Study and Application of a Recurrence Relationship

Estudo e Aplicação de uma Relação de Recorrência

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ABSTRACT
Recurrence relationships allow us to accurately determine logarithms, square roots, factorials, several sequences (like Fibonacci), roots of equations and more. This paper aims to describe a recurrence relationship and present some applications. It presents the concept of reciprocal matrix and allowed the approximation of non perfect square roots of positive integers and their representation in continuous fraction. In addition, there was a correspondence between this study and the so-called Fibonacci polynomials. As it is a new job, more studies are needed to expand its concepts and applications.

Keywords: Recurrence relationship, Fibonacci Sequence, Reciprocal Matrix, Continued Fraction.

RESUMO
Relações de recorrência permitem determinar com precisão logaritmos, raízes quadradas, fatoriais, diversas sequências (a Example da Fibonacci), raízes de equações e outros mais. O principal objetivo deste trabalho é descrever uma relação de recorrência e apresentar algumas aplicações. Ele apresenta o conceito de matriz recíproca e permitiu a aproximação de raízes quadradas não exatas de números inteiros positivos e sua representação em fração contínua. Além disso, houve a correspondência entre

este estudo e os chamados polinômios de Fibonacci. Por ser um trabalho novo, são necessários mais estudos para ampliar seus conceitos e aplicações.

Palavras-chave: Relação de recorrência, Sequência de Fibonacci, Matriz Recíproca, Fração Contínua.

1 INTRODUCTION

A recurrence relationship is one where the first or first terms are given and the rest are obtained according to the previous terms. There are several examples of recurrence relationships such as the sequence of odd numbers, Fibonacci sequence and many others (ROSA, 2017; PEREIRA, 2014). Among the several problems that can be solved using the recurrence relation we have the approximation of non perfect square root of positive integers through the Pell's equation (TEKCAN, 2007; KAPLAN & WILLIAMS, 1986).

This paper aims to study a new recurrence relation that, like the Pell's equation, allowed us to calculate with non perfect square root approximations of positive integers as well as their continuous fraction, Fibonacci polynomials, among other things.

2 RESULTS

Consider that $a_{k,0} = b$, $a_{0,i} = 0$ and the recurrence relationship below:

$$a_{k,i} = \frac{h}{a_{k,i-1} + a_{k-1,i}} \quad (\boldsymbol{I})$$

Condition of existence: $h \geq -\frac{b^2}{4}$ (h and $b \in \mathbb{R}^*$)

We can organize the terms $a_{k,i}$ in matrix form $H_{k \times i+1}$ not necessarily square:

$$H_{k,i+1} = \begin{bmatrix} a_{1,0} & a_{1,1} & a_{1,2} & & a_{1,i} \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots & a_{2,i} \\ a_{3,0} & a_{3,1} & a_{3,2} & & a_{3,i} \\ \vdots & & \ddots & & \vdots \\ a_{k,0} & a_{k,1} & a_{k,2} & \cdots & a_{k,i} \end{bmatrix} = \begin{bmatrix} b & a_{1,1} & a_{1,2} & & a_{1,i} \\ b & a_{2,1} & a_{2,2} & \cdots & a_{2,i} \\ b & a_{3,1} & a_{3,2} & & a_{3,i} \\ \vdots & & \ddots & & \vdots \\ b & a_{k,1} & a_{k,2} & \cdots & a_{k,i} \end{bmatrix}$$

Example 1: We will use $b = 1$ and $h = 2$ until the term $a_{3,2}$.

$$a_{1,1} = \frac{2}{a_{1,0} + a_{0,1}} = \frac{2}{1+0} = 2$$

$$a_{2,1} = \frac{2}{a_{2,0} + a_{1,1}} = \frac{2}{1+2} = \frac{2}{3}$$

$$a_{3,1} = \frac{2}{a_{3,0} + a_{2,1}} = \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$$

$$a_{1,2} = \frac{2}{a_{1,1} + a_{0,2}} = \frac{2}{2+0} = 1$$

$$a_{2,2} = \frac{2}{a_{2,1} + a_{1,2}} = \frac{2}{\frac{2}{3} + 1} = \frac{6}{5}$$

$$a_{3,2} = \frac{2}{a_{3,1} + a_{2,2}} = \frac{2}{\frac{6}{5} + \frac{6}{5}} = \frac{5}{6}$$

$$H_{3,3} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & \frac{2}{3} & \frac{6}{5} \\ 1 & \frac{6}{5} & \frac{5}{6} \end{bmatrix}$$

Example 2: Now if we do $b = 1$ and $h = 1$ until the term $a_{5,4}$ we will have the following matrix:

$$H_{5\times 5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{2}{3} & \frac{3}{5} & \frac{5}{8} \\ 1 & \frac{2}{3} & \frac{3}{4} & \frac{20}{27} & \frac{216}{295} \\ 1 & \frac{3}{5} & \frac{20}{27} & \frac{27}{40} & \frac{2360}{3321} \\ 1 & \frac{5}{8} & \frac{216}{295} & \frac{2360}{3321} & \frac{3321}{4720} \end{pmatrix}$$

We can observe the following property in the matrix $H_{k\times i+1}$.

P1: The numbers $a_{k,1}$ $(k \to \infty)$ converge to:

$$\lim_{k\to\infty} a_{k,1} = \frac{\sqrt{b^2 + 4h} - b}{2} \quad (\boldsymbol{II})$$

Example 3: If $b = 1$ and $h = 2$, so:

$$\lim_{k\to\infty} a_{k,1} = \frac{\sqrt{1^2 + 4.2} - 1}{2} = 1$$

2.1 Reciprocal Matrix

If $b = 1$ and $h = 1$, the matrix $H_{k\times i+1}$ is denominated by reciprocal matrix because each term in the matrix not belonging to the first row and first column is equal to the reciprocal of the sum between the number on the left and the number immediately above of the term (and vice versa). As an example, in the reciprocal matrix $H_{5\times 5}$ seen in example 2 the term $\frac{20}{27} = \frac{1}{\frac{3}{5}+\frac{3}{4}} = \frac{1}{\frac{3}{4}+\frac{3}{5}}$. Based in (***II***), we have:

$$\lim_{k\to\infty} a_{k,1} = \lim_{k\to\infty} \frac{1}{1 + a_{k-1,1}} = \frac{\sqrt{5} - 1}{2} = \frac{1}{\phi}, \phi = golden\, ratio.$$

In fact, in matrix$H_{5\times5}$ seen previously, we see that the sequence $a_{k,1} \in \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \dots\right\}$ is the ratio between a Fibonacci number (F_k) and its successor Fibonacci (F_{k+1}), that is:

$$\lim_{k\to\infty} a_{k,1} = \frac{F_k}{F_{k+1}} = \frac{1}{\phi}$$

Similarly, we have the limit:

$$\lim_{k\to\infty} a_{k,2} = \frac{\sqrt{4\phi^2+1}-1}{2\phi}$$

2.2.1 Properties of Reciprocal Matrix

P2: If the reciprocal matrix $H_{k\times i+1}$ is square, there will be a palindrome sequence in inverted L.

Example 4: In the matrix $H_{5\times5}$ in the example 2 we have:

$$H_{5\times5} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{2}{3} & \frac{3}{5} & \frac{5}{8} \\ 1 & \frac{2}{3} & \frac{3}{4} & \frac{20}{27} & \frac{216}{295} \\ 1 & \frac{3}{5} & \frac{20}{27} & \frac{27}{40} & \frac{2360}{3321} \\ 1 & \frac{5}{8} & \frac{216}{295} & \frac{2360}{3321} & \frac{3321}{4720} \end{pmatrix}$$

P3: The product of the terms of $a_{1,1}$ until the term $a_{k,1}$ is equal to:

$$P = \prod_{p=1}^{k} a_{p,1} = \frac{1}{F_{k+1}} \ (\boldsymbol{III})$$

The demonstration of $(\boldsymbol{III})$ is easy because it is enough to note that each term $a_{k,1} = \frac{F_k}{F_{k+1}}$. Therefore:

$$P = a_{1,1}.a_{2,1}.a_{3,1}.a_{4,1} \dots a_{k,1} = \frac{F_1}{F_2}.\frac{F_2}{F_3}.\frac{F_3}{F_4}.\frac{F_4}{F_5} \dots \frac{F_k}{F_{k+1}} = \frac{F_1}{F_{k+1}} = \frac{1}{F_{k+1}}$$

P4: Each term $a_{k,2}$ of reciprocal matrix is given by:

$$a_{k,2} = \frac{F_{k+1}.q(k+1)}{q(k+2)} \ (\boldsymbol{IV})$$

Where $q(1) = 0, q(2) = 1, q(3) = 1$ and $(4) = 3$. The remaining values of $q(k)$ are recursively found by the formula:

$$q(k+2) = F_k.[q(k+1) + q(k).F_{k+1}](\boldsymbol{V})$$

Example 5: The term $a_{4,2} = \frac{20}{27}$ can also be calculated as follows:

$$a_{4,2} = \frac{F_5.q(5)}{q(6)} \to q(5) = F_3.[q(4) + q(3).F_4] = 2.(3 + 1.3) = 12$$

$$a_{4,2} = \frac{F_5.q(5)}{q(6)} = \frac{F_5.q(5)}{F_4.[q(5) + q(4).F_5]} = \frac{5.12}{3.(12 + 3.5)} = \frac{20}{27}$$

Proof of (IV).

Each term $a_{k,2}$ is constructed as follows:

Columns	$a_{k,0}$	$a_{k,1}$	$a_{k,2}$
Row k-1	1	$\frac{F_{k-1}}{F_k}$	$a_{k-1,2}$
Row k	1	$\frac{F_k}{F_{k+1}}$	$a_{k,2}$

$$a_{k,2} = \frac{1}{\frac{F_k}{F_{k+1}} + a_{k-1,2}} = \frac{F_{k+1}}{F_k + F_{k+1}.a_{k-1,2}} \; (\boldsymbol{VI})$$

Let's use the mathematical induction in ***(IV)***. For $k = 1$ we have:

$$a_{k,2} = \frac{F_{k+1}.q(k+1)}{q(k+2)} \to a_{1,2} = \frac{F_2.q(2)}{q(3)} = \frac{1.1}{1} = 1 \; (True)$$

If ***(IV)*** is valid for any $k(k > 1)$, so it is valid for every $k + 1$. Therefore:

$$a_{k+1,2} = \frac{F_{k+2}.q(k+2)}{q(k+3)} \; (\boldsymbol{VII})$$

Of $(\boldsymbol{VI})$ we have:

$$a_{k+1,2} = \frac{F_{k+2}}{F_{k+1} + F_{k+2}.a_{k,2}} (\boldsymbol{VIII})$$

Like $a_{k,2} = \frac{F_{k+1}.q(k+1)}{q(k+2)}$, so:

$$a_{k+1,2} = \frac{F_{k+2}}{F_{k+1} + F_{k+2}.a_{k,2}} = \frac{F_{k+2}}{F_{k+1} + F_{k+2}.\frac{F_{k+1}.q(k+1)}{q(k+2)}}$$

$$a_{k+1,2} = \frac{F_{k+2}.q(k+2)}{F_{k+1}.[q(k+2) + F_{k+2}.q(k+1)]} = \frac{F_{k+2}.q(k+2)}{F_{k+1}.[q(k+2) + q(k+1).F_{k+2}]}$$

$$a_{k+1,2} = \frac{F_{k+2}.q(k+2)}{q(k+3)} \; (\boldsymbol{X}) \; (q.e.d)$$

Proof of (V).

We have already proved to be valid (IV). From dividing numerator and denominator of ***(IV)*** by $q(k+1)$, we have:

$$a_{k,2} = \frac{F_{k+1}.q(k+1)}{q(k+2)} = \frac{F_{k+1}}{\frac{q(k+2)}{q(k+1)}} = \frac{F_{k+1}}{\frac{F_k.[q(k+1)+q(k).F_{k+1}]}{q(k+1)}} = \frac{F_{k+1}}{F_k + \frac{q(k)}{q(k+1)}.F_k.F_{k+1}} \; (\boldsymbol{XI})$$

Comparing ***(XI)*** with ***(VI)***, we have:

$$a_{k,2} = \frac{F_{k+1}}{F_k + \frac{q(k)}{q(k+1)}.F_k.F_{k+1}} = \frac{F_{k+1}}{F_k + F_{k+1}.a_{k-1,2}} \ (\boldsymbol{XII})$$

From the above equation it follows that:

$$F_k + \frac{q(k)}{q(k+1)}.F_k.F_{k+1} = F_k + F_{k+1}.a_{k-1,2}$$

$$\frac{q(k)}{q(k+1)}.F_k.F_{k+1} = F_{k+1}.a_{k-1,2}$$

$$a_{k-1,2} = \frac{F_k.q(k)}{q(k+1)} \to a_{k,2} = \frac{F_{k+1}.q(k+1)}{q(k+2)}$$

The above result (equation (***IV***)) has already been proven to be valid and therefore proves that (***V***) is also true.

P5: The product of terms of $a_{1,2}$ until $a_{k,2}$ is given by:

$$P = \prod_{p=1}^{k} a_{p,2} = \frac{\prod_{n=1}^{k} F_{n+1}}{F_k.[q(k+1) + q(k).F_{k+1}]} \ (\boldsymbol{XIV})$$

Proof of $(\boldsymbol{XIV})$.

$$P = \prod_{p=1}^{k} a_{p,2} = a_{1,2}.a_{2,2}.a_{3,2}.a_{4,2} \dots a_{k,2}$$

$$P = \frac{F_2.q(2)}{q(3)}.\frac{F_3.q(3)}{q(4)}.\frac{F_4.q(4)}{q(5)}.\frac{F_5.q(5)}{q(6)} \dots \frac{F_{k+1}.q(k+1)}{q(k+2)}$$

$$P = \frac{F_2.q(2)}{q(3)}.\frac{F_3.q(3)}{q(4)}.\frac{F_4.q(4)}{q(5)}.\frac{F_5.q(5)}{q(6)} \dots \frac{F_{k+1}.q(k+1)}{q(k+2)} = \frac{F_2.F_3.F_4.F_5 \dots F_{k+1}.q(2)}{q(k+2)}$$

$$P = \frac{F_2.F_3.F_4.F_5 \dots F_{k+1}.1}{q(k+2)} = \frac{\prod_{n=1}^{k} F_{n+1}}{F_k.[q(k+1) + q(k).F_{k+1}]}$$

Example 6: The product of terms of $a_{1,2}$ until $a_{5,2}$ is:

$$P = \frac{\prod_{n=1}^{5} F_{n+1}}{F_5.[q(6) + q(5).F_6]} = \frac{F_2.F_3.F_4.F_5.F_6}{F_5.(81 + 12.8)} = \frac{1.2.3.5.8}{5.(81 + 96)} = \frac{16}{59}$$

$$\text{Really, } P = 1.\tfrac{2}{3}.\tfrac{3}{4}.\tfrac{20}{27}.\tfrac{216}{295} = \tfrac{16}{59}$$

3 APPLICATIONS OF RECURRENCE RELATIONSHIP

3.1 Determine of approximation of non perfect square roots of positive integers

To approximate non perfect square roots of positive integers we can use the term $a_{k,1}$ with $(\boldsymbol{II})$ in a matrix $H_{k\times 2}$.

Example 7: Let's use the term $a_{5,1}$ of a matrix $H_{5\times 2}$ to approximate $\sqrt{2}$.

Solution: We know that $\lim_{k\to\infty} a_{k,1} = \frac{\sqrt{b^2+4h}-b}{2}$. We must choose satisfactory numbers b and h to make $\sqrt{2}$ appear in the expression. Let's use $b = 2$ and $h = 1$. Doing this we have:

$$a_{k,1} \approx \frac{\sqrt{b^2+4h}-b}{2} \to a_{5,1} \approx \frac{\sqrt{2^2+4.1}-2}{2} = \sqrt{2}-1$$

So, let's calculate the term $a_{5,1}$which will be a good approximation for $\sqrt{2}-1$.

$$a_{1,0} = 2\,, a_{0,1} = 0 \text{ and } a_{k,1} = \frac{1}{a_{k,0}+a_{k-1,1}}$$

$$a_{1,1} = \frac{1}{a_{1,0}+a_{0,1}} = \frac{1}{2+0} = \frac{1}{2}$$

$$a_{2,1} = \frac{1}{a_{2,0}+a_{1,1}} = \frac{1}{2+\frac{1}{2}} = \frac{2}{5}$$

$$a_{3,1} = \frac{1}{a_{3,0}+a_{2,1}} = \frac{1}{2+\frac{2}{5}} = \frac{5}{12}$$

$$a_{4,1} = \frac{1}{a_{4,0}+a_{3,1}} = \frac{1}{2+\frac{5}{12}} = \frac{12}{29}$$

$$a_{5,1} = \frac{1}{a_{5,0}+a_{4,1}} = \frac{1}{2+\frac{12}{29}} = \frac{29}{70}$$

The value $a_{5,1} = \frac{29}{70}$ is a good approximation to $\sqrt{2}-1$. Therefore, we can do:

$$\sqrt{2}-1 \approx \frac{29}{70} \to \sqrt{2} \approx \frac{99}{70} \approx 1{,}4142.$$

Example 8: Let's use the term $a_{5,1}$ of a matrix $H_{5\times 2}$ to approximate $\sqrt{0{,}3}$.

Solution: A good possibility is to use $b = 1$ and $h = -\frac{7}{40}$.

$$a_{k,1} \approx \frac{\sqrt{b^2+4h}-b}{2} \to a_{3,1} \approx \frac{\sqrt{0{,}3}-1}{2}$$

$$a_{1,0} = 1\,, a_{0,1} = 0 \text{ and } a_{k,1} = \frac{-7}{40(a_{k,0}+a_{k-1,1})}$$

$$a_{1,1} = \frac{-7}{40(a_{1,0}+a_{0,1})} = \frac{-7}{40}$$

$$a_{2,1} = \frac{-7}{40(a_{2,0}+a_{1,1})} = \frac{-7}{40\left(1-\frac{7}{40}\right)} = -\frac{7}{33}$$

$$a_{3,1} = \frac{-7}{40(a_{3,0}+a_{2,1})} = \frac{-7}{40\left(1-\frac{7}{33}\right)} = -\frac{231}{1040}$$

$$a_{4,1} = \frac{-7}{40(a_{4,0} + a_{3,1})} = \frac{-7}{40\left(1 - \frac{231}{1040}\right)} = -\frac{182}{809}$$

$$a_{5,1} = \frac{-7}{40(a_{5,0} + a_{4,1})} = \frac{-7}{40\left(1 - \frac{182}{809}\right)} = -\frac{5563}{25080}$$

Assim:

$$a_{5,1} \approx \frac{\sqrt{0,3} - 1}{2} \approx -\frac{5563}{25080} \to \sqrt{0,3} \approx \frac{6877}{12540} \approx 0,5484.$$

3.2 Continued Fraction of non perfect square root of positive integers

With the non perfect square root approximations of a positive integer as in the previous topic, it is easy to determine the continuous fractions.

Example 9: Using the example 7, let's determine the continued fractions of $\sqrt{2}$ remembering that in this example $a_{k,1} \approx \sqrt{2} - 1$.

Solution:

$$a_{1,1} = \frac{1}{2} = [0; 2]$$

$$a_{2,1} = \frac{2}{5} = \frac{1}{2 + \frac{1}{2}} = [0; 2,2]$$

$$a_{3,1} = \frac{5}{12} = \frac{1}{2 + \frac{2}{5}} = \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = [0; 2,2,2]$$

$$a_{4,1} = \frac{12}{29} = \frac{1}{2 + \frac{5}{12}} = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} = [0; 2,2,2,2]$$

$$a_{5,1} = \frac{29}{70} = \frac{1}{2 + \frac{12}{29}} = \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}}} = [0; 2,2,2,2,2]$$

If we continue this indefinitely, we will always obtain as a continuous fraction $[0; \bar{2}]$. Therefore:

$$\sqrt{2} - 1 = [0; \bar{2}] \to \sqrt{2} = 1 + [0; \bar{2}] = [1; \bar{2}]$$

Example 10: Let's calculate the continued fractions of $\sqrt{5}$.

Solution: We have already seen that in the reciprocal matrix $\lim_{k \to \infty} a_{k,1} = \frac{\sqrt{5}-1}{2}$ which allows us to state that $\sqrt{5} \approx 2. a_{k,1} + 1$. So just calculate the terms $a_{k,1}$ and observe the behavior of a possible period, remembering that $b = h = 1$.

$$\sqrt{5} \approx 2. a_{1,1} + 1 = 2.1 + 1 = 3 = [3]$$

$$\sqrt{5} \approx 2.a_{2,1} + 1 = 2.\frac{1}{2} + 1 = 2 = [2]$$

$$\sqrt{5} \approx 2.a_{3,1} + 1 = 2.\frac{2}{3} + 1 = \frac{7}{3} = [2;3]$$

$$\sqrt{5} \approx 2.a_{4,1} + 1 = 2.\frac{3}{5} + 1 = \frac{11}{5} = [2;5]$$

$$\sqrt{5} \approx 2.a_{5,1} + 1 = 2.\frac{5}{8} + 1 = \frac{18}{8} = [2;4]$$

In doing so, we find the next continuous fractions to be $[2;4,3]$, $[2;4,5]$, $[2;4,4]$, $[2;4,4,3]$, $[2;4,4,5]$, $[2;4,4,4]$, $[2;4,4,4,3]$, $[2;4,4,4,5]$, $[2;4,4,4,4]$, $[2;4,4,4,4,3]$, $[2;4,4,4,4,5]$, $[2;4,4,4,4,4]$. Following these steps definitely, we find as a continuous fraction $\sqrt{5} = [2;\overline{4}]$.

Based on this, it's possible to obtains the following expressions $\forall k \in \mathbb{N}^*$ where $[a_0; a_1, a_2, \dots, a_n]$ represent continuous fractions and $F_k$is the Fibonacci number.

$$2.\frac{F_{3k}}{F_{3k+1}} = \left[1; \underbrace{4,4,\dots,4}_{k-1\ vezes}, 3\right] \quad (\boldsymbol{XV})$$

$$2.\frac{F_{3k+1}}{F_{3k+2}} = \left[1; \underbrace{4,4,\dots,4}_{k-1\ vezes}, 5\right] (\boldsymbol{XVI})$$

$$2.\frac{F_{3k+2}}{F_{3k+3}} = \left[1; \underbrace{4,4,\dots,4}_{k\ vezes}\right] \quad (\boldsymbol{XVII})$$

3.3 Fibonacci Polynomials

Fibonacci polynomials (generalization of Fibonacci numbers) are defined by the second order linear recurrence relation:

$$F_n(x) = \begin{cases} 0, if\ n = 0 \\ 1, if\ n = 1 \\ x.F_{n-1}(x) + F_{n-2}(x), if\ n \geq 2 \end{cases} = \frac{\left(x + \sqrt{x^2+4}\right)^n - \left(x - \sqrt{x^2+4}\right)^n}{2^n.\sqrt{x^2+4}}$$

We have $F_n(1) = F_n$.

Example 11: Let's determine the 6th $F_n(x)$.

$$F_1(x) = 1$$

$$F_2(x) = x$$

$$F_3(x) = x^2 + 1$$

$$F_4(x) = x^3 + 2x$$

$$F_5(x) = x^4 + 3x^2 + 1$$

$$F_6(x) = x^5 + 4x^3 + 3x$$

Fibonacci polynomials relate to the recurrence relationship of this work as follows:

$$\lim_{n\to\infty} \frac{F_n(x)}{F_{n+1}(x)} = \frac{\sqrt{x^2+4} - x}{2}$$

This is equivalent to finding $\lim_{n\to\infty} a_{n,1}$, when we have $b = x$ and $h = 1$ in the recurrence relationship $a_{n,1} = \frac{1}{a_{n,0}+a_{n-1,1}}$, $a_{n,0} = b = x$, $a_{0,1} = 0$. So we have:

$$\lim_{n\to\infty} \frac{F_n(1)}{F_{n+1}(1)} = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi} \to automatic\ demo$$

$$\lim_{n\to\infty} \frac{F_n(2)}{F_{n+1}(2)} = \sqrt{2} - 1$$

4 FINAL CONSIDERATIONS

The recurrence relationship studied in this work provided us with tools to determine approximation for non perfect square roots of positive integers as well as their representation by continuous fraction and relation to Fibonacci polynomials. The concept of reciprocal matrix proved to be innovative and with intriguing properties when it is closely related to the Fibonacci sequence and golden ratio. However, as it is a new tool, further studies can still be done to expand the fields of application of these approaches.

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