On p-biharmonic equations with critical growth

Sobre equações p-biharmónicas com crescimento crítico

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ABSTRACT
We study p-biharmonic problems dealing with concave-convex nonlinearitiesin the critical case with both Navier and Dirichlet boundary conditions in a bounded, smooth domain and some $f \in C(\Omega)$, which is either a positive or a change-sign function. By applying Nehari's minimization method, we prove the existence of two nontrivial solutions for the problems. If f is positive, both solutions of the problem with Navier boundary condition are positive.

Keywords: p-biharmonic operator, Navier and Dirichlet boundary conditions, concave-convex nonlinearities, critical growth.

RESUMO
Estudamos problemas p-biharmónicos que lidam com não-linearidades côncavo-convexas no caso crítico, tanto com Navier como com Dirichlet em condições de fronteira num domínio delimitado e suave e alguns $f \in C(\Omega)$, que é ou uma função positiva ou uma função de sinal de mudança. Ao aplicar o método de minimização de Nehari, provamos a existência de duas soluções não triviais para os problemas. Se f for positivo, ambas as soluções do problema com a condição de limite de Navier são positivas.

Palavras-chave: operador p-biharmonic, condições de limite de Navier e Dirichlet, não-linearidades côncavo-convexas, crescimento crítico

1. Introduction

In this work we study the following fourth-order problems

$$\begin{cases} \Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x)|u|^{q-2}u + |u|^{p^*-2}u & \text{in} \quad \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

and

$$\begin{cases} \Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x)|u|^{q-2}u + |u|^{p^*-2}u & \text{in} \quad \Omega, \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain. We suppose that the exponents p and q are such that $1 < p < \infty$, $N > 2p$, $1 < q < p$, and that $p^* = \frac{Np}{N-2p}$ denotes the Sobolev critical exponent for fourth-order problems. The parameter λ is positive and $f : \overline{\Omega} \to \mathbb{R}$ is either a positive or a changing-sign function.

The study of critical growth in semilinear and quasilinear problems are a major subject of study, since the seminal work of Brézis and Nirenberg [4]. See also [7] and [8] and references therein.

The p-biharmonic operator Δ_p^2 has recently attracted the attention of many researchers (see [1], [2], [5], [12], [13], [17] and references therein). Looking for positive solutions $u, v > 0$ defined in a bounded, smooth domain Ω, problem (2), it is sometimes associated with Hamiltonian systems with Dirichlet boundary conditions (see [11]).

When $p = 2$, the biharmonic operator frequently appears in Navier-Stokes equations as a viscosity coefficient, but was also used to describe the failure of the Tacoma Narrows bridge, see [16]. The biharmonic equation $\Delta^2 u = 0$ appears in quantum mechanics and also in the theory of linear elasticity modelling Stokes' flows.

Existence of positive solution for semilinear biharmonic and p-biharmonic problems in a smooth, bounded domain $\Omega \subset \mathbb{R}^N$ with Navier boundary conditions are extensively studied (see [3, 17]), while a series of works proving existence of solutions for problem (2) is also available, see [3] and [6].

This is not the case of p-biharmonic operators: results about existence of solutions are mostly restricted to Steklov and Navier boundary conditions, see [13] and [17]; existence and multiplicity of solutions for problems with Dirichlet boundary conditions in bounded, smooth domains are not so common.

The main motivation for the present work comes from Bernis, García-Azorero and Peral [3], who in 1996 studied the following problems for the biharmonic operator,

$$\begin{cases} \Delta^2 u = \lambda|u|^{q-2}u + |u|^{2^*-2}u & \text{in} \quad \Omega, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta^2 u = \lambda|u|^{q-2}u + |u|^{2^*-2}u & \text{in} \quad \Omega, \\ u = \Delta u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain, $\lambda > 0$, $1 < q < 2$ e $2^* = \frac{2N}{N-4}$ is the critical exponent of Sobolev for the biharmonic operator. The authors proved results of existence and multiplicity by applying both the sub- and supersolution method and Ljusternik-Schnirelmann theory. In particular, they proved that the functionals associated with these problems satisfy a local Palais-Smale condition. So, our study can be considered a generalization of these results for the p-biharmonic operator.

Another motivation for this work is the article [12], where the existence of two nontrivial solutions for (1) in the *subcritical case* is proved, if $\lambda > 0$ is small enough.

To handle problems (1) and (2) simultaneously, we apply a result by Gazzola, Grunau and Sweers [9], which proves that the best constant for the immersion $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ equals the best constant for the immersion $W_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

In this work we prove the existence of two nontrivial solution for problems (1) and (2), if λ is small enough. We state our main result:

Theorem 1. *There exists $\lambda_0 > 0$ such that, for all $\lambda \in (0, \lambda_0)$, problems* (1) *and* (2) *have two distinct nontrivial solutions. If f is positive, the two solutions of problem* (2) *are positive.*

To obtain our result, we consider the "energy" functional

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\Delta u|^p \mathrm{d}x - \frac{\lambda}{q}\int_\Omega f(x)|u|^q \mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x. \tag{3}$$

In the case of problem (1), J_λ is defined in $W_0^{2,p}(\Omega)$; in (2), J_λ is defined in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. So, let $\mathbf{E} = \mathbf{E}(\Omega)$ stand for the space $W_0^{2,p}(\Omega)$ *or* the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, according to the problem we deal with, both considered with the norm $\|u\| = \|\Delta u\|_p$, where $\|\cdot\|_p$ denotes the usual norm of $L^p(\Omega)$. Critical points of J_λ are weak solutions of problems (1) and (2).

From now on we denote by

$$S = \inf\{\|u\|^p \ : \ u \in \mathbf{E} \text{ and } \|u\|_{p^*} = 1\}$$

the best constant for the Sobolev's immersion of $\mathbf{E}$ into L^{p^*}. So, by definition, $\|u\|_{p^*} \leq S^{-1/p}\|u\|$. When $\Omega = \mathbb{R}^N$ the extremal function $U(x)$ attains the constant S: $\|U\|_{p^*} = S^{-\frac{1}{p}}\|U\|$.

We consider, for each $\lambda > 0$, Nehari's minimization problem:

$$m_\lambda(\Omega) = \inf\{J_\lambda(u) \,:\, u \in \mathcal{N}_\lambda(\Omega)\},$$

where

$$\mathcal{N}_\lambda(\Omega) = \{u \in \mathbf{E} \setminus \{0\} \,:\, \langle J'_\lambda(u), u\rangle = 0\}.$$

We define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle = \|u\|^p - \lambda \int_\Omega f(x)|u|^q \mathrm{d}x - \int_\Omega |u|^{p^*} \mathrm{d}x.$$

So, for each $u \in \mathcal{N}_\lambda(\Omega)$, we have

$$\langle \psi'_\lambda(u), u\rangle = p\|u\|^p - q\lambda \int_\Omega f(x)|u|^q \mathrm{d}x - p^* \int_\Omega |u|^{p^*} \mathrm{d}x. \tag{4}$$

(In order to guarantee that $\mathcal{N}_\lambda(\Omega)$ is really a manifold, is enough to have $\psi'_\lambda(u) \neq 0$. See Lemma 2.)

Since the immersion of $\mathbf{E}$ into $L^{p^*}(\Omega)$ is not compact, we apply P.L. Lions' lemma (see [14]-[15]), which implies that J_λ satisfies a local Palais-Smale (PS) condition below the level $\frac{2}{N}S^{\frac{N}{2p}} - D\lambda^\beta$. (We denote by S the best constant for the immersion of $\mathbf{E}$ into L^{p^*} and $\beta = \frac{p^*}{p^*-q}$; the constant D will be defined later on.) In order to do that, we obtain estimates for $m^+_\lambda(\Omega)$ and $m^-_\lambda(\Omega)$, see Lemmas 6 and 10.

In the *subcritical case*, Ji and Wang [12] considered a subset $\Xi \subset \Omega$ where the weight function f is positive and found a solution for the auxiliary problem

$$\begin{cases} \Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u) = |u|^{r-2}u & \text{in} \quad \Xi, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Xi, \end{cases} \tag{5}$$

thus establishing the existence of a solution for the minimization problem

$$m_0(\Xi) = \inf_{\mathcal{N}_0(\Xi)} J_0(u),$$

where J_0 is the functional naturally associated with the problem and $\mathcal{N}_0(\Xi)$ is the Nehari manifold related to Ξ. This solution was used to prove that $m^+_\lambda(\Omega) < 0$.

A similar approach is not possible to deal with $m_\lambda^+(\Omega)$ in our case: the solution of (5) is not available in the critical case: when $r = p^* = \frac{Np}{N-2p}$, we have

$$J_0(u) \geq \frac{2}{N} S^{\frac{N}{2p}}, \quad \forall\, u \in \mathcal{N}_0(\Xi)$$

and it turns out to be difficult to obtain compactness by applying the result of Lions.

We overcome this issue by noting that the functional J_0 has the mountain pass geometry. So, although not finding a solution for problem (5) when $r = p^* = \frac{Np}{N-2p}$, information about the sign of $m_0(\Xi)$ was enough to guarantee that

$$m_\lambda^+(\Omega) < 0 < \frac{2}{N} S^{\frac{N}{2p}} - D\lambda^\beta,$$

thus obtaining the aimed compactness.

In order to show that

$$m_\lambda^-(\Omega) < \frac{2}{N} S^{\frac{N}{2p}} - D\lambda^\beta$$

we applied the extremal function for the p-biharmonic operator, as in [17].

The outline of this article is as follows. In Section 2 we introduce the framework of this problem and state some preliminary results. In Section 3, applying Ekeland's variational principle, we prove the existence of a Palais-Smale sequence for the functional J, *via* implicit function theorem. In Section 4 we prove a *local* Palais-Smale condition, by using the concentration-compactness principle. Section 5 is devoted to the proof of our main result (Thm. 1).

2. The Nehari manifold $\mathcal{N}_\lambda(\Omega)$

We start proving that, for λ small enough, $\mathcal{N}^0(\Omega) = \emptyset$. This fact implies that $\mathcal{N}_\lambda(\Omega)$ is really a manifold.

Lemma 2. *There exists $\lambda_1 > 0$ (depending on p, q, N and f) such that, for all $\lambda \in (0, \lambda_1)$, we have $\mathcal{N}_\lambda^0(\Omega) = \emptyset$.*

Proof. We start noting that, if $u \in \mathcal{N}_\lambda^0(\Omega)$, then we have

$$\lambda \int_\Omega f(x)|u|^q \mathrm{d}x = \frac{p^* - p}{p - q} \int_\Omega |u|^{p^*} \mathrm{d}x = \frac{p^* - p}{p^* - q} \|u\|^p,$$

that is, for fixed q and p, the value $\lambda \int_\Omega f(x)|u|^q \mathrm{d}x$ determines $\|u\|_{p^*}$ e $\|u\|$.

If $\beta = p^*/(p^* - q)$, it is true that

$$\lambda \int_\Omega f(x)|u|^q \mathrm{d}x \leq \lambda \|f\|_\beta \|u\|_{p^*}^q \leq \lambda \|f\|_\beta S^{-\frac{q}{p}} \|u\|^q,$$

from what follows, for all $u \in \mathcal{N}_\lambda^0(\Omega)$,

$$\|u\| \leq \left[\lambda \left(\frac{p^* - q}{p^* - p} \right) \|f\|_\beta S^{-\frac{q}{p}} \right]^{\frac{1}{p-q}}. \tag{6}$$

We define $I_\lambda(u) \colon \mathbf{E} \setminus \{0\} \to \mathbb{R}$ by

$$I_\lambda(u) = K(p^*, q) \left(\frac{\|u\|^{p^*}}{\int_\Omega |u|^{p^*} \mathrm{d}x} \right)^{\frac{p}{p^*-p}} - \lambda \int_\Omega f(x)|u|^q \mathrm{d}x, \tag{7}$$

where $K(p^*, q) = \left(\frac{p^*-p}{p^*-q} \right) \left(\frac{p-q}{p^*-q} \right)^{\frac{p}{p^*-p}}$ e $\lambda \in [0, \infty)$. It follows that $I_\lambda(u) = 0$ for all $u \in \mathcal{N}_\lambda^0(\Omega)$.

However, for all $u \in \mathbf{E} \setminus \{0\}$, by Sobolev inequality and (6)

$$\begin{aligned} I_\lambda(u) &\geq \|u\|_{p^*}^q \left(K(p^*, q) \frac{1}{S^{-\frac{q(p^*-p)+p^*p}{p(p^*-p)}}} \|u\|^{-q} - \lambda \|f\|_\beta \right) \\ &\geq \|u\|_{p^*}^q \left\{ K(p^*, q) \frac{1}{S^{-\frac{q(p^*-p)+p^*p}{p(p^*-p)}}} \lambda^{\frac{-q}{p-q}} \left[\left(\frac{p^* - q}{p^* - p} \right) \|f\|_\beta S^{-\frac{q}{p}} \right]^{\frac{-q}{p-q}} - \lambda \|f\|_\beta \right\}. \end{aligned} \tag{8}$$

By considering the function

$$\begin{aligned} g(\lambda) &= \frac{K(p^*, q)}{S^{-\frac{q(p^*-p)+p^*p}{p(p^*-p)}}} \lambda^{\frac{-q}{p-q}} \left[\left(\frac{p^* - q}{p^* - p} \right) \|f\|_\beta S^{-\frac{q}{p}} \right]^{\frac{-q}{p-q}} - \lambda \|f\|_\beta \\ &= \frac{A}{\lambda^{q/(p-q)}} - \lambda \|f\|_\beta, \end{aligned}$$

where A is a positive constant, we observe that $g(\lambda)$ is decreasing for $\lambda > 0$ and satisfies both $g(\lambda) \to +\infty$ when $\lambda \to 0^+$ and $g(\lambda) \to -\infty$ when $\lambda \to +\infty$. The constant

$$\lambda_1 = K(p^*, q)^{\frac{p-q}{p}} \left(\frac{p^* - q}{p^* - p} \right) S^{\frac{p^*-q}{p^*-p}} \|f\|_\beta^{-\frac{q+p}{p}}$$

is the unique positive solution of $g(\lambda_1) = 0$. Therefore, if there exists $u \in \mathcal{N}_\lambda^0(\Omega)$ for $\lambda \in (0, \lambda_1)$, we would have $g(\lambda) > 0$, thus contradicting $I_\lambda(u) = 0$. □

Lemma 3. *If* $u \in \mathcal{N}^+(\Omega)$, *then* $\int_\Omega f(x)|u|^q dx > 0$.

Proof. Follows from the definition. □

In the sequel we study extremal properties the real function

$$t \mapsto J_\lambda(tu) = \frac{1}{p} t^p \|u\|^p - \frac{\lambda}{q} t^q \int_\Omega f(x)|u|^q \mathrm{d}x - \frac{1}{p^*} t^{p^*} \int_\Omega |u|^{p^*} \mathrm{d}x,$$

for $t \geq 0$ and any $u \in \mathbf{E} \setminus \{0\}$.

We have

$$\frac{d}{dt} J_\lambda(tu) = t^{q-1} \left(t^{p-q} \|u\|^p - \lambda \int_\Omega f(x)|u|^q \mathrm{d}x - t^{p^*-q} \int_\Omega |u|^{p^*} \mathrm{d}x \right),$$

from what follows that critical points of $t \mapsto J_\lambda(tu)$ occur when the function

$$s(t) = t^{p-q} \|u\|^p - t^{p^*-q} \int_\Omega |u|^{p^*} \mathrm{d}x$$

equals $\lambda \int_\Omega f(x)|u|^q \mathrm{d}x$, a value that does not depend on t.

The function s satisfies $s(0) = 0$ and $\lim_{t\to\infty} s(t) = -\infty$. Its maximum is attained at $t_{\max}$ (depending on u) given by

$$t_{\max} = \left[\frac{(p-q)\|u\|^p}{(p^*-q) \int_\Omega |u|^{p^*} \mathrm{d}x} \right]^{\frac{1}{p^*-p}}$$

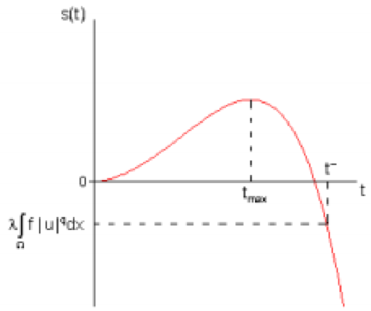
and (see Figure 1)

$$s(t_{\max}) = \left(\frac{(p-q)\|u\|^p}{(p^*-q) \int_\Omega |u|^{p^*} \mathrm{d}x} \right)^{\frac{p-q}{p^*-p}} \|u\|^p - \left(\frac{(p-q)\|u\|^p}{(p^*-q) \int_\Omega |u|^{p^*} \mathrm{d}x} \right)^{\frac{p^*-q}{p^*-p}} \int_\Omega |u|^{p^*} \mathrm{d}x. \quad (9)$$

An estimate for $s(t_{\max})$ is given by

$$s(t_{\max}) \geq \|u\|^q \left(\frac{p-q}{p^*-q} \right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-p}{p^*-q} \right) \left(S^{\frac{p^*}{p}} \right)^{\frac{p-q}{p^*-p}} > 0. \quad (10)$$

Figure 1: The graph of $s(t) = t^{p-q}\|u\|^p - t^{p^*-q} \int_\Omega |u|^{p^*} \mathrm{d}x$ is displayed, exhibiting the unique point $t^- = t_u^-$ tal que $s(t_u^-) = \lambda \int_\Omega f(x)|u|^q \mathrm{d}x$, if $\int_\Omega f(x)|u|^q \mathrm{d}x \leq 0$.



Lemma 4. *For* $\beta = \frac{p^*}{p^*-q}$ *and* $\lambda_2 = \left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-p}{p^*-q}\right) S^{\frac{p^*-q}{p^*-p}} \|f\|_\beta^{-1}$ *there exists, for each fixed* $u \in \mathbf{E} \setminus \{0\}$ *and* $\lambda \in (0, \lambda_2)$,

(*i*) *a unique* t_u^- *(depending on* u*) such that* $t_u^- u \in \mathcal{N}_\lambda^-(\Omega)$*. Furthermore,* $t_u^- > t_{\max}$ *and*

$$J_\lambda(t_u^- u) = \max_{t \geq t_{\max}} J_\lambda(tu).$$

If $\int_\Omega f(x)|u|^q \mathrm{d}x > 0$*, there also exists*

(*ii*) *a unique* t_u^+ *(depending on* u*) such that* $t_u^+ u \in \mathcal{N}_\lambda^+(\Omega)$*. Furthermore,* $0 < t_u^+ < t_{\max}$ *and*

$$J_\lambda(t_u^+ u) = \min_{0 \leq t \leq t_u^-} J_\lambda(tu).$$

Proof. Let us initially suppose that $\int_\Omega f(x)|u|^q \mathrm{d}x \leq 0$. Then there exists a unique $t_u^- > t_{\max}$ such that $s(t_u^-) = \lambda \int_\Omega f(x)|u|^q \mathrm{d}x$. We clearly have $s'(t_u^-) < 0$. See Figure 1.

We claim that $t_u^- u \in \mathcal{N}_\lambda^-(\Omega)$. In fact, since $\langle J_\lambda'(t_u^- u), t_u^- u\rangle = 0$, we have that $t_u^- u \in \mathcal{N}_\lambda(\Omega)$. But $\langle \psi_\lambda'(t_u^- u), t_u^- u\rangle = (t_u^-)^{q+1} s'(t_u^-) < 0$ proves our claim.

Notice that $\frac{d}{dt} J_\lambda(tu) = 0$ if, and only if $t = 0$ or $t = t_u^-$. And that $\frac{d^2}{dt^2} J_\lambda(tu)\big|_{t=t_u^-} < 0$ and we conclude that $J_\lambda(t_u^- u) = \max_{t \geq t_{\max}} J_\lambda(tu)$. [1]

Suppose now that $\int_\Omega f(x)|u|^q \mathrm{d}x > 0$. We claim that $\lambda \int_\Omega f(x)|u|^q \mathrm{d}x < s(t_{\max})$, if $0 < \lambda < \lambda_2$. In fact, we have $s(0) = 0 < \lambda \int_\Omega f(x)|u|^q \mathrm{d}x \leq$

$\lambda \|f\|_\beta \|u\|_{p^*}^q \leq \lambda \|f\|_\beta S^{-\frac{q}{p}} \|u\|^q$ and the definition of λ_2 guarantees that $s(0) =$

$0 < \lambda \int_\Omega f(x)|u|^q \mathrm{d}x < \|u\|^q \left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} \left(\frac{p^*-p}{p^*-q}\right) \left(S^{\frac{p^*}{p}}\right)^{\frac{p-q}{p^*-p}} \leq s(t_{\max})$, according to (10).

So, there exist unique t_u^+ and t_u^- such that $s(t_u^+) = \lambda \int_\Omega f(x)|u|^q \mathrm{d}x = s(t_u^-)$, such that $0 < t_u^+ < t_{\max} < t_u^-$ and $s'(t_u^+) > 0 > s'(t_u^-)$.

Since $\langle J_\lambda'(t_u^- u), t_u^- u\rangle = 0$ and $\langle \psi_\lambda'(t_u^- u), t_u^- u\rangle < 0$, we conclude that $t_u^- u \in \mathcal{N}_\lambda^-(\Omega)$. A similar argument shows that $t_u^+ u \in \mathcal{N}_\lambda^+(\Omega)$. Notice that t_u^+ and t_u^- are local extrema of $t \mapsto J_\lambda(tu)$. The derivative $\frac{d}{dt} J_\lambda(tu)$ shows that $J_\lambda(tu)$ is decreasing if $0 < t < t_u^+$ and increasing if $t_u^+ < t < t_u^-$, we conclude that

$$J_\lambda(t_u^+ u) = \min_{0 \leq t \leq t_u^-} J_\lambda(tu).$$

We conclude that $J_\lambda(t_u^- u) = \max\limits_{t \geq t_{\max}} J_\lambda(tu)$ as in the case $\int_\Omega f(x)|u|^q \mathrm{d}x \leq 0$, and we are done. □

Observe that, to apply Lemma 4(*ii*), we need that $\int_\Omega f(x)|u|^q dx > 0$, a conclusion that we can not infere from our hypotheses: the weight function f might change sign so that $\int_\Omega f(x)|u|^q dx \leq 0$. So, we restrict our domain to a subset of Ω where f is positive and consider the auxiliary problem (5) in the case $r = p^*$. This restriction aims to obtain an upper bound for $m_\lambda^+(\Omega)$ (see Lemma 6).

2.1. The restricted problem

Since $f : \overline{\Omega} \to \mathbb{R}$ is continuous and change signs in Ω, we have that

$$\Xi = \{x \in \Omega \, : \, f(x) > 0\} \neq \emptyset$$

is an open set of $\mathbb{R}^N$, so it has positive measure.

Therefore, we consider the functional J_0: $W_0^{2,p}(\Xi) \to \mathbb{R}$ defined by

$$J_0(u) = \frac{1}{p}\int_{\Xi} |\Delta u|^p \mathrm{d}x - \frac{1}{p^*}\int_{\Xi} |u|^{p^*} \mathrm{d}x$$

and the minimization problem

$$m_0(\Xi) = \inf\{J_0(u) \, : \, u \in \mathcal{N}_0(\Xi)\},$$

where, as before, $\mathcal{N}_0(\Xi) = \big\{u \in W_0^{2,p}(\Xi) \setminus \{0\} \, : \, \langle J_0'(u), u\rangle = 0\big\}$.

The manifold $\mathcal{N}_0(\Xi)$ splits $\mathbf{E}(\Xi) \setminus \{0\}$ into two components:

$$C_a = \{u \in W_0^{2,p}(\Xi) \setminus \{0\} \, : \, \langle J_0'(u), u\rangle > 0\}$$

and

$$C_b = \{u \in W_0^{2,p}(\Xi) \setminus \{0\} \, : \, \langle J_0'(u), u\rangle < 0\}.$$

Notice that if $u \in C_a$, then $J_0(u) > 0$; and if $\langle J_0'(u), u\rangle < 0$ we have that $u \in C_b$. The geometry of the functional J_0 implies that the component C_a is a neighborhood of the origin.

In order to prove that $m_0(\Xi) > 0$ we define

$$c_1 := \inf_{u \in W_0^{2,p}(\Xi)\setminus\{0\}} \max_{t \geq 0} J_0(tu) \quad \text{and} \quad c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_0(\gamma(t)),$$

where $\Gamma = \{\gamma \in C\left([0,1], W_0^{2,p}(\Xi)\right) \, : \, \gamma(0) = 0, \; J_0(\gamma(1)) < 0\}$.

By using the definition and standard arguments, we can show that

Lemma 5. $m_0(\Xi) = c_1 = c > 0$.

We have obtained some properties and results about the Nehari manifold $\mathcal{N}_\lambda(\Omega)$. We have proved that, for each $u \not\equiv 0$ fixed, extremal properties of the functional $J_\lambda(tu)$ in $\mathcal{N}_\lambda(\Omega)$. When restraining our study to the open set Xi where f is positive, we have considered the minimization problem

$$m_0(\Xi) = \inf\{J_0(u) \, : \, u \in \mathcal{N}_0(\Xi)\},$$

where $\mathcal{N}_0(\Xi)$ is the Nehari manifold associated to J_0. The last result proves that $m_0(\Xi) > 0$. We now apply this result to obtain an upper bound to $m_\lambda^+(\Omega)$.

2.2. Back to the original problem

We observe that, for each $u \in W_0^{2,p}(\Xi) \setminus \{0\}$, by defining $u = 0$ in $\Omega \setminus \Xi$, we obtain a function in $\mathbf{E} \setminus \{0\}$, since $u \in W_0^{2,p}(\Omega) \subset (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$.

Lemma 6. *The following statements are true:*

(*i*) *there exists* $\tilde{t} > 0$ *such that*

$$m_\lambda(\Omega) \leq m_\lambda^+(\Omega) < \frac{q-p}{q}\,(\tilde{t})^p\, m_0(\Xi) < 0, \quad \forall\ \lambda \in (0, \lambda_2);$$

(*ii*) J *is coercive and bounded from below in* $\mathcal{N}_\lambda(\Omega)$ *for any* $\lambda \in (0, \lambda_3)$.

Proof. For each $u \in W_0^{2,p}(\Xi) \setminus \{0\} \subset \mathbf{E} \setminus \{0\}$, we have

$$\int_\Omega f(x)|u|^q \mathrm{d}x = \int_\Xi f(x)|u|^q \mathrm{d}x > 0.$$

Consider t_u^+ as defined in Lemma 4 (*ii*). So, $t_u^+ u \in \mathcal{N}_\lambda^+(\Omega)$ and

$$\begin{aligned} J_\lambda(t_u^+ u) &= \frac{(t_u^+)^p}{p}\int_\Omega |\Delta u|^p \mathrm{d}x - \lambda\frac{(t_u^+)^q}{q}\int_\Omega f(x)|u|^q \mathrm{d}x - \frac{(t_u^+)^{p^*}}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x \\ &= \left(\frac{q-p}{q}\right)\tilde{t}^p\left[\frac{1}{p}\int_\Omega |\Delta u|^p dx - \left(\frac{q-p^*}{q-p}\right)\frac{\tilde{t}^{p^*-p}}{p^*}\int_\Omega |u|^{p^*}dx\right]. \end{aligned}$$

Since $\tilde{t} < t_{\max} = \left[\frac{(p-q)\|u\|^p}{(p^*-q)\int_\Omega |u|^{p^*}dx}\right]^{\frac{1}{p^*-p}}$ and $u \in \mathcal{N}(\Xi)$, we conclude that

$$J_\lambda(t_u^+ u) < \left(\frac{q-p}{q}\right)(t_u^+)^p\left[\frac{1}{p}\int_\Omega |\Delta u|^p \mathrm{d}x - \left(\frac{q-p^*}{q-p}\right)\frac{(t_{\max})^{p^*-p}}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x\right].$$

from what follows

$$J_\lambda(t_u^+ u) < \frac{q-p}{q}(t_u^+)^p m_0(\Xi) < 0,$$

proving the first claim for $\tilde{t} = t_u^+$.

Since $u \in \mathcal{N}_\lambda(\Omega)$ guarantees that $\int_\Omega |\Delta u|^p \mathrm{d}x = \lambda\int_\Omega f(x)|u|^q \mathrm{d}x + \int_\Omega |u|^{p^*}\mathrm{d}x$ estimating $J_\lambda(u)$, by Young inequality we obtain

$$\begin{aligned} J_\lambda(u) &\geq \frac{p^*-p}{pp^*}\int_\Omega |\Delta u|^p \mathrm{d}x - \lambda\frac{p^*-q}{qp^*}\|f\|_\beta S^{-\frac{q}{p}}\|u\|^q \\ &\geq \frac{1}{pp^*}\left[(p^*-p) - \lambda(p^*-q)\right]\|u\|^p - \lambda\frac{(p^*-q)(p-q)}{pqp^*}\left(\|f\|_\beta S^{-\frac{q}{p}}\right)^{\frac{p}{p-q}} \end{aligned}$$

proving that $J_\lambda(u)$ is coercive in $\mathcal{N}_\lambda(\Omega)$, for all $\lambda \in (0, \lambda_3)$ where $\lambda_3 = \frac{p^*-p}{p^*-q}$. Note also that from $\lambda \in (0, \lambda_3)$, follows

$$J_\lambda(u) \geq -\lambda\frac{(p^*-q)(p-q)}{pqp^*}\left(\|f\|_\beta S^{-\frac{q}{p}}\right)^{\frac{p}{p-q}}, \tag{11}$$

showing that J_λ is bounded from below in $\mathcal{N}_\lambda(\Omega)$. □

We now obtain an estimate from below for $m_\lambda^-(\Omega)$, if $\lambda \in (0, \lambda_1)$, where λ_1 was defined in Lemma 2.

Lemma 7. *For $\lambda \in (0, \lambda_1)$, the set $\mathcal{N}_\lambda^-(\Omega)$ is closed.*

Proof. If $u \in \mathcal{N}_\lambda^-(\Omega)$, we have

$$(p-q)\int_\Omega |\Delta u|^p dx - (p^*-q)\int_\Omega |u|^{p^*} dx < 0,$$

from what follows $\|u\|^{p^*-p} > \left(\frac{p-q}{p^*-q}\right) S^{\frac{p^*}{p}}$. Thus,

$$\|u\| > \left(\frac{p-q}{p^*-q}\right)^{\frac{1}{p^*-p}} S^{\frac{p^*}{p(p^*-p)}} = \left(\frac{p-q}{p^*-q}\right)^{\frac{1}{p^*-p}} S^{\frac{N}{2p^2}}, \tag{12}$$

an inequality valid for all $u \in \mathcal{N}_\lambda^-(\Omega)$.

Consider a sequence $\{u_n\} \subset \mathcal{N}_\lambda^-(\Omega)$ such that $u_n \to u$ in $\mathbf{E}$. It follows from (12) that $u \in \mathbf{E} \setminus \{0\}$. The C^1 regularity of J_λ implies that

$$\psi_\lambda(u) := \langle J_\lambda'(u), u\rangle = 0,$$

thus showing that $u \in \mathcal{N}_\lambda(\Omega)$. Since $\langle \psi_\lambda'(u), u\rangle$ is continuous, we have $\langle \psi_\lambda'(u), u\rangle \le 0$. But $\mathcal{N}_\lambda^0(\Omega) = \emptyset$, if $\lambda \in (0, \lambda_1)$, what implies that $\langle \psi_\lambda'(u), u\rangle < 0$, proving that $u \in \mathcal{N}_\lambda^-(\Omega)$. □

Lemma 8. *There exists a constant $C > 0$ such that, for all $\lambda \in (0, \lambda_4)$ we have*

$$m_\lambda^-(\Omega) \ge C > 0,$$

where $\lambda_4 = \frac{q}{2p}\left(\frac{p^*-p}{p^*-q}\right)\left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} \|f\|_\beta^{-1} S^{\frac{Np}{2p^2}}$.

Proof. It follows from (11) and (12) that

$$\begin{aligned} J_\lambda(u) &\ge \frac{p^*-p}{pp^*}\int_\Omega |\Delta u|^p \mathrm{d}x - \lambda\frac{p^*-q}{qp^*}\|f\|_\beta S^{-\frac{q}{p}}\|u\|^q \\ &> \|u\|^q\left[\left(\frac{p^*-p}{pp^*}\right)\left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}} S^{\frac{N(p-q)}{2p^2}} - \lambda\frac{p^*-q}{qp^*}\|f\|_\beta S^{-\frac{q}{p}}\right] \end{aligned}$$

(Observe that λ_4 was chosen to guarantee that the bracket is positive.)

So,

$$J_\lambda(u) > \left(\frac{1}{2}\right)\left(\frac{p^*-p}{pp^*}\right)\left(\frac{p-q}{p^*-q}\right)^{\frac{p}{p^*-p}} S^{\frac{N}{2p}} =: C.$$

Taking the infimum of $J_\lambda(u)$ for $u \in \mathcal{N}_\lambda^-(\Omega)$, we obtain

$$m_\lambda^-(\Omega) \ge C > 0.$$

□

We now aim to obtain an upper bound for $m_\lambda^-(\Omega)$, which will be used to prove the Palais-Smale condition in this level. The same proof is valid in both spaces $\mathbf{E} = W_0^{2,p}(\Omega)$ and $\mathbf{E} = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, thanks to the following result showed by Gazzola, Grunau and Sweers (veja [9, Thm.1, p. 2]).

Theorem 9. *The best constant for the immersion $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is equal to the best constant for the immersion $W_0^{2,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.*

Lemma 10. *For any $D > 0$ there exists $\lambda_5 > 0$ (depending on f, p, q, Ω and D) such that, for all $\lambda \in (0, \lambda_5)$, we have*

$$m_\lambda^-(\Omega) < \frac{2}{N} S^{\frac{N}{2p}} - D\lambda^\beta,$$

where $\beta = \frac{p^*}{p^*-q}$.

Proof. Take $x_0 \in \Omega$ such that $f(x_0) > 0$ and ρ_0, $0 < \rho_0 < 1$, such that $f > 0$ in $B_{2\rho_0}(x_0) \subset \Omega$.

Again, we consideremos the functional $J_0 \colon \mathbf{E} \to \mathbb{R}$ given by $J_0(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*}\int_\Omega |u|^{p^*} \mathrm{d}x$ and a cut-off function $\eta \in C_0^\infty(\Omega)$ satisfying

$$\begin{aligned} \eta &\equiv 1 \quad \text{in } B_{\rho_0}(x_0) \\ \eta &\equiv 0 \quad \text{out } B_{2\rho_0}(x_0) \\ 0 \le \eta \le 1 &\quad \text{and} \quad |\nabla \eta| \le C. \end{aligned}$$

For $\epsilon > 0$, consider

$$u_\epsilon(x) = \eta(x) U\left(\frac{x}{\epsilon}\right),$$

where U is a radially symmetric minimizer of $\left\{\frac{\|u\|^p}{\|u\|_{p^*}^p}\right\}$, for $u \in \mathbf{E}(\mathbb{R}^N) \setminus \{0\}$.

The following estimates are true (see [17]):

$$\begin{aligned} \left(\int_\Omega |u_\epsilon|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}} &= \epsilon^{-\frac{N-2p}{p}} \|U\|_{p^*}^p + O(\epsilon) \\ \int_\Omega |\Delta u_\epsilon|^p \mathrm{d}x &= \epsilon^{-\frac{N-2p}{p}} \|U\|^p + O(1) \\ \frac{\int_\Omega |u_\epsilon|^{p^*} \mathrm{d}x}{\left(\int_\Omega |u_\epsilon|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}}} &= S + O(\epsilon^{\frac{N-2p}{p}}), \end{aligned} \tag{13}$$

where

$$\frac{\|U\|^p}{\|U\|_{p^*}^p} = S = \inf_{u \in \mathbf{E}(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_{p^*}^p}.$$

Define u_0 by $u_0(x) = u_\epsilon(x - x_0)$. For $A, B > 0$, by applying the identity

$$\sup_{t \ge 0}\left(\frac{t^p}{p}A - \frac{t^{p^*}}{p^*}B\right) = \frac{2}{N}\left(\frac{A}{B^{\frac{p}{p^*}}}\right)^{\frac{N}{2p}}$$

to $J_0(tu_0)$ and also (13), we obtain

$$\begin{aligned}\sup_{t\geq 0} J_0(tu_0) &= \sup_{t\geq 0}\left(\frac{t^p}{p}\|u_0\|^p - \frac{t^{p^*}}{p^*}\int_\Omega |u_0|^{p^*}\mathrm{d}x\right) \\ &= \frac{2}{N}S^{\frac{N}{2p}} + O(\epsilon^{\frac{N-2p}{p}}).\end{aligned} \tag{14}$$

It follows from the definition of J_λ and u_0 that

$$\begin{aligned}\sup_{t\geq t_{\max}} J_\lambda(tu_0) &= \sup_{t\geq t_{\max}}\left(J_0(tu_0) - t^q\frac{\lambda}{q}\int_\Omega f|u_0|^q\mathrm{d}x\right) \\ &\leq \frac{2}{N}S^{\frac{N}{2p}} + O(\epsilon^{\frac{N-2p}{p}}) - (t_{\max})^q\frac{\lambda}{q}\int_{B_{\rho_0}(0)} f|u_\epsilon|^q dx \end{aligned} \tag{15}$$

For $0 < \epsilon < \rho_0^{\frac{p}{p-1}}$, we estimate the integral in the last inequality:

$$\int_{B_{\rho_0}(0)} f|u_\epsilon|^q\mathrm{d}x = \int_{B_{\rho_0}(0)} \frac{f}{(\epsilon + |x|^{\frac{p}{p-1}})^{\frac{N-2p}{p}q}}\mathrm{d}x \geq \int_{B_{\rho_0}(0)} \frac{f}{(2\rho_0^{\frac{p}{p-1}})^{\frac{N-2p}{p}q}}\mathrm{d}x = C_1, \tag{16}$$

where C_1 is a positive constant that does not depend on ϵ.

Since $\beta > 1$, we can choose $\delta > 0$ such that, for all $\lambda \in (0, \delta)$, we have

$$O(\lambda^\beta) + D\lambda^\beta - C_2\lambda < 0, \tag{17}$$

where D is a positive constant and $C_2 > 0$ will be chosen in the sequel.

We define $\lambda_5^\beta = \min\{\rho_0^{\frac{N-2p}{p-1}}, \delta^\beta\}$ and $\epsilon = (\lambda^\beta)^{\frac{p}{N-2p}}$. If $\lambda \in (0, \lambda_5)$, we can substitute (16) and (17) into (15) to obtain

$$\sup_{t\geq t_{\max}} J_\lambda(tu_0) \leq \frac{2}{N}S^{\frac{N}{2p}} + O(\lambda^\beta) - C_2\lambda,$$

where $C_2 = (t_{\max})^q\frac{C_1}{q}$. Since (17) implies that $O(\lambda^\beta) - C_2\lambda < -D\lambda^\beta$, we conclude that

$$\sup_{t\geq t_{\max}} J_\lambda(tu_0) < \frac{2}{N}S^{\frac{N}{2p}} - D\lambda^\beta. \tag{18}$$

Lemma 4 proves the existence of $t_{u_0}^- > t_{\max} > 0$ such that $t_{u_0}^- u_0 \in \mathcal{N}_\lambda^-(\Omega)$ and $J_\lambda(t_{u_0}^- u_0) = \max_{t\geq t_{\max}} J_\lambda(tu_0)$. Thus, it follows from (18) that

$$m_\lambda^-(\Omega) \leq J_\lambda(t_{u_0}^- u_0) = \max_{t\geq t_{\max}} J_\lambda(tu_0) < \frac{2}{N}S^{\frac{N}{2p}} - D\lambda^\beta$$

for $\lambda \in (0, \lambda_5)$. □

3. The local PS-condition *via* concentration-compactness principle

We now prove that the functional J_λ satisfies the Palais-Smale condition for levels below a certain constant.

Theorem 11. *There exists a positive constant D such that all* Palais-Smale *sequence $\{u_n\} \subset \mathbf{E}$ for J_λ in the level c, has a strongly convergent subsequence, if*

$$c < \frac{2}{N}S^{\frac{N}{2p}} - D\lambda^\beta.$$

Proof. The sequence $\{u_n\}$ is bounded. In fact, for any $\delta > 0$, we have

$$\begin{aligned} c+\delta &\geq J(u_n) - \frac{1}{p^*}\langle J'(u_n), u_n\rangle + \frac{1}{p^*}\langle J'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |\Delta u_n|^p \mathrm{d}x - \lambda\left(\frac{1}{q} - \frac{1}{p^*}\right)\|f\|_\beta S^{-\frac{q}{p}}\left(\int_\Omega |\Delta u_n|^p \mathrm{d}x\right)^{\frac{q}{p}} \\ &\quad -\frac{1}{p^*}\|J'(u_n)\|\left(\int_\Omega |\Delta u_n|^p \mathrm{d}x\right)^{\frac{1}{p}}. \end{aligned} \tag{19}$$

So, by supposing $\{u_n\}$ unbounded, we reach a contradiction with (19), since $1 < q < p$. So, we suppose that

$$u_n \rightharpoonup u \quad \text{in} \quad \mathbf{E} \tag{20}$$

and

$$\left.\begin{array}{l} |\Delta u_n|^p \rightharpoonup \mu \\ \\ |u_n|^{p*} \rightharpoonup \nu \end{array}\right\} \text{weakly-* in the sense of measures,} \tag{21}$$

where μ and ν are bounded measures. It follows from Lions' lemma (see [14]-[15]) that (passing to subsequences if necessary)

$$\begin{cases} u_n \to u \quad \text{in } L^r(\Omega) \text{ and a.e. in } \overline{\Omega}, \text{ if } 1 < r < p^*, \\ |\Delta u_n|^p \rightharpoonup^* \mu \geq |\Delta u|^p + \displaystyle\sum_{k\in I} \mu_k \delta_{x_k}, \\ |u_n|^{p^*} \rightharpoonup^* \nu = |u|^{p^*} + \displaystyle\sum_{k\in I} \nu_k \delta_{x_k}, \\ \nu_k^{\frac{p}{p^*}} \leq \mu_k S^{-1}, \end{cases} \tag{22}$$

for some set I, finite or empty.

Enough to show that $I = \emptyset$. Supposing the contrary, fix $k \in I$ and define $\psi_\epsilon \in C^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} \psi_\epsilon \equiv 1 \text{ in } B_\epsilon(x_k), \\ \psi_\epsilon \equiv 0 \text{ out } B_{2\epsilon}(x_k), \\ |\nabla \psi_\epsilon| \leq \dfrac{2}{\epsilon}, \quad |\Delta \psi_\epsilon| \leq \dfrac{2}{\epsilon^2}. \end{cases} \tag{23}$$

Now, consider the bounded sequence given by $\{\phi_\epsilon u_n\}$, where $\phi_\epsilon(x) = \psi_\epsilon(x)\chi_\Omega(x)$. It follows that $\lim_{n\to\infty}\langle J'(u_n), \phi_\epsilon u_n\rangle = 0$, thus implying

$$\lim_{n\to\infty} \int_\Omega |\Delta u_n|^{p-2}\Delta u_n \Delta(\phi_\epsilon u_n)\mathrm{d}x \;=\; \lambda\int_\Omega f(x)|u|^q\phi_\epsilon dx + \int_\Omega \phi_\epsilon \mathrm{d}\nu. \tag{24}$$

Expanding the left side of the above equation and taking the limit when $\epsilon \to 0$, we have that $\nu_k = \mu_k$.

It follows from Lions' lemma, that $\mu_k \geq S\nu_k^{\frac{p}{p^*}}$ and, since $\nu_k > 0$, we conclude that $\nu_k \geq S^{\frac{N}{2p}}$.

Therefore,

$$\begin{aligned} c &= \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty}\left\{J_\lambda(u_n) - \frac{1}{p}\langle J'_\lambda(u_n), u_n\rangle\right\} \\ &= \lim_{n\to\infty}\inf\left\{\lambda\left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega f(x)|u_n|^q \mathrm{d}x + \left(\frac{1}{p}-\frac{1}{p^*}\right)\int_\Omega |u_n|^{p^*}\mathrm{d}x\right\} \\ &\geq \frac{2}{N}S^{\frac{N}{2p}} + \frac{2}{N}\int_\Omega |u|^{p^*}\mathrm{d}x - \lambda\left(\frac{1}{q}-\frac{1}{p}\right)\|f\|_\beta\left(\int_\Omega |u|^{p^*}\mathrm{d}x\right)^{\frac{q}{p^*}} \end{aligned}$$

We now define $g(x) = \kappa_1 x^{p^*} - \lambda\kappa_2 x^q$, for

$$\kappa_1 = \frac{2}{N} \qquad \text{e} \qquad \kappa_2 = \left(\frac{1}{q}-\frac{1}{p}\right)\|f\|_\beta.$$

The function g attains an absolute minimum for $x > 0$ at $x_0 = \left(\frac{\lambda\kappa_2 q}{p^*\kappa_1}\right)^{\frac{1}{p^*-q}}$. Thus,

$$g(x) \geq g(x_0) = -D\lambda^{\frac{p^*}{p^*-q}},$$

where

$$D = \kappa_2\left(\frac{\kappa_2 q}{p^*\kappa_1}\right)^{\frac{q}{p^*-q}} - \kappa_1\left(\frac{\kappa_2 q}{p^*\kappa_1}\right)^{\frac{p^*}{p^*-q}} > 0.$$

Therefore

$$c \geq \frac{2}{N}S^{\frac{N}{2p}} - D\lambda^{\frac{p^*}{p^*-q}},$$

thus contradicting that $c < \frac{2}{N}S^{\frac{N}{2p}} - D\lambda^\beta$. This means that $I = \emptyset$ and the proof is complete.

4. Proof of Theorem 1

Proposition 12. *There exists $\overline{\lambda} > 0$ such that, for all $\lambda \in (0, \overline{\lambda})$, the functional J_λ has a minimizer $u_0^+ \in \mathcal{N}_\lambda^+(\Omega)$ satisfying*

(*i*) $J_\lambda(u_0^+) = m_\lambda(\Omega) = m_\lambda^+(\Omega)$;

(*ii*) *u_0^+ is a critical point for J_λ.*

Proof. We define $\overline{\lambda} = min\{\lambda_1, \lambda_2, \lambda_3\}$. For all $\lambda \in (0, \overline{\lambda})$ according to Lemma 6, we have $m_\lambda(\Omega) < 0$. Decreasing $\overline{\lambda}$ if necessary, we obtain

$$m_\lambda(\Omega) < 0 \leq \frac{2}{N}S^{\frac{N}{2p}} - D\lambda^\beta, \quad \forall\, \lambda \in (0, \overline{\lambda}).$$

By using Ekeland's variational principle we obtain a Palais-Smale sequence in the level $m_\lambda(\Omega)$. The Theorem 11 ensure that there exists a sequence $\{u_n\} \subset \mathcal{N}_\lambda(\Omega)$ such that (passing to a subsequence if necessary) $u_n \to u_0^+$ in $\mathbf{E}$. Since $J_\lambda \in C^1$, we have

$$J_\lambda(u_0^+) = m_\lambda(\Omega) \quad \text{and} \quad J'_\lambda(u_0^+) = 0.$$

We now claim that $u_0^+ \in \mathcal{N}_\lambda^+(\Omega)$. In fact, since $m_\lambda(\Omega) < 0$, we have $u_0^+ \in \mathbf{E} \setminus \{0\}$. Taking limits in the identity

$$\langle J_\lambda'(u_n), u_n\rangle = \int_\Omega |\Delta u_n|^p \mathrm{d}x - \lambda \int_\Omega f(x)|u_n|^q \mathrm{d}x - \int_\Omega |u_n|^{p^*} \mathrm{d}x = 0 \qquad (25)$$

we conclude that $u_0^+ \in \mathcal{N}_\lambda(\Omega) = \mathcal{N}_\lambda^+(\Omega) \cup \mathcal{N}_\lambda^-(\Omega)$.

We claim that $\int_\Omega f(x)|u_0^+|^q \mathrm{d}x > 0$. Supposing the contrary,

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{p}\int_\Omega |\Delta u_n|^p \mathrm{d}x - \frac{\lambda}{q}\int_\Omega f(x)|u_n|^q \mathrm{d}x - \frac{1}{p^*}\int_\Omega |u_n|^{p^*}\mathrm{d}x \\ &\to -\left(\frac{1}{q} - \frac{1}{p}\right)\lambda \int_\Omega f(x)|u_0^+|^q \mathrm{d}x + \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |u_0^+|^{p^*}\mathrm{d}x \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_\Omega |u_0^+|^{p^*}\mathrm{d}x \end{aligned}$$

but $J_\lambda(u_n) \to m_\lambda(\Omega) < 0$, a contradiction. This proves the claimed.

Again by contradiction, suppose that $u_0^+ \in \mathcal{N}_\lambda^-(\Omega)$. According to Lemma 4, there exist unique t_0^+ and t_0^- such that $t_0^+ u_0^+ \in \mathcal{N}_\lambda^+(\Omega)$, $t_0^- u_0^+ \in \mathcal{N}_\lambda^-(\Omega)$ and $0 < t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_\lambda(t_0^+ u_0^+) = 0 \quad e \quad \frac{d^2}{dt^2}J_\lambda(t_0^+ u_0^+) > 0,$$

there exists $\bar{t} \in t_0^+, t_0^-]$, such that $J_\lambda(t_0^+ u_0^+) < J_\lambda(\bar{t} u_0^+)$, from what follows

$$J_\lambda(t_0^+ u_0^+) < J_\lambda(\bar{t} u_0^+) \leq J_\lambda(t_0^- u_0^+) = J_\lambda(u_0^+) = m_\lambda(\Omega)$$

and we have reached a contradiction. This proves (i) and (ii).

□

Since $\mathcal{N}_\lambda^-(\Omega)$ is closed, the proof of the next result is similar, but simpler:

Proposition 13. *The exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, the functional J_λ has a minimizer $u_0^- \in \mathcal{N}_\lambda^-(\Omega)$ such that*

- (i) *$J_\lambda(u_0^-) = m_\lambda^-(\Omega)$;*
- (ii) *u_0^- is a critical point for J_λ.*

We take $\lambda_0 = min\{\overline{\lambda}, \lambda^*\}$ and the Theorem 1 is then obtained collecting the results already proved.

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