




Arithmetic Progressions and Chaos in Linear Dynamics

Rodrigo Cardeccia and Santiago Muro 

Abstract. We characterize chaotic linear operators on reflexive Banach spaces in terms of the existence of long arithmetic progressions in the sets of return times. We also show that this characterization does not hold for arbitrary Banach spaces. To achieve this, we study \mathcal{F} -hypercyclicity for a family of subsets of the natural numbers associated to the existence of arbitrarily long arithmetic progressions.

Mathematics Subject Classification. 47A16, 37B20, 37A45, 11B25, 47B37.

Keywords. Hypercyclic operators, Chaotic operators, Furstenberg families, Arithmetic progressions, Small periodic sets.

1. Introduction

A linear operator T is said to be hypercyclic provided that there is $x \in X$ such that $\text{Orb}_T(x) := \{T^n(x) : n \in \mathbb{N}\}$ is dense in X and chaotic if it is hypercyclic and has a dense set of periodic points. The notion of chaos was introduced by Devaney [15] and first developed by Godefroy and Shapiro [17] in the context of linear dynamics. Since then it was one of the most important concepts in the dynamics of linear operators. Linear dynamics has experienced a lively development in the last decades, see [6, 20]. For instance we know that every infinite dimensional and separable Banach space supports a hypercyclic operator [2, 7] while there are Banach spaces without chaotic operators [10], there are hypercyclic operators T such that $T \oplus T$ is no longer hypercyclic [14], etc. Over the last years much of the attention was given to *frequent hypercyclicity* (see e.g. [4, 5, 25]) and more recently to \mathcal{F} -hypercyclicity [8, 9, 11–13], for more general families \mathcal{F} of subsets of \mathbb{N} .

Given a *hereditary upward* family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ (also called Furstenberg family) we say that an operator is \mathcal{F} -hypercyclic if there is $x \in X$ for which the sets $N_T(x, U) := \{n \in \mathbb{N} : T^n(x) \in U\}$ of return times belong to \mathcal{F} for any nonempty open set $U \subseteq X$. Thus, for example, if we consider Inf , the family

of infinite sets, *Inf*-hypercyclicity is simply hypercyclicity and if $\underline{\mathcal{D}}$ denotes the family of sets with positive lower density, then $\underline{\mathcal{D}}$ -hypercyclicity is frequent hypercyclicity. Over the last years several notions of \mathcal{F} -hypercyclicity were introduced such as upper frequent hypercyclicity [28], reiterative hypercyclicity [8] and more recently piecewise-syndetic hypercyclicity [26].

Although most of the main concepts in linear dynamics are known to fit within the framework of \mathcal{F} -hypercyclicity, there is one notable exception: chaos.

In this context, the following question was posed by Bonilla and Grosse-Erdmann.

Question 1.1. ([11]). *Does there exist a hereditary upward family \mathcal{F} such that \mathcal{F} -hypercyclicity is equivalent to chaos?*

A related (weaker) question is the following.

Question 1.2. *Is it possible to characterize chaos in terms of the behavior of a single orbit?*

In the present note, we introduce a notion of \mathcal{F} -hypercyclicity related to the existence of long arithmetic progressions and study its connection with chaos and other concepts in linear dynamics. We answer Question 1.1 affirmatively on separable reflexive Banach spaces and Question 1.2 on arbitrary separable Fréchet spaces.

The motivation to study the relationship between arithmetic progressions and chaotic operators is simple: if T is a chaotic operator and U is a nonempty open set then the existence of periodic points in U implies that $N_T(x, U)$ must have arbitrarily long arithmetic progressions for any hypercyclic vector x .

The study of sets having arbitrarily long arithmetic progressions (or sets in \mathcal{AP}) had a great development over the last century and is a central task in number theory and additive combinatorics. For instance, the celebrated Szemerédi Theorem [29] and the Green-Tao Theorem [18] establish that the sets having positive lower density and the set of prime numbers belong to \mathcal{AP} . On the other hand, there isn't, up to our knowledge, a systematic investigation on sets having arbitrarily long arithmetic progressions with bounded common difference. Nevertheless, as we shall see, these sets play an important role in linear dynamics. We will denote by \mathcal{AP}_b this family of subsets, and we will use it to answer Question 1.1 for weak*-weak* continuous operators: such an operator is chaotic if and only if it is \mathcal{AP}_b -hypercyclic (Theorem 3.1). For arbitrary operators the family \mathcal{AP}_b can still be used to characterize chaos in terms of a single orbit: T is chaotic if and only if there is $x \in X$ such that for every nonempty open set U , the return times set $N(x, U)$ contains a subsequence $(n_k)_k \in \mathcal{AP}_b$ for which the set $\{T^{n_k}(x)\}$ is weakly precompact. As a corollary, we obtain a Transitivity Theorem (Theorem 3.14) for chaotic operators.

The paper is organized as follows. In Sect. 2 we fix notation and recall some facts about hereditary upward families and \mathcal{F} -hypercyclicity. In Sect.

3 we study \mathcal{AP}_b -hypercyclic operators, operators having dense small periodic sets and their connection to chaos. We prove that these concepts are equivalent for weak*-weak* continuous operators (Theorems 3.1 and 3.12) and we show the existence of an \mathcal{AP}_b -hypercyclic weighted shift on c_0 which is not chaotic (Theorem 3.19). We also prove that hypercyclic weighted shifts with dense small periodic sets are chaotic (Theorem 3.17) and that \mathcal{AP}_b -hypercyclic operators cannot have isolated points in the spectrum (Corollary 3.25). In Sect. 4 we close the article with some comments and open questions.

2. Preliminaries

A family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is called a *hereditary upward family* or a *Furstenberg family* if $A \subseteq B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$ (see for example [1,11]). Given a Furstenberg family \mathcal{F} we will say that T is \mathcal{F} -hypercyclic provided that there is $x \in X$ such that for every nonempty open set U , $N_T(x, U) := \{n : T^n(x) \in U\} \in \mathcal{F}$. Such an x is called an \mathcal{F} -hypercyclic vector.

The following hereditary upward families and notions of \mathcal{F} -hypercyclicity are the most widely studied in the literature:

1. A is said to have positive lower density (or $A \in \underline{\mathcal{D}}$) if

$$\underline{dens}(A) := \liminf_n \frac{\#\{k \leq n : k \in A\}}{n} > 0,$$

and an operator is said to be frequently hypercyclic if T is $\underline{\mathcal{D}}$ hypercyclic.

2. A is said to have positive upper density (or $A \in \overline{\mathcal{D}}$) if

$$\limsup_n \frac{\#\{k \leq n : k \in A\}}{n} > 0,$$

and an operator is said to be upper or (\mathcal{U} -) frequently hypercyclic if T is $\overline{\mathcal{D}}$ -hypercyclic.

3. A is said to have positive upper Banach density (or $A \in \overline{\mathcal{BD}}$) if

$$\lim_n \limsup_k \frac{\#A \cap [k, k + n]}{n} > 0,$$

and an operator is said to be reiteratively hypercyclic if T is $\overline{\mathcal{BD}}$ -hypercyclic.

A hereditary upward family is said to be *upper* provided that $\emptyset \notin \mathcal{F}$ and \mathcal{F} can be written as

$$\bigcup_{\delta \in D} \mathcal{F}_\delta, \quad \text{with } \mathcal{F}_\delta = \bigcap_{m \in M} \mathcal{F}_{\delta,m},$$

where D is arbitrary but M is countable and such that the families $\mathcal{F}_{\delta,m}$ and \mathcal{F} satisfy

- each $\mathcal{F}_{\delta,m}$ is *finitely hereditary upward*, that means that for each $A \in \mathcal{F}_{\delta,m}$, there is a finite set F such that $F \cap A \subseteq B$, then $B \in \mathcal{F}_{\delta,m}$;
- \mathcal{F} is *uniformly left invariant*, that is, if $A \in \mathcal{F}$ then there is $\delta \in D$ such that for every n , $A - n \in \mathcal{F}_\delta$.

The families \mathcal{Inf} , $\overline{\mathcal{D}}$, $\overline{\mathcal{BD}}$ are upper while $\underline{\mathcal{D}}$ is not upper (see [11]).

Theorem 2.1. (*Bonilla-Grosse Erdmann [11]*). *Let \mathcal{F} be an upper hereditary upward family and T be a linear operator on a separable Fréchet space. Then the following are equivalent:*

1. T is \mathcal{F} -hypercyclic.
2. For any nonempty open set V there is $\delta \in D$ such that for any nonempty open set U there is $x \in U$ with $N_T(x, V) \in \mathcal{F}_\delta$.
3. For any nonempty open set V there is $\delta \in D$ such that for every nonempty open set U and $m \in M$ there is $x \in U$ with $N_T(x, V) \in \mathcal{F}_{\delta, m}$.
4. The set of \mathcal{F} -hypercyclic points is residual.

3. \mathcal{AP}_b -hypercyclic operators and chaotic operators

In this section we study \mathcal{AP}_b -hypercyclic operators and their relationship with chaotic operators. In Sect. 3.1 we prove our main result, which shows that the weak*-weak* continuous chaotic operators on dual Banach spaces are exactly the \mathcal{AP}_b -hypercyclic operators. To prove it, we will need some preliminary results and we also have to study the concept of dense small periodic sets. In Sect. 3.2 we study \mathcal{AP}_b -hypercyclic weighted shifts operators and show that the assumption on weak*-weak* continuity cannot be dropped, by exhibiting an \mathcal{AP}_b -hypercyclic operator on c_0 that is not chaotic (Theorem 3.19). In Sect. 3.3 we study the spectrum of \mathcal{AP}_b -hypercyclic operators. Unless explicitly stated, all Banach spaces considered may be real or complex.

3.1. The main result

Theorem 3.1. *Let X be a separable Banach space which is a dual space and let T be a weak*-weak* continuous operator on X . Then T is chaotic if and only if there exists $x \in X$ such that for each nonempty open set U , $N_T(x, U)$ contains arbitrarily long arithmetic progressions of common difference k , for some $k \in \mathbb{N}$.*

Note that the above equivalence holds for arbitrary operators on reflexive spaces. Theorem 3.1 is a direct consequence of Theorem 3.12 below. Let us first define the Furstenberg family \mathcal{AP}_b . Recall that the arithmetic progression of length $m + 1$ ($m \in \mathbb{N}$), common difference $k \in \mathbb{N}$ and initial term $a \in \mathbb{N}$ is the subset of \mathbb{N} of the form $\{a, a + k, a + 2k, \dots, a + mk\}$.

Definition 3.2. We will denote by \mathcal{AP}_b the family of subsets of the natural numbers that contain arbitrarily long arithmetic progressions of bounded common difference (i.e. there are arbitrarily long arithmetic progressions of common difference bounded by k , for some fixed $k \in \mathbb{N}$).

The family \mathcal{AP}_b is an upper Furstenberg family: it is the union of the families $(\mathcal{AP}_b)_n$ of subsets having arbitrarily long arithmetic progressions with fixed step n , and $(\mathcal{AP}_b)_n$ is the intersection of the families $(\mathcal{AP}_b)_{n, m}$ of subsets having arithmetic progressions of fixed step n with length m .

The next proposition is similar to Theorem 2.1, but simpler as the proof does not rely on Baire’s category Theorem. For the sake of the completeness, we include the proof here.

Proposition 3.3. *Let T be a mapping on a separable Fréchet space. Then the following assertions are equivalent.*

- (1) T is hypercyclic and every hypercyclic vector is \mathcal{AP}_b -hypercyclic.
- (2) There is an \mathcal{AP}_b -hypercyclic vector.
- (3) T is hypercyclic and for every nonempty open set U there is k such that for every m , $\bigcap_{j=0}^m T^{-jk}(U) \neq \emptyset$.
- (4) For every nonempty open set V , there is k such that for every nonempty open set U and m , there are $x_m \in U$ and k_m with $T^{k_m+jk}(x_m) \in V$ for every $0 \leq j \leq m$.
- (5) For every pair of nonempty open sets U and V there is k such that for every m there are k_m and $x_m \in U$ with $T^{k_m+jk}(x_m) \in V$ for every $0 \leq j \leq m$.
- (6) The set of \mathcal{AP}_b -hypercyclic vectors is residual.

Proof. (1) \Rightarrow (2) is immediate.

(2) \Rightarrow (3). Let x be an \mathcal{AP}_b -hypercyclic vector and U be a nonempty open set. Thus there is k such that, for every m there is k_m with $T^{k_m+jk}(x) \in U$ for every $0 \leq j \leq m$. In particular, $T^{k_m}(x) \in \bigcap_{j=0}^m T^{-jk}(U)$.

(3) \Rightarrow (4). Let V be a nonempty set and k such that for every m , $\bigcap_{j=0}^m T^{-jk}(V) \neq \emptyset$ is a nonempty open set. Let U be a nonempty open, $x \in U$ be a hypercyclic vector and $m \in \mathbb{N}$. Thus, there is $k_m \in \mathbb{N}$ such that $T^{k_m}(x) \in \bigcap_{j=0}^m T^{-jk}(V)$. Equivalently, $T^{k_m+jk}(x) \in V$ for every $0 \leq j \leq m$.

(4) \Rightarrow (5) is immediate.

(5) \Rightarrow (3). Clearly T is transitive and hence hypercyclic. Let U be a nonempty open set. Thus, there is k such that for every m , there are k_m and $x_m \in U$ with $T^{k_m+jk}(x_m) \in U$ for every $0 \leq j \leq m$. In particular, $T^{k_m}(x_m) \in \bigcap_{j=1}^m T^{-jk}(U)$.

(3) \Rightarrow (1). Let x be a hypercyclic vector and U be a nonempty open set. Let k such that $\bigcap_{j=0}^m T^{-jk}(U)$ is a nonempty open set for every m . Let $m \in \mathbb{N}$, then since x is hypercyclic there is k_m such that $T^{k_m}(x) \in \bigcap_{j=0}^m T^{-jk}(U)$. Equivalently, $T^{k_m+jk}(x) \in U$ for every $0 \leq j \leq m$.

(1) \Rightarrow (6) follows from the fact that the set of hypercyclic vectors is residual.

(6) \Rightarrow (2) is immediate. □

In [25] it was shown that chaotic operators are reiteratively hypercyclic. The proof given there essentially proves the following (which is also an immediate consequence of equivalence (3) in the above proposition).

Proposition 3.4. *Let T be a chaotic operator. Then T is \mathcal{AP}_b -hypercyclic.*

On the other hand, there are subsets of the natural numbers (for instance the square free numbers) that have positive lower density but do not belong to \mathcal{AP}_b . So we cannot conclude that frequently hypercyclic operators are \mathcal{AP}_b -hypercyclic. There are also chaotic operators that are not upper frequently hypercyclic [25]. Therefore there are \mathcal{AP}_b -hypercyclic operators that are not upper frequent hypercyclic. Moreover, since there are frequently

hypercyclic operators on Hilbert spaces that are not chaotic [6, Section 6.5], by Theorem 3.1 frequent hypercyclicity does not imply \mathcal{AP}_b -hypercyclicity.

Very recently, it has been proven in [16, Theorem 2.5] that, for some Furstenberg families \mathcal{F} , if T is \mathcal{F} -hypercyclic then $T \oplus T$ is also \mathcal{F} -hypercyclic. That theorem may be applied for \mathcal{AP}_b , but in this special case we have a simpler proof.

Proposition 3.5. *Let T be an \mathcal{AP}_b -hypercyclic operator. Then*

- (i) T is reiteratively hypercyclic and weakly mixing.
- (ii) $\underbrace{T \oplus T \dots \oplus T}_n$ is \mathcal{AP}_b -hypercyclic for every n .

Proof. Since \mathcal{AP}_b sets have positive upper Banach density, T is reiteratively hypercyclic. Moreover, by [8], reiteratively hypercyclic operators are weakly mixing. This proves (i).

Let $n \in \mathbb{N}$. Since T is weakly mixing, then $\underbrace{T \oplus T \dots \oplus T}_n$ is hypercyclic.

By Proposition 3.3, to prove (ii) it suffices to show that, for every tuple of nonempty open sets $U_1 \dots, U_n$, there is k , such that for every m and i , $\bigcap_{j=0}^m T^{-jk} U_i$ is nonempty.

Let $U_1 \dots U_n$ be nonempty open sets. Since T is \mathcal{AP}_b -hypercyclic there is for every i some k_i such that for every m , $\bigcap_{j=0}^m T^{-jk_i}(U_i)$ is nonempty. In particular, if we consider $k = \prod_{j=0}^n k_j$, we have that for every m and every i , $\bigcap_{j=0}^m T^{-jk}(U_i)$ is nonempty. □

Remark 3.6. In [13] the authors studied \mathcal{F} -hypercyclicity for the family $\mathcal{F} = \mathcal{AP}$ of sets having arbitrarily large arithmetic progressions (of any common difference). Although at first sight it may seem to be a concept very similar to \mathcal{AP}_b -hypercyclicity, it is not. It turns out that while \mathcal{AP}_b -hypercyclicity is related to chaos, \mathcal{AP} -hypercyclicity is equivalent to multiple recurrence plus hypercyclicity. It is shown there, for example, that such operators may even fail to be weakly mixing (see [13, Theorem 5.1]).

In order to prove Theorem 3.1, we need the concept of dense small periodic sets, which is a natural generalization of density of periodic points. The notion was introduced by Huan and Ye in [21] for non linear dynamics on compact spaces. We will say that a nonempty subset Y is a periodic set for T if $T^k(Y) \subseteq Y$ for some $k > 0$.

Definition 3.7. A mapping T has dense small periodic sets provided that for every nonempty open set U there is a closed periodic set $Y \subseteq U$.

Proposition 3.8. *A mapping T has dense small periodic sets if and only if for every nonempty open set U there is k such that $\bigcap_{j=1}^\infty T^{-jk}(U) \neq \emptyset$.*

In particular if T is hypercyclic and has dense small periodic sets then it is \mathcal{AP}_b -hypercyclic.

Proof. Let U be a nonempty open set and consider a nonempty open set $V \subseteq U$ such that $V \subseteq \bar{V} \subseteq U$. Let $x \in \bigcap_{j=1}^\infty T^{-jk}(V)$. Then the set $Y =$

$\overline{Orb_{T^k}(x)}$ is a closed subset of U which satisfies $T^k(Y) \subseteq Y$. Reciprocally given a nonempty open set U and $Y \subseteq U$ a closed subset which is T^k -invariant, every $x \in Y$ belongs to $\bigcap_{j=1}^{\infty} T^{-jk}(U)$.

The last assertion follows from Proposition 3.3. □

In Sect. 3.2 we will present an example of an \mathcal{AP}_b -hypercyclic operator that does not have dense small periodic sets.

The next lemma, which is purely linear as it exploits the linearity of both the operator and the space, is an important ingredient for the proof of the main theorem.

Lemma 3.9. *Let Y be a k -periodic set for an operator T on a Fréchet space X such that either*

- i) Y is weakly compact or
- ii) X is a dual space, Y is weak*-compact and T is weak*-weak* continuous.

Then there is a k -periodic vector in $\overline{co(Y)}^\tau$, where τ denotes weak or weak star topology, respectively.

Proof. The proof is an elementary application of the Schauder-Tychonoff fixed point Theorem for locally convex spaces [30].

Let Y be a k -periodic set. Then $\overline{co(Y)}^\tau$ is T^k -invariant. Moreover, $\overline{co(Y)}^\tau$ is τ -compact (by either the Krein-Šmulian Theorem [22] or [27, Chapter II, 4.3]). Therefore, the Schauder-Tychonoff Theorem assures the existence of a fixed point of T^k in $\overline{co(Y)}^\tau$. This fixed point is a k -periodic vector for T . □

For a subset A of a Banach space, $\overset{\circ}{A}$ denotes the norm interior of A .

Proposition 3.10. *Let T be a hypercyclic operator on a separable Banach space. Consider the following statements.*

- i) T is \mathcal{AP}_b -hypercyclic.
- ii) For each nonempty open set $U \subseteq X$ there is a closed periodic set of T^{**} contained in $\overline{U}^{w*} \subseteq X^{**}$.
- iii) For each nonempty open set $U \subseteq X$ there is k such that

$$\bigcap_{j=1}^{\infty} (T^{**})^{-jk} \left(\overline{U}^{\overset{\circ}{w*}} \right) \neq \emptyset.$$

- iv) The norm closure of the periodic points of T^{**} contains X .

Then i) \Rightarrow ii) \Leftrightarrow iii) \Leftrightarrow iv).

Proof. i) \Rightarrow ii). Let $U \subseteq X$ be a nonempty open set. If $V = B_s(x_0) \subseteq B_r(x_0) \subseteq U$, with $0 < s < r$, then $\overline{V} \subseteq U$ and $\overline{V}^{w*} \subseteq \overline{U}^{w*}$ is a weak*-compact set in X^{**} . Let $x \in V$ such that $N(x, V) \in \mathcal{AP}_b$. Thus, there are $k \in \mathbb{N}$ and a sequence $(a_n)_n$, such that $T^{a_n+ik}(x) \in V$ for every $i \leq n$. There is a weak*-limit point $y \in \overline{V}^{w*}$ of the sequence $(T^{a_n}(x))_n$

Let $Y = \overline{Orb_{(T^{**})^k}(y)}^{w*}$, the weak*-closure of the orbit of y under $(T^{**})^k$. This set is clearly $(T^{**})^k$ -invariant, so we only need to show that

$Y \subseteq \overline{U}^{w^*}$. It suffices to show that for every m , $(T^{**})^{km}(y) \in \overline{V}^{w^*}$. Fix $m \in \mathbb{N}$ and notice that since T^{**} is weak*-weak* continuous then $(T^{**})^{km}(y)$ is a weak*-limit point of $((T^{**})^{a_n+km}(x))_n = (T^{a_n+km}(x))_n$. Since for any $n \geq m$, we have that $T^{a_n+km}(x) \in V$, we conclude that $T^{mk}(y) \in \overline{V}^{w^*}$.

ii) ⇔ iii) Follows as in the proof of Proposition 3.8.

ii) ⇔ iv) Since by ii) any ball of the bidual X^{**} centered at a point of X contains a weak*-compact periodic set, statement *iv)* holds by Lemma 3.9. The converse is immediate. □

Note that, in particular, the above proposition proves Theorem 3.1 for reflexive spaces. Let us see that we can push this argument a little further.

In [21, Proposition 3.2] Huang and Ye studied the relationship between compact dynamical systems having dense small periodic sets and sets $N_f(x, U)$ having arbitrarily long arithmetic progressions with fixed step (see also [24]). The following lemma is a generalization of their result to dynamical systems on infinite dimensional spaces.

Lemma 3.11. *Let X be a separable Banach space which is a dual space and let $T : X \rightarrow X$ be a weak*-weak* continuous mapping. Then T is \mathcal{AP}_b -hypercyclic if and only if T is hypercyclic and has dense small periodic sets.*

Proof. One implication is Proposition 3.8. For the converse, using the same argument as in the proof of *i) ⇒ ii)* in Proposition 3.10, it may be seen that if T is a weak*-weak* continuous mapping, then for each nonempty open set $U \subseteq X$ there is a closed periodic set of T contained in $U \subseteq X$. □

We can now prove our main theorem, which can be restated as follows.

Theorem 3.12. *Let X be a separable Banach space which is a dual space and let $T : X \rightarrow X$ be a weak*-weak* continuous linear operator. The following assertions are equivalent:*

1. T is \mathcal{AP}_b -hypercyclic.
2. T is hypercyclic and has dense small periodic sets.
3. T is chaotic.

Proof. (1)⇔ (2) is Lemma 3.11 and (3)⇒(2) is immediate.

(2)⇒ (3). Let U be a nonempty open set. We must show that T has a periodic point in U . Consider $V \subseteq U$ such that V is nonempty, open, convex, weak*-precompact and such that $\overline{V} \subseteq U$. Let $Y \subseteq V$ be a k -periodic set. Then by Lemma 3.9, T has a k -periodic point in $\overline{co(Y)}^{w^*} \subseteq U$. □

In a similar way we have.

Corollary 3.13. *Let X be a Fréchet space and T a linear operator that has dense small weakly compact periodic sets. Then T has a dense set of periodic points.*

If we apply Proposition 3.3 we obtain a transitivity theorem for chaotic operators.

Theorem 3.14. (A transitivity Theorem for chaotic operators.). *Let X be a separable Banach space which is a dual space and let $T : X \rightarrow X$ be weak*-weak* continuous. Then the following are equivalent:*

1. T is chaotic.
2. For every pair of nonempty open sets U, V , there is k such that for every m there are $x \in U$ and k_m with $T^{k_m+jk}(x) \in V$ for every $0 \leq j \leq m$.
3. T is hypercyclic and for every nonempty open set U there is k such that $\bigcap_{j=1}^m T^{-jk}(U) \neq \emptyset$ for every m .

If the operator is not weak*-weak* continuous we still can characterize chaos in terms of the behavior of a single orbit.

Theorem 3.15. (A characterization in terms of a single orbit). *Let X be separable Fréchet space. The following are equivalent:*

1. Every hypercyclic vector x satisfies that for every nonempty open set U there is $(a_n)_n \in \mathcal{AP}_b$, such that $(T^{a_n}x)_n$ converges to an element in U .
2. Every hypercyclic vector x satisfies that for every nonempty open set U there is $(a_n)_n \in \mathcal{AP}_b$, such that $(a_n)_n \subseteq N_T(x, U)$ and $(T^{a_n}x)_n$ is contained in a compact set of X .
3. There exists a hypercyclic vector x such that for every nonempty open set U there is $(a_n)_n \in \mathcal{AP}_b$, such that $(a_n)_n \subseteq N_T(x, U)$ and $(T^{a_n}x)_n$ is contained in a weakly compact set of X .
4. T is chaotic.

Proof. (1) \Rightarrow (2) \Rightarrow (3) is immediate.

(3) \Rightarrow (4). By Corollary 3.13, it suffices to show that T has dense small weakly compact periodic sets. So let U be a nonempty open set and consider a nonempty convex open set V such that $\overline{V} \subseteq U$. By assumption there is $k > 0$ and a sequence $(k_n)_n$ such that for each n , $T^{k_n+ik}x \in V$ for every $i \leq n$ and such that $K := \{T^{k_n+jk}x : 0 \leq j \leq n\}$ is weakly precompact. Let y be a weak accumulation point of $\{T^{k_n}x : n \in \mathbb{N}\} \subseteq K$. Then $y \in \overline{K}^w$. Since V is convex it follows that $y \in U$. Proceeding as in the proof of Lemma 3.11 (but using that T is weak-weak continuous) we prove that $Y = \overline{Orb_{T^k}(y)}^w$ is a periodic set contained in U . Moreover, Y is weakly compact because $T^{mk}(y) \in \overline{K}^w$ for every m .

(4) \Rightarrow (1). Let $(\rho_n)_n$ be an increasing fundamental system of seminorms. Let x be a hypercyclic vector and U be a nonempty open set. Let $y \in U$ be a k -periodic vector. Since y is periodic, there is, for each m , a nonempty open set U_m around y such that $\rho_m(T^{jk}z - y) < \frac{1}{m}$ for every $0 \leq j \leq m$ and $z \in U_m$.

For each m let $k_m \in N(x, U_m)$. Thus, $k_m + jk \in N(x, U)$ for every $0 \leq j \leq m$ and $\rho_m(T^{k_m+jk}(x) - y) < \frac{1}{m}$ for every $0 \leq j \leq m$. Let $(a_n)_n$ be the sequence formed by $\bigcup_{m, 0 \leq j \leq m} \{k_m + jk\}$. It follows that $(a_n)_n \in \mathcal{AP}_b$ and that $T^{a_n}(x) \rightarrow y$. \square

Remark 3.16. According to [23, Definition 1.25] a vector $x \in X$ is said to be quasiperiodic if for every $\varepsilon > 0$ there is $p \in \mathbb{N}$ such that $\|T^{np}(x) - x\| < \varepsilon$ for every $n \in \mathbb{N}$. On the other hand, the concept of \mathcal{F} -recurrence was

recently introduced in [12]. A vector x is said to be \mathcal{F} -recurrent if for every neighborhood U of x , $N(x, U) \in \mathcal{F}$. Then a vector is quasiperiodic if and only if it is \mathcal{F} -recurrent with respect to the family \mathcal{F} of sets of the natural numbers that contain $p\mathbb{N}$ for some $p \geq 1$. It is immediate that operators having a dense set of quasiperiodic vectors have dense small periodic sets. Therefore, the proof of Theorem 3.12 shows that on dual Banach spaces, a linear weak*-weak* continuous operator has a dense set of quasiperiodic vectors if and only if it has a dense set of periodic vectors.

3.2. Weighted Shifts

In this subsection we show that every hypercyclic weighted shift with dense small periodic sets is chaotic. We also show the existence of a weighted shift operator on c_0 that is \mathcal{AP}_b -hypercyclic but does not have dense small periodic sets and hence it is not chaotic (Theorem 3.19).

It is well known [19, Theorem 8] that a backward shift defined on a Fréchet space with unconditional basis $\{e_n\}$ is chaotic if and only if

$$\sum_{n=1}^{\infty} e_n \in X. \tag{1}$$

Theorem 3.17. *Let $\{e_n\}_n$ be an unconditional basis on a Fréchet space X and let $B : X \rightarrow X$ be the backward shift defined in $\{e_n\}$. Then B is chaotic if and only if it has dense small periodic sets.*

Proof. By (1) it is enough to show that $\sum_{n=1}^{\infty} e_n$ is convergent.

Let ρ be a continuous seminorm such that for every x , $|x_1| \leq \rho(x)$. Since B has dense small periodic sets there is $k \in \mathbb{N}$ and x such that $B^{kn}(x) \in \frac{1}{4}\{y : \rho(y) < 1\} + e_1$ for every $n \geq 0$. Thus, $|x_1| \geq 1 - \frac{1}{4}$ and $\rho(B^{nk}(x) - x) < \frac{1}{2}$ for every $n \in \mathbb{N}$. Then we have that $|x_1 - x_{1+nk}| = |e_1^*(x - B^{nk}(x))| < \frac{1}{2}$ for every n . Thus, $x_{1+nk} = (x_1 + \delta_n)$, where δ_n is a number of modulus less than $\frac{1}{2}$. Note that, in particular we get that $|x_{1+nk}| > 1 - \frac{1}{4} - \frac{1}{2}$ and hence $|\frac{1}{x_{1+nk}}| < 4$ for every n .

We consider now the series $\sum_{n=1}^{\infty} e_{1+nk} = \sum_{n=1}^{\infty} \frac{1}{x_1 + \delta_n} x_{1+nk} e_{1+nk}$, which is (unconditionally) convergent by the unconditionality of $\{e_n\}$. Finally we notice that

$$\sum_{n=1}^{\infty} e_n = \sum_{j=0}^{k-1} \sum_{n \geq 1} e_{1+nk-j},$$

which is convergent, because $\sum_{n \geq 1} e_{1+nk-j} = B^j(\sum_{n=1}^{\infty} e_{1+nk})$ for each j . Therefore B is chaotic. □

Corollary 3.18. *Let $\{e_n\}_n$ be an unconditional basis on a Fréchet space X and let $B_w : X \rightarrow X$ be a weighted backward shift defined in $\{e_n\}$. Then B_w is chaotic if and only if it has dense small periodic sets.*

On the other hand we show next that there are weighted backward shifts on c_0 that are \mathcal{AP}_b -hypercyclic but that are neither upper frequently hypercyclic nor chaotic. In [8] the authors exhibited an example of a reiteratively

hypercyclic weighted shift on c_0 that is not upper frequently hypercyclic. A closer look at their proof will allow us to show that their operator is \mathcal{AP}_b -hypercyclic.

Theorem 3.19. *Let $S = \bigcup_{l,j} [l10^j - j, l10^j + j]$ and (w_n) the sequence of weights defined by*

$$w_n = \begin{cases} 2 & \text{if } n \in S \\ \prod_{l=1}^{n-1} w_l^{-1} & \text{if } n \in (S + 1) \setminus S \\ 1 & \text{else.} \end{cases}$$

Then $T := B_w : c_0 \rightarrow c_0$ is \mathcal{AP}_b -hypercyclic and does not have dense small periodic sets. In particular it is not chaotic.

The main argument used by the authors to prove that T is reiteratively hypercyclic is that T satisfies an \mathcal{F} -hypercyclicity criterion applied to the family of sets with positive upper Banach density. Let us recall the criterion restricted to weighted shifts on ℓ_p or c_0 .

Theorem 3.20. (Bès, Menet, Peris, Puig). *Let \mathcal{F} be a Furstenberg family such that there exist disjoint sets $(A_k)_k \subseteq \mathcal{F}$ such that*

- (i) *for any $j \in A_k$, any $j' \in A_{k'}$, $j \neq j'$ we have that $|j - j'| \geq \max\{k, k'\}$;*
- (ii) *for any $k' \geq 0$ and any $k > k'$*

$$\sum_{n \in A_k + k'} \frac{e_n}{\prod_{v=1}^n w_v} \in X \text{ and } \sum_{n \in A_k + k'} \frac{e_n}{\prod_{v=1}^n w_v} \xrightarrow{k \rightarrow \infty} 0;$$

- (iii) *There are $(C_{k,l})_{k,l}$ such that for every $k' \geq 0$, any $k > k'$ and any $l \geq 1$,*

$$\sup_{j \in A_l} \left\| \sum_{n \in A_k - j} \frac{e_{n+k'}}{\prod_{v=1}^n w_{v+k'}} \right\| \leq C_{k,l}$$

and such that $\sup_l C_{k,l} \rightarrow 0$ when $k \rightarrow \infty$ and such that for any k , $C_{k,l} \rightarrow 0$ when $l \rightarrow \infty$.

Then B_w is \mathcal{F} -hypercyclic in $X = \ell_p$ or c_0 .

Proof of Theorem 3.19. Bès et. al. [8] proved that the operator is not upper frequently hypercyclic. Since a weighted backward shift on c_0 is chaotic if and only if it has dense small periodic sets and since chaotic weighted backward shifts are upper frequently hypercyclic, we conclude that the operator does not have dense small periodic sets.

In [8], sets $(A_k)_k$ satisfying *i – iii*) of the above criterion and of positive upper Banach density were constructed. To prove that T is \mathcal{AP}_b -hypercyclic, it suffices to show that the sets $(A_k)_k$ chosen by the authors belong to \mathcal{AP}_b .

Each A_k is defined as $\bigcup_{j \in \phi^{-1}(k)} F_j$, where the $\phi^{-1}(k)$ are disjoint infinite subsets of \mathbb{N} and the F_j are defined as $F_{j+1} := \{10^{j_0} + 10^{2k}l : 0 \leq l \leq l_0\}$, where $l_0 > j$ and j_0 is large enough (they are defined inductively). Thus, for each $j \in \phi^{-1}(k)$ each set F_{j+1} is an arithmetic progression of length greater than j with step 10^{2k} . Since the set $\phi^{-1}(k)$ is infinite, we conclude that the sets A_k have arbitrarily long arithmetic progressions with fixed step 10^{2k} . □

It is worth mentioning that in [11, Theorem 5.3] a simpler characterization of \mathcal{F} -hypercyclic unilateral weighted backward shifts for upper families was proven.

3.3. The spectrum of an \mathcal{AP}_b -hypercyclic operator

In this subsection we study the spectrum of \mathcal{AP}_b -hypercyclic operators. Recall that chaotic operators are easily seen to have perfect spectrum. In [28], Shkarin presented a very ingenious argument to prove that frequently hypercyclic operators share the same property. We will see that \mathcal{AP}_b -hypercyclic operators also have perfect spectrum.

Recall that an operator is said to be quasinilpotent provided that $\|T^n\|^{\frac{1}{n}} \rightarrow 0$. The proof of the next lemma is a modification of an analogous result for frequently hypercyclic operators (see [28] or [20, Lemma 9.38]).

Lemma 3.21. *Let X be a real or complex Banach space. Let S be an operator, $x^* \in X^* \setminus \{0\}$ and $U = \{y : \operatorname{Re}(\langle y, x^* \rangle) > 0, \operatorname{Re}(\langle S(y), x^* \rangle) < 0\}$. Suppose that for some $x \in U$,*

$$\liminf_{k \rightarrow \infty} \frac{|N_S(x, U) \cap [0, k]|}{k + 1} = \mu > 0. \tag{2}$$

Then $S - Id$ is not quasinilpotent.

Proof. Replacing x^* by $\frac{x^*}{\operatorname{Re}(\langle x, x^* \rangle)}$ we can suppose that $\operatorname{Re}(\langle x, x^* \rangle) = 1$.

Suppose that $S - Id$ is quasinilpotent. Then, given $\varepsilon > 0$, there is some constant $M > 0$ such that $\|(S - Id)^k\| \leq M\varepsilon^k$, for every k . Thus we have for $z \in \mathbb{C}$, and $|z| \leq R$ that

$$\begin{aligned} & \sum_{k=0}^{\infty} |\operatorname{Re}(\langle (S - Id)^k x, x^* \rangle)| \left| \frac{z(z - 1) \dots (z - k + 1)}{k!} \right| \\ & \leq M \|x\| \|x^*\| \sum_{k=0}^{\infty} \varepsilon^k \frac{R(R + 1) \dots (R + k - 1)}{k!} = \frac{M \|x\| \|x^*\|}{(1 - \varepsilon)^R} \end{aligned}$$

where, for the last equality, we have used the generalized binomial theorem.

This implies that

$$f(z) = \sum_{k=0}^{\infty} \operatorname{Re}(\langle (S - Id)^k x, x^* \rangle) \frac{z(z - 1) \dots (z - k + 1)}{k!}$$

defines an entire function of exponential type 0, such that $f(0) = \operatorname{Re}(\langle x, x^* \rangle) = 1$. Therefore, as a consequence of Jensen’s formula, the number of zeros of f on the disk $\{|z| < R\}$, $N(R)$ is bounded above by

$$\frac{\log(M \|x\| \|x^*\| (1 - \varepsilon)^{-2R})}{\log 2} = c - 2R \frac{\log(1 - \varepsilon)}{\log 2}.$$

Thus, we have

$$\frac{N(k + 1)}{k + 1} \leq \frac{c}{k + 1} - \frac{2(k + 1) \log(1 - \varepsilon)}{(k + 1) \log 2} \rightarrow -\frac{2 \log(1 - \varepsilon)}{\log 2}.$$

This contradicts (2) because ε can be chosen arbitrarily close to 0, and $|N_S(x, U) \cap [0, k]| \leq N(k + 1)$. Indeed, since

$$\begin{aligned} f(n) &= \sum_{k=0}^n \operatorname{Re}(\langle (S - Id)^k x, x^* \rangle) \frac{n(n-1)\dots(n-k+1)}{k!} \\ &= \operatorname{Re}(\langle \sum_{k=0}^n \binom{n}{k} (S - Id)^k Id^{n-k} x, x^* \rangle) = \operatorname{Re}(\langle S^n x, x^* \rangle), \end{aligned}$$

we have that $n \in N_S(x, U)$ if and only if $f(n) > 0$ and $f(n + 1) < 0$. Finally, since $f|_{\mathbb{R}}$ is real valued, f must have at least a zero in the open interval $(n, n + 1)$. □

We show now that \mathcal{AP}_b -hypercyclic operators satisfy Ansari’s property.

Proposition 3.22. *Let $p \in \mathbb{N}$. Then, an operator T on a Fréchet space is \mathcal{AP}_b -hypercyclic if and only if T^p is \mathcal{AP}_b -hypercyclic. Moreover they share the \mathcal{AP}_b -hypercyclic vectors.*

Proof. Assume that T is \mathcal{AP}_b -hypercyclic, it follows by Ansari’s Theorem that T^p is hypercyclic. Since, by Proposition 3.3, for each nonempty open set U there is k such that for every m , $\bigcap_{j=1}^m T^{-jk}(U) \neq \emptyset$, we have that for every m , $\bigcap_{j=1}^m T^{-jkp}(U) \neq \emptyset$. Applying again Proposition 3.3, we conclude that T^p is \mathcal{AP}_b -hypercyclic. The other implication is immediate.

By Proposition 3.3 every hypercyclic vector of an \mathcal{AP}_b -hypercyclic operator is an \mathcal{AP}_b -hypercyclic vector and by Ansari’s Theorem the hypercyclic vectors of T and T^p coincide. We conclude that T and T^p have the same \mathcal{AP}_b -hypercyclic vectors. □

Note that, in contrast to the case of powers, the rotations of \mathcal{AP}_b -hypercyclic operators need not to be \mathcal{AP}_b -hypercyclic.

Remark 3.23. There exist an \mathcal{AP}_b -hypercyclic operator on a complex Hilbert space and $|\lambda| = 1$ such that λT is not \mathcal{AP}_b -hypercyclic.

Proof. It is known that there are a chaotic operator T in a Hilbert space and $\lambda \in \mathbb{T}$ such that λT is not chaotic, see [3]. Hence, T is \mathcal{AP}_b -hypercyclic and by Theorem 3.12, λT is not \mathcal{AP}_b -hypercyclic. □

Note that the above remark in particular implies that the family \mathcal{AP}_b has neither the Ramsey property nor the CuSP property, because for families \mathcal{F} having one of these two properties, rotations and powers of \mathcal{F} -hypercyclic operators are again \mathcal{F} -hypercyclic (see [12, 28]).

Theorem 3.24. *Let T be an \mathcal{AP}_b -hypercyclic operator on a real or complex Banach space. Then $T - \lambda Id$ is not quasinilpotent for any $|\lambda| = 1$.*

Proof. Let $\lambda = e^{2\pi i\theta}$. Note that in the real case, we have just to prove the cases $\theta = 1$ and $\theta = \frac{1}{2}$.

Suppose first that $\theta = \frac{p}{q}$ is a rational angle. Note that if $T - \lambda Id$ is quasinilpotent and $q \in \mathbb{N}$ (or $q \in \{1, 2\}$ for the real case), then $(T^{**})^q -$

$\lambda^q Id_{X^{**}} = (T^q)^{**} - Id_{X^{**}}$ is a quasinilpotent operator on X^{**} . Indeed, if G_q denotes the set of q -th roots of 1, then

$$(T^q)^{**} - Id_{X^{**}} = \prod_{\xi \in G_q} (T^{**} - \xi Id_{X^{**}}).$$

Therefore, as $n \rightarrow \infty$,

$$\begin{aligned} \|((T^q)^{**} - Id_{X^{**}})^n\|^{\frac{1}{n}} &\leq \prod_{\xi \in G_q} \|(T^{**} - \xi Id_{X^{**}})^n\|^{\frac{1}{n}} \\ &\leq \|(T^{**} - \lambda Id_{X^{**}})^n\|^{\frac{1}{n}} \cdot \prod_{\xi \in G_q, \xi \neq \lambda} \|T^{**} - \xi Id_{X^{**}}\| \\ &= \|(T - \lambda Id)^n\|^{\frac{1}{n}} \cdot \prod_{\xi \in G_q, \xi \neq \lambda} \|T^{**} - \xi Id_{X^{**}}\| \rightarrow 0. \end{aligned}$$

We will apply the above lemma to $S = (T^q)^{**}$. Let $x^* \in X^* \setminus \{0\}$ and $U = \{y \in X^{**} : Re(\langle y, x^* \rangle) > 0, Re(\langle S(y), x^* \rangle) < 0\}$. Note that since T^q is hypercyclic, $U \neq \emptyset$ and, moreover, it contains a nonempty open ball V of X such that $\bar{V}^{w^*} \subseteq U$.

Then, since T^q is \mathcal{AP}_b -hypercyclic, Proposition 3.10 implies that there are $x \in \bar{V}^{w^*}$ and $m \in \mathbb{N}$ for which $S^{jm}(x) \in \bar{V}^{w^*}$ for every $j \in \mathbb{N}$. In particular $\liminf_{k \rightarrow \infty} \frac{|N_S(x, U) \cap [0, k]|}{k+1} \geq \frac{1}{m} > 0$.

Therefore we have that $S - Id$ and hence $T - \lambda Id$ is not quasinilpotent for $|\lambda| = 1$ with rational angle.

It remains to prove the case when X is a complex space and θ is an irrational angle. Note that it suffices to prove that $S - Id$ is not quasinilpotent, where $S = e^{-2\pi i \theta} T^{**}$.

Let $x^* \in X^* \setminus \{0\}$ and $U = \{y \in X^{**} : Re(\langle y, x^* \rangle) > 0, Re(\langle S(y), x^* \rangle) < 0\}$ be the open set of X^{**} defined in Lemma 3.21 for S . Since $e^{-2\pi i \theta} T$ is hypercyclic, U is nonempty and contains a nonempty open ball V of X such that $\bar{V}^{w^*} \subseteq U$.

For $\delta > 0$, let $V_\delta := \{x \in V : d(x, V^c) > \delta \text{ and } \|x\| < 1/\delta\}$, which is a nonempty open subset of V for δ small enough. Since T is \mathcal{AP}_b -hypercyclic, by Proposition 3.10, there is some $x \in \bar{V}_\delta^{w^*}$ and m such that $T^{jm}x \in \bar{V}_\delta^{w^*} \subseteq U$ for every $j \in \mathbb{N}$.

We claim that if $\varepsilon < \delta^2/4\pi$, $y \in \bar{V}_\delta^{w^*}$ and $\varphi \in p + (-\varepsilon, \varepsilon)$ for some $p \in \mathbb{Z}$, then $e^{2\pi i \varphi} y \in \bar{V}^{w^*} \subseteq U$. Indeed, if $z \notin \bar{V}^{w^*}$,

$$\|e^{2\pi i \varphi} y - z\| \geq \|y - z\| - \|y(1 - e^{2\pi i \varphi})\| \geq \delta - \frac{1}{\delta} \varepsilon 2\pi > \delta/2.$$

Define now

$$A := \{j : -jm\theta \in p + (-\varepsilon, \varepsilon) \text{ for some } p \in \mathbb{Z}\}.$$

Since $m\theta$ is irrational, $\underline{dens}(A) > 0$, and by the claim,

$$A \subseteq \{j : S^{mj}x \in U\}.$$

Thus

$$0 < \underline{\text{dens}}(A) \leq m \cdot \underline{\text{dens}}(N_S(x, U)).$$

Therefore by Lemma 3.21, $S - Id$ is not quasinilpotent. \square

Corollary 3.25. *The spectrum of an \mathcal{AP}_b -hypercyclic operator cannot have isolated points.*

Proof. If λ is an isolated point of the spectrum of a hypercyclic operator then by the Riesz decomposition Theorem and the fact that the property of having dense small periodic sets is preserved under quasiconjugacies, we may construct an operator S having dense small periodic sets and such that $\sigma(T) = \lambda$. Since T is hypercyclic, it would be of the form $T = S + \lambda I$ for some $|\lambda| = 1$. By the spectral radius formula, S would be quasinilpotent, contradicting Theorem 3.24. \square

Corollary 3.26. *There are no \mathcal{AP}_b -hypercyclic operators on hereditarily indecomposable Banach spaces.*

4. Final comments and questions

We would like to end this note with some questions related to the results discussed in the preceding paragraphs.

The proof of Theorem 3.12 relies on the normability of the space.

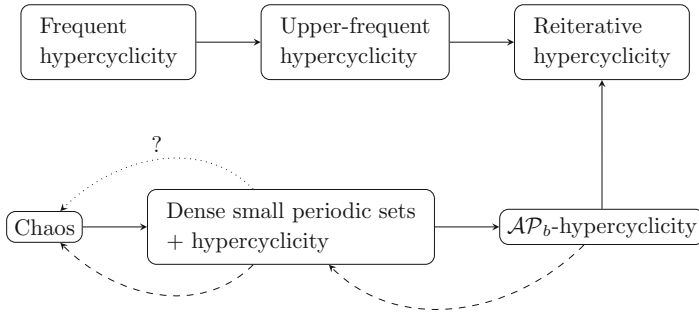
Question 4.1. *Does Theorem 3.12 hold on arbitrary Fréchet spaces?*

In Theorem 3.19 we showed the existence of an \mathcal{AP}_b -hypercyclic operator that is not chaotic. By Theorem 3.17 the operator does not have dense small periodic sets. In fact, we could not come up with an operator that has dense small periodic sets and does not have a dense set of periodic points.

Question 4.2. *Is any hypercyclic operator with dense small periodic sets necessarily chaotic. Or, more generally, does any operator with dense small periodic sets have a dense set of periodic points?*

We answered Question 1.1 for a wide class of operators and spaces. However the general question whether there exists a Furstenberg family \mathcal{F} for which \mathcal{F} -hypercyclicity is equivalent to chaos remains open.

The following diagram shows the known implications between the concepts appearing in this article. A solid arrow means that the implication holds. A dashed arrow means that the implication holds with some extra hypothesis (here in both cases weak*-weak* continuity of the operator suffices). For the dotted line we don't know if the implication holds in general (Question 4.2) and all other implications are known to be strict.



Acknowledgements

We wish to thank Daniel Carando for fruitful conversations at the beginning of the project. We are also indebted to the referee for a very careful reading of the manuscript and for many comments and corrections that helped to improve our work considerably.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Akin, E.: Recurrence in topological dynamics: furstenberg families and Ellis actions. Plenum Press, New York, NY (1997)
- [2] Ansari, S.I.: Existence of hypercyclic operators on topological vector spaces. *J. Funct. Anal.* **148**(2), 384–390 (1997)
- [3] Bayart, F., Bermúdez, T.: Semigroups of chaotic operators. *Bull. Lond. Math. Soc.* **41**(5), 823–830 (2009)
- [4] Bayart, F., Grivaux, S.: Frequently hypercyclic operators. *Trans. Am. Math. Soc.* **358**(11), 5083–5117 (2006)
- [5] Bayart, F., Grivaux, S.: Invariant Gaussian measures for operators on Banach spaces and linear dynamics. *Proc. Lond. Math. Soc.* (3) **94**(1), 181–210 (2007)
- [6] Bayart, F., Matheron, E.: Dynamics of Linear Operators. Cambridge Tracts in Mathematics, vol. 198. Cambridge University Press, Cambridge (2009)
- [7] Bernal-González, L.: On hypercyclic operators on Banach spaces. *Proc. Am. Math. Soc.* **127**(4), 1003–1010 (1999)
- [8] Bès, J., Menet, Q., Peris, A., Puig, Y.: Recurrence properties of hypercyclic operators. *Math. Ann.* **366**(1–2), 545–572 (2016)
- [9] Bès, J., Menet, Q., Peris, A., Puig, Y.: Strong transitivity properties for operators. *J. Differ. Equ.* **266**(2–3), 1313–1337 (2019)
- [10] Bonet, J., Martínez-Giménez, F., Peris, A.: A Banach space which admits no chaotic operator. *Bull. Lond. Math. Soc.* **33**(2), 196–198 (2001)
- [11] Bonilla, A., Grosse-Erdmann, K.-G.: Upper frequent hypercyclicity and related notions. *Rev. Mat. Complut.* **31**(3), 673–711 (2018)
- [12] Bonilla, A., Grosse-Erdmann, K.-G., López-Martínez, A., Peris, A.: Frequently recurrent operators. *arXiv preprint arXiv:2006.11428* (2020)

- [13] Cardeccia, R., Muro, S.: Multiple recurrence and hypercyclicity. *arXiv preprint arXiv:2104.15033* (2021)
- [14] De La Rosa, M., Read, C.: A hypercyclic operator whose direct sum $T \oplus T$ is not hypercyclic. *J. Oper. Theory* **61**(2), 369–380 (2009)
- [15] Devaney, R.L.: *An introduction to chaotic dynamical systems*. Addison-Wesley Publishing Company, Inc., Redwood City, CA etc. (1989)
- [16] Ernst, R., Esser, C., Menet, Q.: \mathcal{U} -frequent hypercyclicity notions and related weighted densities. *Isr. J. Math.* **241**(2), 817–848 (2021)
- [17] Godefroy, G., Shapiro, J.H.: Operators with dense, invariant, cyclic vector manifolds. *J. Funct. Anal.* **98**(2), 229–269 (1991)
- [18] Green, B., Tao, T.: The primes contain arbitrarily long arithmetic progressions. *Ann. Math. (2)* **167**(2), 481–547 (2008)
- [19] Grosse-Erdmann, K.-G.: Hypercyclic and chaotic weighted shifts. *Stud. Math.* **139**(1), 47–68 (2000)
- [20] Grosse-Erdmann, K.-G., Peris Manguillot, A.: *Linear chaos*. Springer, Berlin (2011)
- [21] Huang, W., Ye, X.: Dynamical systems disjoint from any minimal system. *Trans. Am. Math. Soc.* **357**(2), 669–694 (2005)
- [22] Krein, M., Smulian, V.: On regularly convex sets in the space conjugate to a Banach space. *Ann. Math. (2)* **41**, 556–583 (1940)
- [23] Kurka, P.: *Topological and symbolic dynamics*, vol. 11. Société Mathématique de France, Paris (2003)
- [24] Li, J.: Transitive points via Furstenberg family. *Topol. Appl.* **158**(16), 2221–2231 (2011)
- [25] Menet, Q.: Linear chaos and frequent hypercyclicity. *Trans. Am. Math. Soc.* **369**(7), 4977–4994 (2017)
- [26] Puig, Y.: Frequent hypercyclicity and piecewise syndetic recurrence sets. *arXiv preprint arXiv:1703.09172* (2017)
- [27] Schaefer, H.H., Wolff, M.P.: *Topological vector spaces*, vol. 3, Springer, New York (1999)
- [28] Shkarin, S.: On the spectrum of frequently hypercyclic operators. *Proc. Am. Math. Soc.* **137**(1), 123–134 (2009)
- [29] Szemerédi, E.: On sets of integers containing no k elements in arithmetic progression. *Acta Arith.* **27**, 199–245 (1975)
- [30] Tychonoff, A.: Ein Fixpunktsatz. *Math. Ann.* **111**, 767–776 (1935)

Rodrigo Cardeccia
Instituto Balseiro-CNEA
Universidad Nacional de Cuyo, CONICET
San Carlos de Bariloche
Argentina
e-mail: rodrigo.cardeccia@ib.edu.ar

Santiago Muro (✉)

CIFASIS: Centro Internacional Franco Argentino de Ciencias de la Informacion y de Sistemas - CONICET

Rosario

Argentina

e-mail: muro@cifasis-conicet.gov.ar

Received: December 18, 2020.

Revised: January 17, 2022.

Accepted: January 18, 2022.