cdot\vec{\tau}}$$

Proof. We pick up where proposition 7.43 left off. Take $m \gg 0$ and assume $m \in -\mu^2 + O_K$. We will continue to simplify the quantity

$$(-1)^n 2^{2n} \pi^n \sqrt{N(m)} |Disc(K)|^{-1} \frac{L\left(\left(\frac{-m}{\cdot}\right), 1\right)}{\zeta_K(2)} \rho e^{2\pi i \vec{m}\cdot\vec{\tau}}$$

By lemmas 7.44 and 7.45, this quantity is equal to

$$\left(\frac{|Disc(K)|^{3/2}}{(-2\pi^2)^n}\right)\left(\frac{\sqrt{|disc(K[\sqrt{-m}])|}}{2^n\sqrt{|N(m)||disc(K)|}}\right) \cdot$$

$$(-1)^n 2^{2n} \pi^n \sqrt{N(m)} |Disc(K)|^{-1} \frac{L\left(\left(\frac{-m}{\cdot}\right), 1\right)}{\zeta_K(-1)} \rho_h(K[\sqrt{-m}]) e^{2\pi i \vec{m}\cdot\vec{\tau}} \tag{7.47}$$

A whole bunch of cancellation later and we have

$$\pi^{-n} |Disc(K)|^{-1/2} |disc(K[\sqrt{-m}])|^{1/2} \frac{L\left(\left(\frac{-m}{\cdot}\right), 1\right)}{\zeta_K(-1)} \rho_h(K[\sqrt{-m}]) e^{2\pi i \vec{m}\cdot\vec{\tau}}$$

An application of lemma 7.45 gives the alternate formulation mentioned above. To get the first formulation, we instead write the L function as a ratio of two zeta functions and apply the class number formula. Writing this out, we have

$$L\left(\left(\frac{-m}{\cdot}\right), 1\right) = \lim_{t\to 1} \frac{\zeta_{K[\sqrt{-m}]}(t)}{\zeta_K(t)} = \left(\frac{(2\pi)^n Reg(K[\sqrt{-m}]) h_{K[\sqrt{-m}]}}{\# w_{K[\sqrt{-m}]} \sqrt{|Disc(K[\sqrt{-m}])|}}\right)\left(\frac{2^n Reg(K) h_K}{2\sqrt{|Disc(K)|}}\right)^{-1}$$

Plugging this in and canceling gives

$$\frac{1}{\zeta_K(-1)} \frac{H(m)}{h_K} \frac{Reg(K[\sqrt{-m}])}{Reg(K)}$$

[Was97] Proposition 4.16 and Theorem 4.12 tell us that the ratio of regulators is of the form $2^{n-1}/Q_{K[\sqrt{-m}]}$ for some $Q_{K[\sqrt{-m}]} \in \{1,2\}$, which gives us the desired result. □

Corollary 7.48. *Take $V = K$ with $Q = -x^2$. Take $\mu \in (1/2)O_K$. Then we get*

$$E^{3/2,\mu}(\vec{\tau}, 1/2) = 1 + \frac{1}{\zeta_K(-1)} \frac{2^{n-1}}{h_K} \sum_{\substack{m \gg 0 \\ m \in -\mu^2 + O_K}} \frac{H(m)}{Q_{K[\sqrt{-m}]}} e^{2\pi i \vec{m}\cdot\vec{\tau}}$$

If we plug in $4\vec{\tau}$ and let $\mu' = 2\mu$ and $m' = 4m$, we may define a version of the series where these parameters are integral. For $\mu' \in O_K$, we have

$$E^{3/2,\mu'/2}(4\vec{\tau}, 1/2) = 1 + \frac{1}{\zeta_K(-1)} \frac{2^{n-1}}{h_K} \sum_{\substack{m' \gg 0 \\ m' \in -\mu'^2 + 4O_K}} \frac{H(m)}{Q_{K[\sqrt{-m'}]}} e^{2\pi i \vec{m'} \cdot \vec{\tau}}$$

We may now give our form an explicit level structure. We recall the statement of proposition 5.19, since we will be evaluating the formula therein. Let $\Gamma = SL_2(K) \cap (\Gamma_f \times SL_2(\mathbb{R})^n)$ for the group Γ_f guaranteed by proposition 5.12. For $\gamma_0 \in \Gamma$,

$$E^{l,\mu}(\gamma_0 \vec{\tau}, s) = \epsilon_\mu([\gamma_{0,\mathfrak{p}}, \epsilon(\gamma_0)^{-1}]) \prod_{\mathfrak{p} | \infty} \beta_{\mathfrak{p}}(\gamma_{0,\mathfrak{p}}) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_R, \vec{\tau})^{2l} E^{l,\mu}(\vec{\tau}, s) \tag{7.49}$$

where ϵ is the character giving the splitting of $SL_2(K)$ given in proposition 3.42, $\epsilon_{\phi_{\mu,\mathfrak{p}}}$ is the character given by proposition 4.16 (and its argument is in Leray coordinates for $\mathfrak{p}$ even and normalized coordinates for $\mathfrak{p}$ odd), and $\beta_{\mathfrak{p}}(g) = \gamma_w\left(x(g), 1/2\right)^{-1} \gamma_w(1/2)^{-j(g)}$ is from definition 3.22.

Proposition 7.50. 7.5.4 Finding the Level

Let

$$\Gamma_\mu = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K) \;\middle|\; a, d \in O_K^\times,\ b \in \frac{1}{4}\partial^{-1},\ c \in 4\partial,\ (a-1)\mu \in O_K \right\}$$

(i) Consider the case $l = 1/2, \kappa = 1$ (and hence $\chi(x) = \langle x, 2 \rangle_{\mathbb{A}}$), $V = K$, $Q(x) = x^2$ (hence $(x, y)_Q = 2xy$), and $L = O_K$ (in which case we must take $\mu \in O_K/2$). Then, $E^{1/2,\mu}(4\vec{\tau}, s)$ transforms as in equation (7.49) with $\Gamma = \Gamma_\mu$. $E^{1/2,\mu}(4\vec{\tau}, s)$ transforms like a modular form of parallel weight 1/2 and level $\Gamma_\mu \cap \mathcal{K}_0(4)$).

(ii) Consider the case $l = 3/2, \kappa = -1$ (and hence $\chi(x) = \langle x, -2 \rangle_{\mathbb{A}}$), $V = K$, $Q(x) = -x^2$ (hence $(x, y)_Q = -2xy$), and $L = O_K$ (in which case we must take $\mu \in O_K/2$). Then, $E^{3/2,\mu}(4\vec{\tau}, s)$ transforms as in equation (7.49) with $\Gamma = \Gamma_\mu$. $E^{3/2,\mu}(4\vec{\tau}, s)$ times the theta function θ_K transforms like a modular form of parallel weight 2 and level $\Gamma_\mu \cap \mathcal{K}_0(4)$.

In either case, it is not difficult to determine the automorphy factor on all of Γ_μ by considering how the action of a matrix $n(b)$ with $b \in \frac{1}{4}\partial^{-1}$ affects the Fourier series.

Proof. In either case (i) or (ii), since $\mu \in O_K/2$, proposition 4.15 tells us the set of places with nonspherical $\Phi_{\mu,\mathfrak{p}}$ is a subset of the even places. However, since even places cannot be spherical (by definition), it follows that the nonspherical places are exactly the even places. So, we get

$$E^{l,\mu}(\gamma_0\vec{\tau}, s) = \epsilon(\gamma_0)\prod_{\mathfrak{p}|2}\epsilon_{\phi_{\mu,\mathfrak{p}}}([\gamma_{0,\mathfrak{p}}, 1]_L)\prod_{\mathfrak{p}|\infty}\beta_{\mathfrak{p}}(\gamma_{0,\mathfrak{p}}) \cdot \tilde{j}_\infty([\gamma_{0,\infty}, 1]_R, \vec{\tau})^{2l}E^{l,\mu}(\vec{\tau}, s)$$

Our next step is to determine the group Γ as a function of μ. Looking at proposition 4.16, we will need to determine for an even place $\mathfrak{p}$ which elements of $k' \in \mathcal{K}'_{0,\mathfrak{p}}(4)$ take $\phi_{\mu,\mathfrak{p}}$ as an eigenfunction. We now proceed by casework.

Case 1: (i)

The Weil representation is given by

$$\omega_V([g,z]_L)\phi(t) = z\langle x(g), 2\rangle_{\mathfrak{p}}\,\gamma_w\left(\frac{1}{2}\right)^{j(g)}\gamma_w\,(1)^{-j(g)}\,r_V(g)\phi(t)$$

$$r_V(g)\phi(t) = \int_{y\in cV}\psi\left(abt^2 + 2bcty + cdy^2\right)\phi\left(at + cy\right)d_g y$$

To check whether ϕ is an eigenfunction under $\omega_V(k')$, it suffices to check whether it is an eigenfunction under $r_V(k)$. Let $k = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathcal{K}_{0,\mathfrak{p}}(4)$. (Recall this means $a, d \in O_{K_\mathfrak{p}}^\times$, $b \in \partial^{-1}$, and $c \in 4\partial$.) Plugging in ϕ_μ we have

$$r_V(k)\phi_\mu(t) = \int_{y\in cV}\psi\left(abt^2 + 2bcty + cdy^2\right)\phi_\mu\left(at + cy\right)d_g y$$

Case 1a: $c = 0$

In this case,

$$r_V(k)\phi_\mu(t) = \psi\left(abt^2\right)\phi_\mu\left(at\right)\mu(g)$$

for some positive real $\mu(g)$ (unrelated to the variable μ) to make the operation unitary. We may ignore the $\mu(g)$ since if ϕ_μ is an eigenfunction, then $\mu(g)$ will only serve to make sure the eigenvalue is in $\mathbb{T}$. As such, once we compute an eigenvalue we can always normalize it to unit length at the end.

It is clear we must have $\phi_\mu(at) = \phi_\mu(t)$, which is equivalent to asking $a\mu \equiv \mu \bmod O_{K_\mathfrak{p}}$, or alternatively $(a-1)\mu \in O_{K_\mathfrak{p}}$. We also need the ψ term to be constant on $\mu + O_{K_\mathfrak{p}}$. Writing $t = \mu + x$ for $x \in O_{K_\mathfrak{p}}$, we get $\psi(abt^2) = \psi(ab\mu^2)\psi(2\mu abx + abx^2)$. Since $2\mu \in O_{K_\mathfrak{p}}$ and $ab \in \partial^{-1}$, we get $\psi(abt^2) = \psi(ab\mu^2)$, so ψ will always be constant. It follows that in this case ϕ_μ is an eigenfunction iff $a \equiv 1 \bmod \pi^{v_\mu}$ and the eigenvalue under $r_V(k)$ will be $\psi(ab\mu^2)$.

Case 1b: $c \neq 0$

In this case, change variables to $y_{\text{new}} = at + cy_{\text{old}}$ to get

$$r_V(k)\phi_\mu(t) = \int_{y\in\mu+O_{K_\mathfrak{p}}} \psi\left(abt^2 + 2bt(y-at) + \frac{d}{c}(y-at)^2\right) d_g y$$

where we absorbed the Jacobian into $d_g y$. This simplifies to

$$\int_{y\in\mu+O_{K_\mathfrak{p}}} \psi\left(\left(ab - 2ab + \frac{a^2 d}{c}\right)t^2 + 2\left(b - \frac{ad}{c}\right)ty + \frac{d}{c}y^2\right) d_g y$$
$$= \int_{y\in\mu+O_{K_\mathfrak{p}}} \psi\left(\frac{a}{c}t^2 - \frac{2}{c}ty + \frac{d}{c}y^2\right) d_g y \quad (7.51)$$

A further change of variables $y_{\text{old}} = \mu + y_{\text{new}}$ gives the following where to save space, we will use C to denote the ψ term out front.

$$\psi\left(\frac{a}{c}t^2 - \frac{2}{c}t\mu + \frac{d}{c}\mu^2\right)\int_{y\in O_{K_\mathfrak{p}}} \psi\left(\frac{d}{c}y^2 + 2\frac{\mu d - t}{c}y\right) d_g y$$
$$= C\int_{y\in O_{K_\mathfrak{p}}} \psi\left(\frac{d}{c}y^2 + 2\frac{\mu d - t}{c}y\right) d_g y \quad (7.52)$$

We know that $c \in 4\partial$, so if we define $c' = c\tau/(4\pi^r)$, we will have $c' \in O_K$. We may rewrite the above quantity as

$$C\int_{y\in O_{K_\mathfrak{p}}} \psi\left(\frac{\tau d}{4\pi^r c'}y^2 + \frac{\tau(\mu d - t)}{2\pi^r c'}y\right) d_g y$$

Change variables via $y_{\text{old}} = y_{\text{new}}/\alpha$ and rewrite using ψ' to get

$$C\int_{y\in O_{K_\mathfrak{p}}} \psi'\left(\frac{d}{\pi^{2e}c'}y^2 + \frac{\mu d - t}{\pi^e c'}y\right) d_g y$$

Now let $c' = u_{c'}\pi^{v_\pi(c')}$ to get

$$C\int_{y\in O_{K_\mathfrak{p}}}\psi'\left(\frac{d}{\pi^{2e+v_\pi(c')}u_{c'}}y^2+\frac{\mu d-t}{\pi^{e+v_\pi(c')}u_{c'}}y\right)d_gy$$

We may evaluate this using corollary 6.35. In the notation of that formula, we have the parameters

$$(a,u,t)=\left(2e+v_\pi(c'),\ \frac{d}{u_{c'}},\ \frac{\mu d-t}{\pi^{(v_\pi(c')-\overline{v_\pi(c')})/2}u_{c'}}\right)$$

We ignore the g_0 term since we can always normalize at the end (as noted in case 1a). The g_1 term is given by

$$\begin{cases}1 & \square_{1-\overline{v_\pi(c')}}(d/u_{c'})\equiv\frac{(\mu d-t)^2}{\pi^{v_\pi(c')-\overline{v_\pi(c')}}u_{c'}^2}\bmod \pi^{-v_\pi(c')+\overline{v_\pi(c')}}\\0 & \text{else}\end{cases}$$

If we multiply through by $u_{c'}^2\pi^{v_\pi(c')-\overline{v_\pi(c')}}$, the condition becomes

$$\square_1(c'd)\equiv(\mu d-t)^2 \bmod O_{K_\mathfrak{p}}$$

The left hand side is in $O_{K_\mathfrak{p}}$, and so we get the condition $t\equiv\mu d \bmod O_{K_\mathfrak{p}}$. Since we want to get the characteristic function ϕ_μ back out at the end, we must have $\mu\equiv\mu d \bmod O_{K_\mathfrak{p}}$, or $(d-1)\mu\in O_{K_\mathfrak{p}}$. (Note that this is equivalent to the earlier condition $(a-1)\mu\in O_{K_\mathfrak{p}}$!)

In the case this condition does hold, we can plug in the rest of the terms and get

$$C\psi'\left(\frac{1}{4\pi^{\overline{v_\pi(c')}}}\frac{\square_{1-\overline{v_\pi(c')}}(d/u_{c'})-(\mu d-t)^2/(\pi^{v_\pi(c')-\overline{v_\pi(c')}}u_{c'}^2)}{d/u_{c'}}\right)g_3(2e+v_\pi(c'))$$

Multiplying top and bottom by $u_{c'}^2\pi^{v_\pi(c')-\overline{v_\pi(c')}}$ gives

$$C\psi'\left(\frac{1}{4}\frac{\square_1(dc')-(\mu d-t)^2}{dc'}\right)g_3(2e+v_\pi(c'))$$

We must now determine when the above function is constant for $t\in\mu+O_{K_\mathfrak{p}}$. By assumption, this is the same as $t\in\mu d+O_{K_\mathfrak{p}}$, so write $t=\mu d+x$ for $x\in O_{K_\mathfrak{p}}$ and plug back in for C to get

$$\psi\left(\frac{a}{c}(\mu d+x)^2-\frac{2}{c}(\mu d+x)\mu+\frac{d}{c}\mu^2\right)g_3(2e+v_\pi(c'))\psi'\left(\frac{1}{4}\frac{\square_1(dc')}{dc'}\right)\psi'\left(\frac{-x^2}{4dc'}\right)$$

We can rewrite the last term to $\psi(-x^2/cd)$. Rearranging then gives

$$g_3(2e+v_\pi(c'))\psi'\left(\frac{1}{4}\frac{\square_1(dc')}{dc'}\right)\psi\left(bd\mu^2\right)\psi\left(\left(\frac{a}{c}-\frac{1}{cd}\right)x^2+2\left(\frac{ad}{c}-\frac{1}{c}\right)\mu x\right)$$

This simplifies to

$$\begin{aligned} g_3(2e+v_\pi(c'))\psi'\left(\frac{1}{4}\frac{\square_1(dc')}{dc'}\right)\psi\left(bd\mu^2\right)\psi\left(\frac{b}{d}x^2+2b\mu x\right) \\ = g_3(2e+v_\pi(c'))\psi'\left(\frac{1}{4}\frac{\square_1(dc')}{dc'}\right)\psi\left(bd\mu^2\right) \end{aligned} \tag{7.53}$$

where the last term is 1 since $b/d \in \partial^{-1}$ and $2b\mu \in \partial^{-1}$. Rewrite this one more time as

$$= g_3(2e+v_\pi(c'))\psi'\left(\frac{1}{4}\frac{\square_{1-\overline{v_\pi(c')}}(d/uc')}{d/u_{c'}}\right)\psi\left(bd\mu^2\right)$$

We may now recognize the product of the first two terms as a Gauss sum $\gamma'(d/(\pi^{2e}c'))$ (while still disregarding the factor γ_0 since we will normalize at the end). This gives us

$$= \gamma'\left(\frac{d}{\pi^{2e}c'}\right)\psi\left(bd\mu^2\right) = \gamma\left(\frac{d}{4c}\right)\psi\left(bd\mu^2\right)$$

This proves that we may take $\Gamma = \Gamma_\mu$ in equation (7.49). If one wishes, one may explicitly determine the characters $\epsilon_{\phi_{\mu,\mathfrak{p}}}$ from the calculation above.

We will now show that the automorphy factors for the $E^{1/2,\mu}$ are the same as that of the theta function θ_K. We know that $\theta_K(\vec{\tau}) = \sum_{\mu\in(1/2)O_K/O_K} E^{1/2,\mu}(4\vec{\tau},-1/2)$. The crucial point here is that the entire process wherein one takes a Schwartz function, forms a section and builds a modular form out of it is linear. In particular, this implies that if we start with Schwartz functions $\phi_\mathfrak{p} = \mathbb{1}_{(1/2)O_{K_\mathfrak{p}}}$ and let Φ_f denote the product of their associated sections, then $E(4\vec{\tau}, s, \Phi_f \times \Phi_\infty^l) = \theta_K(\vec{\tau})$.

We now write $\theta_K(\vec{\tau}) = E^{1/2,\mu}(4\vec{\tau},-1/2) + (\theta_K(\vec{\tau}) - E^{1/2,\mu}(4\vec{\tau},-1/2))$. So that we have written $\theta_K(\vec{\tau})$ as a sum of two functions. All three functions obey equation (7.49) under the group $\Gamma_\mu \cap \mathcal{K}_0(4)$ and at the level of Schwartz functions, this sum corresponds to the disjoint sum $\mathbb{1}_{(1/2)O_{K_\mathfrak{p}}} = \mathbb{1}_{\mu+O_{K_\mathfrak{p}}} + (\mathbb{1}_{(1/2)O_{K_\mathfrak{p}}} - \mathbb{1}_{\mu+O_{K_\mathfrak{p}}})$.

Given an element $\gamma_0 \in \Gamma_\mu \cap \mathcal{K}_0(4)$, it will act on the Schwartz functions of $E^{1/2,\mu}(4\vec{\tau},-1/2)$ and $\theta_K(\vec{\tau})$ with some eigenvalue, and hence from the disjointness of the sum the eigenvalues must be the same. It follows that we get the same factor of automorphy in equation (5.12).

Case 2: (ii)

In this case,

$$r_V(g)\phi(t) = \int_{y\in cV} \psi\left(-(abt^2 + 2bcty + cdy^2)\right)\phi\left(at + cy\right) d_g y$$

so we can see that $r_V(k)\phi_\mu(t)$ will be exactly the complex conjugate of whatever we got in case (i). Hence, the conditions on Γ are the same and so we can still take $\Gamma = \Gamma_\mu$.

To argue how they vary under $\Gamma_\mu \cap \mathcal{K}_0(4)$, it suffices to prove the result for $\sum_{\mu\in(1/2)O_K/O_K} E^{3/2,\mu}(4\vec{\tau}, 1/2)$, since the rest will follow from the same "disjoint Schwartz functions" argument as above. However, this level is provided by [Su16]. See page 3 where he sets $\Gamma = \mathcal{K}_0(4)$, page 23 for the statement that the modular forms E he computes are in a set $M_{3/2}(\Gamma)$. [Su16] Theorem 10.3 with trivial twisting character χ' gives exactly the Fourier series for $\sum_{\mu\in(1/2)O_K/O_K} E^{3/2,\mu}(4\vec{\tau}, 1/2)$, showing it is indeed part of $M_{3/2}(\Gamma)$. Finally, a look at pages 1 and 2 makes it clear that elements of $M_{3/2}(\Gamma)$ have the desired transformation property. (One should be careful since Ren's $j^{1/2}$ is our j_θ.)

$\square$

7.6 Computing $Q_{K[\sqrt{-m}]}$

Finally, given $m \in O_K$, $m \gg 0$, we determine a formula for $Q_{K[\sqrt{-m}]}$. Let $L = K[\sqrt{-m}]$. By our setup, L/K is a degree 2 extension where K is totally real but L has no real embeddings. This is known as a CM extension. Every CM extension arises in this way, and so we are essentially computing the value of $Q_{K[\sqrt{-m}]}$ (and hence the ratio of regulators) for an arbitrary CM extension.

We start by collecting some useful observations.

Lemma 7.54. *(i)* $Q_{K[\sqrt{-m}]} = Q_{K[\sqrt{-mx^2}]}$ *for any* $x \in O_K$ *so* m *only matters up to a square factor.*

(ii) If $K[\sqrt{a}] = K[\sqrt{b}]$ *then* $a = bx^2$ *for some* $x \in K$.

(iii) L *contains a root of unity of order divisible by 4 iff* $L = K[i]$.

(iv) Let d *be the largest integer so that* L *contains the cyclotomic field* $\mathbb{Q}(\zeta_{2^d})$. *Let* $\zeta \in L$ *be some root of unity. Then there exists a root of unity* $\zeta' \in L$ *so that* $\zeta\zeta'^2$ *is either 1 or a primitive* 2^d*th root of unity. In particular, if* $L \neq K[i]$ *then* $\zeta\zeta'^2 = \pm 1$.

Proof. (i) and (iii) are trivial. (ii) is mostly trivial - if $\sqrt{b} \in K[\sqrt{a}]$, then we may write $\sqrt{b} = y\sqrt{a} + z$ for some $y, z \in K$. Then we have $b = z^2 + ay^2 + 2yz\sqrt{a}$ from which it follows $yz = 0$. Hence either $b = z^2$ or $b = ay^2$ and some trivial casework finishes the proof.

For (iv), first observe that $\mu(L) \cong \mathbb{Z}/n$ for some n. Let $H = \mathbb{Z}/n$. We may decompose $H = \mathbb{Z}/2^d \times \mathbb{Z}/n'$ where n' is some odd integer. It then follows that $H/2H \cong \mathbb{Z}/2$ and that the two congruence classes have representatives of the desired forms.

In the case that $L \neq K[i]$, we have $d = 1$ and this part of the claim follows easily. □

The following theorem is [Was97] theorem 4.12 and gets us most of the way to an answer. It says that Q is given by a particular index and that this index is always either 1 or 2.

Theorem 7.55. *For a CM extension* $L = K[\sqrt{-m}]/K$, *one has*

$$Q_{K[\sqrt{-m}]} = [O_L^\times : \mu(L)O_K^\times] \in \{1, 2\}$$

where $\mu(L)$ *is the group of roots of unity of* L.

We now give conditions for when Q takes on each possible value. In the case m is not a perfect square (that is, when $L \neq K[i]$), the conditions are clean and easy to use. If $L = K[i]$, one can still write down an interesting condition, although it is much more complicated to verify. Fortunately, this is only a single extension where the more difficult condition needs to be checked. We start with the easier cases.

Proposition 7.56. *Continue to let K be totally real and $m \gg 0$.*

(i) If m cannot be written in the form $m = ux^2$ for some unit u, then $Q_{K[\sqrt{-m}]} = 1$.

(ii) If we can write $m = ux^2$ for a unit u and u is not itself a perfect square, then $Q_{K[\sqrt{-m}]} = 2$.

Proof. We prove (i) by contrapositive. In the case that $[O_L^\times : \mu(L)O_K^\times] = 2$, let u' denote some unit in $O_L^\times - \mu(L)O_K^\times$. Since the index is 2, we can write $u'^2 = \zeta u$ for some root of unity ζ and some unit $u \in O_K^\times$.

Let d be the largest integer so that L contains a 2^dth root of unity. Using part (iv) of the lemma, we may assume our choice of u' is such that either $\zeta = 1$ or ζ is a primitive 2^dth root of unity.

Case 1: $\zeta = 1$

Let us consider the case that for our chosen u' we have $\zeta = 1$. Then, we have $u'^2 = u$ for $u \in O_K^\times$ and it follows that $L = K[\sqrt{u}]$. By part (ii) of the lemma, we know$-m = ux^2$. Hence, $m = (-u)x^2$ and so m may be written as a unit times a square.

Case 2: ζ is a primitive 2^dth root of unity

If $d \geq 2$ then part (iii) of the lemma tells us $L = K[\sqrt{-1}]$. Part (ii) of the lemma allows us to conclude this case in the same way as the last one. On the other hand if $d = 1$, then $L = K[\sqrt{-u}]$ and the argument again concludes in the same way.

Now we prove (ii). In this case we have $m = ux^2$ for a totally positive unit u and $u, x \in O_K$. It follows that $L = K[\sqrt{-m}] = K[\sqrt{-u}]$. Since u is assumed nonsquare, we know that $L \neq K[i]$. Our plan is to show that $\sqrt{-u} \in O_L^\times - \mu(L)O_K^\times$, so that $[O_L^\times : \mu(L)O_K^\times] \neq 1$ and then we will be done by theorem 7.55.

Since it is already obviously a unit of L, to prove $\sqrt{-u} \notin \mu(L)O_K^\times$, assume for contradiction that we may write $\sqrt{-u} = \zeta u'$ for $\zeta \in \mu(L)$ and $u' \in O_K^\times$. Then we have $-u = \zeta^2 u'^2$ and it follows that $\zeta^2 \in K$. Since by assumption $L \neq K[i]$, we have $\zeta = \pm 1$. However, this tells us that $\sqrt{-u} = \zeta u' = \pm u' \in K$ which is a contradiction. □

For the case $L = K[i]$, we will need some lemmas.

Lemma 7.57. *If we can write $u'^2 = \zeta u$ for $u' \in O_L^\times$ (we do not exclude $u' \in \mu(L)O_K^\times$!), $\zeta \in \mu(L)$, and $u \in O_K^\times$ then one of $\pm u$ is totally positive. (And in particular we may take u totally positive by swapping ζ for $-\zeta$.)*

Proof. In the proof of [Was97] Theorem 4.12, it is shown that for a CM extension L/K and any unit $u' \in O_L^\times$, $u'/\bar{u'} \in \mu(L)$. Write $u' = \zeta' \bar{u'}$ so that $\zeta' u' \bar{u'} = u'^2 = \zeta u$. It then follows that $u = |u'|^2 \zeta' \zeta^{-1}$. Writing $u' = x + iy$ for $x, y \in O_K$, we see $u = (x^2+y^2)\zeta'\zeta^{-1}$. It follows that $\zeta'\zeta^{-1} \in K$ and hence is ± 1. Since $x^2 + y^2$ is totally positive, the result follows. □

Lemma 7.58. *Let $n \geq 2$ be an integer. There is an element $\alpha \in \mathbb{Q}(\zeta_{2^n})$ such that $\alpha = \beta \zeta_{2^{n+1}}$ where $\beta, \zeta_{2^{n+1}} \in \mathbb{Q}(\zeta_{2^{n+1}})$, β is totally real and $\zeta_{2^{n+1}}$ is a 2^{n+1}st root of unity. In particular, we may take $\beta = \cos(2\pi/2^{n+1})^{-1}$.*

Proof. By the tangent half angle formula,

$$\tan(2\pi/2^{n+1}) = \frac{\sin(2\pi/2^n)}{1 + \cos(2\pi/2^n)} \in \mathbb{Q}(\zeta_{2^n})$$

and so we take $\alpha = 1 + i\tan(2\pi/2^{n+1})$. The desired factorization is given by $\alpha = \cos(2\pi/2^{n+1})^{-1}(\cos(2\pi/2^{n+1}) + i\sin(2\pi/2^{n+1}))$ with $\beta = \cos(2\pi/2^{n+1})^{-1}$ being totally real. (Note $n = 1$ does not work since $\cos(2\pi/2^{n+1})^{-1}$ would be undefined.) □

Let $\mathbb{Q}(\zeta_{2^n})^+$ denote the maximal totally real subfield of $\mathbb{Q}(\zeta_{2^n})$. We introduce the notation $v_2(x)$ as a type of 2-adic valuation of an element $x \in K$. In particular, $v_2(x) = n$ will mean that $v_\mathfrak{p}(x) = n * v_\mathfrak{p}(2)$ for every even place $\mathfrak{p}$. In any other case

$v_2(x)$ is undefined. Finally, we will use the notation $x \sim 2^{1/n}$ to denote the equality of ideals $(x)^n = (2)$. In particular this is equivalent to $v_2(x) = 1/n$ and $v_{\mathfrak{p}}(x) = 0$ for all odd places. As such one could refer to such an element as a valuational nth root of 2.

Lemma 7.59. *Let $n \geq 2$ be an integer. Then $\cos^2(2\pi/2^{n+1}) \in \mathbb{Q}(\zeta_{2^n})^+$ and $4\cos^2(2\pi/2^{n+1}) \sim 2^{1/2^{n-2}}$*

Proof. That $\cos^2(2\pi/2^{n+1}) \in \mathbb{Q}(\zeta_{2^n})^+$ is just the double angle formula. To prove the ideal equality, define

$$f_+(n) = 2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}, \quad f_-(n) = 2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}$$

where there are n 2s in each expression and all signs are + except for a single - at the start of f_-. We use the formula

$$4\cos^2(2\pi/2^{n+1}) = f_+(n-1)$$

This formula may be proven by observing the $n = 2$ case to be $4\cos^2(2\pi/8) = 2$ and then inducting using the half angle formula $2\cos(\theta) = \sqrt{2 + 2\cos(2\theta)} \implies 4\cos^2(\theta) = 2 + \sqrt{4\cos^2(2\theta)}$. By direct calculation one may also observe the equations $f_+(n) + f_-(n) = 4$ and $f_+(n)f_-(n) = f_-(n-1)$.

Since $f_-(1) = 2$ and all $f_\pm(n)$ are integral, the relation $f_+(n)f_-(n) = f_-(n-1)$ implies that $f_+(n'-1) = 4\cos^2(2\pi/2^{n'+1})$ divides 2 for any $n' \geq 2$. It follows $4\cos^2(2\pi/2^{n+1})$ is not divisible by any odd primes, so if we can prove that $v_2(4\cos^2(2\pi/2^{n+1})) = 1/2^{n-2}$ then we will be done.

Observe that $f_\pm(2) = 2 \pm \sqrt{2} = \sqrt{2}(\sqrt{2} \pm 1)$. Since $(\sqrt{2} \pm 1)$ are units, we have $v_2(f_\pm(2)) = 1/2$.

For any even prime $\mathfrak{p}$, let a, b such that $v_{\mathfrak{p}}(f_+(3)) = av_{\mathfrak{p}}(2)$ and $v_{\mathfrak{p}}(f_-(3)) = bv_{\mathfrak{p}}(2)$. Since all $f_\pm(n)$ are integral, we know $a, b \geq 0$. Since $f_+(3)f_-(3) = f_-(2)$, we have $a + b = 1/2$. However, $f_+(3) + f_-(3) = 4$, so we must have $a = b = 1/4$. (This is true since if $a \neq b$ then $v_\mathfrak{p}(f_+(3)+f_-(3)) = min(v_\mathfrak{p}(f_+(3)), v_\mathfrak{p}(f_-(3))) \leq 1/2v_\mathfrak{p}(2)$. However, we know $v_\mathfrak{p}(f_+(3)+f_-(3)) = v_\mathfrak{p}(4) = 2v_\mathfrak{p}(2)$.) Hence, $v_2(f_\pm(3)) = 1/4$. Repeating this argument shows that $v_2(f_\pm(n)) = 1/2^{n-1}$. It follows $v_2(4\cos^2(2\pi/2^{n+1}) = f_+(n-1)) = 1/2^{n-2}$.

$\square$

Proposition 7.60. *If m is a perfect square (in other words, if $L = K[i]$) let $\mathbb{Q}(\zeta_{2^d})$ be the largest power-of-2 cyclotomic field contained in $K[i]$. Then, $Q_{K[\sqrt{-m}]} = 2$ iff K contains an element x such that $x \sim 2^{1/2^{d-1}}$.*

Remark 7.61. *By lemma 7.59, K always contains an element $x \sim 2^{1/2^{d-2}}$ and so $v_2(x) = 1/2^{d-2}$. A rather loose phrasing of the above condition is that we are looking for "a valuational diadic root of 2 that does not come from the cyclotomic field".*

Proof. Since $L = K[i]$ we have $d \geq 2$. Let us start with the forward implication. Assume $[O_L^\times : \mu(L)O_K^\times] = 2$. Let $u' \in O_L^\times - \mu(L)O_K^\times$ and since the index is 2 write $u'^2 = \zeta u$ with $\zeta \in \mu(L)$ and $u \in O_K^\times$. Just as in the proof of proposition 7.56, we may use lemma 7.54 part (iv) make such a choice of $u' \in O_L^\times - \mu(L)O_K^\times$ so that either $\zeta = 1$ or ζ is a primitive 2^dth root of unity. We may further use lemma 7.57 to take $u \gg 0$.

The case $\zeta = 1$ is impossible - in this case $u'^2 = u$. However, they only way an element $u' = x + iy \in K[i]$ can square into K is if x or y is zero. In either case, we would have $u' \in \mu(L)O_K^\times$.

So, we have $u'^2 = \zeta u$ for ζ a primitive 2^dth root of unity. By lemma 7.58, there exists an $\alpha \in \mathbb{Q}(\zeta_{2^d}) \subset L$ so that $\alpha = \cos(2\pi/2^{d+1})^{-1}\sqrt{\zeta}$ where $\cos(2\pi/2^{d+1})^{-1} \in \mathbb{Q}(\zeta_{2^{d+1}})$ is totally real and $\sqrt{\zeta}$ is a 2^{d+1}st root of unity. Let $x' = u'/\alpha \in L$ so that we get

$$x'^2/\cos^2(2\pi/2^{d+1}) = u \implies (2x')^2 = 4\cos^2(2\pi/2^{d+1})u$$

Since $\cos(2\pi/2^{d+1})$ is totally real, it follows that $\cos^2(2\pi/2^{d+1}) \in L$ is totally positive. We also know $u \gg 0$, and so we see that $(2x')^2 \gg 0$. It follows that $2x'$ is totally real and lies in L, so it therefore lies in K. Lemma 7.59 tells us that $4\cos^2(2\pi/2^{d+1})u \sim 2^{1/2^{d-2}}$, and so it follows that $2x' \sim 2^{1/2^{d-1}}$ is the element we were looking for.

For the reverse implication, assume there exists some element $2x' \in K$ such that $2x' \sim 2^{1/2^{d-1}}$. Then, $(2x')^2/(4\cos^2(2\pi/2^{d+1})) \sim 2^0$ and is also an element of K (note that the $\cos^2$ term is in K since it is totally real and in L). Hence, it must be some unit $u \in O_K^\times$. Letting $u' = \alpha x'$, we get $u'^2 = \zeta u$. Since the right hand side is a unit, so is the left hand side and $u' \in O_L^\times$.

For the sake of contradiction, assume that we can write $u' = \zeta' u_2$ for $\zeta' \in \mu(L)$ and $u_2 \in O_K^\times$. Then we get $\zeta'^2 u_2^2 = \zeta u$. It follows that $\zeta\zeta'^{-2} \in K$ and hence $\zeta'^2 = \pm\zeta$. However, this is impossible since this would imply that ζ'^2 is a primitive 2^dth root of unity. $\square$

Corollary 7.62. *If K has narrow class number 1, then $Q_{K[\sqrt{-m}]} = 1$ always.*

Proof. This is actually a corollary to lemma 7.57. As we have done before, since $[O_L^\times : \mu(L)O_K^\times] \in \{1, 2\}$, any $u' \in O_L^\times$ satisfies $u'^2 = \zeta u$ for $\zeta \in \mu(L)$ and $u \in O_K^\times$. By lemma 7.57, we may choose u totally positive. By the narrow class number 1 assumption, it follows that u is a perfect square and so we write $u = v^2$ giving $u'^2 = \zeta v^2$. Rearranging yields $(u'/v)^2 = \zeta$ and so u'/v is some root of unity ζ'. Hence $u' = v\zeta'$ and $u' \in \mu(L)O_K^\times$. $\square$

References

[BS20] Ajit Bhand and Ranveer Kumar Singh, *Zagier's weight* 3/2 *mock modular form*, arXiv preprint arXiv:2012.00539 (2020).

[Coh75] Henri Cohen, *Sums involving the values at negative integers of l-functions of quadratic characters*, Mathematische Annalen **217** (1975), 271–285.

[Fre90] Eberhard Freitag, *Hilbert modular forms*, Springer, 1990.

[Gel06] Stephen S Gelbart, *Weil's representation and the spectrum of the metaplectic group*, vol. 530, Springer, 2006.

[HI13] Kaoru Hiraga and Tamotsu Ikeda, *On the kohnen plus space for hilbert modular forms of half-integral weight i*, Compositio Mathematica **149** (2013), no. 12, 1963–2010.

[HZ76] Friedrich Hirzebruch and Don Zagier, *Intersection numbers of curves on hilbert modular surfaces and modular forms of nebentypus*, Inventiones mathematicae **36** (1976), 57–113.

[KRY04] Stephen S Kudla, Michael Rapoport, and Tonghai Yang, *Derivatives of eisenstein series and faltings heights*, Compositio Mathematica **140** (2004), no. 4, 887–951.

[KRY06] ______, *Modular forms and special cycles on shimura curves.(am-161)*, vol. 161, Princeton university press, 2006.

[Kud96] Stephen Kudla, *Notes on the local theta correspondence*, unpublished notes, available online (1996).

[Kud97] Stephen S Kudla, *Central derivatives of eisenstein series and height pairings*, Annals of mathematics (1997), 545–646.

[KY10] Stephen S Kudla and TongHai Yang, *Eisenstein series for sl (2)*, Science China Mathematics **53** (2010), 2275–2316.

[LSL65] Nikolai Nikolaevich Lebedev, Richard A Silverman, and DB Livhtenberg, *Special functions and their applications*, American Institute of Physics, 1965.

[Miz84] Shin-Ichiro Mizumoto, *On the second l-functions attached to hilbert modular forms*, Mathematische Annalen **269** (1984), 191–216.

[RR93] R Ranga Rao, *On some explicit formulas in the theory of weil representation*, Pacific Journal of Mathematics **157** (1993), no. 2, 335–371.

[Shi82] Goro Shimura, *Confluent hypergeometric functions on tube domains*, Mathematische Annalen **260** (1982), no. 3, 269–302.

[Shi85] ______, *On eisenstein series of half-integral weight.*

[Shi87] ———, *On hilbert modular forms of half-integral weight.*

[Su16] Ren-He Su, *Eisenstein series in the kohnen plus space for hilbert modular forms*, International Journal of Number Theory **12** (2016), no. 03, 691–723.

[W$^+$64] André Weil et al., *Sur certains groupes d'opérateurs unitaires*, Acta math **111** (1964), no. 143-211, 14.

[Was97] Lawrence C Washington, *Introduction to cyclotomic fields, volume 83 of*, Graduate Texts in Mathematics (1997), 104.

[Woo93] Jay A Wood, *Witt's extension theorem for mod four valued quadratic forms*, Transactions of the American Mathematical Society **336** (1993), no. 1, 445–461.