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Persuading sincere and strategic voters

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Abstract

A sender wants to persuade multiple homogeneous receivers to vote in favor of a proposal. Before the vote sender commits to a signal which sends private, potentially correlated, messages to receivers that are contingent on the true state of the world. The best equilibrium for sender in the resulting incomplete information game is unappealing: all receivers vote in favor of sender's preferred outcome, irrespective of their message. We therefore focus on the equilibrium where receivers vote sincerely, that is they vote in favor of the outcome that is optimal given their posterior. We characterize the optimal public and the optimal private signal, both for the case where receivers are behavioral and vote sincerely as well as the case where such behavior is a Bayes–Nash equilibrium (BNE). For the optimal public signal, sincere voting is a BNE, but the optimal private signal is subject to the swing voter's curse. Imposing the constraint that sincere voting be a BNE leads to an optimal signal where receivers are never pivotal.

KEYWORDS

choice, economic theory, political systems, voting

1 | INTRODUCTION

Suppose a country is holding a referendum to decide whether or not to adopt a certain policy. Typically, government has a strict preference for either the proposal or the status quo. For instance, in the run-up to the Brexit referendum, a majority of the cabinet campaigned against leaving the European Union (see https://www.bbc.com/news/uk-politics-eu-referendum-35616946), that is, in favor of the status quo; whereas in the 2017 Turkish constitutional referendum government campaigned for replacing the existing parliamentary system with a presidential system, that is, in favor of the proposal. Either way, the aim of a political campaign run by government is to increase the probability that their preferred outcome obtains a majority of the votes, that is, to persuade the electorate.

The voting literature since Condorcet (1785) has focused on and extensively analyzed behavioral individuals who vote sincerely, which is an implicit assumption of the Condorcet Jury Theorem. It has been argued by Austen-Smith (1989) and Austen-Smith and Banks (1996), however, that presuming sincere voting is not reasonable in legislative elections and that sincere behavior by all agents might fail to be a Nash equilibrium even when their preferences are aligned. Others have pointed out that strategic voting is observationally equivalent to sincere voting when agents vote over endogenous agendas (Austen-Smith, 1987) and that sophisticated agents vote sincerely given multiple voter types and voting rules (Acharya & Meirowitz, 2017; Feddersen et al., 1990; McLennan, 1998). In this paper, we analyze both types of voting behavior in a Bayesian persuasion framework and characterize the optimal signal by sender for a number of different scenarios regarding the availability of public or private signals, the threshold needed to make the proposal pass, and the prior of the voters.

1.1 | Overview of results

We consider a sender who wishes to persuade multiple homogeneous receivers, who want the implemented outcome to match the true state of the world, to vote in favor of a proposal. Receivers' utility is determined by the unknown state of the world and the outcome that results after voting. Sender commits to a signal and reveals information about the true state, after which each receiver votes either in favor of or against the proposal. The proposal is implemented if it is approved by at least k out of n receivers.

In the Bayesian persuasion literature, it is common to focus on the best equilibrium for sender. In voting games, this is the one where, regardless of their information, all receivers vote in favor of the proposal. The reason is that for $k < n$ no receiver can affect the outcome by changing their vote. Bardhi and Guo (2018) show that sender can achieve the best equilibrium outcome with probability 1 even when $k < n$ and receivers condition on being pivotal, and therefore, restrict attention to $k = n$, that is, the unanimity voting rule. However, receivers in their model are voting against their beliefs. In contrast, we focus on the case $k < n$ and on equilibria where receivers vote sincerely, that is, the voting rule is nonunanimous and receivers vote only according to their belief about the state. We thereby follow other papers in the literature that also use sincere voting to select among the large multiplicity of equilibria, see Banks and Duggan (2000), Levy (2004), and Osborne and Slivinski (1996).

We first consider the case where sender uses a *public* signal, that is, all receivers observe the same message under sender's signal. The optimal public signal when receivers are assumed to vote sincerely can be derived from the analysis in Kamenica and Gentzkow (2011). If the true state is favorable to sender (henceforth state X), this is revealed with probability one; but when the proposal is harmful (henceforth state Y), then this is communicated with a probability strictly less than one. This probability is independent of the voting threshold k and is decreasing in the prior probability that the true state is X. We show that sincere voting is a Bayes–Nash equilibrium (BNE) for receivers when sender uses the optimal public signal.

Sender can implement the proposal with a higher probability under *private* communication, where receivers know the joint distribution of message profiles, but only observe their own private message. When receivers are simply assumed to vote sincerely, then the optimal signal can be derived from Arieli and Babichenko (2019). In their model of product adoption by receivers, there is no voting structure and agents only want their own actions to match the true state. In our model agents want *the outcome of the vote* to match the true state. However, under the assumption of sincere voting, receivers act as if their vote is decisive and the two optimization problems lead to the same result. Sender's optimal private signal prescribes sending a recommendation to vote in favor of the proposal to all receivers if the state is X. If the state is Y, then the recommendation to vote in favor of the proposal is sent either only to k agents (i.e., to a minimal winning coalition) or to none. The probability that the recommendation to vote in favor of the proposal is sent to k agents is increasing in the prior probability that the state is X and is decreasing in the threshold k.

Under the optimal private signal just described, whenever $k < n$, sincere voting is not a BNE. The driving force behind this finding is the swing voter's curse (Feddersen & Pesendorfer, 1996): a receiver who observed the recommendation to vote in favor of the proposal is pivotal if and only if the state is Y. Thus, he has an incentive to vote against the proposal: if the state is X, his vote is irrelevant; if the state is Y, he will have prevented the wrong outcome. Since its introduction, the swing voter's curse has been a prominent topic in the voting literature and has been explored both experimentally and theoretically (e.g., see Battaglini et al., 2010; Buechel & Mechtenberg, 2019; Grosser & Seebauer, 2016).

There are two obvious ways to make sincere voting a BNE and avoid the swing voter's curse: (i) by increasing the probability of being pivotal in state X or (ii) by decreasing the probability of being pivotal in state Y. It turns out that the optimal private signal which makes sincere voting a BNE follows the second route and sends a recommendation to vote in favor of the proposal to $k + 1$ receivers or none in state Y. As a single receiver cannot change the outcome of the vote, he has no incentive to deviate from his recommendation and thus, sincere voting is a BNE. It is interesting to note that while the initial signal persuades sincere voters only, the BNE adjusted signal persuades voters regardless of their behavior. Designing such an information structure is, in spirit, similar to designing a social choice rule that encourages voters to vote sincerely (Brams & Fishburn, 1978, 2002; Wolitzky, 2009).

The rest of the paper is organized as follows. In Section 1.2 we provide an illustrative example and in Section 1.3 we give a review of related literature. In Section 2 we introduce the necessary notation, define the voting game as well as the concept of BNE and sender's optimization problem. Section 3 presents the optimal public signal and establishes that under this signal sincere voting is a BNE. Section 4 presents the optimal private signal when receivers are behavioral and vote sincerely. In Section 5 the swing voter's curse is shown to emerge, so that sincere voting is not a BNE. In Section 6, we derive the optimal signal subject to sincere voting being a BNE. Finally, we present our conclusions in Section 7. All proofs can be found in the Appendix.

1.2 | Illustrating example

Consider a government (sender) that proposes for environmental reasons to ban cars on certain days of the year. For simplicity, assume that changing the status quo requires the approval of three out of five committee members (receivers). Committee members are not certain whether the proposal will *benefit* (X) or *harm* (Y) society and vote in favor of the proposal if they think that the chance of X is sufficiently high, say at least 50%. They vote *sincerely*: each one acts as if his own vote were decisive. Government wants the proposal to pass independently of its benefit or harm and, hence, wants to persuade the committee members to approve the proposal. Government and committee members have a common prior belief that the ban will be beneficial with 30% probability. If committee members vote sincerely, then, without intervention, no committee member will vote in favor of the ban and the ban will not be implemented. Sincere voting is a BNE as no committee member has an incentive to unilaterally change their vote: if they do so, this will have no effect on the outcome.

1.2.1 | Fully informative research

Suppose that before the vote, government attempts to influence the committee by conducting research and truthfully sharing the results with the committee members. Government can, for example, ask political scientists and economists to report their findings on the possible implications of implementing the ban. Given the common prior, there is a 30% chance that this research will prove the ban on cars to be beneficial. Hence, by conducting this research, government can increase the ex ante probability of implementing the ban to 30%. As the proposal is either endorsed by all committee members or by none of them, sincere voting is a BNE.

1.2.2 | Persuasion with public messages

Government can, however, improve upon this by conducting a partially informative research whose results will be shared publicly. This could be done, for example, by hiring a partisan economist whose research only points out the benefits of implementing the ban. Since all members' observations are the same, either all members are persuaded or none of them is. The optimal public signal sends message x with probability 1 in state X and with probability 3/7 in Y and it sends message y with probability 4/7 in state Y (see Kamenica & Gentzkow, 2011). This increases the acceptance probability of the proposal to 60%. For the same reason as before, sincere voting is a BNE.

1.2.3 | Persuasion with private correlated messages

Capabilities of government do not end here; it can do better by using correlated private messages. Consider the signal, formally described by $\pi^*(\cdot|X)$ and $\pi^*(\cdot|Y)$ on the set of message profiles $\{x, y\}^5$ that is obtained as follows. Government hires two economists: one being honest and one partisan. The honest economist conducts a fully informative study and reports to two randomly selected committee members. The partisan conducts a biased study that will send x if

TABLE 1 Signal π^*.

π^*	$\omega = X$	$\omega = Y$
(x, x, x, x, x)	1	0
(x, x, x, y, y)	0	$\frac{1}{14}$
(x, x, y, y, x)	0	$\frac{1}{14}$
(x, y, y, x, x)	0	$\frac{1}{14}$
(y, y, x, x, x)	0	$\frac{1}{14}$
(y, x, y, x, x)	0	$\frac{1}{14}$
(y, x, x, y, x)	0	$\frac{1}{14}$
(y, x, x, x, y)	0	$\frac{1}{14}$
(x, y, x, y, x)	0	$\frac{1}{14}$
(x, x, y, x, y)	0	$\frac{1}{14}$
(x, y, x, x, y)	0	$\frac{1}{14}$
(y, y, y, y, y)	0	$\frac{2}{7}$

X is the true state and send x with probability 5/7 if Y is the true state. He reports his results to the remaining three committee members. The signal π^* presented Table 1 provides the conditional probabilities for each of the 12 possible message profiles being sent. In this case, government can increase the probability of implementing its preferred outcome to 80%. We will show later on that π^* is indeed optimal when committee members vote sincerely.

1.2.4 | The swing voter's curse

In the previous scenario committee members know that the ban will be harmful to society upon receiving y, but regard the states equally likely upon receiving x. Given signal π^* and sincere voting, a voter becomes pivotal if and only if two conditions are satisfied: (i) *they have received* x and (ii) *the ban is harmful*. Thus, upon observing x choosing y is optimal if all other voters vote sincerely. This means that sincere voting is not a BNE.

It is, then, natural to ask: what can sender achieve under the additional constraint that sincere voting be a BNE? One possibility to ensure that sincere voting is a BNE is to make receivers pivotal with sufficiently high probability when the true state is X. Another possibility is to reduce receivers' probability of being pivotal when the true state is Y to zero. This can be achieved by sending x to four out of five committee members if the state is Y, as opposed to sending it to a minimal winning coalition. Then a single committee member cannot change the outcome of the vote and sincere voting is a BNE. Table 2 presents a signal π^{*e} that has these features. With π^{*e}, government can implement its preferred outcome with 67.5% probability. Observe that this is higher than the probability under the optimal public signal. Thus,

TABLE 2 Signal π^{*e}.

π^{*e}	$\omega = X$	$\omega = Y$
(x, x, x, x, x)	1	0
(x, x, x, x, y)	0	$\frac{3}{28}$
(x, x, x, y, x)	0	$\frac{3}{28}$
(x, x, y, x, x)	0	$\frac{3}{28}$
(x, y, x, x, x)	0	$\frac{3}{28}$
(y, x, x, x, x)	0	$\frac{3}{28}$
(y, y, y, y, y)	0	$\frac{13}{28}$

government benefits from private communication. It turns out, perhaps surprisingly, that π^{*e} is the optimal signal under the constraint that sincere voting is a BNE.

1.3 | Literature review

There are several papers in the Bayesian persuasion literature which extend Kamenica and Gentzkow (2011) by considering private communication between a sender and multiple receivers. In a closely related paper, Chan et al. (2019) consider a setting with costly voting and show that the optimal signal targets minimal winning coalitions; in particular, the receivers who are easiest to persuade. It follows that in equilibrium all receivers are pivotal with positive probability when voting for the sender's preferred alternative. This is in stark contrast to our optimal BNE-constrained signal: while all receivers are pivotal when voting is costly, no receiver is pivotal when there is no cost of voting. Without the requirement that sincere voting is a BNE, the optimal signal comes closer to the optimal signal of Chan et al. (2019) in the sense that receivers are pivotal with positive probability in the "bad" state.

There are several studies that explore why and how voters vote sincerely. While Poole and Rosenthal (2000) and Groseclose and Milyo (2010) show that strategic voting is rare in Congress, Felsenthal and Brichta (1985) and Degan and Merlo (2007) provide evidence that sincere voting is consistent with voter behavior in larger elections. Numerous other papers also provide explanations for behavioral voters (Austen-Smith, 1992; Denzau et al., 1985; Ferrari, 2016; Ginzburg, 2017; Kleiner & Moldovanu, 2017, 2022; Krishna & Morgan, 2012; Laslier & Weibull, 2013).

Our paper also relates to the literature on voting games and public communication. Schnakenberg (2015) considers an "expert" who can conceal information after she is privately informed about the state. He shows that the removal of the commitment assumption leads to stricter conditions on persuasion and that experts can manipulate information such that voters' ex ante expected utilities are reduced. Alonso and Câmara (2016) consider a symmetric information voting model. They show how the optimal signal exploits the heterogeneity of voter preferences by targeting minimal winning coalitions. Gitmez and Molavi (2022) consider media bias in the context of Bayesian persuasion, where a biased sender is trying to persuade receivers who have heterogeneous preferences and beliefs, and show that the sender becomes less biased as society becomes more polarized. In a slightly different context, Hennigs (2021) also considers heterogeneous receivers who have private information about their types and shows that employing uncorrelated messages is optimal. In contrast, we show that the sender is better off employing correlated messages.

As we extend public communication by allowing for private messages, our paper relates to papers which consider private communication under collective decision making. Wang (2013) studies both public and private communication with general voting rules. Under private communication, she focuses on uncorrelated messages and shows that the sender is weakly worse off under private persuasion. In contrast, we allow for correlated messages, which results in asymmetrically informed receivers and a nontrivial equilibrium selection problem. Heese and Lauermann (2021) consider private communication in large elections and allow for exogenous private messages for receivers. They show that any state-contingent outcome can be implemented in some BNE. Bardhi and Guo (2018) consider voters with correlated payoff states and unanimity voting, while we allow for more general voting rules.

Our paper is also linked to a literature on information design in more general games. Bergemann and Morris (2016) study a game of incomplete information with multiple receivers and show that the set of outcomes that can arise in a BNE corresponds to Bayes correlated equilibrium outcomes. They allow receivers to observe additional private messages, whereas in a related paper Taneva (2019) focuses on a sender who has full control over the information gathered by the receivers. Mathevet et al. (2020) consider an epistemic approach and incorporate higher-order beliefs into persuasion. For further connections to different extensions and applications of Bayesian persuasion, we refer the reader to the literature surveys Kamenica (2019) and Bergemann and Morris (2019).

2 | MODEL AND EQUILIBRIUM CONCEPT

2.1 | Signals and beliefs

Let $N = \{1, \ldots, n\}$ be the set of receivers and let $\Omega = \{X, Y\}$ be the set of *states* of the world. Let S_i be a set of *messages* sender can send to receiver $i \in N$ and let $S = \prod_{i \in N} S_i$ be the set of *message profiles*. A *signal* is a function $\pi : \Omega \to \Delta(S)$ which maps each state of the world to a joint probability distribution over a finite set of message profiles. Denote the set of all signals by Π. For each message profile $s \in S$, let $s_i \in S_i$ denote the message for receiver i. For each message $m \in S_i$ and $\omega \in \Omega$, let

$$\pi_i(m|\omega) = \sum_{s \in S : s_i = m} \pi(s|\omega)$$

be the probability that receiver i observes message m given that the true state is ω, which is a marginal probability. For every $\pi \in \Pi$, define

$$S^{\pi} = \{s \in S | \exists\ \omega \in \Omega : \pi(s|\omega) > 0\},$$

which is the set of message profiles that are sent with positive probability by π. Similarly, for each $\pi \in \Pi$ and $i \in N$, define

$$S_i^\pi = \{s_i \in S_i | \exists\ \omega \in \Omega : \pi_i(s_i|\omega) > 0\}$$

to be the set of messages receiver i observes with positive probability under π.

Throughout the paper we assume that senders and receivers share a common prior belief $\lambda^0 \in \Delta^\circ(\Omega)$ about the true state of the world, where $\Delta^\circ(\Omega)$ denotes the set of strictly positive probability distributions on Ω. Given $\lambda^0 \in \Delta^\circ(\Omega)$ and $\pi \in \Pi$, a message profile $s \in S^\pi$ generates the posterior belief vector $\lambda^s \in \Delta(\Omega)^n$ defined by

$$\lambda_i^s(\omega) = \frac{\pi_i(s_i|\omega)\lambda^0(\omega)}{\sum_{\omega' \in \Omega} \pi_i(s_i|\omega')\lambda^0(\omega')}, \quad i \in N, \omega \in \Omega. \tag{1}$$

That is, $\lambda_i^s(\omega)$ denotes i's updated belief upon receiving s_i that the true state is ω.

2.2 | Voting problems

A *voting problem* is a tuple $P = (n, k, \lambda^0)$ of the number of receivers n, integer *voting threshold* $k \geq 1$, and a common prior λ^0 over states in Ω. We assume that there are two possible voting outcomes, which we represent by $Z = \{x, y\}$, and that voters can cast a vote either for x or for y, represented by their action sets $A_i = \{x, y\}$ for all $i \in N$. Let $A = \prod_{i \in N} A_i$ be the set of action profiles.

In our motivating example of a government wanting to ban cars on certain days of the year, the messages receivers observe are recommendations to either vote in favor of or against the proposal, that is, for each $i \in N, S_i^\pi \subseteq A_i$. However, it is also possible that messages are statistics about the pollution caused by excess car usage on which receivers base their decision, rather than a recommended action, so that S_i^π would not be a subset of A_i. For the remainder of the paper, we capture both possibilities and assume that, for every $i \in N, \{x, y\} = A_i \subseteq S_i$.

We focus on anonymous and monotonic voting rules that determine for any action profile a the selected outcome. They are defined by functions of the form $z^k : A \to Z$, where for each $a \in A$ we have

$$z^k(a) = \begin{cases} x & \text{if } |\{i \in N : a_i = x\}| \geq k, \\ y & \text{otherwise.} \end{cases}$$

If $k = n$, then z^k is the *unanimity rule.*

We assume that sender's utility function $v : Z \to \mathbb{R}$ has the form

$$v(z) = \begin{cases} 1 & \text{if } z = x, \\ 0 & \text{otherwise.} \end{cases}$$

That is, sender wants to implement x and does not care about the true state.

We assume that receiver i's utility function $u_i : Z \times \Omega \to \mathbb{R}$ is given by

$$u_i(z, \omega) = \begin{cases} 1 & \text{if } [z = x \text{ and } \omega = X] \text{ or } [z = y \text{ and } \omega = Y], \\ 0 & \text{otherwise.} \end{cases}$$

That is, receivers want the implemented outcome to match the true state of the world.

2.3 | Bayes–Nash equilibrium

A voting problem P together with a signal π defines a game of incomplete information $G(P,\pi)$. We shall refer to $G(P,\pi)$ as the *voting game* associated with P and π. A strategy of receiver $i \in N$ is a map $\beta_i^\pi : S_i^\pi \to A_i$ and a strategy profile is denoted by $\beta^\pi = \left(\beta_i^\pi\right)_{i\in N}$. With a slight abuse of notation we shall write $\beta_{-i}^\pi(s_{-i})$ for $\left(\beta_j^\pi(s_j)\right)_{j\neq i}$. The standard equilibrium concept for voting games is BNE.

Definition 2.1. Let $P=(n,k,\lambda^0)$ be a voting problem and $\pi \in \Pi$ be a signal. A strategy profile β^π constitutes a *BNE* of $G(P,\pi)$ if for all $i \in N$, $s' \in S^\pi$, and $a_i \in A_i$, it holds that

$$\begin{aligned}\sum\nolimits_{\omega\in\Omega} \lambda_i^{s'}(\omega)\sum\nolimits_{s\in S^\pi:s_i=s_i'} \frac{\pi\left(\left(s_i',s_{-i}\right)\middle|\omega\right)}{\pi_i\left(s_i'\middle|\omega\right)}u_i\left(z^k\left(\beta_i^\pi\left(s_i'\right),\beta_{-i}^\pi(s_{-i})\right),\omega\right)\\ \geq \sum\nolimits_{\omega\in\Omega} \lambda_i^{s'}(\omega)\sum\nolimits_{s\in S^\pi:s_i=s_i'} \frac{\pi\left(\left(s_i',s_{-i}\right)\middle|\omega\right)}{\pi_i\left(s_i'\middle|\omega\right)}u_i\left(z^k\left(a_i,\beta_{-i}^\pi(s_{-i})\right),\omega\right).\end{aligned} \tag{2}$$

Let $G(P,\pi)$ be a voting game and β^π be a strategy profile. We say that receiver $i \in N$ is *pivotal* at $s \in S^\pi$ if, for every $a_i \in A_i$, $z^k\left(a_i,\beta_{-i}^\pi(s_{-i})\right)=a_i$. That is, i is pivotal at message profile s if i's vote determines the voting outcome in the voting game $G(P,\pi)$ given that all receivers $j\neq i$ vote according to their respective strategies β_j^π. For every $m \in S_i^\pi$, define

$$T^{i,m}(\pi,\beta^\pi)=\{s\in S^\pi | s_i=m \text{ and } i \text{ is pivotal at } s\}$$

as the set of message profiles where receiver i observes message m and is pivotal.

One easily observes that receiver i's utilities on both sides of Equation (2) differ only if i is pivotal at s'. The following lemma uses this insight.

Lemma 2.2. *Let $P=(n,k,\lambda^0)$ be a voting problem and $\pi\in\Pi$ be a signal. A strategy profile β^π is a BNE of the voting game $G(P,\pi)$ if and only if, for every $i\in N$, for every $m\in S_i^\pi$ such that $\beta_i^\pi(m)=x$, it holds that*

$$\sum_{s\in T^{i,m}(\pi,\beta^\pi)} \lambda^0(X)\pi(s|X) \geq \sum_{s\in T^{i,m}(\pi,\beta^\pi)} \lambda^0(Y)\pi(s|Y) \tag{3}$$

and, for every $m\in S_i^\pi$ such that $\beta_i^\pi(m)=y$, it holds that

$$\sum_{s\in T^{i,m}(\pi,\beta^\pi)} \lambda^0(Y)\pi(s|Y) \geq \sum_{s\in T^{i,m}(\pi,\beta^\pi)} \lambda^0(X)\pi(s|X). \tag{4}$$

Lemma 2.2 points towards a myriad of BNEs. For instance, whenever $P=(n,k,\lambda^0)$ with $2\leq k\leq n-1$, there are always two trivial equilibria in which no receiver is pivotal: namely, those with $\beta_i^\pi(s_i)=x$ for all $i\in N$ and all $s_i\in S_i^\pi$, and those with $\beta_i^\pi(s_i)=y$ for all $i\in N$ and all $s_i \in S_i^\pi$. In particular, the sender preferred outcome can always be implemented as a BNE in which all receivers ignore their private information and simply vote in favor of sender's preferred outcome x. As this is rather unsatisfying—it renders the entire problem of persuasion unnecessary—our first objective is to select a meaningful equilibrium.

We follow the voting literature (Banks & Duggan, 2000; Levy, 2004; Osborne & Slivinski, 1996) and consider equilibria with sincere voting. For each $i \in N$, let $\alpha_i^\pi : S_i^\pi \to A_i$ be the strategy such that receiver i votes for the state that is most likely conditional on observing message s_i. More precisely, let

$$\alpha_i^\pi(s_i) = \begin{cases} x & \text{if } \lambda_i^s(X) \geq \frac{1}{2}, \\ y & \text{otherwise.} \end{cases}$$

We call α_i^π the *sincere voting strategy* of receiver i. Note that α_i^π is the "sender-preferred" sincere voting strategy: receivers vote in favor of sender's preferred outcome in case they deem both states equally likely.

Let $P = (n, k, \lambda^0)$ be a voting problem and $\pi \in \Pi$ be a signal. For every $i \in N$, let

$$\begin{aligned} M_i^x(\pi) &= \left\{ m \in S_i^\pi \middle| \, \alpha_i^\pi(m) = x \right\}, \\ M_i^y(\pi) &= \left\{ m \in S_i^\pi \middle| \, \alpha_i^\pi(m) = y \right\}, \end{aligned}$$

which are the sets of messages in S_i^π after which states x and y, respectively, are most likely. Define further

$$Z^x(\pi) = \{ s \in S^\pi | z^k(\alpha^\pi(s)) = x \}$$

as the set of message profiles in S^π that lead to x as the voting outcome under α^π. Sender's expected payoff under sincere voting is given by

$$V^\pi(\lambda^0) = \mathbb{E}_{\lambda^0}[\mathbb{E}_\pi[v(z^k(\alpha^\pi(s)))]] = \lambda^0(X) \sum_{s \in Z^x(\pi)} \pi(s|X) + \lambda^0(Y) \sum_{s \in Z^x(\pi)} \pi(s|Y), \tag{5}$$

which is the probability that x will be the outcome of the voting game under receivers' strategy profile α^π. For $\Pi' \subseteq \Pi$ the signal $\pi' \in \Pi'$ is *optimal in* Π' if $V^{\pi'}(\lambda^0) = \max_{\pi \in \Pi'} V^\pi(\lambda^0)$.

3 | OPTIMAL PUBLIC COMMUNICATION

A signal $\pi \in \Pi$ is *public* if for all $s \in S^\pi$ and all $i, j \in N$, $s_i = s_j$, that is, if all receivers observe the same message under π. Denote by Π^{p} the set of all public signals, and let $\mathbf{x}, \mathbf{y} \in A$ be the vectors $(x, ..., x)$ and $(y, ..., y)$. Since agents have homogeneous preferences, a public signal either persuades all receivers or none. Thus, the analysis of a multiple receiver model with public signals is very similar to the analysis of the single receiver case when the entire electorate is considered a single receiver, so that we omit the proof of the following proposition, which follows easily from Kamenica and Gentzkow (2011).

Proposition 3.1. *Let $P = (n, k, \lambda^0)$ be a voting problem and let $\pi^{*\mathrm{p}} \in \Pi$ be given by*

$$\pi^{*\mathrm{p}}(s|\omega) = \begin{cases} 1 & \text{if } s = \mathbf{x} \text{ and } \omega = X, \\ \min\left\{\frac{\lambda^0(X)}{\lambda^0(Y)}, 1\right\} & \text{if } s = \mathbf{x} \text{ and } \omega = Y, \\ \max\left\{1 - \frac{\lambda^0(X)}{\lambda^0(Y)}, 0\right\} & \text{if } s = \mathbf{y} \text{ and } \omega = Y. \end{cases}$$

Then $\pi^{*\mathrm{p}}$ is optimal in Π^{p}. In particular, it holds that

$$V^{\pi^{*\mathrm{p}}}(\lambda^0) = \min\{2\lambda^0(X), 1\}.$$

Under $\pi^{*\mathrm{p}}$, the set of messages receiver $i \in N$ can observe with positive probability is a subset of his action set; thus, the signal sends a "recommended" action to each receiver. Moreover, these recommendations are convincing in the sense that the recommended action corresponds to the most likely state conditional on receiving this recommendation. Such signals are called straightforward, formally defined as follows.

Definition 3.2. A signal $\pi \in \Pi$ is *straightforward* if for all $i \in N$

(i) $S_i^\pi \subseteq A_i$,
(ii) For all $a_i \in S_i^\pi$, $\alpha_i^\pi(a_i) = a_i$.

Denote the set of all straightforward signals by Π^{s}. A signal $\pi \in \Pi$ with $S^\pi \subseteq \{x, y\}^n$ is, by the definition of α_i^π, straightforward if and only if for any $i \in N$ and $s, t \in S^\pi$ with $s_i = x$ and $t_i = y$, we have $\lambda_i^s(X) \geq 1/2$ and $\lambda_i^t(Y) > 1/2$. Equivalently, for all $i \in N$,

$$\lambda^0(X)\pi_i(x|X) \geq \lambda^0(Y)\pi_i(x|Y), \tag{6}$$

$$\lambda^0(Y)\pi_i(y|Y) > \lambda^0(X)\pi_i(y|X). \tag{7}$$

If $\lambda^0(X) < 1/2$, then Equation (6) is actually necessary and sufficient as it implies inequality (7), see Lemma A.1 in the appendix for a proof. As in the single receiver case, straightforwardness of the optimal signal can be assumed without loss of generality. We provide a proof of this observation in Lemma A.2.

If the prior satisfies $\lambda^0(X) < 1/2$, then the optimal public signal $\pi^{*\mathrm{p}}$ in Proposition 3.1 is straightforward as it satisfies (6) with equality. This immediately implies the optimality of $\pi^{*\mathrm{p}}$: any attempt to increase the probability of the x-message in state Y would lead to a violation of (6), so that the posterior belief that the true state is X would be less than 1/2 and x would not be chosen anymore.

Observe that the strategy profile $\alpha^{\pi^{*\mathrm{p}}}$ is a BNE of the voting game $G(P, \pi^{*\mathrm{p}})$. Indeed, since $\pi^{*\mathrm{p}}(\mathbf{y}|X) = 0$, inequality (4) is trivially satisfied. To verify inequality (3) let $i \in N$ and observe that $m \in S_i^{\pi^{*\mathrm{p}}}$ is such that $\alpha_i^{\pi^{*\mathrm{p}}}(m) = x$ if and only if $s = \mathbf{x}$. If $k < n$, then receiver i is not pivotal at message profile $\mathbf{x}$ and inequality (3) holds trivially. If $k = n$, then receiver i is pivotal at message profile $\mathbf{x}$. Since $\pi^{*\mathrm{p}}(\mathbf{x}|X) = 1$ and $\pi^{*\mathrm{p}}(\mathbf{x}|Y) = \min\{\lambda^0(X)/\lambda^0(Y), 1\}$, inequality (3) is equivalent to $\lambda^0(X) \geq \min\{\lambda^0(X), \lambda^0(Y)\}$, which is clearly satisfied.

4 | OPTIMAL PRIVATE SIGNALS FOR SINCERE VOTERS

In this section, our focus lies on behavioral voters who vote sincerely; we are not yet imposing that sincere voting be a BNE. Our aim is to find a signal which maximizes sender's expected utility if sender can use arbitrary message profiles, that is, sender chooses the best signal in Π.

Recall that the voting rule z^k is anonymous: only the total number of votes in favor of each outcome matters, not the identities of the receivers who voted for each alternative. Hence, we might expect that signals need not discriminate between receivers either. To make this claim formal, let B denote the set of all permutations on N, that is the set of all bijections $b : N \to N$. For each $s \in S^\pi$ and each $b \in B$, denote by s^b the unique message profile with, for every $i \in N$, $s_i^b = s_{b(i)}$.

A signal $\pi \in \Pi$ is called *anonymous* if $\pi(s|\omega) = \pi(s^b|\omega)$ for all bijections $b \in B$ and all $\omega \in \Omega$. The set of all anonymous signals is denoted by Π^{a} and the set of all straightforward and anonymous signals by Π^{sa}.

In the following, define for $\ell = 0, \ldots, n$,

$$S_\ell^x = \{s \in \{x, y\}^n : |\{i \in N : s_i = x\}| = \ell\}.$$

That is, S_ℓ^x is the set of message profiles in which exactly ℓ receivers observe x. An anonymous and straightforward signal π can be represented by weights $q = (q_0, \ldots, q_n) \geq 0$ and $r = (r_0, \ldots, r_n) \geq 0$ with $\sum_{\ell=0}^n q_\ell = \sum_{\ell=0}^n r_\ell = 1$, where q_ℓ is the probability that x is sent to exactly ℓ receivers if the state is X and r_ℓ is the probability that x is sent to exactly ℓ receivers if the state is Y. For every $\ell = 0, \ldots, n$, for every $s \in S_\ell^x$, we define

$$\pi(s|X) = \binom{n}{\ell}^{-1} q_\ell, \quad \pi(s|Y) = \binom{n}{\ell}^{-1} r_\ell.$$

The probabilities that an agent observes x given that the state is X and Y, respectively, are

$$\pi_i(x|X) = \sum_{\ell=0}^n \sum_{s \in S_\ell^x : s_i = x} \binom{n}{\ell}^{-1} q_\ell = \sum_{\ell=0}^n \binom{n-1}{\ell-1}\binom{n}{\ell}^{-1} q_\ell = \sum_{\ell=0}^n \frac{\ell}{n} q_\ell, \tag{8}$$

$$\pi_i(x|Y) = \sum_{\ell=0}^n \sum_{s \in S_\ell^x : s_i = x} \binom{n}{\ell}^{-1} r_\ell = \sum_{\ell=0}^n \binom{n-1}{\ell-1}\binom{n}{\ell}^{-1} r_\ell = \sum_{\ell=0}^n \frac{\ell}{n} r_\ell. \tag{9}$$

The straightforwardness constraints (6) and (7) are therefore equivalent to

$$\lambda^0(X) \sum_{\ell=0}^n \frac{\ell}{n} q_\ell \geq \lambda^0(Y) \sum_{\ell=0}^n \frac{\ell}{n} r_\ell, \tag{10}$$

$$\lambda^0(Y) \sum_{\ell=0}^n \frac{n-\ell}{n} r_\ell > \lambda^0(X) \sum_{\ell=0}^n \frac{n-\ell}{n} q_\ell, \tag{11}$$

respectively. We demonstrate the weights q and r in the next example.

Example 4.1. Recall our illustrative example from Section 1.2 with $n = 5$, $k = 3$, $\lambda^0(X) = 0.3$, and π^* in Table 1. The signal π^* is represented by weights $q_5^* = 1$, $r_0^* = 2/7$, and $r_3^* = 5/7$. The straightforwardness constraint in (10) is satisfied as

$$\lambda^0(X)\sum_{\ell=0}^{5}\frac{\ell}{5}q_\ell^* = \lambda^0(X)q_5^* = 0.3 = 0.7\cdot\frac{3}{5}\cdot\frac{5}{7} = \lambda^0(Y)\frac{3}{5}r_3^* = \lambda^0(Y)\sum_{\ell=0}^{5}\frac{\ell}{5}r_\ell^*.$$

Since $\lambda^0(X) < 1/2$, the constraint (11) is satisfied as well.

The restriction to signals that are anonymous is without loss of generality as we show in Lemma A.3. An anonymous and straightforward signal that equals sender's highest expected payoff of signals in Π is given in Proposition 4.2. This proposition implies that the signal in the previous example is indeed optimal.

Proposition 4.2. *Let $P = (n, k, \lambda^0)$ be a voting problem. Let $\pi^* \in \Pi^{\text{sa}}$ be represented by (q^*, r^*), where*

$$\left(q_n^*; r_0^*, r_k^*\right) = \begin{cases}(1; 0, 1) & \text{if } \lambda^0(X) \geq \frac{k}{n+k},\\ \left(1; 1 - \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k}, \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k}\right) & \text{if } \lambda^0(X) < \frac{k}{n+k}.\end{cases}$$

Then, π^* is optimal in Π. In particular, it holds that

$$V^{\pi^*}(\lambda^0) = \min\left\{\frac{n+k}{k}\lambda^0(X), 1\right\}.$$

The idea behind π^* is quite simple: if the true state of the world is X, send x to everybody. If the state of the world is Y, then maximize the probability that at least k agents observe an x, so vote in favor of the proposal, while ensuring that the probability of observing an x is not too high for any individual, so that it remains sufficiently persuasive. Targeting more than k agents would be a "waste" as there is no need to persuade more than k voters, and r_k^* cannot be further increased without individual posterior beliefs of state X falling below 1/2.

Proposition 4.2 can be derived from Corollary 2 of Arieli and Babichenko (2019). This might not be immediate since in their model there is no voting problem and agents only want their action to match the true state, whereas in our model agents want *the outcome of the vote* to match the true state. However, under the sincere voting strategy, receivers operate as if their vote were decisive and the two optimization problems are equivalent. We therefore present Proposition 4.2 without proof. Interested readers can find a detailed proof in our working paper (Kerman et al., 2020).

5 | THE SWING VOTER'S CURSE

Recall Example 4.1 and consider some fixed receiver $i \in N$. If the true state is X, all receivers observe x and, as they act sincerely, receiver i is not pivotal. If, on the other hand, the state is Y, he is pivotal upon observing x as his vote will determine the outcome. Therefore, i is better off choosing action y upon observing x: either the true state is indeed X and his vote does not matter, or the state is Y in which case voting for y is strictly beneficial. This phenomenon has been coined the *swing voter's curse*: even though an agent believes that the true state is X, his rational action is to vote in favor of y.

The following example presents the formal analysis of the swing voter's curse by calculating receiver's expected utility from voting x, respectively y, after having received message x.

Example 5.1. Recall the signal π^* in Example 4.1 and Table 1. Suppose receiver i observes message x. Then his belief that the remaining players have observed s_{-i}, conditional on having observed x and the state being ω is $\pi^*((x, s_{-i})|\omega)/\pi_i^*(x|\omega)$. Thus, his expected utility from voting x provided that all other receivers vote according to α^{π^*} is given by

$$\begin{aligned}
&\lambda_i^{\mathbf{x}}(X)\textstyle\sum_{s\in S^{\pi^*}:s_i=x} \frac{\pi^*((x,s_{-i})\mid X)}{\pi_i^*(x\mid X)}u_i\left(z^k\left(x,\alpha_{-i}^{\pi^*}(s_{-i})\right),X\right)\\
&\quad+\lambda_i^{\mathbf{x}}(Y)\textstyle\sum_{s\in S^{\pi^*}:s_i=x} \frac{\pi^*((x,s_{-i})\mid Y)}{\pi_i^*(x\mid Y)}u_i\left(z^k\left(x,\alpha_{-i}^{\pi^*}(s_{-i})\right),Y\right)\\
&\quad=\tfrac{1}{2}\cdot 1\cdot 1+0=\tfrac{1}{2}.
\end{aligned}$$

Note that since the utility of receiver i depends on the implemented outcome, it depends on s_{-i} through $\alpha_{-i}^{\pi^*}(s_{-i})$. His expected utility from voting for y is given by

$$\begin{aligned}
&\lambda_i^{\mathbf{x}}(X)\textstyle\sum_{s\in S^{\pi^*}:s_i=x} \frac{\pi^*((x,s_{-i})\mid X)}{\pi_i^*(x\mid X)}u_i\left(z^k\left(y,\alpha_{-i}^{\pi^*}(s_{-i})\right),X\right)\\
&\quad+\lambda_i^{\mathbf{x}}(Y)\textstyle\sum_{s\in S^{\pi^*}:s_i=x} \frac{\pi^*((x,s_{-i})\mid Y)}{\pi_i^*(x\mid Y)}u_i\left(z^k\left(y,\alpha_{-i}^{\pi^*}(s_{-i})\right),Y\right)\\
&\quad=\tfrac{1}{2}\cdot 1\cdot 1+\tfrac{1}{2}\cdot\binom{4}{2}\frac{\binom{5}{3}^{-1}r_3^*}{\binom{4}{2}\binom{5}{3}^{-1}r_3^*}\cdot 1=1.
\end{aligned}$$

Hence, choosing action y upon observing message x is optimal for receiver i.

Because of the swing voter's curse, α^{π^*} is not a BNE of the voting game $G(P, \pi^*)$ if $k \le n-1$: upon observing x, receivers are pivotal with probability 1 if the state is Y and not pivotal if the state is X. Hence, voting for y is optimal. However, α^{π^*} is a BNE of the voting game $G(P, \pi^*)$ if z^k is the unanimity voting rule, that is, if $k = n$.

Proposition 5.2. *Let $P = (n, k, \lambda^0)$ be a voting problem and let $\pi^* \in \Pi$ be the optimal private signal in Proposition 4.2. Then α^{π^*} is a BNE of $G(P, \pi^*)$ if and only if $k = n$.*

For $k \le n-1$ there are two simple options to ensure sincere voting is a BNE: either increase voters' probability of being pivotal upon observing x if the state is X, or decrease voters' probability of being pivotal upon observing x if the state is Y. We illustrate both approaches using our initial example.

Example 5.3. Recall Example 4.1 with $n = 5, k = 3, \lambda^0(X) = 0.3$ and the signal π^* in Table 1. The signal π in Table 3 is chosen such that the probability r_3 is as high as possible subject to α^{π} being a BNE of the voting game $G(P, \pi)$. The constraint in (3) implies

TABLE 3 Signal π.

π	$\omega = X$	$\omega = Y$
(x, x, x, x, x)	0	0
(x, x, x, y, y)	$\frac{1}{10}$	$\frac{3}{70}$
(x, x, y, y, x)	$\frac{1}{10}$	$\frac{3}{70}$
(x, y, y, x, x)	$\frac{1}{10}$	$\frac{3}{70}$
(y, y, x, x, x)	$\frac{1}{10}$	$\frac{3}{70}$
(y, x, y, x, x)	$\frac{1}{10}$	$\frac{3}{70}$
(y, x, x, y, x)	$\frac{1}{10}$	$\frac{3}{70}$
(y, x, x, x, y)	$\frac{1}{10}$	$\frac{3}{70}$
(x, y, x, y, x)	$\frac{1}{10}$	$\frac{3}{70}$
(x, x, y, x, y)	$\frac{1}{10}$	$\frac{3}{70}$
(x, y, x, x, y)	$\frac{1}{10}$	$\frac{3}{70}$
(y, y, y, y, y)	0	$\frac{40}{70}$

$0.3q_3 \geq 0.7r_3$, so that $r_3 \leq 3/7$ as $q_3 \leq 1$. We verify next that π is straightforward and that α^π is a BNE of the voting game $G(P, \pi)$. For any $i \in N$, we have $\lambda_i^{\mathbf{x}}(X) = 1/2$ and $\lambda_i^{\mathbf{y}}(X) = 3/16$, so that $\alpha_i^\pi(x) = x$ and $\alpha_i^\pi(y) = y$. A simple calculation shows that upon receiving x, (3) holds with equality, so receivers are indifferent between following their recommendation and deviating. Moreover, agents are never pivotal upon observing y, so (4) is trivially satisfied. Therefore, α^π is a BNE of $G(P, \pi)$. Yet, π does not improve upon $\pi^{*\mathrm{p}}$:

$$\begin{aligned} V^\pi(0.3) &= 0.3 \sum_{s \in Z^x(\pi)} \pi(s|X) + 0.7 \sum_{s \in Z^x(\pi)} \pi(s|Y) \\ &= 0.3 \cdot 1 + 0.7 \cdot \frac{3}{7} = 0.6 = V^{\pi^{*\mathrm{p}}}(0.3). \end{aligned}$$

In the signal $\pi^{*\mathrm{e}}$ in Table 2 any voter's probability of being pivotal upon observing x if the state is Y is reduced to 0. Indeed, in state Y message x is sent to $k + 1 = 4$ agents rather than to a minimal winning coalition, so that $T^{i,x}(\pi, \alpha^\pi) = T^{i,y}(\pi, \alpha^\pi) = \varnothing$. Thus, (3) and (4) hold with equality, so that $\alpha^{\pi^{*\mathrm{e}}}$ is a BNE of $G(P, \pi^{*\mathrm{e}})$. For all $i \in N$, we have $\lambda_i^{\mathbf{x}}(X) = 1/2$ and $\lambda_i^{\mathbf{y}}(X) = 0$. So, $\pi^{*\mathrm{e}}$ is straightforward as well. Moreover,

$$\begin{aligned} V^{\pi^{*\mathrm{e}}}(0.3) = 0.3 \sum_{s \in Z^x(\pi')} \pi'(s|X) + 0.7 \sum_{s \in Z^x(\pi')} \pi'(s|Y) = 0.3 \cdot 1 + 0.7 \cdot 5 \cdot \frac{3}{28} = \frac{27}{40} \\ > 0.6 = V^{\pi^{*\mathrm{p}}}(0.3), \end{aligned}$$

that is, π^{*e} improves upon π^{*p}.

6 | EQUILIBRIUM OPTIMAL PRIVATE COMMUNICATION

For each voting problem $P = (n, k, \lambda^0)$, let $\Pi^e(P)$ be the set of all signals under which sincere voting constitutes a BNE, that is,

$$\Pi^e(P) = \{\pi \in \Pi | \alpha^\pi \text{ is a BNE of } G(P, \pi)\}.$$

Sender's problem is to find $\pi \in \Pi^e(P)$ which maximizes her expected utility. If $\lambda^0(X) \geq 1/2$, persuasion is not needed to ensure that receivers vote in favor of outcome x. We will therefore only consider the case $\lambda^0(X) < 1/2$.

Assuming straightforwardness and anonymity of a signal is without loss of generality if there is no BNE constraint. We now show that this is true even if such a constraint is imposed. Let $\Pi^{sae}(P)$ denote the set of signals in $\Pi^e(P)$ that are straightforward and anonymous.

Lemma 6.1. *Let $P = (n, k, \lambda^0)$ be a voting problem with $\lambda^0(X) \in (0, 1/2)$ and $\hat{\pi} \in \Pi^e(P)$. There is $\pi \in \Pi^{sae}(P)$ such that $V^\pi(\lambda^0) = V^{\hat{\pi}}(\lambda^0)$.*

We can express the BNE constraints in Lemma 2.2 of a straightforward anonymous signal by means of the parameters q and r. Consider the voting problem $P = (n, k, \lambda^0)$, a signal $\pi \in \Pi^{sa}$, the strategy profile α^π, and a message profile $s \in S^\pi$. Voter $i \in N$ with $\alpha_i^\pi(s_i) = x$ is pivotal at s if and only if $s \in S_k^x$. Similarly, if $\alpha_i^\pi(s_i) = y$, then i is pivotal if and only if $s \in S_{k-1}^x$. Thus, the following corollary is an immediate consequence of Lemma 2.2.

Corollary 6.2. *Let $P = (n, k, \lambda^0)$ be a voting problem and let $\pi \in \Pi^{sa}$ have representation (q, r). Then α^π is a BNE of $G(P, \pi)$ if and only if*

$$\lambda^0(X) q_k \geq \lambda^0(Y) r_k, \tag{12}$$

$$\lambda^0(Y) r_{k-1} \geq \lambda^0(X) q_{k-1}. \tag{13}$$

A trivial way to satisfy these constraints is to choose $q_n = 1$, so that $q_{k-1} = 0$, and $r_k = 0$. The next lemma shows that this can be done without loss of generality. Intuitively, given a signal in Π^{sae}, this can be achieved by (i) shifting in state X the probabilities of all message profiles to $\mathbf{x}$, and (ii) shifting in state Y the probabilities of all message profiles in which at least k agents observe x to r_{k+1} and the probabilities of all message profiles in which at most $k - 1$ agents observe x to r_0.

Lemma 6.3. *Let $P = (n, k, \lambda^0)$ be a voting problem with $\lambda^0(X) \in (0, 1/2)$, $\hat{\pi} \in \Pi^{sae}$, and $k \leq n - 1$. Then there exists $\pi \in \Pi^{sae}(P)$ with representation (q, r) such that $q_n = 1$, $r_\ell = 0$ for all $\ell \neq 0, k + 1$, and $V^\pi(\lambda^0) \geq V^{\hat{\pi}}(\lambda^0)$.*

Lemma 6.3 is in stark contrast to Proposition 4.2: without the BNE constraint it is in the sender's best interest to make receivers pivotal upon observing x if the state is Y, that is, $r_k > 0$.

In the presence of the BNE constraint, however, sender achieves the highest expected utility by never making any player pivotal—in which case sincere voting trivially is a BNE.

Example 5.3 illustrates how this very naïve approach ensures that a voter who observes x is never pivotal. From a mechanism design perspective, this idea is well known: for instance, a Vickrey auction is incentive compatible exactly because the highest bidder's payment is independent of his own bid.

The next proposition provides the optimization problem of sender as a linear program.

Proposition 6.4. *Let $P = (n, k, \lambda^0)$ be a voting problem with $k \leq n - 1$. Then the highest expected utility achieved by a signal in $\Pi^{\mathrm{e}}(P)$ is the solution to*

$$\max_{r_0, r_{k+1}} \quad \lambda^0(X) + \lambda^0(Y) r_{k+1} \qquad \text{s. t.} \quad r_0 \geq 0, \tag{14}$$

$$r_{k+1} \geq 0, \tag{15}$$

$$r_0 + r_{k+1} = 1, \tag{16}$$

$$\lambda^0(X) - \lambda^0(Y)\frac{k+1}{n} r_{k+1} \geq 0. \tag{17}$$

The objective function in Proposition 6.4 corresponds to the probability that sincere voting results in outcome x. Optimization takes place over variables r_0 and r_{k+1} and any choice for these variables trivially implies that sincere voting is a BNE. Constraint (17) ensures that the signal is straightforward.

As the objective function is continuous and all constraints are weak inequalities, and because of Lemma 6.1, for any voting problem $P = (n, k, \lambda^0)$, sender's optimization problem has a solution. An optimal signal is given in Theorem 6.5.

Theorem 6.5. *Let $P = (n, k, \lambda^0)$ be a voting problem with $k \leq n - 1$. Then the signal $\pi^{*\mathrm{e}} \in \Pi^{\mathrm{sae}}$ with representation $(q^{*\mathrm{e}}, r^{*\mathrm{e}})$ given by*

$$\left(q_n^{*\mathrm{e}}; r_0^{*\mathrm{e}}, r_{k+1}^{*\mathrm{e}}\right) = \begin{cases} (1; 0, 1) & \text{if } \lambda^0(X) \geq \frac{k+1}{n+k+1}, \\ \left(1; 1 - \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k+1}, \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k+1}\right) & \text{if } \lambda^0(X) < \frac{k+1}{n+k+1} \end{cases}$$

is optimal in Π^{e}. In particular, it holds that

$$V^{\pi^{*\mathrm{e}}}(\lambda^0) = \min\left\{\frac{n+k+1}{k+1}\lambda^0(X), 1\right\}.$$

Theorem 6.5 covers the case that the voting rule is not unanimous. If the state is X, then it is optimal to send x to all receivers. If $\lambda^0(X) \geq (k+1)/(n+k+1)$, then it is optimal to send x with probability 1 to a coalition of $k+1$ receivers if the state is Y. Receivers are never pivotal and x is implemented with probability 1. If $\lambda^0(X) < (k+1)/(n+k+1)$, then $r_0^{*\mathrm{e}} > 0$. As there is a positive probability that x is not implemented, the sender's expected utility of the optimal signal is strictly less than 1.

The reason why it is optimal to ensure voters are never pivotal rather than to make them pivotal at some message profiles lies in the fact that the latter can only be achieved by targeting minimal winning coalitions if the state is X, so that voters will not observe x with probability 1 if the state is X. But the straightforwardness constraint in (6) then implies that the probability of observing x in state Y must not be too high either. Theorem 6.5 shows that the negative effect on the success probability in state Y due to making voters pivotal at some message profiles if the state is X outweighs the negative effect due to the lower probability by which x is implemented if the state is Y.

If $k \geq n-1$, Proposition 5.2 and Theorem 6.5 together imply that a public signal is optimal. Yet, if $k = n-1$, there is another, very different, optimal signal. Recall Example 5.3, where we presented in Table 3 a signal π which had the property that agents were always pivotal when observing x, and we showed that this signal gives sender the same expected utility as the optimal public signal $\pi^{*\mathrm{p}}$, that is $V^{\pi}(\lambda^0) = V^{\pi^{*\mathrm{p}}}(\lambda^0)$. This equality motivates the following corollary, which can be proven by verifying that the straightforwardness constraints (6) and (7) and the BNE constraints (12) and (13) are satisfied and that this signal performs as well as $\pi^{*\mathrm{p}}$.

Corollary 6.6. *Let $P = (n, k, \lambda^0)$ be a voting problem with $k = n-1$. Then the signal represented by $q_k = 1$, $r_k = \min\{\lambda^0(X)/\lambda^0(Y), 1\}$, and $r_0 = 1 - r_k$ is optimal in Π^{e}.*

The corollary highlights a fundamentally different way to achieve the optimal value of sender's expected utility if $k = n-1$: sender does not "tell the truth" to all Receivers even in her preferred state, as $q_n = 0$. This might be interpreted as making the decision of receivers more difficult: they are no longer sure that the true state is Y upon observing y, as opposed to $\pi^{*\mathrm{e}}$.

In the following we shall compare the sender's optimal value functions when the set of possible signals is unrestricted, restricted to implement a BNE, or restricted to be public. The corresponding (indirect) value functions shall be denoted V^*, $V^{*\mathrm{e}}$, and $V^{*\mathrm{p}}$, respectively. If $k \leq n-1$, designing the signal so that sincere behavior constitutes a BNE reduces the probability of implementing sender's preferred outcome. Yet, the additional gains that sender could obtain from relaxing the BNE constraints are bounded. The following corollary shows that the ratio of $V^*(\lambda^0)$ to $V^{*\mathrm{e}}(\lambda^0)$ is bounded by $1 + n/k(n+k+1)$.

Corollary 6.7. *Let $P = (n, k, \lambda^0)$ be a voting problem.*

(i) *If $k \leq n-1$ then* $V^{*\mathrm{p}}(\lambda^0) \leq V^{*\mathrm{e}}(\lambda^0) \leq V^*(\lambda^0) \leq \left(1 + \frac{n}{k(n+k+1)}\right) V^{*\mathrm{e}}(\lambda^0)$.

(ii) *If $k = n$ then* $V^*(\lambda^0) = V^{*\mathrm{p}}(\lambda^0) = V^{*\mathrm{e}}(\lambda^0)$.

In Figure 1, we plot the functions V^*, $V^{*\mathrm{e}}$, and $V^{*\mathrm{p}}$ for $n = 5$ and $k = 3$. Sender's maximum expected utility is highest when sincere voting can be taken for granted, which is given by the solid line in Figure 1. The broken line represents the case where the BNE constraint is imposed and the dotted line corresponds to the case with public signals. Under the BNE constraint she cannot prevent a decrease in the probability of implementing her preferred outcome and ends

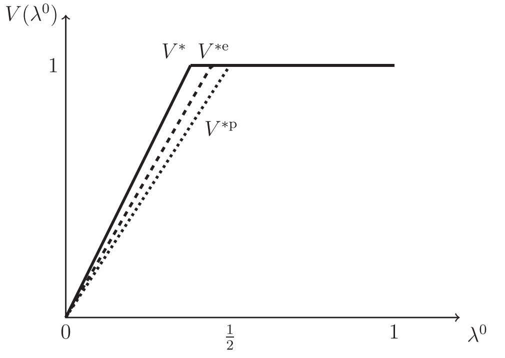

FIGURE 1 Values of π^*, π^{*e}, and π^{*p} for $n = 5$ and $k = 3$.

up with a lower expected utility, that is, $V^*(\lambda^0) > V^{*e}(\lambda^0)$ for $\lambda^0(X) < 4/9$. Finally, for $\lambda^0(X) < 1/2$, we have $V^{*e}(\lambda^0) > V^{*p}(\lambda^0)$.

It is easy to see that sender's expected utility of an optimal signal which implements sincere voting as BNE is decreasing in the quota k. This is intuitive, since persuasion becomes more difficult as the number of receivers that have to be convinced increases.

Corollary 6.8. *Let $P = (n, k, \lambda^0)$ be a voting problem. Then $V^{*e}(\lambda^0)$ is weakly decreasing in k.*

The probability that x is implemented in state Y, that is, that the wrong outcome is implemented, is exactly r^{*e}_{k+1}. This number decreases in k and is minimal for $k = n - 1$ and $k = n$. This contrasts a result by Feddersen and Pesendorfer (1998) who show that strategic voting by jurors may lead to a high probability of convicting an innocent defendant when the unanimity rule is used.

If k is fixed, then sender can implement x with probability 1 when the number of receivers tends to infinity, that is, if $\lim_{n\to\infty} V^{*e}(\lambda^0) = 1$. If the quota is a fixed ratio of the number of voters as in case of majority voting, Theorem 6.5 reveals that the cost of implementing sincere voting as BNE decreases as the population increases. In particular, if $k = \lceil pn \rceil$ for some $p \in (0, 1]$, one has, for all $\lambda^0 \in \Delta^\circ(\Omega)$,

$$\lim_{n\to\infty} V^{*e}(\lambda^0) = \lim_{n\to\infty} V^*(\lambda^0) = \begin{cases} 1 & \text{if } \lambda^0(X) \geq \frac{p}{1+p}, \\ \frac{1+p}{p}\lambda^0(X) & \text{if } \lambda^0(X) < \frac{p}{1+p}. \end{cases}$$

The above equality also shows that in the limit, sender's expected utility of π^* and π^{*e} coincide, that is, the cost of the BNE constraint vanishes.

7 | CONCLUSION

This paper investigates Bayesian persuasion where before a vote, a sender attempts to persuade receivers to vote for her favorite outcome. The best equilibrium for sender is unappealing as all receivers vote in favor of sender's favorite outcome, irrespective of the information they possess.

We therefore study sincere voting by receivers, that is, receivers vote as if they were pivotal. The optimal public signal is derived from Kamenica and Gentzkow (2011), and for this signal, sincere voting constitutes a BNE.

Motivated by social media that allow for targeted communication with individual receivers, we allow sender to use private messages that may be correlated, and we characterize the optimal private signals. While it is indeed beneficial for sender to employ a private signal with correlated messages rather than a public signal, this might lead to the swing voter's curse, that is, situations in which sincere voting is not a BNE. We characterize the optimal private signal such that sincere voting is a BNE and we show that in this case, instead of persuading minimal winning coalitions, sender targets slightly larger coalitions.

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DATA AVAILABILITY STATEMENT

No data are used.

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APPENDIX A: AUXILIARY RESULTS

Lemma A.1. *Let $P = (n, k, \lambda^0)$ be a voting problem with $\lambda^0(X) \in (0, 1/2)$. Then $\pi \in \Pi$ with $S^\pi \subseteq \{x, y\}^n$ is straightforward if and only if* (6) *holds.*

Proof. It is sufficient to show that (6) implies (7). So, let (6) be satisfied. Then

$$\begin{aligned}\lambda^0(X)\pi_i(y|X) &= \lambda^0(X)(1 - \pi_i(x|X))\\ &\leq \lambda^0(X) - \lambda^0(Y)\pi_i(x|Y)\\ &= \lambda^0(X) - \lambda^0(Y) + \lambda^0(Y)\pi_i(y|Y)\\ &< \lambda^0(Y)\pi_i(y|Y),\end{aligned}$$

where the first inequality follows from (6) and the last inequality holds since $\lambda^0(X) < 1/2 < \lambda^0(Y)$. So, π satisfies (7). Hence, π is straightforward. □

Define $\sigma : \Pi \to \Pi$ as follows. Let $\hat{\pi} \in \Pi$. For each action profile $a \in A$, define $S^a(\hat{\pi}) = \{s \in S^{\hat{\pi}} | \alpha^{\hat{\pi}}(s) = a\}$, which is the set of message profiles in $S^{\hat{\pi}}$ that lead to action profile a under strategy profile $\alpha^{\hat{\pi}}$. Note that for $a \neq b$, we have $S^a(\hat{\pi}) \cap S^b(\hat{\pi}) = \emptyset$. We define $\sigma(\hat{\pi}) = \pi$, where

$$\pi(a|\omega) = \sum_{s \in S^a(\hat{\pi})} \hat{\pi}(s|\omega), \quad \omega \in \Omega, a \in A.$$

Note that $S_i^\pi \subseteq A_i$ for all $i \in N$. Signal π is obtained from $\hat{\pi}$ by replacing each message profile by the resulting action profile when voting takes place according to $\alpha^{\hat{\pi}}$.

Lemma A.2. *Let $P = (n, k, \lambda^0)$ be a voting problem with $\lambda^0(X) \in (0, 1/2)$ and let $\pi' \in \Pi$ and $\pi = \sigma(\pi')$. It holds that $\pi \in \Pi^s$ and $V^\pi(\lambda^0) = V^{\pi'}(\lambda^0)$.*

Proof. Fix $i \in N$. Then, for any $\omega \in \Omega$ and $a_i \in S_i^\pi$,

$$\pi_i(a_i|\omega) = \sum_{t \in S^\pi : t_i = a_i} \pi(t|\omega) = \sum_{t \in S^\pi : t_i = a_i} \sum_{s \in S^t(\pi')} \pi'(s|\omega).$$

Recall that, for each $i \in N$, $M_i^x(\pi') = \left\{m \in S_i^{\pi'} | \alpha_i^{\pi'}(m) = x\right\}$. Since $\lambda^0(X) \in (0, 1/2)$, by Lemma A.1 it is sufficient to show that π satisfies (6).

Claim. It holds that

$$\bigcup_{t \in S^\pi : t_i = x} S^t(\pi') = \bigcup_{m \in M_i^x(\pi')} \left\{t' \in S^{\pi'} : t_i' = m\right\}, \tag{A1}$$

and both unions are over disjoint sets.

Proof. Let $s \in \bigcup_{t \in S^\pi : t_i = x} S^t(\pi')$. Then, there exists $t \in S^\pi$ with $t_i = x$ such that $s \in S^t(\pi')$. Thus, by the definition of $S^t(\pi)$, $\alpha_i^{\pi'}(s_i) = x$, so that $s_i \in M_i^x(\pi')$. In particular, $s \in \{t' \in S^{\pi'} : t_i' = s_i\} \subseteq \bigcup_{m \in M_i^x(\pi')} \{t' \in S^{\pi'} : t_i' = m\}$. For the converse, suppose $s \in \bigcup_{m \in M_i^x(\pi')} \{t' \in S^{\pi'} : t_i' = m\}$. Then there exists $m \in M_i^x(\pi')$ with $s_i = m$. Let $t = \alpha^{\pi'}(s) \in A$ and observe that $t_i = \alpha_i^{\pi'}(m) = x$. Since by construction $s \in S^t(\pi')$ and $t \in S^\pi$, it holds that $s \in \bigcup_{t \in S^\pi : t_i = x} S^t(\pi')$.

As noted before, for any $a, b \in S^\pi$ with $a \neq b$ we have $S^a(\pi') \cap S^b(\pi') = \emptyset$. Moreover, for any $m, m' \in M_i^x(\pi')$ with $m \neq m'$, we have $\{t' \in S^{\pi'} : t_i' = m\} \cap \{t' \in S^{\pi'} : t_i' = m'\} = \emptyset$. Thus, both unions are over disjoint sets. This proves the claim. □

Let $M_i^x(\pi') = \{m_1, \ldots, m_L\}$ for some $L \geq 1$. For each $\ell \in \{1, \ldots, L\}$, $m_\ell \in M_i^x(\pi')$, and $s \in S^{\pi'}$ with $s_i = m_\ell$, let

$$c_\ell = \sum_{t \in S^{\pi'} : t_i = m_\ell} \pi'(t|X)\lambda^0(X),$$
$$d_\ell = \sum_{t \in S^{\pi'} : t_i = m_\ell} \pi'(t|Y)\lambda^0(Y),$$

and note that $c_\ell \geq d_\ell$ since $\alpha_i^{\pi'}(m_\ell) = x$. Let $t \in S^\pi$ be such that $t_i = x$. We have that

$$\begin{aligned} \lambda^0(X)\pi_i(x|X) &= \lambda^0(X) \sum_{t \in S^\pi : t_i = x} \sum_{s \in S^t(\pi')} \pi'(s|X) \\ &\overset{(A1)}{=} \lambda^0(X) \sum_{m \in M_i^x(\pi')} \sum_{t \in S^{\pi'} : t_i = m} \pi'(t|X) \\ &= c_1 + \cdots + c_L \geq d_1 + \cdots + d_L = \lambda^0(Y)\pi_i(x|Y). \end{aligned}$$

Thus, for any $m \in M_i^x(\pi')$, we have $\alpha_i^{\pi}(x) = \alpha_i^{\pi'}(m) = x$. Hence, π satisfies (6). What remains to be shown is that $V^{\pi}(\lambda^0) = V^{\pi'}(\lambda^0)$. By the definition of $Z^x(\pi)$

$$\sum_{a \in Z^x(\pi)} \pi(a|\omega) = \sum_{a \in Z^x(\pi)} \sum_{s \in S^a(\pi')} \pi'(s|\omega) = \sum_{s \in Z^x(\pi')} \pi'(s|\omega).$$

Thus, we have

$$\begin{aligned} V^{\pi}(\lambda^0) &= \lambda^0(X) \sum_{s \in Z^x(\pi)} \pi(s|X) + \lambda^0(Y) \sum_{s \in Z^x(\pi)} \pi(s|Y) \\ &= \lambda^0(X) \sum_{s \in Z^x(\pi')} \pi'(s|X) + \lambda^0(Y) \sum_{s \in Z^x(\pi')} \pi'(s|Y) \\ &= V^{\pi'}(\lambda^0) \end{aligned}$$

as required. □

Define $\tau : \Pi \to \Pi^a$ as follows. Let $\hat{\pi} \in \Pi$. For each $\omega \in \Omega$, $s \in S$, and $b \in B$, define π^b by $\pi^b(s|\omega) = \hat{\pi}(s^b|\omega)$, so that $S^{\pi^b} = \{s^{b^{-1}} | s \in S^{\hat{\pi}}\}$. We define the signal $\tau(\hat{\pi})$ by

$$\tau(\hat{\pi})(s|\omega) = \frac{1}{n!} \sum_{b \in B} \pi^b(s|\omega), \omega \in \Omega, s \in S. \tag{A2}$$

Clearly, $\tau(\hat{\pi})$ is anonymous.

Lemma A.3. *Let $P = (n, k, \lambda^0)$ be a voting problem with $\lambda^0(X) \in (0, 1/2)$ and let $\pi' \in \Pi$ and $\pi = \tau(\pi')$. It holds that $V^{\pi}(\lambda^0) = V^{\pi'}(\lambda^0)$.*

Proof. It holds that

$$\begin{aligned} V^{\pi}(\lambda^0) &= \lambda^0(X) \sum_{s \in Z^x(\pi)} \pi(s|X) + \lambda^0(Y) \sum_{s \in Z^x(\pi)} \pi(s|Y) \\ &= \lambda^0(X) \sum_{s \in Z^x(\pi)} \frac{1}{n!} \sum_{b \in B} \pi^b(s|X) + \lambda^0(Y) \sum_{s \in Z^x(\pi)} \frac{1}{n!} \sum_{b \in B} \pi^b(s|Y) \\ &= \lambda^0(X) \frac{1}{n!} \sum_{b \in B} \sum_{s \in Z^x(\pi^b)} \pi^b(s|X) + \lambda^0(Y) \frac{1}{n!} \sum_{b \in B} \sum_{s \in Z^x(\pi^b)} \pi^b(s|Y) \\ &= \lambda^0(X) \frac{1}{n!} \sum_{b \in B} \sum_{s \in Z^x(\pi')} \pi'(s|X) + \lambda^0(Y) \frac{1}{n!} \sum_{b \in B} \sum_{s \in Z^x(\pi')} \pi'(s|Y) \\ &= \lambda^0(X) \sum_{s \in Z^x(\pi')} \pi'(s|X) + \lambda^0(Y) \sum_{s \in Z^x(\pi')} \pi'(s|Y) \\ &= V^{\pi'}(\lambda^0), \end{aligned}$$

where the fourth equation holds because $Z^x(\pi^b) = \{s^{b^{-1}} | s \in Z^x(\pi')\}$. □

APPENDIX B: PROOFS

Proof of Lemma 2.2. Let $i \in N$ and let $s' \in S^\pi$ be such that $\beta_i^\pi\left(s_i'\right) = x$. Then, for $a_i = y$, it holds that

$$
\begin{aligned}
&\sum_{\omega\in\Omega} \lambda_i^{s'}(\omega) \sum_{s\in S^\pi : s_i = s_i'} \frac{\pi\left(\left(s_i', s_{-i}\right) \mid \omega\right)}{\pi_i\left(s_i' \mid \omega\right)} u_i\left(z^k\left(\beta_i^\pi\left(s_i'\right), \beta_{-i}^\pi(s_{-i})\right), \omega\right) \\
&- \sum_{\omega\in\Omega} \lambda_i^{s'}(\omega) \sum_{s\in S^\pi : s_i = s_i'} \frac{\pi\left(\left(s_i', s_{-i}\right) \mid \omega\right)}{\pi_i\left(s_i' \mid \omega\right)} u_i\left(z^k\left(a_i, \beta_{-i}^\pi(s_{-i})\right), \omega\right) \\
&= \sum_{\omega\in\Omega} \frac{\lambda_i^{s'}(\omega)}{\pi_i\left(s_i' \mid \omega\right)} \sum_{s\in T^{i,s_i'}(\pi,\beta^\pi)} \pi(s|\omega)(u_i(x,\omega) - u_i(y,\omega)) \\
&= \frac{1}{\sum_{\omega'\in\Omega} \pi_i\left(s_i' \mid \omega'\right)\lambda^0(\omega')} \sum_{\omega\in\Omega} \sum_{s\in T^{i,s_i'}(\pi,\beta^\pi)} \lambda^0(\omega)\pi(s|\omega)(u_i(x,\omega) - u_i(y,\omega)) \\
&= \frac{1}{\sum_{\omega'\in\Omega} \pi_i\left(s_i' \mid \omega'\right)\lambda^0(\omega')} \Big(\sum_{s\in T^{i,s_i'}(\pi,\beta^\pi)} \lambda^0(X)\pi(s|X) \\
&\qquad - \sum_{s\in T^{i,s_i'}(\pi,\beta^\pi)} \lambda^0(Y)\pi(s|Y)\Big).
\end{aligned}
$$

In this case one observes that (2) is equivalent to (3). Similarly, one shows that (2) is equivalent to (4) for $s' \in S^\pi$ with $\beta_i^\pi(s_i') = y$. □

Proof of Proposition 5.2. Let $i \in N$ and assume first that $k \le n - 1$. Then upon receiving x, i is pivotal if the state is Y but not pivotal if the state is X. Thus, we have $\sum_{s\in T^{i,x}(\pi^*,\alpha^{\pi^*})} \pi^*(s|X) = 0$ and $\sum_{s\in T^{i,x}(\pi^*,\alpha^{\pi^*})} \pi^*(s|Y) > 0$. If α^{π^*} were a BNE at π^*, (3) would imply that $0 \ge \lambda^0(Y)\sum_{s\in T^{i,x}(\pi^*,\alpha^{\pi^*})} \pi^*(s|Y) > 0$, which is impossible.

Let $k = n$. Since i never observes y if the state is X, (4) is trivially satisfied. By Proposition 4.2 we have $q_n^* = 1$ and $\lambda^0(Y)r_n^* = \lambda^0(Y)\min\{\lambda^0(X)/\lambda^0(Y), 1\} \le \lambda^0(X)$. Thus,

$$\sum_{s\in T^{i,x}(\pi^*,\alpha^{\pi^*})} \lambda^0(X)\pi^*(s|X) = \lambda^0(X) \ge \lambda^0(Y)r_n^* = \sum_{s\in T^{i,x}(\pi^*,\alpha^{\pi^*})} \lambda^0(Y)\pi^*(s|Y).$$

Hence, (3) is satisfied as well and α^{π^*} is a BNE. □

Proof of Lemma 6.1. Assume without loss of generality that for all $i, j \in N$ with $i \ne j$, we have $S_i^{\hat\pi} \cap S_j^{\hat\pi} = \varnothing$, that is, different receivers observe different messages. Let $\pi' = \tau(\hat\pi)$, (as defined before Lemma A.3), so that $\pi' \in \Pi^{\mathrm{a}}$. Then $V^{\pi'}(\lambda^0) = V^{\hat\pi}(\lambda^0)$ by Lemma A.3. We show that $\alpha^{\pi'}$ is a BNE of $G(P, \pi')$.

Let $i \in N$ and $m \in M_i^x(\pi')$. Let $j \in N$ be the unique receiver such that $m \in S_j^{\hat\pi}$. It holds that

$$
\begin{aligned}
\sum_{s\in T^{i,m}(\pi',\alpha^{\pi'})} \sum_{b\in B} \hat\pi(s^b|X) &= \sum_{s\in T^{i,m}(\pi',\alpha^{\pi'})} \sum_{b\in B : b(i) = j} \hat\pi(s^b|X) \\
&= (n-1)! \sum_{s\in T^{j,m}(\hat\pi,\alpha^{\hat\pi})} \hat\pi(s|X).
\end{aligned} \tag{B1}
$$

Therefore,

$$\begin{aligned}\lambda^0(X)\sum_{s\in T^{i,m}(\pi',\alpha^{\pi'})}\pi'(s|X) &= \lambda^0(X)\frac{1}{n!}\sum_{s\in T^{i,m}(\pi',\alpha^{\pi'})}\sum_{b\in B}\hat{\pi}(s^b|X)\\ &\overset{(B1)}{=} \lambda^0(X)\frac{1}{n}\sum_{s\in T^{j,m}(\hat{\pi},\alpha^{\hat{\pi}})}\hat{\pi}(s|X)\\ &\geq \lambda^0(Y)\frac{1}{n}\sum_{s\in T^{j,m}(\hat{\pi},\alpha^{\hat{\pi}})}\hat{\pi}(s|Y)\\ &= \lambda^0(Y)\sum_{s\in T^{i,m}(\pi',\alpha^{\pi'})}\pi'(s|Y),\end{aligned}$$

where the inequality comes from Lemma 2.2 and the fact that $\alpha^{\hat{\pi}}$ is a BNE of $G(P,\hat{\pi})$. A similar argument holds for $m' \in M_i^y(\pi')$. It follows that $\alpha^{\pi'}$ is a BNE of $G(P,\pi')$.

Let $\pi = \sigma(\pi')$ (as defined before Lemma A.2). Clearly, it holds that $\pi \in \Pi^{\text{a}}$. It follows from Lemma A.2 that $\pi \in \Pi^{\text{s}}$ and $V^{\pi}(\lambda^0) = V^{\pi'}(\lambda^0)$. So, what remains to be shown is that α^{π} is a BNE of $G(P,\pi)$.

Claim. For all $i \in N$ it holds that

$$\bigcup_{a\in T^{i,x}(\pi,\alpha^{\pi})} S^a(\pi') = \bigcup_{m\in M_i^x(\pi')} T^{i,m}(\pi',\alpha^{\pi'}). \tag{B2}$$

and the unions on both sides of the equation are over disjoint sets.

Proof. Let $s \in \bigcup_{a\in T^{i,x}(\pi,\alpha^{\pi})} S^a(\pi')$. Then, there exists $a \in S^{\pi}$ with $a_i = x$ such that $\alpha^{\pi'}(s) = a$ and i is pivotal in s. Since $a_i = x$, we have $\alpha_i^{\pi'}(s_i) = a_i = x$, hence $s_i \in M_i^x(\pi')$, so $s \in \bigcup_{m\in M_i^x(\pi')} T^{i,m}(\pi',\alpha^{\pi'})$. For the converse, let $s \in \bigcup_{m\in M_i^x(\pi')} T^{i,m}(\pi',\alpha^{\pi'})$. Then, $s_i \in M_i^x(\pi')$ and i is pivotal in s. Let $a = \alpha^{\pi'}(s)$, so that $s \in S^a(\pi')$. Since $s_i \in M_i^x(\pi')$, we have $a_i = \alpha_i^{\pi'}(s_i) = x$, so that $a \in T^{i,x}(\pi',\alpha^{\pi'})$, hence $s \in \bigcup_{a\in T^{i,x}(\pi,\alpha^{\pi})} S^a(\pi')$. For the same reasons as in the proof of Lemma A.2 both unions are over disjoint sets. □

Since the unions on both sides of (B2) are disjoint, we have

$$\begin{aligned}\lambda^0(X)\sum_{a\in T^{i,x}(\pi,\alpha^{\pi})}\pi(a|X) = \lambda^0(X)&\sum_{a\in T^{i,x}(\pi,\alpha^{\pi})}\sum_{t\in S^a(\pi')}\pi'(t|X)\\ &\overset{(B2)}{=} \lambda^0(X)\sum_{m\in M_i^x(\pi')}\sum_{s\in T^{i,m}(\pi',\alpha^{\pi'})}\pi'(s|X)\\ &\geq \lambda^0(Y)\sum_{m\in M_i^x(\pi')}\sum_{s\in T^{i,m}(\pi',\alpha^{\pi'})}\pi'(s|Y) = \lambda^0\\ &(Y)\sum_{a\in T^{i,x}(\pi,\alpha^{\pi})}\pi(a|Y),\end{aligned}$$

where the inequality holds since $\alpha^{\pi'}$ is a BNE. Showing that choosing action y is optimal upon observing message y is similar. It follows that $\pi \in \Pi^{\text{sae}}(P)$. □

Proof of Lemma 6.3. First, we show that it is without loss of generality to assume that $\hat{q}_k = \hat{r}_k = 0$, where $(\hat{q}, \hat{r})$ is the representation of $\hat{\pi}$. Let $\pi \in \Pi^{\mathrm{a}}$ be defined such that $q_{k-1} = q_k = 0, q_{k+1} = \hat{q}_{k-1} + \hat{q}_k + \hat{q}_{k+1}, q_\ell = \hat{q}_\ell$ for all $\ell \neq k-1, k, k+1, r_k = 0, r_{k+1} = \hat{r}_k + \hat{r}_{k+1}$, and $r_\ell = \hat{r}_\ell$ for all $\ell \neq k, k+1$. Clearly, π satisfies (12) and (13), so $\pi \in \Pi^{\mathrm{e}}$. We next show that π is straightforward. Since $\hat{\pi}$ is straightforward, it holds that $\lambda^0(Y) \sum_{\ell=0}^{n} \frac{\ell}{n} \hat{r}_\ell \leq \lambda^0(X) \sum_{\ell=0}^{n} \frac{\ell}{n} \hat{q}_\ell$. Moreover, as $\alpha^{\hat{\pi}}$ is a BNE, it holds that $\lambda^0(Y)\hat{r}_k \leq \lambda^0(X)\hat{q}_k$. Hence,

$$\begin{aligned}\lambda^0(Y) \sum_{\ell=0}^{n} \frac{\ell}{n} r_\ell &= \lambda^0(Y)\left(\sum_{\ell=0}^{n} \frac{\ell}{n}\hat{r}_\ell + \frac{1}{n}\hat{r}_k\right) \leq \lambda^0(X)\left(\sum_{\ell=0}^{n} \frac{\ell}{n}\hat{q}_\ell + \frac{1}{n}\hat{q}_k + \frac{2}{n}\hat{q}_{k-1}\right)\\ &= \lambda^0(X) \sum_{\ell=0}^{n} \frac{\ell}{n} q_\ell,\end{aligned}$$

which proves that $\pi \in \Pi^{\mathrm{sae}}(P)$. Finally,

$$V^\pi(\lambda^0) = \lambda^0(X) \sum_{\ell=k}^{n} q_\ell + \lambda^0(Y) \sum_{\ell=k}^{n} r_\ell \geq \lambda^0(X) \sum_{\ell=k}^{n} \hat{q}_\ell + \lambda^0(Y) \sum_{\ell=k}^{n} \hat{r}_\ell = V^{\hat{\pi}}(\lambda^0).$$

Thus, we can assume without loss of generality that $\hat{q}_k = \hat{r}_k = 0$.

Now, let $\pi \in \Pi^{\mathrm{a}}$ be defined by $q_n = 1, q_\ell = 0$ for all $\ell \neq n, r_0 = \sum_{\ell=0}^{k-1} \hat{r}_\ell, r_{k+1} = \sum_{\ell=k+1}^{n} \hat{r}_\ell$ and $r_\ell = 0$ for all $\ell \neq 0, k+1$. Then $\pi \in \Pi^{\mathrm{sa}}$ since

$$\begin{aligned}\lambda^0(Y)\pi_i(x|Y) = \lambda^0(Y) \sum_{\ell=0}^{n} \frac{\ell}{n} r_\ell &= \lambda^0(Y)\frac{k+1}{n} r_{k+1}\\ &= \lambda^0(Y)\frac{k+1}{n} \sum_{\ell=k+1}^{n} \hat{r}_\ell \leq \lambda^0(Y) \sum_{\ell=0}^{n} \frac{\ell}{n}\hat{r}_\ell\\ &\leq \lambda^0(X) \sum_{\ell=0}^{n} \frac{\ell}{n}\hat{q}_\ell \leq \lambda^0(X) = \lambda^0(X)\pi_i(x|X),\end{aligned}$$

where the second inequality holds as $\hat{\pi} \in \Pi^{\mathrm{sae}}(P)$. Since $q_{k-1} = q_k = r_k = 0$, (12) and (13) are satisfied. So, $\pi \in \Pi^{\mathrm{sae}}(P)$. Finally,

$$V^\pi(\lambda^0) = \lambda^0(X) + \lambda^0(Y) r_{k+1} \geq \lambda^0(X) \sum_{\ell=k}^{n} \hat{q}_\ell + \lambda^0(Y) \sum_{\ell=k+1}^{n} \hat{r}_\ell = V^{\hat{\pi}}(\lambda^0),$$

where the last equality uses that $\hat{r}_k = 0$. □

Proof of Proposition 6.4. If $\lambda^0(X) \geq 1/2$, then the solution to the linear optimization problem in Proposition 6.4 is given by $(r_0^*, r_{k+1}^*) = (0, 1)$, leading to the desired value of 1 for the objective function. Let $\lambda^0(X) < 1/2$. By Lemmas 6.1 and 6.3, if there is an optimal $\hat{\pi} \in \Pi^{\mathrm{sae}}(P)$, then there is $\pi \in \Pi^{\mathrm{sae}}(P)$ with representation (q, r) such that $q_k = 1, r_\ell = 0$ for all $\ell \neq 0, k+1$, and $V^{\hat{\pi}}(\lambda^0) = \lambda^0(X) + \lambda^0(Y) r_{k+1}$. Such π is straightforward if and only if

$$\lambda^0(X) = \lambda^0(X)\pi_i(x|X) \geq \lambda^0(Y)\pi_i(x|Y) = \lambda^0(Y)\frac{k+1}{n}r_{k+1},$$

which is equivalent to (17). □

Proof of Theorem 6.5. By (17), it holds that

$$\frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k+1} \geq r_{k+1}.$$

Since the objective function is increasing in r_{k+1}, the optimal value is obtained by choosing

$$r_{k+1}^* = \begin{cases} 1 & \text{if } \lambda^0(X) \geq \frac{k+1}{n+k+1}, \\ \frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k+1} & \text{if } \lambda^0(X) < \frac{k+1}{n+k+1}. \end{cases}$$

So, if $\lambda^0(X) \geq \frac{k+1}{n+k+1}$, then $V^{\pi^{*e}}(\lambda^0) = 1$. Otherwise, it holds that

$$V^{\pi^{*e}}(\lambda^0) = \lambda^0(X) + \lambda^0(Y)r_{k+1} = \lambda^0(X) + \lambda^0(Y)\frac{\lambda^0(X)}{\lambda^0(Y)}\frac{n}{k+1} = \lambda^0(X)\frac{n+k+1}{k+1}$$

as required. □

Proof of Corollary 6.7.

(i) These inequalities follow immediately from the simple observation that, for all $k = 1, \ldots, n-1$,

$$2 \leq \frac{n+k+1}{k+1} \leq \frac{n+k}{k}.$$

(ii) This follows immediately from Proposition 3.1, Proposition 4.2, and Theorem 6.5.

□

Proof of Corollary 6.8. This follows immediately from the observation that $(n+k+1)/(k+1)$ is decreasing in k and bounded from below by 2 for all $k \leq n-1$. □