Monsky's Theorem

O Teorema de Monsky

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Guilherme Israel Vedana
Licentiate in Mathematics
MSc student at IMPA – Instituto de Matemática Pura e Aplicada
IMPA – Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil
E-mail: guilherme.israel@impa.br

João Biesdorf
PhD in Mathematics
Universidade Tecnológica Federal do Paraná
Departamento de Matemática – Universidade Tecnológica Federal do Paraná, Pato Branco, PR, Brazil
E-mail: jbiesdorf@utfpr.edu.br

Ivan Italo Gonzales Gargate
PhD in Mathematics
Universidade Tecnológica Federal do Paraná
Departamento de Matemática – Universidade Tecnológica Federal do Paraná, Pato Branco, PR, Brazil
E-mail: ivangargate@utfpr.edu.br

ABSTRACT
The main objective of this paper is to prove Monsky's Theorem, that provides a beautiful application of the 2-adic valuation in order to solve a plane geometry problem. This theorem states that given any dissection of a square into finitely many nonoverlapping triangles of equal area the number of triangles must be even. In order to prove this statement, we will need some previous results from Combinatorial Topology and Algebra.

Keywords: Dissection of a square into triangles of equal area, 2-adic valuation.

RESUMO
O objetivo principal desse trabalho é a demonstração do Teorema de Monsky, o qual fornece uma bela aplicação da valoração 2-ádica na resolução de um problema de Geometria Plana. Esse teorema afirma que dada qualquer dissecção de um quadrado em triângulos não sobrepostos e de mesma área, o número de triângulos deve ser par. Com o objetivo de demonstrar essa afirmação precisaremos de alguns resultados da Topologia Combinatória e da Álgebra.

Palavras-chave: Dissecção de um quadrado em triângulos de mesma área, Valoração 2-ádica.

1 INTRODUCTION

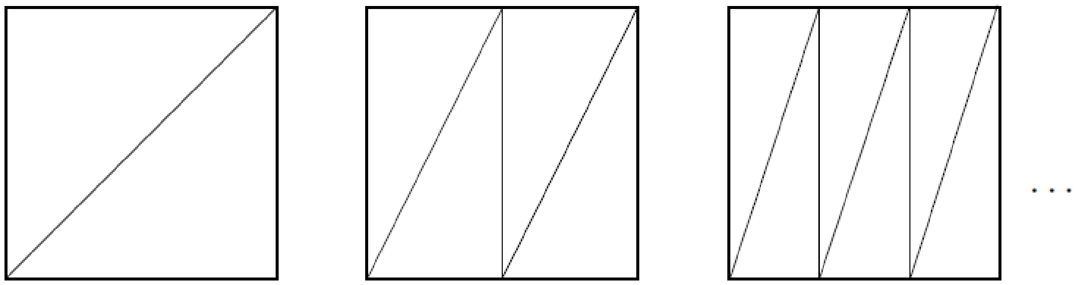
Consider the following situation: we want to dissect a square into n nonoverlapping triangles of equal area. This is trivial if n is even: we can divide the horizontal sides of the square into $\frac{n}{2}$ segments of equal length, which determines a dissection of the square into $\frac{n}{2}$ congruent rectangles and then draw the diagonal of each rectangle (see figure 1 below). What about the case when n is odd? Is such a dissection possible?

This problem arose when Fred Richman was preparing a master's exam, in 1965. In fact, the question was first asked about a rectangle, instead of a square. If we consider the cartesian coordinate plane (the xy −plane), using horizontal or vertical dilations, if necessary, we can assume without loss of generality that our rectangle is a square. Moreover, we can suppose our square to have vertices (0,0), (1,0), (1,1) and (0,1). We will call this square the *unit square*.

At first the problem seemed to be easy, but Richman could not find a complete solution for it. He found that such a dissection is impossible for $n = 3$ and 5, and if it exists for some n, then it exists for $n + 2$. He presented the question to his colleague, John Thomas, who got some more progress toward the solution, but also could not solve it for completely. Thomas found that for the unit square there is no such a dissection where all the vertices of the triangles have rational coordinates with odd denominators [Th68].

The problem was presented to the American Mathematical Monthly as Problem 5479, in 1967. It was only in 1970 that Paul Monsky, working on previous work of Thomas, in a stroke of genius, presented the first (and unique up to now) solution for the problem, that became known as Monsky's Theorem. In this paper we present the solution discovered by Monsky.

Figure 1. [AZ14]



2 VALUATIONS

The first tool we need to prove Monsky's Theorem comes from Algebra: it is the concept of **valuation**.

Definition: Let K be a field. A **real valuation** of K is a mapping $\nu: K \to \mathbb{R}$ satisfying the following properties:

i) $\nu(x) \geq 0, \forall x \in K, \nu(x) = 0 \; if, and \; only \; if, x = 0;$

ii) $\nu(xy) = \nu(x)\nu(y), \forall x, y \in K;$

iii) $\nu(x + y) \leq \nu(x) + \nu(y), \forall x, y \in K.$

Note that if $1_K \in K$ is the identity of K with respect to multiplication, then $\nu(1_K) = \nu(1_K) \cdot \nu(1_K)$ and from i) we conclude that $\nu(1_K) = 1$. Similarly, we have that $1 = \nu(1_K) = \nu(-1_K) \cdot \nu(-1_K)$ and again from i) we get $\nu(-1_K) = 1$. Therefore, $\nu(x) = \nu(-x), \forall x \in K$.

Let $r \in \mathbb{Q}\backslash\{0\}$ be a nonzero rational. Then there exist nonzero integers p, q such that $r = \frac{p}{q}$. From the Fundamental Theorem of Arithmetic we know that there exist unique integers a, b, n with $b \neq 0$ and $n \geq 0$, such that a and b are both odd and $\frac{p}{q} = 2^n \frac{a}{b}$. Consider the function

$$|\cdot|_2: \mathbb{Q} \to \mathbb{R}$$

$r \mapsto 2^{-n}, 0 \mapsto 0.$

It is easy to check that this function is well-defined and satisfies the properties i) to iii) of the above definition. Therefore, it defines a real valuation of the field $\mathbb{Q}$. In fact, the function $|\cdot|_2$ satisfies a stronger property than iii), the so called **non-Archimedean property**:

$$|x+y|_2 \leq \max\{|x|_2, |y|_2\} \text{ with equality whenever } |x|_2 \neq |y|_2.$$

The valuation $|\cdot|_2$ is called the **2-adic valuation**.

Here are some examples:

$|4|_2 = \frac{1}{4}, |6|_2 = \frac{1}{2}, |5|_2 = 1, \left|\frac{1}{20}\right|_2 = 4$ and $\left|\frac{7}{12}\right|_2 = 4$.

Note that given a positive integer n, then $|n|_2 < 1$ if, and only if, n is even.

Once we have our 2-adic valuation $|\cdot|_2$ defined over $\mathbb{Q}$ we ask about the possibility of extending it to a valuation ν of $\mathbb{R}$ still satisfying the non-Archimedean property. Chevalley's Theorem guarantees that such an extension exists.

Summarizing, there exists a real valuation ν of $\mathbb{R}$ that extends the 2-adic valuation and such that $\nu(x+y) \leq \max\{\nu(x), \nu(y)\}, \forall x, y \in \mathbb{R}$, with equality if $\nu(x) \neq \nu(y)$. These are the algebraic results we need.

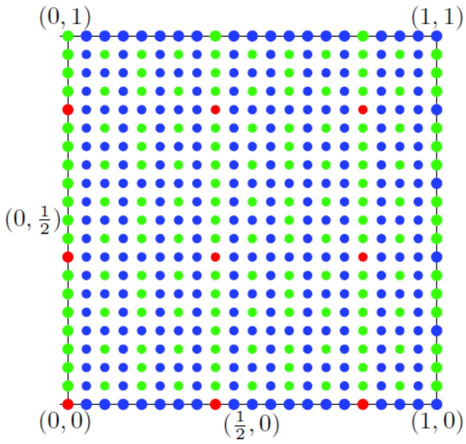
3 COLORING THE PLANE

We now proceed to the topological part of Monsky's proof. We start constructing a three coloring of the xy −plane in the following way: given any point (x, y) in the plane, we consider the ordered triple $(x, y, 1)$ and we look at the maximum of the set $\{\nu(x), \nu(y), \nu(1)\}$. We will color the point (x, y) blue, green or red if the maximal ν −value occurs first respectively at the first, second or third coordinate of the triple $(x, y, 1)$. Thus, we have

- (x, y) is blue if $\nu(x) \geq \nu(y)$ and $\nu(x) \geq 1 = \nu(1)$;
- (x, y) is green if $\nu(x) < \nu(y)$ and $\nu(y) \geq 1$;
- (x, y) is red if $\nu(x) < 1$ and $\nu(y) < 1$.

It is easy to see that this rule assigns a unique color to each point in the xy −plane. Figure 2 shows the coloring of the points in the unit square whose coordinates are rational numbers of the form $\frac{p}{20}, 0 \leq p \leq 20$.

Figure 2. [AZ14]



This coloring has very interesting properties and is the key-step of the proof. We now derive some of these properties.

Lemma 1: Let $p_b = (x_b, y_b)$ be a blue point, $p_g = (x_g, y_g)$ a green point and $p_r = (x_r, y_r)$ a red point in the plane and let

$$d \coloneqq \det\begin{bmatrix} x_b & y_b & 1 \\ x_g & y_g & 1 \\ x_r & y_r & 1 \end{bmatrix} \qquad (1)$$

Then $\nu(d) \geq 1$.

Proof: Expanding the determinant, we get

$$d = x_b y_g \cdot 1 + y_b \cdot 1 \cdot x_r + 1 \cdot x_g y_r + \left(-1 \cdot y_g x_r\right) + \left(-x_b \cdot 1 \cdot y_r\right) + \left(-y_b x_g \cdot 1\right) \ (2)$$

In the first term, the product of the main diagonal terms, we have that $\nu(x_b) \geq 1$, $\nu(y_g) \geq 1$ and $\nu(1) = 1$. It follows that $\nu(x_b y_g \cdot 1) = \nu(x_b) \cdot \nu(y_g) \cdot \nu(1) \geq 1$. The other five terms have a ν −value that is strictly smaller than $\nu(x_b y_g \cdot 1)$. This can be obtained by comparing each factor with the term of the main diagonal in the same row, for each term in the sum and the fact that $\nu(-1) = 1$. For example, for the second term we have that $\nu(y_b) \leq \nu(x_b)$, $\nu(1) \leq \nu(y_g)$ and $\nu(x_r) < \nu(1)$. Thus $\nu(y_b \cdot 1 \cdot x_r) = \nu(y_b) \cdot \nu(1) \cdot \nu(x_r) < \nu(x_b) \cdot \nu(y_g) \cdot \nu(1) = \nu(x_b y_g \cdot 1)$.

Applying the non-Archimedean property of the valuation ν to the sum (2) we obtain that $\nu(d) = \nu(x_b y_g \cdot 1) \geq 1$.

■

Let us fix some notation. Given a line segment in the plane, we will call it a *red-blue segment* if its endpoints are colored red and blue. A triangle in the plane will be called a *complete triangle* if its vertices have all three colors.

Corollary 1: Given any line in the plane, it contains points of at most two different colors.

Proof: Suppose the lemma is false. Then, there exists a line l that contains points p_b, p_g and p_r colored blue, green and red, respectively. Then, one of the three points lie on the line segment determined by the other two. Thus, the first point is a linear combination of the last and, therefore, the determinant in (1) would be zero. But this is a contradiction with Lemma 1, because $\nu(0) = 0$.

■

Corollary 2: The ν −value of the area of any complete triangle is strictly larger than 1.

Proof: Consider a complete triangle in the plane. Let p_b, p_g and p_r be its vertices, colored respectively blue, green and red, and with coordinates as stated in Lemma 1. Then, the area A of this triangle is given by $A = \frac{1}{2}|d|$, where d is the determinant (1). Then, Lemma 1 implies

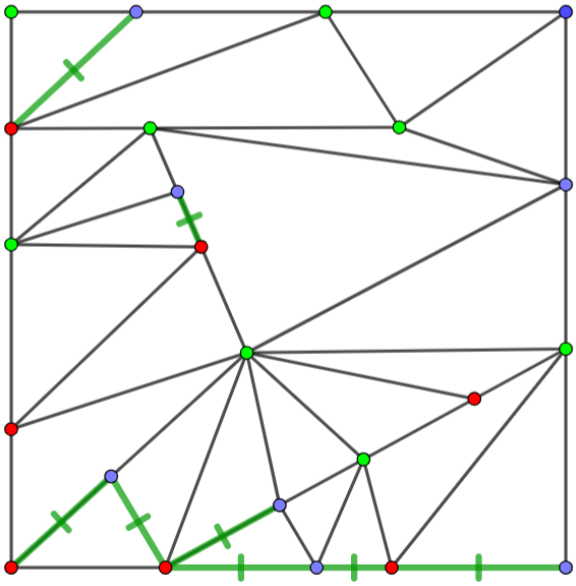
$$\nu(A) = \nu\left(\frac{1}{2}|d|\right) = \nu\left(\frac{1}{2}\right)\nu(|d|) = \left|\frac{1}{2}\right|_2 \nu(d) \geq \left|\frac{1}{2}\right|_2 = 2 > 1.$$

■

Lemma 2: In any dissection of the unit square into finitely many nonoverlapping triangles (not necessarily having equal area) there exists at least one complete triangle.

Proof: We will follow the idea of the proof of Sperner's Lemma, due to Emanuel Sperner. Suppose we are given such a dissection of the unit square. Consider the set of vertices of the triangles. We will say that two vertices are *adjacent* if they are both in the same edge of some triangle and the line segment joining them contains no other vertices. By a *segment* we will be referring to a line segment joining two adjacent vertices (see figure 3 below). A red-blue segment can be on the boundary of the square or in its interior. We will count modulo 2 the number of the red-blue segments.

Figure 3. The red-blue segments are draw in green.



Step 1: (red-blue segments on the boundary of the square)

Figure 2 together with Corollary 1 imply that only the bottom edge of the square contains red-blue segments. Because the points (0,0) and (1,0) are colored red and blue, respectively, it follows that the number of red-blue segments on the bottom edge of the square, and therefore on the whole boundary of the square, is odd. Therefore,

$$\sum_{\substack{red-blue\ segments\ on \\ the\ boundary\ of\ the\ square}} 1 \equiv 1 \ (mod\ 2).$$

Step 2: (red-blue segments in the boundary of each triangle)

Let us fix some triangle in the dissection and count modulo 2 the number of red-blue segments on its boundary. Fix some edge e of the triangle:

i) If e contains a green point, then by Corollary 1, it does not have any red-blue segment;

ii) If e does not contain any green point, then if the endpoints of e have the same color – blue or red – the number of red-blue segments on e will be even;

iii) If e does not contain any green point and if its endpoints have different colors, then the number of red-blue segments on e will be odd.

It follows that a complete triangle has an odd number of red-blue segments on its boundary and a triangle that is not complete has an even number of red-blue segments on its boundary.

Step 3: (counting modulo 2 the total number of red-blue segments)

Suppose our square is dissected in n triangles $T_1, \dots, T_n$. For each $i, 1 \leq i \leq n$, let m_i be the number of red-blue segments on the boundary of T_i. We know, from step 2, that $m_i \equiv 1\ (mod\ 2)$ if, and only if, T_i is complete.

Counting the red-blue segments directly and using the triangles, we obtain the following:

$$\sum_{\substack{i\in\{1,\dots,n\} \\ T_i\ is\ complete}} m_i + \sum_{\substack{i\in\{1,\dots,n\} \\ T_i\ is\ not\ complete}} m_i = \sum_{\substack{red-blue\ segments \\ on\ the\ boundary \\ of\ the\ square}} 1 + 2\left(\sum_{\substack{red-blue\ segments \\ in\ the\ interior \\ of\ the\ square}} 1\right)$$

where the second summand in the right-hand side of the above equality is multiplied by 2 because in the left-hand side each red-blue segment in the interior of the square lies on the common boundary of two different triangles, therefore it is counted twice.

Reducing this equality modulo 2 and using step 1, we get

$$\sum_{\substack{i\in\{1,\dots,n\} \\ T_i\ is\ complete}} 1 \equiv \sum_{\substack{red-blue\ segments \\ on\ the\ boundary \\ of\ the\ square}} 1 \equiv 1\ (mod\ 2)$$

If there were no complete triangles in the dissection, we would have $\sum_{\substack{i\in\{1,\dots,n\} \\ T_i\ is\ complete}} 1 \equiv 0\ (mod\ 2)$, which is not the case. Therefore, there exist some complete triangle in the given dissection. This concludes the proof.

■

4 PROOF OF MONSKY'S THEOREM

We are now in a position to prove Monsky's Theorem.

Suppose we are given a dissection of the unit square into n nonoverlapping triangles of equal area. We claim that n must be even. Because the area of the unit square is 1, it follows that the area of each triangle is $\frac{1}{n}$. Lemma 2 ensures us the existence of a complete triangle in the given dissection. Corollary 2 implies that the area $\frac{1}{n}$ of such a complete triangle satisfies $\left|\frac{1}{n}\right|_2 = \nu\left(\frac{1}{n}\right) > 1$. Because

$$1 = |1|_2 = \left|n \cdot \frac{1}{n}\right|_2 = |n|_2 \left|\frac{1}{n}\right|_2$$

it follows that $|n|_2 < 1$ implying that n is even and proving the theorem.

■

5 CONCLUSIONS AND FURTHER RESULTS

This dissection problem was an easy to understand and very specific question that brought a very beautiful solution evidencing the power of Mathematics in explaining characteristics, patterns and properties of our universe, from the most complex to the "simplest" and innocent things. Monsky's Theorem is one of a vast number of examples that demonstrate the profound, smooth and harmonious way the different areas of Mathematics are interwoven.

Although there is no known application of Monsky's Theorem in other areas of Mathematics or the real world, it encourages us to study the different areas of Mathematics and seek for relations between them. Richman and Thomas' problem together with Monsky's proof led to some generalizations. Here we present only three of them, with some references:

1) In 1979, David Mead generalized Monsky's Theorem for higher dimensions. He proved that an n −dimensional cube can be divided into m simplices of equal volume if, and only if, m is a multiple of $n!$. In his proof Mead used a version of Sperner's Lemma in higher dimensions and the p −adic valuations for all primes p dividing $n!$. (The p −adic valuation, where p is a prime integer, is defined in the same way as the 2 −adic valuation). See Mead's paper [Me79];

2) In 1989, Elaine Kasimatis proved the following theorem: Given any integer $n \geq 5$, a regular n −gon is dissectable into m triangles of equal area if, and only if, m is a multiple of n. See [Kas89];

3) Sherman Stein conjectured that any centrally symmetric polygon has the same property than the square, namely that in any dissection of the polygon into finitely many nonoverlapping triangles of equal area, the number of triangles must be even. The result was proved by Paul Monsky in 1990. See Monsky's paper [Mon90].

There are several proved results and a vast number of open problems arisen from Monsky's Theorem.

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