Computational simulation of one-dimensional waves with the Multigrid Method

Simulação computacional de ondas unidimensionais com o Método Multigrid

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ABSTRACT
Several Engineering problems are modeled computationally, these simulations involve large systems, which are commonly difficult to solve. This paper deals with the simulation of one-dimensional waves, where the system resulting from the discretization by the Finite Difference Method is solved using the Multigrid Method with the conventional Gauss-Seidel solver, in order to decrease the computational time. Temporal discretization using the Time-Stepping method, where the system of equations is solved at each time step sequentially.

Key-words: Hyperbolic equations, Multigrid, computational simulations and Time-Stepping.

RESUMO
Vários problemas de Engenharia são modelados computacionalmente, estas simulações envolvem grandes sistemas, que são normalmente difíceis de resolver. Este artigo trata da simulação de ondas unidimensionais, onde o sistema resultante da discretização pelo Método da Diferença Finita é resolvido usando o Método Multigrid com o solucionador convencional Gauss-Seidel, a fim de diminuir o tempo computacional. Discretização

temporal usando o método TimeStepping, onde o sistema de equações é resolvido em cada passo sequencialmente.

Palavras-chave: Equações hiperbólicas, Multigrid, simulações computacionais e Time-Stepping.

1 INTRODUCTION

There are several Engineering problems modeled by differential equations, which can be classified as elliptical, parabolic or hyperboilic, in which it is generally not possible to obtain an analytical solution. Therefore, approximate methods are the main strategy Kohut et al. [29]. According to Dehghan and Dehghan and Mohebbi [1], hyperbolic partial differential equations (PDE) model several physical phenomena, from macrocosmic models, such as the vibration of structures, to microscopic models, which are used in atomic physics. Thus, Engineering problems involving this type of equation are recurrent in the literature. Some of these applications are found in Devi et al. [2] in the telegraphic equation De Sterck et al. [3] in systems of equations for advection and linear elasticity, Dehghan and Mohebbi [1] in wave propagation in the linear case, Rincon and Quintino [4] in the non-linear case, Avalos and Lasiecka [5] in applications in structural acoustics, Metaxas and Meredith [6] in the propagation of electric charges, and also in Bailly and Juve [7] in sound waves.

In this context, an approximate solution for the wave equation is presented, which consists of a hyperbolic PDE, with several applications that are linked to the propagation of different types of mechanical and electromagnetic waves, such as sound waves, vibration of membranes, strings, sea waves, geological prospecting, x-rays, radar, microwaves, among others. According to Cuminato and Meneguette [8], hyperbolic equations differ from parabolic and elliptical equations, since their solution is often less smooth than the problem data, so the
discontinuities are carried without smoothing, which may cause the formation of singularities, even with wellbehaved
initial data.

This work presents computational simulations of the one-dimensional wave equation (1D). The domain discretization is performed using the Finite Difference Method (FDM), presented by Cuminato and Meneguette [8], which results in a system of implicit equations, which when solved produces a fourth order method. Some of the variations in the approach to the wave equation are found in the works of Gopal et al.

[9] on polar coordinates, Melo [10] on elastic waves, and Brandt and Livishits [11] on stationary form.

The approximate solution to the wave equation is obtained by sweeping time with the Time-Stepping Method (TS). In this method, the discretized system in the spatial direction is solved in each time step until reaching a stopping criterion. The process is repeated until the final time step, using the solution in the previous time step as an initial estimate, Burden and Faires [12], Lent [13]. The Multigrid Method (MG) is used to accelerate the convergence of the system of equations in each time step, since it allows the smoothing of the oscillatory modes when using a set of meshes, Trottenberg et al. [14], Brandt and Livne [15].

2 Mathematical and Numerical Model

The 1D vibration problem of a string fixed at its ends is summarized in finding the displacement u(x; t) with the independent variables x and t respectively representing position and time. Assuming a positive scalar $\alpha^2 = 1/V^2$, such that, where V is related to the linear density and tension in the string, we define the wave equation, Olver [16], as

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$u(x,0) = f(x), 0 \leq x \leq 1, \tag{2}$$

$$u_t(x,0) = g(x), 0 \leq x \leq 1, \tag{3}$$

$$u(0,t) = u(1,t) = 0, t > 0, \tag{4}$$

where f(x) is the initial string configuration, g(x) is the initial velocity, u(0; t) and u(1; t) are dirichlet boundary conditions. Given the above, the analytical solution of the problem is given by

$$u(x,t) = \frac{1}{2}[f(x+\alpha t) + f(x-\alpha t)] + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(y)dy. \tag{5}$$

2.1 DISCRETIZATION 1D

Consider the problem defined by the eqs. (1) to (4). The size of each spatial element is defined by $h = L/N_x$ and an increase in time $\tau = t_f/N_t$, where $N_x > 0$ and $N_t > 0$, respectively, the number of spatial and temporal intervals, with final time $t_f > 0$ and string length L. Admitting an approximation v for the solution u, with $x_i = i.h$ and $t_k = k.\tau$, for the node i and the time k. Then the problem can be discretized by using the FDM.

$$\frac{v_i^{k-1} - 2v_i^k + v_i^{k+1}}{\tau^2} = \frac{\alpha^2}{h^2}[(\eta)(v_{i-1}^{k+1} - 2v_i^{k+1} + v_{i+1}^{k+1}) + (1-2\eta)(v_{i-1}^k - 2v_i^k + v_{i+1}^k) + (\eta)(v_{i-1}^{k-1} - 2v_i^{k-1} + v_{i+1}^{k-1})], \tag{6}$$

where η is a parameter of the method. By adopting $\eta > 0{,}25 >$ we have an unconditionally stable approach, Cuminato and Meneguette [8]. In this work, $\eta = 0{,}5$ was adopted, as it was empirically verified that this choice reduces discretization error and computational time. By reordering the terms of eq. (6), we have

$$v_i^{k-1} - 2v_i^k + v_i^{k+1} = \frac{\alpha^2\tau^2}{h^2}\left[\frac{1}{2}(v_{i-1}^{k+1} - 2v_i^{k+1} + v_{i+1}^{k+1} + v_{i-1}^{k-1} - 2v_i^{k-1} + v_{i+1}^{k-1})\right], \tag{7}$$

or still, assuming that $\lambda = \alpha\tau/h$, we obtain

$$a_p v_i^{k+1} = a_w v_{i-1}^{k+1} + a_e v_{i+1}^{k+1} + bp, \tag{8}$$

thus, the system of linear equations generated has its coefficients and source term given by

$$a_p = 1 + \lambda^2, \tag{9}$$

$$a_w = a_e = \frac{\lambda^2}{2}, \tag{10}$$

$$b_p = (-1 - \lambda^2)(v_i^{k-1}) + 2v_i^k + \frac{\lambda^2}{2}(v_{i-1}^{k-1} + v_{i+1}^{k-1})). \tag{11}$$

In order to calculate v_i^{k+1} it is necessary to know the solution in two previous time steps $v_i^k$and v_i^{k-1}. To start the process, v_i^{k-1} i is given by the initial configuration and v_i^k is given in Burden and Faires [12] by

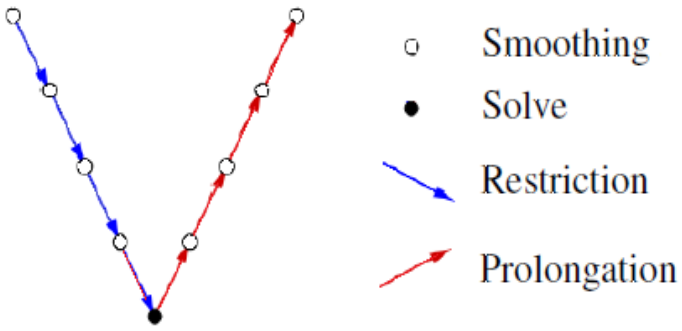
$$v_i^k = (1-\lambda^2)f(x_i) + \frac{\lambda^2}{2}f(x_{i+1}) + \frac{\lambda^2}{2}f(x_{i-1}) + \tau g(x_i). \quad (12)$$

3 Multigrid Method

The Multigrid method is a technique commonly adopted by a number of researchers when solving $Au = f$ equation systems, which regularly require high computational cost, Fedorenko [17] [18], Franco et al. [19], Wesseling [20], Pinto et al. [21], Malacarne et al. [30]. The Multigrid uses a basic iterative method named solver to smooth high and low frequency errors. This is possible due to the use of several meshes, as some of the modes that are smooth in fine meshes become oscillatory in coarse meshes,Trottenberg et al. [14].

In this work, the V-cycle is used to go through the grids, Briggs et al. [22]. An equation is smoothed v_1 times (pre-smoothing) in the fine grid. Then, its residue is restricted to the coarse grid with the restriction operator by full weighting, in which the residual equation is smoothed. The process is repeated until reaching the coarsest grid, where the problem is solved. In this case, a standard coarsening ratio is used, that is, $q = 2$. After that, the ascent process begins and the residual equation solution is prolonged by using a linear interpolation operator. The solution is then corrected and smoothed v_2 times (post-smoothing), and the process is repeated until reaching the finest grid h, Briggs et al [22], Oliveira et al. [23]. This process is illustrated by Fig. 1.

Figure 1. Multigrid V-cycle



This approach allows the iterative process of the Multigrid method to always act on the oscillating componentes of the errors, Franco et al. [24]. In this work $v_1 = v_2 = 2$ was adopted, as according to Dehghan and Mohebbi [1] this is a choice that produces satisfactory results to the wave equation.

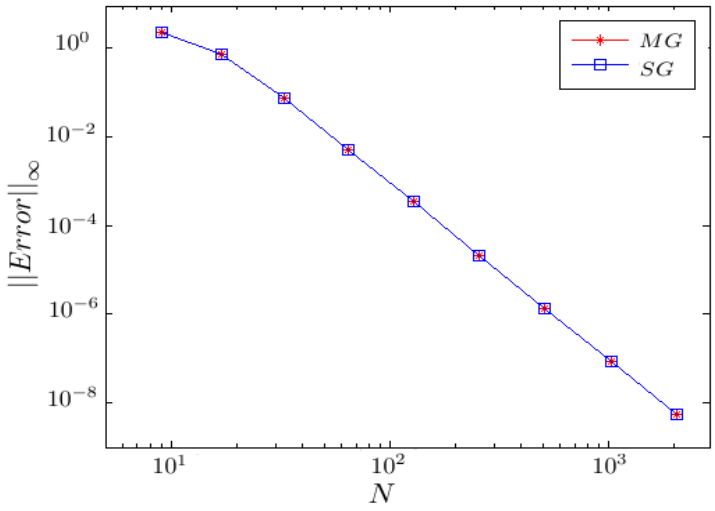
4 RESULTS

This section presents code verification techniques based on numerical simulations and a posteriori analysis of the results obtained by using the Multigrid and Singlegrid (SG), with the Gauss-Seidel solver in lexicographical ordering, Adms et al. [25]. The problem of wave propagation on a string, modeled by the eqs. (1) to (4) is solved by admitting $\alpha = 2$, with the initial configuration $f(x) = \sin(\pi x)$ and initial speed $g(x) = 0$. The tests were performed on a computer with an Intel Core i3 1:5 GHz processor, 4 GB of RAM, and 64 bit Windows 10
operating system.

4.1 DISCRETIZATION ERROR

The same number of points is adopted in the spatial and temporal discretization $N = N_x = N_t$, parameter $\eta = 0{,}5$, time $t_f = 1{,}0$ s, with double precision and the stop criterion based on rounding error in order to minimize iteration error. In order to verify the behavior of discretization errors, the infinity norms of the errors, see Burden and Faires [12], with $\text{Error} = |u - v|$ are presented in Fig. 2.

Figure 2. Discretization error for Singlegrid and Multigrid methods versus N.



4.2 EFFECTIVE AND APPARENT ORDERS

Numerical solutions can be verified using the effective order P_E and the apparent order P_U, Dehghan and Mohebbi [1], Anunciacão et al. [26]. Richardson's estimator is also used based on the apparent order of the numerical error, given by.

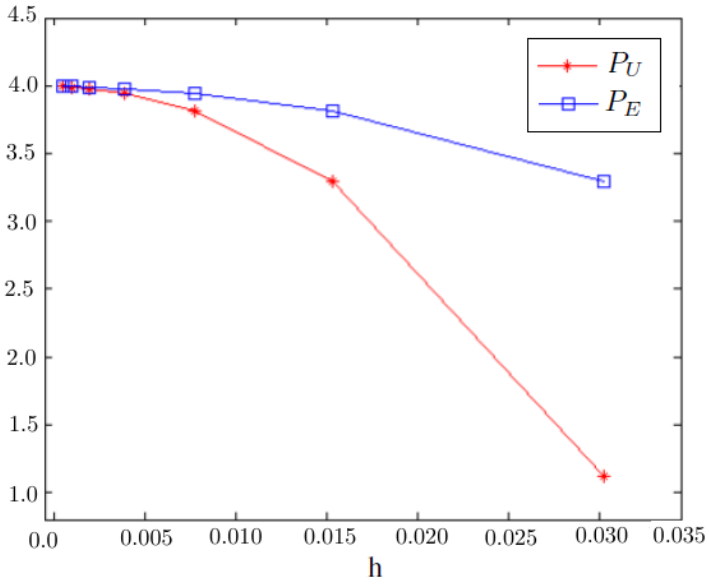
$$P_U = \frac{log\left(\frac{\phi_2 - \phi_3}{\phi_1 - \phi_2}\right)}{log(q)} \tag{13}$$

where ϕ_1, ϕ_2 and ϕ_3 respectively indicate the solutions in the fine, coarse and extra coarse grids, for point x = 0:5. When the analytical solution is known, P_E is calculated by

$$P_E = \frac{log\left(\frac{Error(\phi_2)}{Error(\phi_1)}\right)}{log(q)} \tag{14}$$

where $\text{Error}(\phi_2)$ and $\text{Error}(\phi_1)$ respectively represent the errors in the coarse and fine grids in the $x = 0.5$. In Fig.3 depicts the values of the effective and apparent orders for different grids.

Figure 3. Effective and apparent orders.



The Fig. 3 shows that P_U and P_E tend to 4,0. Therefore, the model presented in eqs. (6) to (11) generates a high order method, which is a desired characteristic in approximate methods. By refining the mesh at a $q = 2$ coarsening rate the discretization error is reduced 16 times.

4.3 COMPLEXITY ORDER

According to Burden and Faires [12], by taking into account the results of the computational t_{cpu} it is possible to perform a geometric (or non-linear) fit in order to verify the complexity of the algorithm used, where

$$t_{cpu} = c.N^p \tag{15}$$

and c is the coefficient related to the method, p represents the order of complexity of the solver associated with the slope of the fit curve, and N is the dimension of the problem. Theoretically, according to Trottenberg et al. [14], p must be close to the unit when employing the Multigrid method, indicating its linear behavior. Table 1 shows the results of these parameters for different values of λ in Singlegrid and Multigrid.

Table 1. Parameters of the geometric fit

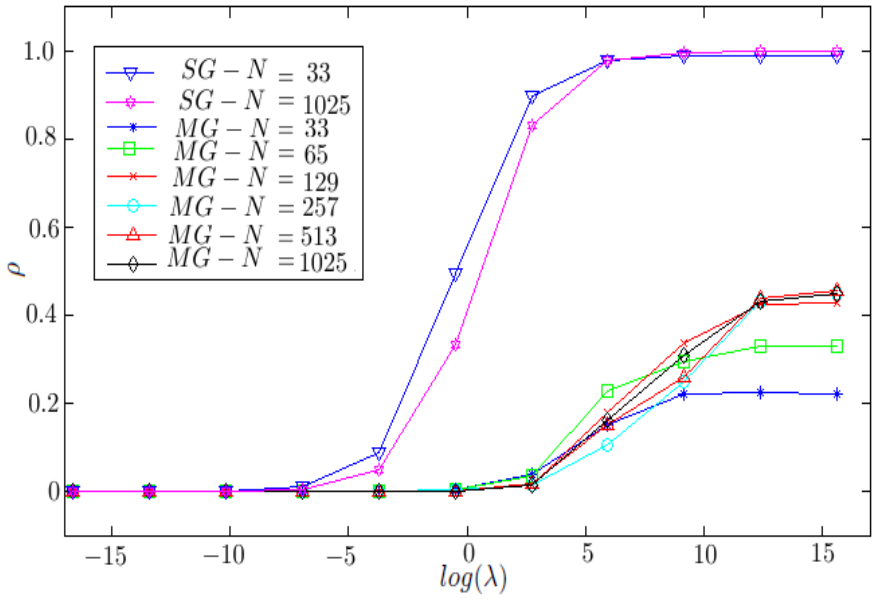
λ	SG		MG	
	c	p	c	p
10^0	1.00E-05	0.9723	1.00E-05	0.8941
10^1	2.00E-05	1.0364	7.00E-06	0.9559
10^2	4.00E-05	1.1175	2.00E-05	0.9249
10^3	5.00E-05	1.2558	3.00E-05	0.9161
10^4	7.00E-06	1.5312	4.00E-05	0.9012
10^5	3.00E-06	1.7894	5.00E-05	0.9421

It can be seen that as λ increases, p of Singlegrid presents values close to 2 and p of Multigrid tends to 1.

4.4 CONVERGENCE FATOR

In order to verify which intervals and values of the variables were used in the calculations, the parameter λ and the convergence factor ρ were considered, with $\rho = ||res(it)||/||res(it-1)||$, with $= ||res(it)||$ the infinity norm of residue in iteration it, see Briggs et al. [22]. According to Horton and Vandewalle [27], Thole and Trottenberg [28], λ can be considered as a measure of the degree of anisotropy in the operator discretized in a given mesh. Such anisotropy can affect the performance of solvers. This approach can also be used to find the intervals of λ, where Multigrid and Singlegrid are most efficient. This indicator is calculated for different meshes, as shown in Fig 4.

Figure 4. Convergence fator.



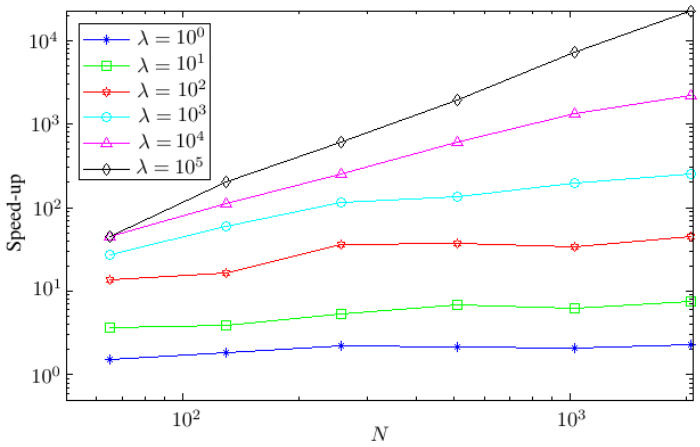
It can be verified that for log(λ) values inferior to 0, both Multigrid and Singlegrid present small values of ρ, which implies a high convergence. As log(λ) becomes greater than 0, Singlegrid presents ρ values close to 1, that is, it is inefficient in this interval, Franco et al. [24]. However, the convergence factor of the Multigrid presents ρ values close to 0.45, making it more efficient.

It is also observed that for all the parameters that compose α, the Multigrid method presents values of the convergence factor ρ inferior to 0.5. Therefore, this method can be considered robust for the solution of the wave equation.

4.5 SPEED-UP

The ratio between the computational time of the Singlegrid (t_{cpuSG}) and that of the Multigrid (t_{cpuMG}) is given by $\text{Speed} - \text{up} = t_{cpuSG}/t_{cpuMG}$, see in Fig. 5.

Figure 5. Ratio between the t_{cpu} of the Multigrid and Singlegrid methods



It is noted that t_{cpuMG} is lower than tcpuSG and that the Speed-up increases considerably for large values of λ. For instance, for $\lambda = 10^4$ in a mesh with $N = 2049$, there is a $t_{cpuMG} = 30.1$ s and $t_{cpuSG} = 65977.3$ s, that is, $\text{Speed} - \text{up} = 2191.9$.

5 CONCLUSIONS

When solving the 1D problem of wave propagation on a string, with discretization given by the eqs. (9) to (11), there is a fourth order scheme confirmed by the orders P_U and P_E. When solving the resulting equation system with the Singlegrid and Multigrid methods with the lexicographic Gauss-Seidel solver, discretization errors close to the analytical solution may be found. By analyzing the complexity, convergence factors and the computational time of the Multigrid and Singlegrid methods, it can be said that the use of the Multigrid is more appropriate in cases where the problem has $\log(\lambda) > 0$, whereas the Singlegrid is recommended for cases in which $\log(\lambda) < 0$. Thus, a robust algorithm is generated for the solution of the wave equation, regardless of the parameters adopted. It is suggested that further studies extend the approach proposed in this work to other hyperbolic problems.

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