Scalar product according to Klein-Fock-Gordon equation in the context of light cone coordinates

Produto escalar associado a equação de Klein-Fock-Gordon no contexto de coordenadas de cone de luz

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ABSTRACT
The objective of the present work is to didactically analyze, in the context of relativistic quantum mechanics, the algebraic structuring of the dot product between two quantum wavefunctions relative to a spin-0 quantum particle, assuming the light cone coordinates. For that, the linear momentum tensor is used in four dimensions, considering the description of the operational and quantum character of energy and linear momentum in three dimensions, as well as defining a pseudo volumetric probability density from which the product is reached scalar, properly written in terms of light cone coordinates.

Keywords: scalar product, light cone coordinates.

RESUMO
O objetivo do presente trabalho é analisar didaticamente, no âmbito da mecânica quântica relativística, a estruturação algébrica do produto escalar entre duas funções de onda quântica relativas a uma partícula quântica de spin-0, assumindo as coordenadas do cone de luz. Para tanto, utiliza-se do tensor momento linear em quatro dimensões, considerando a descrição de caráter operatorial e quântico da energia e do momento linear em três dimensões, bem como se define uma pseudo-densidade volumétrica de probabilidade a partir da qual se obtém o produto escalar, devidamente escrito em termos das coordenadas do cone de luz.

Palavras-chave: produto escalar, coordenadas do cone de luz.

1 INTRODUCTION

The dynamics of a spin-0 quantum particle endowed with properties such as rest mass and electric charge, moving in a regime described by Einstein's Special Relativity has been analyzed and studied in the context of Relativistic quantum mechanics aiming at a better understanding of the formalisms of Quantum Electrodynamics and Quantum Field Theory (GOMES, 2015). In this scenario, the present work proposes to present a didactic text on how to write the dot product (or inner product) between two quantum wavefunctions associated with a relativistic spin-0 quantum particle, appropriating of the formalism of light cone coordinates.

To properly achieve this dot product, it initially considers the covariant and contravariant components of the position tensor and the linear momentum tensor, both in four dimensions. Then, within the ambit of Minkowski spacetime, it uses the relativistic expression of the energy of the particle through the tensor contraction between the covariant component and the contravariant component of the linear momentum four-vector, performing the substitution of algebraic structure of the quantum energy operator and the three-dimensional linear momentum operator. Additionally, a pseudo volumetric probability density is obtained from which the dot product between two quantum wavefunctions is defined, finally writing said inner product in terms of light cone coordinates in four dimensions.

2 ANALYSIS OF THE PROBABILITY DENSITY TENSOR

Consider the following contravariant component of the order-1 tensor, which describes the position of an event or structure in 4-dimensional spacetime:

$$x^{\mu} = (ct, x, y, z) \tag{1}$$

It is still possible to write its respective covariant component as

$$x_{\mu} = g_{uv}x^{\nu} \tag{2}$$

Being the components of the order-2 tensor, called metric tensor associated with the 4-dimensional Minkowski spacetime, given by:

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \tag{3}$$

Thus, we can write the covariant components in terms of the temporal variable and the spatial Cartesian coordinates, as (BASSALO, 2008):

$$x_{\mu} = (ct, -x, -y, -z) \tag{4}$$

Analyzing about linear momentum in its 4-dimensional tensor form and considering its contravariant component, there is:

$$p^{\mu} = [p^0, p^1, p^2, p^3] = \left[\frac{E}{c}, p_x, p_y, p_z\right] \tag{5}$$

In this case, it is also possible to write the covariant components of the linear momentum four-vector, as follows:

$$p_\mu = [p_0, p_1, p_2, p_3] = \left[\frac{E}{c}, -p_x, -p_y, -p_z\right] \tag{6}$$

The multiplication of the tensor contraction type between the linear momentum tensors of order-1 in 4-dimensions is given by (BASSALO, 2006):

$$\hat{p}^\mu \hat{p}_\mu \psi = m_0^2 c^2 \psi \tag{7}$$

Making use of the properties of the order-2 metric tensor for the 4-dimensional Minkowski spacetime, it is verified:

$$\begin{aligned} p^\mu p_\mu = g_{\mu\nu} p^\mu p^\nu &= p^0 p^0 - p_x^2 - p_y^2 - p_z^2 \\ &= (p^0)^2 - p_x^2 - p_y^2 - p_z^2 \\ &= \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 \\ &= \frac{E^2}{c^2} - \vec{p} \cdot \vec{p} \end{aligned} \tag{8}$$

Wherein $\vec{p} = \hat{\imath} p_x + \hat{\jmath} p_y + \hat{k} p_z$.

Identifying the quantum energy operator as:

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \tag{9}$$

Consequently, it is possible to write the following expression:

$$\frac{\hat{E}\hat{E}}{c^2} = -\hbar^2 \frac{\partial^2}{\partial (ct)^2} \tag{10}$$

Defining the linear momentum operator in 3-dimensional space

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}, \qquad \hat{p}_y = -i\hbar \frac{\partial}{\partial y}, \qquad \hat{p}_z = -i\hbar \frac{\partial}{\partial z} \tag{11}$$

We can, conveniently, write the relations below:

$$\hat{p}_x\hat{p}_x = -\hbar^2\frac{\partial^2}{\partial x^2}\,, \qquad \hat{p}_y\hat{p}_y = -\hbar^2\frac{\partial^2}{\partial y^2}\,, \qquad \hat{p}_z\hat{p}_z = -\hbar^2\frac{\partial^2}{\partial z^2} \tag{12}$$

Developing the first term of equation (8), as follows:

$$\hat{p}^\mu\hat{p}_\mu\psi = \frac{\hat{E}^2}{c^2}\psi - \hat{\vec{p}}.\hat{\vec{p}}\psi$$

$$= -\hbar^2\frac{\partial^2\psi}{\partial(ct)^2} + \hbar^2\frac{\partial^2\psi}{\partial x^2} + \hbar^2\frac{\partial^2\psi}{\partial y^2} + \hbar^2\frac{\partial^2\psi}{\partial z^2}$$

$$= -\hbar^2\frac{\partial^2\psi}{\partial(ct)^2} + \hbar^2\vec{\nabla}^2\psi$$

Substituting this result into equation (7), we have:

$$-\hbar^2\frac{\partial^2\psi}{\partial(ct)^2} + \hbar^2\vec{\nabla}^2\psi = m_0^2c^2\psi$$

Therefore, this expression can be rewritten as:

$$\frac{\hbar^2}{c^2}\frac{\partial^2\psi}{\partial t^2} - \hbar^2\vec{\nabla}^2\psi + m_0^2c^2\psi = 0 \tag{13}$$

For the purpose of simplification, it is considered $a = \frac{m_0^2c^2}{\hbar^2}$, so that equation (13) will be presented as:

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \frac{1}{a}\vec{\nabla}^2\psi = \psi \tag{14}$$

And in an equivalent way:

$$-\frac{1}{c^2}\frac{\partial^2\psi^*}{\partial t^2} + \frac{1}{a}\vec{\nabla}^2\psi^* = \psi^* \tag{15}$$

In the meantime, let us consider the quantum wavefunction expressed as:

$$\psi = \exp\left(-\frac{i}{\hbar} p_\mu x^\mu\right) \tag{16}$$

And performing tensor multiplication

$$\begin{aligned} p_\mu x^\mu &= \sum_{i=0}^{3} p_i x^i \\ &= \frac{E}{c} ct - x p_x - y p_y - z p_z \\ &= Et - \vec{p} \cdot \vec{r} \end{aligned} \tag{17}$$

Being $\vec{r} = \hat{\imath} x + \hat{\jmath} y + \vec{k} z$. In this case, the above quantum wavefunction can be rewritten as:

$$\psi = \exp\left[\frac{i}{\hbar}(\vec{p} \cdot \vec{r} - Et)\right] \tag{18}$$

Remembering that $p^\mu p_\mu = m_0^2 c^2$ and the relationship expressed in equation (8), it is concluded that:

$$E = +\, c\sqrt{|\vec{p}|^2 + m_0^2 c^2} \tag{19}$$

$$E = -\, c\sqrt{|\vec{p}|^2 + m_0^2 c^2} \tag{20}$$

Being the equations (19) and (20) corresponding to positive and negative energy, respectively.

Returning to equation (7), it is verified that:

$$\hat{p}^\mu \hat{p}_\mu \psi = m_0^2 c^2 \psi \implies \left(\hat{p}^\mu \hat{p}_\mu - m_0^2 c^2\right)\psi = 0 \tag{21}$$

$$\hat{p}^\mu \hat{p}_\mu \psi^* = m_0^2 c^2 \psi^* \implies \left(\hat{p}^\mu \hat{p}_\mu - m_0^2 c^2\right)\psi^* = 0 \tag{22}$$

In this case, it is possible and convenient to write the following relations:

$$\psi^*\left(\hat{p}^\mu\hat{p}_\mu - m_0^2c^2\right)\psi = 0 \tag{23}$$

$$\psi\left(\hat{p}^\mu\hat{p}_\mu - m_0^2c^2\right)\psi^* = 0 \tag{24}$$

Then, subtracting equation (24) from equation (23)

$$\psi^*\left(\hat{p}^\mu\hat{p}_\mu - m_0^2c^2\right)\psi - \psi\left(\hat{p}^\mu\hat{p}_\mu - m_0^2c^2\right)\psi^* = 0 \tag{25}$$

Considering that:

$$\hat{p}^\mu = i\hbar\frac{\partial}{\partial x_\mu} \text{ e } \hat{p}_\mu = i\hbar\frac{\partial}{\partial x^\mu} \tag{26}$$

And still expressing in detail, we have:

$$\hat{p}^0 = \frac{i\hbar}{c}\frac{\partial}{\partial t}, \quad \hat{p}^1 = -i\hbar\frac{\partial}{\partial x}, \quad \hat{p}^2 = -i\hbar\frac{\partial}{\partial y}, \quad \hat{p}^3 = -i\hbar\frac{\partial}{\partial z} \tag{27}$$

$$\hat{p}_0 = \frac{i\hbar}{c}\frac{\partial}{\partial t}, \quad \hat{p}_1 = +i\hbar\frac{\partial}{\partial x}, \quad \hat{p}_2 = +i\hbar\frac{\partial}{\partial y}, \quad \hat{p}_3 = +i\hbar\frac{\partial}{\partial z} \tag{28}$$

We can substitute the relations (26) in equation (25), obtaining:

$$\psi^*\left[i\hbar\frac{\partial}{\partial x_\mu}\left(i\hbar\frac{\partial}{\partial x^\mu}\right) - m_0^2c^2\right]\psi - \psi\left[i\hbar\frac{\partial}{\partial x_\mu}\left(i\hbar\frac{\partial}{\partial x^\mu}\right) - m_0^2c^2\right]\psi^* = 0$$

$$\psi^*\left(-\hbar^2\frac{\partial^2}{\partial x_\mu\partial x^\mu} - m_0^2c^2\right)\psi - \psi\left(-\hbar^2\frac{\partial^2}{\partial x_\mu\partial x^\mu} - m_0^2c^2\right)\psi^* = 0$$

$$-\psi^*\hbar^2\frac{\partial^2\psi}{\partial x_\mu\partial x^\mu} - \psi^*m_0^2c^2\psi + \psi\hbar^2\frac{\partial^2\psi^*}{\partial x_\mu\partial x^\mu} + \psi m_0^2c^2\psi^* = 0$$

$$-\psi^*\frac{\partial^2\psi}{\partial x_\mu\partial x^\mu} + \psi\frac{\partial^2\psi^*}{\partial x_\mu\partial x^\mu} = 0$$

$$-\psi^*\frac{\partial}{\partial x_\mu}\left(\frac{\partial\psi}{\partial x^\mu}\right) + \psi\frac{\partial}{\partial x_\mu}\left(\frac{\partial\psi^*}{\partial x^\mu}\right) = 0$$

$$-\frac{\partial}{\partial x_\mu}\left(\psi^*\frac{\partial\psi}{\partial x^\mu}\right) + \frac{\partial}{\partial x_\mu}\left(\psi\frac{\partial\psi^*}{\partial x^\mu}\right) = 0 \tag{29}$$

Developing the partial derivatives of equation (29), as follows:

$$\frac{\partial}{\partial x^\mu}\left(\psi^*\frac{\partial\psi}{\partial x_\mu}\right)=\frac{\partial\psi^*}{\partial x^\mu}\frac{\partial\psi}{\partial x_\mu}+\psi^*\frac{\partial}{\partial x^\mu}\left(\frac{\partial\psi}{\partial x_\mu}\right)$$

$$\frac{\partial}{\partial x^\mu}\left(\psi\frac{\partial\psi^*}{\partial x_\mu}\right)=\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi^*}{\partial x_\mu}+\psi\frac{\partial}{\partial x^\mu}\left(\frac{\partial\psi^*}{\partial x_\mu}\right)$$

And substituting them in equation (29), we reach the following relationship:

$$\frac{\partial\psi^*}{\partial x^\mu}\frac{\partial\psi}{\partial x_\mu}=\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi^*}{\partial x_\mu} \tag{30}$$

We can also identify the following relationship:

$$\frac{\partial}{\partial x^\mu}\left(\psi^*\frac{\partial\psi}{\partial x_\mu}\right)-\frac{\partial}{\partial x^\mu}\left(\psi\frac{\partial\psi^*}{\partial x_\mu}\right)=0$$

$$\psi^*\frac{\partial}{\partial x^\mu}\left(\frac{\partial\psi}{\partial x_\mu}\right)-\psi\frac{\partial}{\partial x^\mu}\left(\frac{\partial\psi^*}{\partial x_\mu}\right)=0$$

$$\frac{\partial}{\partial x^\mu}\left(\psi^*\frac{\partial\psi}{\partial x_\mu}-\psi\frac{\partial\psi^*}{\partial x_\mu}\right)=0 \tag{31}$$

In practice, it is possible to define a candidate order-1 tensor to be configured as a 4-dimensional probability current density tensor, relative to the quantum wavefunction, expressed in 4-dimensions as:

$$J^\mu=\frac{i\hbar}{2m_0}\left(\psi^*\frac{\partial\psi}{\partial x_\mu}-\psi\frac{\partial\psi^*}{\partial x_\mu}\right) \tag{32}$$

For reasons of simplification in the notation, it is assumed that $\frac{\partial}{\partial x^\mu}=\partial_\mu$. Based on expression (31) and considering that the term $\frac{i\hbar}{2m_0}$ is a constant, we can highlight:

$$\partial_\mu J^\mu=0 \tag{33}$$

Using the order-2 metric tensor, we have:

$$g_{\mu\nu}\partial^{\mu}J^{\nu} = 0 \;\Rightarrow g_{\mu\nu}\frac{\partial J^{\nu}}{\partial x_{\mu}} = 0 \tag{34}$$

In detail, it is possible to obtain:

$$g_{00}\frac{\partial J^0}{\partial x_0} + g_{11}\frac{\partial J^1}{\partial x_1} + g_{22}\frac{\partial J^2}{\partial x_2} + g_{33}\frac{\partial J^3}{\partial x_3} = 0 \tag{35}$$

Substituting the components, we have:

$$\begin{gathered}\frac{\partial J^0}{\partial(ct)} - \frac{\partial J^1}{\partial(-x)} - \frac{\partial J^2}{\partial(-y)} - \frac{\partial J^3}{\partial(-z)} = 0\\ \frac{1}{c}\frac{\partial J^0}{\partial t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = 0\end{gathered} \tag{36}$$

Using these relations, we easily obtain:

$$\begin{aligned}J^0 &= \frac{i\hbar}{2m_0}\left(\psi^*\frac{\partial\psi}{\partial x_0} - \psi\frac{\partial\psi^*}{\partial x_0}\right)\\ &= \frac{i\hbar}{2m_0}\left(\psi^*\frac{\partial\psi}{\partial(ct)} - \psi\frac{\partial\psi^*}{\partial(ct)}\right)\\ &= \frac{i\hbar}{2cm_0}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\end{aligned} \tag{37}$$

Applying the partial derivative with respect to time about J^0

$$\frac{\partial J^0}{\partial t} = \frac{\partial}{\partial t}\left[\frac{i\hbar}{2cm_0}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right] \tag{38}$$

And following similar procedures, we obtain:

$$J^1 = -\frac{i\hbar}{2m_0}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right) \tag{39}$$

$$J^2 = -\frac{i\hbar}{2m_0}\left(\psi^* \frac{\partial \psi}{\partial y} - \psi \frac{\partial \psi^*}{\partial y}\right) \quad (40)$$

$$J^3 = -\frac{i\hbar}{2m_0}\left(\psi^* \frac{\partial \psi}{\partial z} - \psi \frac{\partial \psi^*}{\partial z}\right) \quad (41)$$

Substituting the expressions (38), (39), (40) and (41) in the expression (36), we obtain:

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] - \frac{i\hbar}{2m_0}\left[\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)\right]$$
$$- \frac{i\hbar}{2m_0}\left[\frac{\partial}{\partial y}\left(\psi^* \frac{\partial \psi}{\partial y} - \psi \frac{\partial \psi^*}{\partial y}\right)\right] - \frac{i\hbar}{2m_0}\left[\frac{\partial}{\partial z}\left(\psi^* \frac{\partial \psi}{\partial z} - \psi \frac{\partial \psi^*}{\partial z}\right)\right] = 0$$

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] - \frac{i\hbar}{2m_0}\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$
$$- \frac{i\hbar}{2m_0}\frac{\partial}{\partial y}\left(\psi^* \frac{\partial \psi}{\partial y} - \psi \frac{\partial \psi^*}{\partial y}\right) - \frac{i\hbar}{2m_0}\frac{\partial}{\partial z}\left(\psi^* \frac{\partial \psi}{\partial z} - \psi \frac{\partial \psi^*}{\partial z}\right) = 0$$

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] - \frac{i\hbar}{2m_0}\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x}\right)$$
$$+ \frac{i\hbar}{2m_0}\frac{\partial}{\partial x}\left(\psi \frac{\partial \psi^*}{\partial x}\right)$$
$$- \frac{i\hbar}{2m_0}\frac{\partial}{\partial y}\left(\psi^* \frac{\partial \psi}{\partial y}\right) + \frac{i\hbar}{2m_0}\frac{\partial}{\partial y}\left(\psi \frac{\partial \psi^*}{\partial y}\right) - \frac{i\hbar}{2m_0}\frac{\partial}{\partial z}\left(\psi^* \frac{\partial \psi}{\partial z}\right)$$
$$+ \frac{i\hbar}{2m_0}\frac{\partial}{\partial z}\left(\psi \frac{\partial \psi^*}{\partial z}\right) = 0$$

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] - \frac{i\hbar}{2m_0}\left[\vec{\nabla}\cdot\left(\psi^*\vec{\nabla}\psi\right)\right] + \frac{i\hbar}{2m_0}\left[\vec{\nabla}\cdot\left(\psi\vec{\nabla}\psi^*\right)\right] = 0$$

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2m_0c^2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] + \vec{\nabla}\cdot\left[-\frac{i\hbar}{2m_0}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right] = 0 \quad (42)$$

Considering that $J^0 = c\rho$, being $\rho = \frac{i\hbar}{2c^2m_0}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)$ a scalar quantity candidate to be configured as volumetric density of probability and $\vec{J} = \hat{\imath}J^1 + \hat{\jmath}J^2 + \hat{k}J^3$ is a vector in 3-dimensions, which is candidate to be configured as volumetric density vector of probability current, which can be written as:

$$\vec{J} = -\frac{i\hbar}{2m_0}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) \tag{43}$$

In such a way that is appropriate to write equation (42) as:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\bullet\vec{J} = 0 \implies \frac{\partial\rho}{\partial t} = -\vec{\nabla}\bullet\vec{J} \tag{44}$$

Making use of the divergence theorem (ARFKEN et al., 2012):

$$\iiint \left(\vec{\nabla}\bullet\vec{J}\right) d^3\vec{r} = \oint\!\!\oint \vec{J}\bullet d^2\vec{r} \tag{45}$$

with respect to which $d^3\vec{r}$ and $d^2\vec{r}$ refer to the infinitesimal element of volume and the infinitesimal element of area of a surface, respectively.

Assuming that $\psi = \psi(x, y, z, ct)$ is a bounded function at all points, this implies that ψ does not tend to infinity at any point (x, y, z, ct) (PIZA, 2009). Reiterating that the quantum wavefunction does not diverge to infinity in 4-dimensional regions of spacetime infinitely distant from the quantum particle, it is possible to write:

$$\lim_{(x,y,z)\to\pm\infty} \psi(x, y, z, ct) = 0\,, \lim_{ct\to+\infty} \psi(x, y, z, ct) = 0 \tag{46}$$

Considering that the limits of the closed surface are located in regions infinitely distant from the probability density current $\vec{J}$, it is appropriate to write (BASSALO e CATTANI, 2010):

$$\lim_{(x,y,z)\to\pm\infty} \oint\!\!\oint \vec{J}\bullet d^2\vec{r} = \lim_{(x,y,z)\to\pm\infty} \oint\!\!\oint \left[-\frac{i\hbar}{2m_0}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)\right]\bullet d^2\vec{r} = 0 \tag{47}$$

Using the relationship given in equation (44), we have:

$$\iiint \left(\frac{\partial \rho}{\partial t}\right) d^3\vec{r} = 0 \;\Longrightarrow\; \frac{\partial}{\partial t}\left(\iiint \rho\, d^3\vec{r}\right) = 0 \tag{48}$$

The result of equation (48) indicates that $\iiint \rho d^3\vec{r}$ is constant in time (ABERS, 2004). In fact, it is natural to interpret the quantity ρ as the volumetric probability density. However, considering an instant t, $\psi(x, y, z, ct)$ and $\frac{\partial\psi(x,y,z,ct)}{\partial t}$ present positive and negative values so that ρ can also assume negative and positive values. Because it is considered that the probability function must satisfy the following properties, namely:

- $0 < P(A_i) < 1$: being A_i random event belonging to or contained in a sample space;
- $P(E.A.) = \sum_i P(A_i) = 1$, being $E.A.$ the sample space (union of all A_i) is trivial to observe that the fact of ρ admitting negative values makes it unfit to play the role or to be defined as a probability density function.

3 SCALAR PRODUCT BETWEEN WAVE FUNCTIONS OF A SPIN-ZERO QUANTUM PARTICLE

Considering the inner product relative to the Klein-Fock-Gordon relativistic quantum equation properly defined as:

$$(\psi_1, \psi_2) = \frac{i\hbar}{2m_0c^2}\iiint \left(\psi_1^* \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1^*}{\partial t}\right) d^3\vec{r} \tag{49}$$

And making use the description presented by (BAGROV and GITMAN, 1990) and (KAMASSURY et al., 2020), it is possible to relate the Cartesian coordinates (cdt,dx,dy,dz) with the light cone coordinates (du^0 ,du^1,du^2,du^3), as follows:

$$dx = du^1,\; dy = du^2,\; dz = \frac{\sqrt{2}}{2}(du^0 - du^3),\; dt = \frac{\sqrt{2}}{2}(du^0 + du^3) \tag{50}$$

Considering $x^0 = ct$, we have (MEIRA FILHO et al., 2021):

$$\frac{\partial}{\partial x^0} = \frac{\sqrt{2}}{2}\frac{\partial}{\partial u^0} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial u^3}, \quad \frac{1}{c}\frac{\partial}{\partial t} = \frac{\sqrt{2}}{2}\frac{\partial}{\partial u^0} + \frac{\sqrt{2}}{2}\frac{\partial}{\partial u^3} \tag{51}$$

$$u^0 = \frac{ct+z}{\sqrt{2}}, \quad u^1 = x, \quad u^2 = y, \quad u^3 = \frac{ct-z}{\sqrt{2}} \tag{52}$$

$$\psi_1^* \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1^*}{\partial t} = c\psi_1^* \left(\frac{\sqrt{2}}{2}\frac{\partial \psi_2}{\partial u^0} + \frac{\sqrt{2}}{2}\frac{\partial \psi_2}{\partial u^3} \right) - c\psi_2 \left(\frac{\sqrt{2}}{2}\frac{\partial \psi_1^*}{\partial u^0} + \frac{\sqrt{2}}{2}\frac{\partial \psi_1^*}{\partial u^3} \right) \tag{53}$$

If the coordinate u^0 is constant, ψ_1, ψ_1^*,ψ_2,ψ_2^* it does not depend on this variable, that is:

$$\psi_1^* \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1^*}{\partial t} = \frac{\sqrt{2}}{2} c\psi_1^* \frac{\partial \psi_2}{\partial u^3} - \frac{\sqrt{2}}{2} c\psi_2 \frac{\partial \psi_1^*}{\partial u^3} \tag{54}$$

Thus, it is convenient to write the expression (49) assuming that the coordinate u^0 is equal to a numerical constant:

$$(\psi_1, \psi_2)_{u^0=\text{cte}} = \frac{i\hbar}{2m_0c^2}\frac{c\sqrt{2}}{2} \iiint \left(\psi_1^* \frac{\partial \psi_2}{\partial u^3} - \psi_2 \frac{\partial \psi_1^*}{\partial u^3} \right) d^3\vec{r} \tag{55}$$

And performing the appropriate substitutions of the coordinates, we have:

$$(\psi_1, \psi_2)_{u^0=\text{cte}} = \frac{i\hbar}{2m_0c^2}\left(\frac{c\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) \iiint \left(\psi_1^* \frac{\partial \psi_2}{\partial u^3} - \psi_2 \frac{\partial \psi_1^*}{\partial u^3} \right) du^1 du^2 du^3$$

$$= -\frac{i\hbar}{4m_0c} \iiint \left(\psi_1^* \frac{\partial \psi_2}{\partial u^3} - \psi_2 \frac{\partial \psi_1^*}{\partial u^3} \right) du^1 du^2 du^3$$

$$= \frac{1}{4m_0c} \iiint \left[\left(i\hbar \frac{\partial \psi_1^*}{\partial u^3} \right) \psi_2 + \psi_1^* \left(-i\hbar \frac{\partial \psi_2}{\partial u^3} \right) \right] du^1 du^2 du^3 \tag{56}$$

Being $\tilde{p}_3 = -i\hbar\frac{\partial}{\partial u^3}$ e $[\tilde{p}_3\psi_1]^* = \left(-i\hbar\frac{\partial\psi_1}{\partial u^3}\right)^* = i\hbar\frac{\partial\psi_1^*}{\partial u^3}$, we can write the inner product between the wavefunctions ψ_1 and ψ_2 under the condition that u^0 is equal to a numerical constant:

$$(\psi_1, \psi_2)_{u^0=cte} = \frac{1}{4m_0c}\iiint du^1du^2du^3\left[\psi_1^*\left(\hat{P}_3\psi_2\right) + \psi_2\left(\hat{P}_3\psi_1\right)^*\right] \qquad (57)$$

4 FINAL CONSIDERATIONS

Throughout this content review work, we present the algebraic procedures necessary to obtain the scalar product (or inner product) between the wavefunctions, which are solutions of the Klein-Fock-Gordon relativistic wave equations, considering a quantum particle of spin-zero immersed in a 4-dimensional Minkowski space-time parameterized by the light cone coordinates.

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