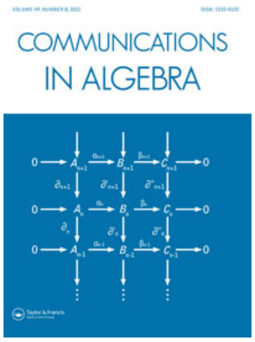

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Twisted tensor products of K^3 with K^3

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ABSTRACT
We describe all the twisted tensor products of K^3 with K^3, where K is an arbitrary field.

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0. Introduction

Let K be a commutative ring with 1, and let A and C be unitary K-algebras. Recall that a twisted tensor product of A with C is an algebra structure defined on $A \otimes_K C$, with unit $1 \otimes 1$, such that the canonical maps $i_A : A \to A \otimes_K C$ and $i_C : C \to A \otimes_K C$ are algebra maps satisfying $a \otimes c = i_A(a) i_C(c)$. This structure, which was introduced independently in [11] and [13], have been study by many people with different motivations (see for instance [2–4, 7, 9, 11, 12, 13] and [14])). It is well known that there is a canonical bijection between the twisted tensor products of A with C and the so called twisting maps $\chi : C \otimes_K A \to A \otimes_K C$, of C with A. So, each twisting map $\chi : C \otimes_K A \to A \otimes_K C$ is associated with a twisted tensor product of A with C over K, which will be denoted by $A \otimes_\chi C$, and the problem of constructing all the twisted tensor products of A with C is equivalent to the problem of finding all the twisting maps of C with A. This problem, even in the simplest cases, turn out to be very hard. To our knowledge, the first paper in which this problem was attacked in a systematic way was [6], in which C. Cibils solve the case $C := K \times K$ when K is a field (hypothesis that we maintain throughout this work). In [8] the case $C := K^n$ was analyzed and some partial classification result were achieved. In [1] we introduced the concept of standard twisting map of K^m with K^n and the more general concept of quasi-standard twisting map of K^m

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with K^n. Moreover, in [1, Remark 10.5 and Corollary 10.15] we give a method for constructing the quasi-standard twisting maps, which we apply in order to obtain all the quasi-standard twisting maps of K^3 with K^3. But there exist twisting maps of K^3 with K^3 that are not quasi-standard, and in this paper we complete the construction of all the twisting maps of K^3 with K^3, describing them.

The paper is organized as follows: In Section 1 first we make a quick review of the notion of twisting map of an algebra C with an algebra A and its relation with the notion of twisted tensor product, with emphasis in the case $A = K^n$ and $C = K^m$, and then we establish two results that we need in our quest of the twisting maps of K^3 with K^3. In Section 2 we describe the twisting maps of K^2 with K^2 and of K^3 with K^2. In the first case this result was obtained in [10], while in the second case a classification in terms of quivers is given in [6], but to carry out our task of finding all the twisted tensor products of K^3 with K^3, we need a direct description. Finally, in Section 3 we compute the twisting maps of K^3 with K^3 that are not quasi-standard.

1. Preliminaries

Let K be a field. From now on we assume implicitly that all the maps whose domain and codomain are K-vector spaces are K-linear maps, that all the algebras are associative and unitary algebras over K, and that all the algebra homomorphisms are unital. We set $K^\times := K \setminus \{0\}$. The tensor product over K is denoted by $\otimes$, without any subscript. Given a matrix X, we let X^{T} denote the transpose matrix of X. Moreover, we denote with a juxtaposition the multiplication of two matrices and with a bullet the multiplication in K^3. So,

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = (a_1b_1, a_2b_2, a_3b_3).$$

Note that an element $\mathbf{a} = (a_1, a_2, a_3)$ is invertible respect to this multiplication map if and only if $\mu_n(\mathbf{a}) := a_1a_2a_3 \neq 0$. In this case we let $\mathbf{a}^{\cdot}$ denote the inverse $(a_1^{-1}, a_2^{-1}, a_3^{-1})$ of $\mathbf{a}$. Finally, for the sake of simplicity we write $\mathbb{1} = \mathbb{1}_3 := \mathbb{1}_{K^3}^T$.

1.1. Twisting maps

Let A and C be two K-algebras and let μ_A, η_A, μ_C and η_C be the multiplication and unit maps of A and C, respectively. A *twisted tensor product* of A with C is an algebra B with underlying vector space $A \otimes C$, such that the canonical maps $i_A : A \to A \otimes C$ and $i_C : C \to A \otimes C$ are algebra homomorphisms and $\mu \circ (i_A \otimes i_C) = \mathrm{id}_{A \otimes C}$, where μ denotes the multiplication map of B. It is well known that given a twisted tensor product of A with C, the map

$$\chi : C \otimes A \to A \otimes C,$$

defined by $\chi := \mu \circ (i_C \otimes i_A)$, satisfies:

1. $\chi \circ (\eta_C \otimes A) = A \otimes \eta_C$,
2. $\chi \circ (C \otimes \eta_A) = \eta_A \otimes C$,
3. $\chi \circ (\mu_C \otimes A) = (A \otimes \mu_C) \circ (\chi \otimes C) \circ (C \otimes \chi)$,
4. $\chi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \chi) \circ (\chi \otimes A)$.

A map that fulfills these conditions is called a *twisting map* of C with A. Conversely, if

$$\chi : C \otimes A \to A \otimes C$$

is a twisting map, then $A \otimes C$ becomes a twisted tensor product, denoted $A \otimes_\chi C$, via

$$\mu_\chi := (\mu_A \otimes \mu_C) \circ (A \otimes \chi \otimes C).$$

Furthermore, these constructions are inverse one to each other.

1.2. Twisting tensor products of K^n with K^m

Let $\chi : K^m \otimes K^n \to K^n \otimes K^m$ be a map and let $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_n\}$ be the canonical bases of K^m and K^n, respectively. There exist unique scalars λ_{ij}^{kl} such that

$$\chi(e_i \otimes f_j) = \sum_{k,l} \lambda_{ij}^{kl} f_k \otimes e_l \qquad \text{for all } e_i \text{ and } f_j. \tag{1.1}$$

For all $i, l \in N_m^*$ and $j, k \in N_n^*$, we let $A_\chi(i, l) \in M_n(K)$ and $B_\chi(j, k) \in M_m(K)$ denote the matrices defined by

$$A_\chi(i, l)_{kj} := \lambda_{ij}^{kl} =: B_\chi(j, k)_{li}. \tag{1.2}$$

Moreover we set

$$\mathcal{A}_\chi := (A_\chi(i, l))_{i, l \in N_m^*} \quad \text{and} \quad \mathcal{B}_\chi := (B_\chi(j, k))_{j, k \in N_n^*}.$$

In [1, Proposition 3.3, Corollary 3.6, Remark 3.7 and Proposition 3.11] we find necessary and sufficient conditions for χ to be a twisting map.

Let

$$\chi : C \otimes A \to A \otimes C \quad \text{and} \quad \chi' : C' \otimes A' \to A' \otimes C'$$

be twisting maps. A *morphism* $F_{gh} : \chi \to \chi'$ is a pair (g, h) of algebra homomorphisms $g : C \to C'$ and $h : A \to A'$ such that the equality $\chi' \circ (g \otimes h) = (h \otimes g) \circ \chi$ holds. In [1, Proposition 3.15] we show that two twisting maps $\chi, \chi' : K^m \otimes K^n \to K^n \otimes K^m$ are isomorphic if and only if there exists $\sigma \in S_m$ and $\varsigma \in S_n$ such that

$$A_{\chi'}(i, l)_{kj} = A_\chi(\sigma(i), \sigma(l))_{\varsigma(k)\varsigma(j)} \quad (\text{or, equivalently}, B_{\chi'}(j, k)_{li} = B_\chi(\varsigma(j), \varsigma(k))_{\sigma(l)\sigma(i)}).$$

If two twisting maps of K^m with K^n are isomorphic, then we also say that they are *equivalent*.

Let χ be a twisting map of K^n with k^m. Recall from [1, Definition 3.12] that the $\mathcal{A}_\chi$-rank matrix $\Gamma_\chi \in M_m(K)$ and the $\mathcal{B}_\chi$-rank matrix $\tilde{\Gamma}_\chi \in M_n(K)$ are the matrices

$$\Gamma_\chi := \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{pmatrix} \quad \text{and} \quad \tilde{\Gamma}_\chi := \begin{pmatrix} \tilde{\gamma}_{11} & \cdots & \tilde{\gamma}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1} & \cdots & \tilde{\gamma}_{nn} \end{pmatrix},$$

where $\gamma_{il} := \text{rk}(A_\chi(i, l))$ and $\tilde{\gamma}_{jk} := \text{rk}(B_\chi(j, k))$.

In this paper we will make extensive use of the previous results and other results of [1] in order to obtain all the twisting maps of K^3 with K^3. We also will need the following propositions:

Proposition 1.1. *If* $\text{rk}(A_\chi(i, i)) = 1$ *for some* $i \in N_m^*$, *then there exists* $j \in N_n^*$ *such that* $\tilde{\gamma}_{jk} \neq 0$ *for all* k. *Moreover, if such* j *is unique, then* $A_\chi(i, i)_{st} = \delta_{tj}$ *for all* $s, t \in N_n^*$. *A similar statement holds for* $B_\chi(j, j)$ *and* Γ_χ.

Proof. Since $\text{Tr}(A_\chi(i, i)) = \text{rk}(A_\chi(i, i)) = 1$, there exists j such that $A_\chi(i, i)_{jj} \neq 0$. Consequently, by [1, Remark 5.13],

$$B_\chi(j, k)_{ii} = A_\chi(i, i)_{kj} = A_\chi(i, i)_{jj} \neq 0 \quad \text{for all } k.$$

This implies that $\tilde{\gamma}_{jk} \neq 0$ for all k. If j is unique, then for each $l \neq j$ there exists k such that $\tilde{\gamma}_{lk} = 0$, and so, again by [1, Remark 5.13], we have

$$A_\chi(i,i)_{hl} = A_\chi(i,i)_{kl} = B_\chi(l,k)_{ii} = 0 \quad \text{for all } h.$$

The argument for $B_\chi(j,j)$ and Γ_χ is the same. □

Proposition 1.2. *Let $\chi : K^m \otimes K^3 \to K^3 \otimes K^m$ be a twisting map and let i_1, i_2 and i_3 be three different elements of N_m^* such that $A_\chi(i_2,i_1) \neq 0 \neq A_\chi(i_3,i_1)$.*

1. *If the i_1-th column of $\mathcal{A}_\chi$ is not quasi-standard and $A_\chi(i_1,i_1)$ is*

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

then $A_\chi(i_2,i_3) \neq 0 \neq A_\chi(i_3,i_2)$ and neither the i_2-th nor the i_3-th column of $\mathcal{A}_\chi$ are quasi-standard columns.

1. *If the i_1-th column of $\mathcal{A}_\chi$ is not quasi-standard and $A_\chi(i_1,i_1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then*

$$A_\chi(i_2,i_1)_{22} = A_\chi(i_2,i_2)_{22} = A_\chi(i_2,i_3)_{22}, \qquad A_\chi(i_3,i_1)_{22} = A_\chi(i_3,i_2)_{22} = A_\chi(i_3,i_3)_{22},$$

and there exist $z \in K^\times$ and $a \in K \setminus \{0,1\}$ such that

$$A_\chi(i_2,i_1) = \begin{pmatrix} 0 & 0 & 0 \\ -a-z & a & z \\ a-1-\dfrac{a(1-a)}{z} & \dfrac{a(1-a)}{z} & 1-a \end{pmatrix}$$

and

$$A_\chi(i_3,i_1) = \begin{pmatrix} 0 & 0 & 0 \\ a+z-1 & 1-a & -z \\ \dfrac{a(1-a)}{z}-a & -\dfrac{a(1-a)}{z} & a \end{pmatrix}.$$

Proof. Assume we are in the hypothesis of item (2). Without loss of generality we can assume that $i_1 = 1, i_2 = 2$ and $i_3 = 3$. By [1, Corollary 3.13], we have $\sum_{i=1}^n \gamma_{i1} = 3$. Hence $A_\chi(i,1) = 0$ for $i > 3$ and $\mathrm{Tr}(A_\chi(i,1)) = \gamma_{i1} = 1$ for $i \leq 3$. Moreover, by items (1) and (3) of [1, Corollary 3.6] we know that

$$A_\chi(1,1)A_\chi(i,1) = 0 \quad \text{for all } i > 1 \qquad \text{and that} \qquad A_\chi(1,1)+A_\chi(2,1)+A_\chi(3,1) = \mathrm{id}_3.$$

Hence there exists $a \in K$ such that

$$A_\chi(2,1) = \begin{pmatrix} 0 & 0 & 0 \\ * & a & * \\ * & * & 1-a \end{pmatrix} \quad \text{and} \quad A_\chi(3,1) = \begin{pmatrix} 0 & 0 & 0 \\ * & 1-a & * \\ * & * & a \end{pmatrix}.$$

Moreover, by [1, Proposition 8.17(b)] we know that $a \notin \{0,1\}$. Let $z := A_\chi(2,1)_{23}$. Since the sum of the entries of each row of $A_\chi(2,1)$ and $A_\chi(3,1)$ is zero,

$$A_\chi(2,1) = \begin{pmatrix} 0 & 0 & 0 \\ -a-z & a & z \\ * & * & 1-a \end{pmatrix} \quad \text{and} \quad A_\chi(3,1) = \begin{pmatrix} 0 & 0 & 0 \\ a+z-1 & 1-a & -z \\ * & * & a \end{pmatrix}.$$

Furthermore, since lower triangular idempotent matrices have 0 or 1 in each diagonal entry, necessarily $z \neq 0$. Now it is clear that, since $\mathrm{rk}(A_\chi(2,1)) = \mathrm{rk}(A_\chi(3,1)) = 1$, both matrices have the desired form. But then the first row of $B_\chi(2,2)$ is $(0, a, 1-a, 0, ..., 0)$, the first row of $B_\chi(1,2)$ is $(1, -(a+z), (a+z)-1, 0, ..., 0)$ and the first row of $B_\chi(3,2)$ is $(0, z, -z, 0, ..., 0)$. An easy computation using these facts, that $B_\chi(1,2)+B_\chi(2,2)+B_\chi(3,2)=\mathrm{id}_n$ since [1, Remark 3.7], and that by [1, Proposition 3.3(2)] the columns of $B_\chi(2,2)$ are orthogonal to the first rows of $B_\chi(1,2)$ and $B_\chi(3,2)$, shows that

$$B_\chi(2,2)=\begin{pmatrix} 0 & a & 1-a & 0 & \dots & 0\\ 0 & a & 1-a & 0 & \dots & 0\\ 0 & a & 1-a & 0 & \dots & 0\\ * & * & * & * & & *\\ \vdots & & & \vdots & & \vdots\\ * & * & * & * & & * \end{pmatrix},$$

which finishes the proof of item (2) via (1.2). Moreover

$$A_\chi(2,3)_{22}=B_\chi(2,2)_{32}=a \quad \text{and} \quad A_\chi(3,2)_{22}=B_\chi(2,2)_{23}=1-a.$$

Item (1) follows immediately from this fact and [1, Proposition 3.15]. □

2. Twisting maps of K^2 with K^2 and of K^3 with K^2

To achieve our objective of constructing the twisting maps of K^3 with K^3 we need first to describe in detail the twisting maps of K^2 with K^2 and the twisting map of K^3 with K^2.

2.1. Twisting maps of K^2 with K^2

We first give a classification of all twisting maps χ of K^2 with K^2 in a direct way. This classification was already obtained in [10]. In the next computation we confirm that (up to isomorphism), there are four different twisted tensor products of K^2 with K^2. By [1, Corollary 3.13 and Proposition 3.15] we can assume that the $\mathcal{A}_\chi$-rank matrix is one of the following:

$$\Gamma_1=\begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix},\quad \Gamma_2=\begin{pmatrix} 2 & 1\\ 0 & 1\end{pmatrix}\quad \text{or}\quad \Gamma_3=\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

2.1.1. First case

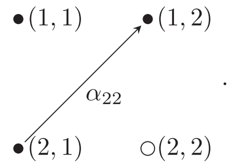
If the $\mathcal{A}_\chi$-rank matrix is Γ_1, then $A_\chi(1,1)=A_\chi(2,2)=\mathrm{id}$. Consequently χ is the flip and $K^2\otimes_\chi K^2\cong K^4$.

2.1.2. Second case

If the $\mathcal{A}_\chi$-rank matrix is Γ_2, then χ is a standard twisting map (use [1, Proposition 5.10 and Remark 8.15], and one verifies readily that χ is equivalent via identical permutations in rows and columns to the standard twisting map χ' with

$$A_{\chi'}(1,1)=\mathrm{id},\quad A_{\chi'}(2,1)=0,\quad A_{\chi'}(2,2)=\begin{pmatrix} 1 & 0\\ 1 & 0\end{pmatrix}\quad \text{and}\quad A_{\chi'}(1,2)=\begin{pmatrix} 0 & 0\\ -1 & 1\end{pmatrix},$$

or, which the same, to the standard twisting map with quiver

${}^{\chi'}Q =$ 

Here the bullets represent the vertices of ${}^{\chi'}Q$ and the white circle in the coordinate (2) indicates that the arrow α_{22} starts at the 2-th row and ends at the 2-th column. By [1, Remark 10.2] the Jacobson radical of $K^2 \otimes_{\chi'} K^2$ has dimension 1.

In the sequel we will simply represent the quivers of this twisting map and of its equivalent twisting maps as

where there is a bullet in the position (j, i) if (j, i) is a vertex (thus $j \in J_i(i)$), and there is a white circle in the position (j, l) if the quiver has an arrow α_{jl} that starts at the j-th row and ends at the l-th column (it is unique). The quivers associated with standard twisting maps of K^3 with K^2 and of K^3 with K^3 will be represented by diagrams constructed following the same instructions.

2.1.3. Third case

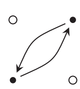
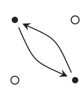
If the $\mathcal{A}_\chi$-rank matrix is Γ_3, then by [1, Corollary 3.6(3) and Remark 4.1] there exist $a, a' \in K$ such that

$$A_\chi(1,1) = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}, \qquad A_\chi(2,1) = \begin{pmatrix} 1-a & a-1 \\ -a & a \end{pmatrix},$$
$$A_\chi(1,2) = \begin{pmatrix} 1-a' & a'-1 \\ -a' & a' \end{pmatrix}, \qquad A_\chi(2,2) = \begin{pmatrix} a' & 1-a' \\ a' & 1-a' \end{pmatrix}.$$

Thus, by (1.2) we have $B_\chi(1,1) = \begin{pmatrix} a & 1-a \\ 1-a' & a' \end{pmatrix}$. Therefore $a' = 1 - a$ by [1, Proposition 6.1 and Remark 4.1], and so

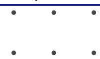
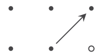
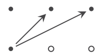
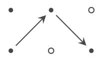
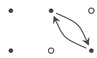
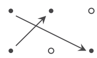
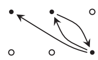
$$A_\chi(1,1) = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}, \qquad A_\chi(2,1) = \begin{pmatrix} 1-a & a-1 \\ -a & a \end{pmatrix},$$
$$A_\chi(1,2) = \begin{pmatrix} a & -a \\ a-1 & 1-a \end{pmatrix}, \qquad A_\chi(2,2) = \begin{pmatrix} 1-a & a \\ 1-a & a \end{pmatrix}.$$

Now a direct computation using (1.2) shows that $B_\chi(i,j) = A_\chi(i,j)$ for $i, j \in \{1, 2\}$, which enable us to check easily that the conditions of [1, Proposition 3.3] are satisfied. Hence we have a family of twisting maps parameterized by $a \in K$. Applying [1, Proposition 3.15] we see that the twisting maps corresponding to a and $1 - a$ are isomorphic. Moreover, using again the same proposition, we check that these are the only isomorphisms between these twisting maps. If $a \in \{0, 1\}$, then the twisting map is standard and the quiver is one of

 or  .

By [1, Remark 10.2] the Jacobson radical of $K^2 \otimes_{\chi'} K^2$ is a two dimensional k-vector space. On the other hand, by [1, Proposition 3.16 and Remark 3.17] we know that for $a \notin \{0, 1\}$, the morphism $\rho_1 : K^2 \otimes_\chi K^2 \to M_2(K)$, given by

Table 1. Standard twisting maps of K^3 with K^2.

#	$\sum \mathrm{Tr}$	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.
1.	6		$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$	1
2.	5		$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$	12
3.	4		$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$	6
4.	4		$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	12
5.	4		$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	6
6.	4		$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	6
7.	3		$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$	12

$$\rho_1(f_j \otimes 1) := E^{jj} \quad \text{and} \quad \rho_1(1 \otimes e_i) := A_\chi(i,1),$$

is an algebra isomorphism.

2.2. Twisting maps of K^3 with K^2

Now we use the results of [1] to classify all the twisting maps

$$\chi : K^3 \otimes K^2 \to K^2 \otimes K^3,$$

distinguishing those that are almost standard, those that are quasi-standard and those that are not quasi-standard (by [1, Remark 3.2 and Propositions 7.9 and 8.20], this immediately gives a similar classification for the twisting maps of K^2 with K^3). By [1, Corollary 3.13 and Proposition 3.15] we can assume that the $\mathcal{A}_\chi$-rank matrix is one of the following:

$$\Gamma_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\Gamma_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Gamma_7 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

By [1, Proposition 5.10], except perhaps in the cases Γ_5 and Γ_6, the matrices $A_\chi(l,l)$ are 0, 1-matrices, which, by [1, Remark 8.15], implies that χ is a standard twisting map. In Table 1 we list all the possible standard twisting maps χ whose $\mathcal{A}_\chi$-rank matrix is one of $\Gamma_1 - \Gamma_7$ (for this we use the method given in [1, Remark 10.5]):

Here $\sum \mathrm{Tr} := \sum_i \mathrm{Tr}(A_\chi(i,i)) = \sum_j \mathrm{Tr}(B_\chi(j,j))$ and # equiv. indicates how many equivalent standard twisting maps there are (we say that two standard twisting maps K^m with K^n are equivalent if they are isomorphic).

If $\Gamma_\chi = \Gamma_5$, then χ is a direct sum of the flip of K with K^2 and a twisting map χ' of K^2 with K^2 such that $\Gamma_{\chi'} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and the twisted tensor product algebra is isomorphic to $K^2 \times A$, where A

is the twisted tensor product $K^2 \otimes_{\chi'} K^2$. So either it is standard (recovering the case #5 in the list), or it corresponds to a value of $a \notin \{0,1\}$ in the third case of Subsection 2.1. Consequently

$$A_\chi(1,1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_\chi(2,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_\chi(3,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
$$A_\chi(1,2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_\chi(2,2) = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}, \quad A_\chi(3,2) = \begin{pmatrix} 1-a & a-1 \\ -a & a \end{pmatrix},$$
$$A_\chi(1,3) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_\chi(2,3) = \begin{pmatrix} a & -a \\ a-1 & 1-a \end{pmatrix}, \quad A_\chi(3,3) = \begin{pmatrix} 1-a & a \\ 1-a & a \end{pmatrix},$$

and we obtain an algebra isomorphic to $K^2 \times M_2(K)$.

If $\Gamma_\chi = \Gamma_6$, then by [1, Proposition 5.10 and Remark 8.15] the first column of $\mathcal{A}_\chi$ is standard column, so that either

$$A_\chi(1,1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_\chi(3,1) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

or

$$A_\chi(1,1) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_\chi(3,1) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

By [1, Proposition 3.15] we can assume, and we do it, that $A_\chi(1,1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Moreover, by [1, Remark 3.14] the matrices $A_\chi(i,j)$ for $i,j \in \{2,3\}$ define a twisting map χ' of K^2 with K^2, such that $\Gamma_{\chi'} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is either standard, or corresponds to a value of a not in {0, 1} in the third case of Subsection 2.1. But [1, Theorem 8.21] shows that

$$\{2\} = F(A_\chi(3,1)) \subseteq F_0(\mathcal{A}_\chi, 3).$$

Consequently $A_\chi(3,3)_{22} = 1$ and the twisting map is standard, corresponding to the sixth case on the list.

If $\Gamma_\chi = \Gamma_7$, then the twisting map is necessarily standard (because the columns of Γ_7 have reduced rank 1), but no standard twisting map χ yields $\Gamma_\chi = \Gamma_7$, and so there is no twisting map in this case.

3. Twisting maps of K^3 with K^3

Our next aim is to determine up to isomorphisms all twisting maps

$$\chi : K^3 \otimes K^3 \to K^3 \otimes K^3$$

that are not quasi-standard. By [1, Proposition 3.15], for this we can and we will assume that the values of the diagonal of Γ_χ are non increasing. So in the rest of this subsection χ denotes an arbitrary twisting map satisfying this restriction and we look for conditions in order that χ be not quasi-standard. We organize our search according to the values of $\sum \mathrm{Tr} := \sum_i \mathrm{Tr}(A_\chi(i,i))$.

3.1. Tr $= 9, 8$ or 7

Theorem 3.1. *If $\sum \mathrm{Tr} \geq 7$ and χ is not standard, then the diagonal of Γ_χ is (2, 3) and χ is a not quasi-standard twisting map with $\mathcal{A}_\chi$ given by*

$$A_\chi(1,1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_\chi(2,1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(3,1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$A_\chi(1,2) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(2,2) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & a & b \end{pmatrix}, \quad A_\chi(3,2) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & -b \\ 0 & -a & a \end{pmatrix},$$
$$A_\chi(1,3) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(2,3) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix}, \quad A_\chi(3,3) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & a \\ 0 & b & a \end{pmatrix},$$

where $a \in K \setminus \{0,1\}$ and $b := 1-a$. Independently of a, we have $K^3 \otimes_\chi K^3 \simeq K^5 \times M_2(K)$.

Proof. Under the hypothesis the diagonal of Γ_χ may be (1–3) or (2, 3). By [1, Proposition 8.22], in the first three cases necessarily χ is a standard twisting map. In the last case Γ_χ is equivalent via identical permutations in rows and columns to one of the following matrices:

$$\begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

By [1, Proposition 5.10] in the two first cases the diagonal matrices are 0, 1-matrices, and so by [1, Remark 8.15] the obtained twisting maps are standard. In the last one χ is a direct sum of the flip of K with K^3 and a twisting map $\chi' : K^2 \otimes K^3 \to K^3 \otimes K^2$. Moreover, the analysis made in Subsection 2.2 shows that if χ' is not quasi-standard, then the matrices $A_\chi(i,j)$ are as in the statement. Since, moreover $K^3 \otimes_{\chi'} K^2$ is isomorphic to the direct product $K^2 \times M_2(K)$, we conclude that $K^3 \otimes_\chi K^3 \simeq K^5 \times M_2(K)$. □

3.2. Tr= 6

Theorem 3.2. *Up to isomorphisms, the unique family of not quasi-standard twisting maps of K^3 with K^3 such that $\sum \mathrm{Tr} = 6$ is obtained taking*

$$A_\chi(1,1) := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_\chi(2,1) := \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(3,1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$A_\chi(1,2) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(2,2) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & a & b \end{pmatrix}, \quad A_\chi(3,2) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & -b \\ 0 & -a & a \end{pmatrix},$$
$$A_\chi(1,3) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(2,3) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix}, \quad A_\chi(3,3) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & a \\ 0 & b & a \end{pmatrix},$$

where $a \in K \setminus \{0,1\}$ and $b := 1-a$.

Proof. Since $\sum \mathrm{Tr} = 6$, the diagonal of Γ_χ is either (2) or (1–3). We treat each case separately:

Diag(Γ χ) $= (2,2,2)$ By [1, Proposition 3.15] we can assume that Γ_χ it is one of the following matrices:

$$\begin{pmatrix}2&1&1\\1&2&0\\0&0&2\end{pmatrix},\quad \begin{pmatrix}2&1&0\\1&2&1\\0&0&2\end{pmatrix},\quad \begin{pmatrix}2&0&0\\1&2&1\\0&1&2\end{pmatrix}\quad \text{or}\quad \begin{pmatrix}2&0&1\\1&2&0\\0&1&2\end{pmatrix}.$$

Moreover, by [1, Proposition 5.10 and Remark 8.15] each twisting map whose rank matrix is the last one is standard, and, again by [1, Proposition 3.15], each twisting map whose rank matrix is the first or the second one is isomorphic to one twisting map whose rank matrix is the third one. So we only must consider the case

$$\Gamma_\chi = \begin{pmatrix}2&0&0\\1&2&1\\0&1&2\end{pmatrix}.$$

Since, by [1, Proposition 5.10 and Remark 8.15] the first column of $\mathcal{A}_\chi$ is standard, the hypothesis of [1, Theorem 8.21] are satisfied. By this theorem we know that χ is twisting map if and only if the matrices $A_\chi(2,2), A_\chi(3,2), A_\chi(3,3)$ and $A_\chi(2,3)$ define a twisting map of K^2 with K^3 and $F(A_\chi(2,1)) \subseteq F_0(\mathcal{A}_\chi, 2)$ (in fact, we also need that $F(A_\chi(i,1)) \subseteq F_0(\mathcal{A}_\chi, i)$ for $i \in \{1,3\}$, but for $i=3$ this is trivial and for $i=1$ this follows from [1, Remark 8.3]). Since we are looking for non quasi-standard twisting maps, by the discussion in Subsection 2.2 we may assume that $A_\chi(2,2), A_\chi(2,3), A_\chi(3,2)$ and $A_\chi(3,3)$ are as in the statement. But then $F_0(\mathcal{A}_\chi, 2) = \{1\}$ and so, necessarily

$$A_\chi(2,1) = \begin{pmatrix}1&-1&0\\0&0&0\\0&0&0\end{pmatrix}\quad \text{or}\quad A_\chi(2,1) = \begin{pmatrix}1&0&-1\\0&0&0\\0&0&0\end{pmatrix}.$$

In both cases setting $A_\chi(1,1) := \mathrm{id}_3 - A_\chi(2,1), A_\chi(3,1) := 0, A_\chi(1,2) := 0$ and $A_\chi(1,3) := 0$ (which is forced), we obtain a twisting map which is not quasi-standard, since $a \notin \{0,1\}$. Note that the twisting map of the first family corresponding to a is equivalent to the twisting map of the second family corresponding to $1-a$. Note also that the first family is the one listed in the statement and that, in this case, $B_\chi(i,j) = A_\chi(i,j)$ for all i, j. So the diagonal of $\tilde{\Gamma}_\chi$ is (2).

Diag$(\boldsymbol{\Gamma_\chi}) = (3,2,1)$ If χ is a not quasi-standard twisting map, then by the last assertion in the previous case we know that $\mathrm{Diag}(\tilde{\Gamma}_\chi) = (3,2,1)$. The rank matrix Γ_χ is one of the following matrices:

$$\begin{pmatrix}3&0&0\\0&2&2\\0&1&1\end{pmatrix},\ \begin{pmatrix}3&0&2\\0&2&0\\0&1&1\end{pmatrix},\ \begin{pmatrix}3&0&1\\0&2&1\\0&1&1\end{pmatrix},\ \begin{pmatrix}3&1&0\\0&2&2\\0&0&1\end{pmatrix},\ \begin{pmatrix}3&1&1\\0&2&1\\0&0&1\end{pmatrix},\ \begin{pmatrix}3&1&2\\0&2&0\\0&0&1\end{pmatrix}.$$

By Proposition 1.1, both Γ_χ and $\tilde{\Gamma}_\chi = \Gamma_{\tilde{\chi}}$ must be one of the last two matrices. But by [1, Corollary 5.11, Proposition 7.9 and Remark 8.15], if Γ_χ or $\tilde{\Gamma}_\chi$ is the last matrix, then χ is a standard twisting map. So the only chance of being not standard for the twisting map χ is that both Γ_χ and $\tilde{\Gamma}_\chi$ be the second last matrix. In that case by [1, Corollary 5.11] we know that $A_\chi(l,l)$ is a 0, 1-matrix for $l \in \{1,2,3\}$. Hence, by [1, Remark 8.15] the first two columns of $\mathcal{A}_\chi$ are standard. Moreover, using [1, Corollary 5.9] and [1, equality (3.2)] we obtain

$$A_\chi(1,3) = \begin{pmatrix}*&0&0\\ *&*&0\\ *&*&*\end{pmatrix},\quad A_\chi(2,3) = \begin{pmatrix}*&0&0\\ *&*&0\\ *&*&*\end{pmatrix}\quad \text{and}\quad A_\chi(1,3) = \begin{pmatrix}1&0&0\\1&*&0\\1&*&*\end{pmatrix}.$$

Since $\mathrm{rk}(A_\chi(1,3)) = \mathrm{rk}(A_\chi(3,3)) = 1$ and $A_\chi(1,3) + A_\chi(2,3) + A_\chi(3,3) = \mathrm{id}_3$, from this it follows easily that $A(i,3)_{kk} \in \{0,1\}$ for $i,k \in \{1,2,3\}$. Thus we can apply [1, Proposition 8.17(b)] in order to obtain that χ is quasi-standard. □

3.3. Tr= 5

Theorem 3.3. *Up to isomorphisms, the unique family of not quasi-standard twisting maps of K^3 with K^3 such that $\sum \mathrm{Tr} = 5$ is obtained taking $A_\chi(1,3)$ and $A_\chi(2,3)$ as in Proposition 1.2(2) with $z \in K^\times, a \in K \setminus \{0,1\}, i_1 = 3, i_2 = 1$ and $i_3 = 2$, taking*

$$A_\chi(1,1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1-a \\ 0 & a & 1-a \end{pmatrix}, \qquad A_\chi(2,1) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-a & a-1 \\ 0 & -a & a \end{pmatrix}$$
$$A_\chi(1,2) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & a-1 & 1-a \end{pmatrix}, \qquad A_\chi(2,2) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-a & a \\ 0 & 1-a & a \end{pmatrix},$$

and taking $A_\chi(3,j) := \mathrm{id}_3 - A_\chi(1,j) - A_\chi(2,j)$ for $j \in \{1,2,3\}$.

Proof. The diagonal of Γ_χ necessarily is either (1, 2) or (1, 3). By [1, Proposition 3.15] we can assume that in the first case the rank matrix Γ_χ is one of

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{3.1}$$

while in the second case the rank matrix Γ_χ is one of

$$\begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{3.2}$$

Since $\tilde{\Gamma}_\chi$ has at least one 1 in the diagonal, by Proposition 1.1 the rank matrix Γ_χ can not be either the first in (3.3) or the first three in (3.4). Moreover, by [1, Proposition 5.10], if Γ_χ is the first or the second matrix of the second row in (3.4), then $A_\chi(l,l)$ is a 0, 1-matrix for $l \in \{1,2,3\}$. So, in the first case by [1, Remark 8.15] we know that χ is a standard twisting map; while, in the second case, again by [1, Remark 8.15] the first two columns of $\mathcal{A}_\chi$ are standard, and by [1, Corollary 5.9] the matrix $A_\chi(3,3)$ is one of

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

Thus, since $A_\chi(2,1) = 0$, applying Proposition 1.2(1) with $i_1 = 3, i_2 = 1$ and $i_3 = 2$ we obtain that the last column of $\mathcal{A}_\chi$ is quasi-standard. Consequently, if χ is not a quasi-standard twisting, then Γ_χ is among the six last matrices in (3.3) and the last matrix in (3.4). Since $\mathrm{Tr}(\tilde{\Gamma}_\chi) = \mathrm{Tr}(\Gamma_\chi)$ by [1, Proposition 7.9], the same thing happens with $\tilde{\Gamma}_\chi$. Furthermore the same arguments as above show that

1. if Γ_χ is the second matrix in (3.3), then the third column of $\mathcal{A}_\chi$ is standard,
2. if Γ_χ is the third or fourth matrix in (3.3), then the first and third columns of $\mathcal{A}_\chi$ are standard,

3. if Γ_χ is the fifth matrix in (3.3), then the second column of $\mathcal{A}_\chi$ is standard and the third column of $\mathcal{A}_\chi$ is quasi-standard,
4. If Γ_χ is the sixth matrix in (3.3), then $A_\chi(3,3)$ is one of (3.5),
5. if Γ_χ is the seventh matrix in (3.3), then the third column of $\mathcal{A}_\chi$ is quasi-standard.

If Γ_χ is the second matrix in (3.3), then χ is quasi-standard. In fact, otherwise by item (1) we know that χ is an extension of a not quasi-standard twisting map χ' of K^2 with K^3. Consequently, by the analysis made in Subsection 2.2 there exists $a \in K \setminus \{0,1\}$ such that

$$A_\chi(1,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & a-1 & 1-a \end{pmatrix}, \quad A_\chi(2,2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-a & a \\ 0 & 1-a & a \end{pmatrix}, \quad A_\chi(3,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore $F_0(\mathcal{A}_\chi,2) = \{1\}$, which is impossible since $F(A_\chi(2,3)) \subseteq F_0(\mathcal{A}_\chi,2)$ by [1, Theorem 8.21(1)] and $\#F(A_\chi(2,3)) = 2$ by [1, Remark 8.4].

If Γ_χ is the third or fourth matrix in (3.3), then by item (2) the first column of $\mathcal{A}_\chi$ is standard and χ is the extension of a twisting map of K^2 with K^3, which is necessarily standard since, otherwise, by [1, Proposition 7.9] it is dual of some twisting map of the unique family of non quasi-standard twisting maps of K^3 with K^2 obtained in the analysis made in Subsection 2.2 (which is impossible because it implies that $\gamma_{23} = 1$). Consequently, in these cases χ is standard.

If Γ_χ is the fifth matrix in (3.3), then χ is quasi-standard. For this, by item (3) we only must prove that the first column of $\mathcal{A}_\chi$ is quasi-standard. By [1, Proposition 3.15] we can assume that

$$A_\chi(2,2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and so} \quad A_\chi(1,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

By equality (1.2) and [1, Proposition 3.3], from this and the fact that $A_\chi(3,2) = 0 = A_\chi(2,1)$ it follows that

$$B_\chi(1,1) = \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix}, \qquad B_\chi(2,2) = \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix},$$
$$B_\chi(3,3) = \begin{pmatrix} * & 0 & * \\ 1 & 0 & 0 \\ * & * & * \end{pmatrix}, \qquad B_\chi(1,3) = \begin{pmatrix} * & 0 & * \\ -1 & 1 & 0 \\ * & * & * \end{pmatrix}.$$

By [1, Remark 5.13] neither $\mathrm{rk}(B_\chi(1,1))$ nor $\mathrm{rk}(B_\chi(2,2))$ can be 1. Hence

$$\mathrm{rk}(B_\chi(1,1)) = 2 = \mathrm{rk}(B_\chi(2,2)) \quad \text{and so} \quad \mathrm{rk}(B_\chi(3,3)) = 1.$$

Moreover we can assume that $\mathrm{rk}(B_\chi(1,3)) = 1$, because, otherwise (modulo equivalence), $\tilde{\Gamma}_\chi$ is one of the cases 2, 3 or 4 of (3.3), and hence the twisting map χ is quasi-standard. Consequently,

$$B_\chi(3,3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_\chi(1,3) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ * & * & 0 \end{pmatrix}.$$

Hence

$$A_\chi(3,3)_{33} = B_\chi(3,3)_{33} = 0 \quad \text{and} \quad A_\chi(3,3)_{31} = B_\chi(1,3)_{33} = 0, \quad \text{and so} \quad A_\chi(3,3) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the main diagonal of $B_\chi(1,1)$ is $(*,1,0)$ and the main diagonal of $B_\chi(2,2)$ is $(*,1,1)$. Since both matrices have rank 2, it follows that the main diagonal of $B_\chi(1,1)$ is (1, 1, 0) and the main diagonal of $B_\chi(2,2)$ is (0, 1, 1). Moreover, $B_\chi(2,1)_{33} = A_\chi(3,3)_{12} = 1$ implies $B_\chi(2,1) \neq 0$, which, together with $\mathrm{rk}(B_\chi(1,1)) = 2$, yields $B_\chi(3,1) = 0$. Hence

$$\begin{aligned} A_\chi(1,1)_{31} &= B_\chi(1,3)_{11} = 0, \\ A_\chi(1,1)_{33} &= B_\chi(3,3)_{11} = 1, \\ A_\chi(1,1)_{11} &= B_\chi(1,1)_{11} = 1, \\ A_\chi(1,1)_{22} &= B_\chi(2,2)_{11} = 0 \end{aligned}$$

and

$$A_\chi(1,1)_{13} = B_\chi(3,1)_{11} = 0.$$

Furthermore, $\mathrm{rk}(B_\chi(2,2)) = 2$ implies that either $B_\chi(1,2) = 0$ or $B_\chi(3,2) = 0$, and so

$$A_\chi(1,1)_{21} = B_\chi(1,2)_{11} = 0 \qquad \text{or} \qquad A_\chi(1,1)_{23} = B_\chi(3,2)_{11} = 0.$$

Using all this and the fact that each row of $A_\chi(1,1)$ sums 1, we obtain that

$$A_\chi(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad A_\chi(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In both cases $A_\chi(1,1)$ is a standard idempotent, and hence, by [1, Remark 8.15], the first column of $\mathcal{A}_\chi$ is standard, as desired.

Assume now that Γ_χ is the sixth matrix in (3.3) and that the twisting map χ is not quasi-standard. By [1, Remark 3.14] we know that χ is an extension of a twisting map χ' of K^2 with K^3. Clearly, if the third column of $\mathcal{A}_\chi$ is quasi-standard, then χ' must be a non quasi-standard twisting map. But by item (4) above, applying Proposition 1.2(1) with $i_1 = 3, i_2 = 1$ and $i_3 = 2$, we obtain that this is also the case if the third column of $\mathcal{A}_\chi$ is not quasi-standard. Thus, by the analysis made in subsection 2.2 we can assume that there exists $a \in K \setminus \{0,1\}$ such that

$$A_\chi(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1-a \\ 0 & a & 1-a \end{pmatrix}, \qquad A_\chi(1,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & a-1 & 1-a \end{pmatrix}$$
$$A_\chi(2,1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-a & a-1 \\ 0 & -a & a \end{pmatrix}, \qquad A_\chi(2,2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-a & a \\ 0 & 1-a & a \end{pmatrix}.$$

If the third column of $\mathcal{A}_\chi$ is quasi-standard, then by [1, Remark 8.4 and Theorem 8.21] we have

$$F(A_\chi(1,3)) \cap F(A_\chi(2,3)) = \emptyset \qquad \text{and} \qquad F(A_\chi(1,3)) \cup F(A_\chi(2,3)) \subseteq F_0(\mathcal{A}_\chi, 2) = \{1\},$$

which is impossible since it implies that $2 = \mathrm{rk}(A_\chi(1,3)) + \mathrm{rk}(A_\chi(2,3)) \leq 1$. Thus it is not quasi-standard. In the two last cases in (3.5) a straightforward computation using [1, Propositions 3.15 and 1.2(2)] leads to the contradiction $A_\chi(1,2)_{11} \neq 0$. Hence $A_\chi(3,3)$ is the first matrix of (3.5), and so necessarily there exists $z \in K^\times$ such that $A_\chi(1,3)$ and $A_\chi(2,3)$ are as in Proposition 1.2(2). Finally $A_\chi(3,1)$ and $A_\chi(3,2)$ are determined by the equality $\sum_{i=1}^{3} A_\chi(i,j) = \mathrm{id}_3$. Since these matrices satisfy the conditions of [1, Corollary 3.6], we obtain a family of not quasi-standard twisting maps parameterized by $a \in K \setminus \{0,1\}$ and $z \in K^{\times}$. Furthermore a direct computation shows that

$$\tilde{\Gamma}_{\chi} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{3.4}$$

The same argument shows that if $\tilde{\chi}$ is not quasi-standard and $\tilde{\Gamma}_{\chi}$ is the sixth matrix in (3.3), then Γ_{χ} is the matrix at the right side of equality (3.6).

Assume now that Γ_{χ} is the seventh matrix in (3.3). We claim that χ is quasi-standard. By item (5) we know that the third column of $\mathcal{A}_{\chi}$ is quasi-standard. Moreover by [1, Definition 8.2(2), Remark 8.4 and Propositions 3.3(3) and 3.15], we can assume that

$$A_{\chi}(1,3) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & \lambda \\ -1 & 0 & 1 \end{pmatrix}, \quad A_{\chi}(2,3) = \begin{pmatrix} 0 & 0 & 0 \\ \lambda-1 & 1 & -\lambda \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{\chi}(3,3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

for some $\lambda \in K$. Using this and the fact that $A_{\chi}(1,2) = A_{\chi}(2,1) = 0$, we obtain

$$B_{\chi}(1,1) = \begin{pmatrix} * & 0 & * \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{\chi}(2,2) = \begin{pmatrix} * & 0 & * \\ * & * & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B_{\chi}(3,3) = \begin{pmatrix} * & 0 & * \\ * & * & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By [1, Remark 5.13] neither $\mathrm{rk}(B_{\chi}(1,1))$ nor $\mathrm{rk}(B_{\chi}(2,2))$ can be 1. Hence

$$\mathrm{rk}(B_{\chi}(1,1) = 2 = \mathrm{rk}(B_{\chi}(2,2)), \quad \text{and so} \quad B_{\chi}(3,3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Moreover $B_{\chi}(1,2) \neq 0$, because $B_{\chi}(1,2)_{33} = A_{\chi}(3,3)_{21} = 1$. Since

$$\mathrm{rk}(B_{\chi}(1,2)) + \mathrm{rk}(B_{\chi}(2,2)) + \mathrm{rk}(B_{\chi}(3,2)) = 3,$$

we have $\mathrm{rk}(B_{\chi}(1,2)) = 1$ and $B_{\chi}(3,2) = 0$. Hence

$$\tilde{\Gamma}_{\chi} = \begin{pmatrix} 2 & 1 & * \\ * & 2 & * \\ * & 0 & 1 \end{pmatrix}.$$

If χ is not quasi-standard, then $\tilde{\Gamma}_{\chi}$ can not be the sixth matrix in (3.3), because otherwise Γ_{χ} is the sixth matrix in (3.4). But we already have proven that in the other cases $\tilde{\chi}$ is quasi-standard and so, by [1, Proposition 8.20], we conclude that χ is also quasi-standard.

Assume finally, that χ is a non quasi-standard twisting map of K^3 with K^3 and Γ_{χ} is the last matrix in (3.4). By [1, Proposition 8.20] we know that $\tilde{\chi}$ is not quasi-standard, and so $\tilde{\Gamma}_{\chi}$ is necessarily the sixth matrix in (3.3) or the last matrix in (3.4). In the first case we obtain a family of not quasi-standard twisting maps χ dual to the family found above when analyzing the case where Γ_{χ} is the sixth matrix in (3.3); while, in the second case, by Proposition 1.1

$$A_{\chi}(2,2) = A_{\chi}(3,3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which, by Proposition 1.2(1) implies that the last two columns of $\mathcal{A}_{\chi}$ are quasi-standard. Since the first column of $\mathcal{A}_{\chi}$ has reduced rank 0, we conclude that χ is quasi-standard. □

3.4 $\sum$ Tr= 4

Theorem 3.4. *All the twisting maps of K^3 with K^3 with $\sum \mathrm{Tr} = 5$ are quasi-standard twisting maps.*

Proof. By [1, Proposition 3.15] in order to prove this it suffices to check that χ is quasi-standard if its rank matrix Γ_χ is one of the following matrices:

$$\begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \qquad (3.5)$$
$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

By Proposition 1.1, the rank matrix Γ_χ can not be the first matrix in the first row. Also Γ_χ can not be the second matrix, because otherwise it would be the extension of a twisting map χ' of K^2 with K^3 with $\Gamma_{\chi'} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, but $\sum \mathrm{Tr} = 2$ is impossible by [1, Remark 5.2]. If Γ_χ is the third, fourth or fifth matrix, then by [1, Remark 3.14] we know that χ is a extension of a twisting map χ' of K^2 with K^3, that is necessarily standard (see the arguments in the analysis of the cases in which Γ_χ is the third or fourth matrix in (3.3)). Moreover, in the first two cases by [1, Proposition 5.10 and Remark 8.5] the third column of $\mathcal{A}_\chi$ is standard; while, if Γ_χ is the fifth matrix, then by [1, Proposition 5.10] the matrix $A_\chi(3,3)$ is one of (3.5), and so, applying Proposition 1.2(1) with $i_1 = 3, i_2 = 1$ and $i_3 = 2$, we obtain that the last column of $\mathcal{A}_\chi$ is quasi-standard (since the first row of $\mathcal{A}_\chi$ is standard). Hence, if χ is not quasi-standard, then necessarily Γ_χ is one of the last four matrices. By [1, Propositions 3.15 and 8.20] the same happens with $\Gamma_{\tilde{\chi}}$ is one of the above matrices. In particular $B_\chi(3,1) = 0$, and so $A_\chi(i,l)_{13} = 0$ for all i, l. Consequently, by [1, Remark 5.13], we have

$$A_\chi(2,2) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \quad \text{and} \quad A_\chi(3,3) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix},$$

Moreover, again by [1, Remark 5.13], from the equality $A_\chi(3,1) = 0$ it follows that

$$B_\chi(2,2) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix} \quad \text{and} \quad B_\chi(3,3) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}.$$

Hence, $A_\chi(3,3)_{22} = B_\chi(2,2)_{33} = 0$ and $B_\chi(3,3)_{22} = A_\chi(2,2)_{33} = 0$. Thus,

$$A_\chi(3,3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_\chi(3,3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (3.6)$$

where we had used again [1, Remark 5.13]. Since

$$A_\chi(3,2)_{22} = B_\chi(2,2)_{23} = 0, \quad A_\chi(3,2)_{33} = B_\chi(3,3)_{23} = 0, \quad A_\chi(3,2)_{13} = 0$$

and $\mathrm{Tr}(A_\chi(3,2)) = \mathrm{rk}(A_\chi(3,2)) = 1$, we have

$$A_\chi(3,2) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix}.$$

Hence,

$$A_\chi(1,2) = \mathrm{id} - A_\chi(2,2) - A_\chi(3,2) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 1 \end{pmatrix},$$

where for the last equality we use that $\mathrm{rk}(A_\chi(1,2)) = 1,$ and so

$$A_\chi(2,2) = \mathrm{id} - A_\chi(1,2) - A_\chi(3,2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ * & * & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where the last equality follows from [1, Remark 5.13]. Consequently, since $A_\chi(3,1) = 0,$ applying Proposition 1.2(1) with $i_1 = 2, i_2 = 1$ and $i_3 = 3,$ we obtain that the second column of $\mathcal{A}_\chi$ is quasi-standard. By this and the first equality in (3.8) we can apply Proposition 1.2(1) with $i_1 = 3, i_2 = 1$ and $i_3 = 2,$ in order to obtain that the last column of $\mathcal{A}_\chi$ also is quasi-standard. Arguing as above we conclude that

$$B_\chi(1,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & 1 \end{pmatrix}, \quad B_\chi(3,2) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ * & * & 0 \end{pmatrix} \quad \text{and} \quad B_\chi(2,2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hence

$$A_\chi(1,1)_{21} = B(1,2)_{11} = 0, \quad A_\chi(1,1)_{23} = B(3,2)_{11} = 1 \quad \text{and} \quad A_\chi(1,1)_{22} = B(2,2)_{11} = 0.$$

Since moreover $A_\chi(1,1)_{13} = 0,$ we have

$$A_\chi(1,1) = \begin{pmatrix} * & * & 0 \\ 0 & 0 & 1 \\ * & * & 1 \end{pmatrix} = \begin{pmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ * & * & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

where in the second equality we use that $\mathrm{Tr}(A_\chi(1,1)) = \mathrm{rk}(A_\chi(1,1)) = 2,$ and in the last one, that $\mathrm{rk}(A_\chi(1,1)) = 2$ and the sum of the element of the each row of $A_\chi(1,1)$ is equal to 1. So, by [1, Remark 8.15] the first column of $\mathcal{A}_\chi$ is standard. □

3.4.1 $\sum$ Tr= 3

Theorem 3.5. *Let $\chi : K^3 \otimes K^3 \to K^3 \otimes K^3$ be a twisting map. Assume that $\sum \mathrm{Tr} = 3$. If χ satisfies the conditions required in items (1, 2) or (3) bellow, then χ is not quasi-standard. Moreover, each not quasi-standard twisting map of K^3 with K^3, such that $\sum \mathrm{Tr} = 3$, is equivalent to one of the described in (1, 2) and (3).*

1. *There exists in invertible vectors $\mathbf{v}_1, \mathbf{v}_2 \in K^3$ with $\det(\mathbf{v}_1^T \mathbf{v}_2^T \mathbf{v}_3^T) = 1$, where $\mathbf{v}_3 := 1$, such that*

$$A_\chi(1,l) := (\mathbf{v}_l^{\cdot} \cdot \mathbf{v}_1)^T (\mathbf{v}_l \cdot (\mathbf{v}_2 \times \mathbf{v}_3)),$$
$$A_\chi(2,l) := -(\mathbf{v}_l^{\cdot} \cdot \mathbf{v}_2)^T (\mathbf{v}_l \cdot (\mathbf{v}_1 \times \mathbf{v}_3))$$

and

$$A_\chi(3,l) := (\mathbf{v}_l^{\cdot} \cdot \mathbf{v}_3)^T (\mathbf{v}_l \cdot (\mathbf{v}_1 \times \mathbf{v}_2))$$

for all l.

1. *There exists* $a \in K \setminus \{0,1\}$ *such that*

$$A_\chi(1,1) = \begin{pmatrix} a & b & 0 \\ a & b & 0 \\ a & b & 0 \end{pmatrix}, \quad A_\chi(2,1) = \begin{pmatrix} b & 0 & -b \\ -a & 0 & a \\ -a & 0 & a \end{pmatrix}, \quad A_\chi(3,1) = \begin{pmatrix} 0 & -b & b \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix},$$
$$A_\chi(1,2) = \begin{pmatrix} a & -a & 0 \\ -b & b & 0 \\ -b & b & 0 \end{pmatrix}, \quad A_\chi(2,2) = \begin{pmatrix} b & 0 & a \\ b & 0 & a \\ b & 0 & a \end{pmatrix}, \quad A_\chi(3,2) = \begin{pmatrix} 0 & a & -a \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix},$$
$$A_\chi(1,3) = \begin{pmatrix} a & -a & 0 \\ -b & b & 0 \\ a & -a & 0 \end{pmatrix}, \quad A_\chi(2,3) = \begin{pmatrix} b & 0 & -b \\ b & 0 & -b \\ -a & 0 & a \end{pmatrix}, \quad A_\chi(3,3) = \begin{pmatrix} 0 & a & b \\ 0 & a & b \\ 0 & a & b \end{pmatrix},$$

where $b := 1 - a$.

1. *There exists* $a \in K \setminus \{0,1\}$ *such that*

$$A_\chi(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_\chi(2,2) = \begin{pmatrix} 0 & a & b \\ 0 & a & b \\ 0 & a & b \end{pmatrix} \quad \text{and} \quad A_\chi(3,3) = \begin{pmatrix} 0 & b & a \\ 0 & b & a \\ 0 & b & a \end{pmatrix},$$

where $b := 1 - a$, *and there exist* $x, y, z \in K^\times$ *with* $y = \frac{a(1-a)}{x}$ *such that*

$$A_\chi(2,1) = \begin{pmatrix} 0 & 0 & 0 \\ -a - z & a & z \\ \dfrac{-b(a+z)}{z} & \dfrac{ab}{z} & b \end{pmatrix}, \quad A_\chi(3,1) = \begin{pmatrix} 0 & 0 & 0 \\ z - b & b & -z \\ \dfrac{a(b-z)}{z} & \dfrac{-ab}{z} & a \end{pmatrix},$$
$$A_\chi(1,2) = \begin{pmatrix} 1 & -a - x & x - b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(3,2) = \begin{pmatrix} 0 & x & -x \\ 0 & b & -b \\ 0 & -a & a \end{pmatrix},$$
$$A_\chi(1,3) = \begin{pmatrix} 1 & y - b & -a - y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_\chi(2,3) = \begin{pmatrix} 0 & -y & y \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix}.$$

Proof. By [1, Proposition 6.1] we know that $\Gamma_\chi = \tilde{\Gamma}_\chi = \mathfrak{J}_3$. By [1, Remark 5.13] each matrix $A_\chi(i,i)$ and each matrix $B_\chi(j,j)$ has their three equal rows and, moreover, the sum of the element of each row is 1. Hence, for each one of these matrices we have the following possibilities: It is equivalent to a standard idempotent 0, 1-matrix via identical permutations in rows and columns, it has all entries non-zero, or it has two non-zero columns and one zero column. Assume that one of the $A_\chi(l,l)$ has all its entries non-zero. By [1, Proposition 6.2] the image of each $A_\chi(i,l)$ is generated by an invertible element $\mathbf{v}_i \in K^3$. Consequently, by [1, Proposition 6.5] the twisting map χ is obtained as in [1, Theorem 6.3] with $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ such that $\mathbf{v}_l = 1$ and $\det(\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2^{\mathrm{T}}\mathbf{v}_3^{\mathrm{T}}) = 1$. Moreover, by [1, Proposition 3.15] we can assume that $l = 3$. Suppose now that two of the matrices $A_\chi(1,1)$, $A_\chi(2,2)$ and $A_\chi(3,3)$ are different 0, 1-matrices. Since $\mathrm{Tr}(B_\chi(j,k)) = \mathrm{rk}(B_\chi(j,k))$, this implies that all the matrices $B_\chi(j,k)$ have zeroes and ones in its diagonal entries. Furthermore, by [1, Remark 5.13] we know that each $B_\chi(j,j)$ is a (0, 1)-matrix. Therefore the hypothesis of [1, Proposition 8.17(b)] are fulfilled by all the columns of $\mathcal{B}_\chi$, and so $\tilde{\chi}$ is a quasi-standard twisting map. Therefore, by [1, Proposition 8.20] the twisting map χ is also. Suppose now that two of $A_\chi(1,1)$, $A_\chi(2,2)$ and $A_\chi(3,3)$ are equal 0, 1-matrices. By [1, Proposition 3.15] we can assume that they are $A_\chi(1,1)$ and $A_\chi(2,2)$ and that

$$A_\chi(1,1) = A_\chi(2,2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Using equality (1.2) and that $\mathrm{Tr}(B_\chi(j,k)) = 1$ for all $j,k \in N_3^*$, we obtain that

$$B_\chi(j,k)_{33} = \begin{cases} -1 & \text{if } j = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Consequently $A_\chi(3,3)_{kj} = B_\chi(jk)_{33} \neq 0$ for all $i,j \in N_3^*$ and we are in the first case considered in this subsection. So there are two cases left:

1. All three matrices $A_\chi(1,1), A_\chi(2,2)$ and $A_\chi(3,3)$ have exactly one zero column.
2. One of them is a 0, 1-matrix and the other two have exactly one zero column.

Consider the first case. We claim that the zero columns of $A_\chi(1,1), A_\chi(2,2)$ and $A_\chi(3,3)$ are different. Suppose this is false. By [1, Proposition 3.15] we can assume that

$$A_\chi(1,1) = \begin{pmatrix} a & 1-a & 0 \\ a & 1-a & 0 \\ a & 1-a & 0 \end{pmatrix} \quad \text{and} \quad A_\chi(2,2) = \begin{pmatrix} b & 1-b & 0 \\ b & 1-b & 0 \\ b & 1-b & 0 \end{pmatrix}$$

with $a,b \in K \setminus \{0,1\}$. Using again equality (1.2) and that $\mathrm{Tr}(B_\chi(j,k)) = 1$ for all $j,k \in N_3^*$, we obtain that

$$B_\chi(j,k)_{33} = \begin{cases} 1-a-b & \text{if } j = 1, \\ a+b-1 & \text{if } j = 2, \\ 1 & \text{if } j = 3. \end{cases}$$

Consequently

$$A_\chi(3,3) = \begin{pmatrix} 1-a-b & a+b-1 & 1 \\ 1-a-b & a+b-1 & 1 \\ 1-a-b & a+b-1 & 1 \end{pmatrix}$$

which contradicts that $A_\chi(3,3)$ has exactly one zero column. So the claim is true. Again by [1, Proposition 3.15] we can assume that

$$A_\chi(1,1) = \begin{pmatrix} a & 1-a & 0 \\ a & 1-a & 0 \\ a & 1-a & 0 \end{pmatrix}, \quad A_\chi(2,2) = \begin{pmatrix} b & 0 & 1-b \\ b & 0 & 1-b \\ b & 0 & 1-b \end{pmatrix} \quad \text{and} \quad A_\chi(3,3) = \begin{pmatrix} 0 & c & 1-c \\ 0 & c & 1-c \\ 0 & c & 1-c \end{pmatrix}$$

with $a,b,c \in K \setminus \{0,1\}$. By equality (1.2) and [1, Remark 5.13] we have

$$B_\chi(1,1) = \begin{pmatrix} a & b & 0 \\ a & b & 0 \\ a & b & 0 \end{pmatrix}, \quad B_\chi(2,1) = \begin{pmatrix} 1-a & * & * \\ * & 0 & * \\ * & * & c \end{pmatrix}, \quad B_\chi(3,1) = \begin{pmatrix} 0 & * & * \\ * & 1-b & * \\ * & * & 1-c \end{pmatrix}$$

$$B_\chi(1,2) = \begin{pmatrix} a & * & * \\ * & b & * \\ * & * & 0 \end{pmatrix}, \quad B_\chi(2,2) = \begin{pmatrix} 1-a & 0 & c \\ 1-a & 0 & c \\ 1-a & 0 & c \end{pmatrix}, \quad B_\chi(3,2) = \begin{pmatrix} 0 & * & * \\ * & 1-b & * \\ * & * & 1-c \end{pmatrix}$$

$$B_\chi(1,3) = \begin{pmatrix} a & * & * \\ * & b & * \\ * & * & 0 \end{pmatrix}, \quad B_\chi(2,3) = \begin{pmatrix} 1-a & * & * \\ * & 0 & * \\ * & * & c \end{pmatrix}, \quad B_\chi(3,3) = \begin{pmatrix} 0 & 1-b & 1-c \\ 0 & 1-b & 1-c \\ 0 & 1-b & 1-c \end{pmatrix},$$

Since $\mathrm{Tr}(B_\chi(j,k)) = 1$ this implies that $b = 1-a$ and $c = a$. Using this and equality (1.2), we obtain that

$$A_\chi(1,2)=\begin{pmatrix} a & * & * \\ * & 1-a & * \\ * & * & 0\end{pmatrix},\quad A_\chi(2,1)=\begin{pmatrix} 1-a & * & * \\ * & 0 & * \\ * & * & a\end{pmatrix},\quad A_\chi(3,1)=\begin{pmatrix} 0 & * & * \\ * & a & * \\ * & * & 1-a\end{pmatrix}$$
$$A_\chi(1,3)=\begin{pmatrix} a & * & * \\ * & 1-a & * \\ * & * & 0\end{pmatrix},\quad A_\chi(2,3)=\begin{pmatrix} 1-a & * & * \\ * & 0 & * \\ * & * & a\end{pmatrix},\quad A_\chi(3,2)=\begin{pmatrix} 0 & * & * \\ * & a & * \\ * & * & 1-a\end{pmatrix}$$

We claim that $A_\chi(1,2)_{13}=A_\chi(1,2)_{23}=0$. Assume for example that $A_\chi(1,2)_{23}\neq 0$. Since the second row of $A_\chi(1,2)$ is not zero and $\mathrm{rk}(A_\chi(1,2))=1$, there exists $\lambda\in K$ such that

$$\lambda A_\chi(1,2)_{23}=A_\chi(1,2)_{33}=0 \quad\text{and}\quad \lambda A_\chi(1,2)_{21}=A_\chi(1,2)_{31}.$$

But then $\lambda=0$ and so $B_\chi(1,3)_{21}=A_\chi(1,2)_{31}=0$, which is impossible since $a\neq 0, b\neq 0$ and $\mathrm{rk}(B_\chi(1,3))=1$. Hence the claim is true. Similarly

$$\begin{aligned} &A_\chi(1,3)_{13}=0,\quad A_\chi(1,3)_{23}=0,\quad A_\chi(2,1)_{12}=0,\quad A_\chi(2,1)_{32}=0,\quad A_\chi(2,3)_{12}=0,\\ &A_\chi(2,3)_{32}=0,\quad A_\chi(3,1)_{21}=0,\quad A_\chi(3,1)_{31}=0,\quad A_\chi(3,2)_{21}=0,\quad A_\chi(3,2)_{31}=0.\end{aligned}$$

Using these facts, that $A_\chi(i,l)1=0$ for all $i\neq l$, and that $\sum_{i=1}^3 A_\chi(i,l)=\mathrm{id}_3$ for all l, we obtain that the matrices $A_\chi(i,j)$ are as in item (2). We consider now the second case. By [1, Proposition 3.15] we can assume that

$$A_\chi(1,1)=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0\end{pmatrix} \tag{3.7}$$

and that the first column is not quasi-standard. By Proposition 1.2(2) and [1, Remark 5.13] there exist $z\in K^\times$ and $a\in K\setminus\{0,1\}$ such that

$$A_\chi(2,1)=\begin{pmatrix} 0 & 0 & 0 \\ -a-z & a & z \\ \dfrac{(a-1)(a+z)}{z} & \dfrac{a(1-a)}{z} & 1-a\end{pmatrix},\quad A_\chi(3,1)=\begin{pmatrix} 0 & 0 & 0 \\ a+z-1 & 1-a & -z \\ \dfrac{a(1-a-z)}{z} & \dfrac{a(a-1)}{z} & a\end{pmatrix}, \tag{3.8}$$

$$A_\chi(2,2)=\begin{pmatrix} * & a & * \\ * & a & * \\ * & a & *\end{pmatrix},\quad A_\chi(3,3)=\begin{pmatrix} * & 1-a & * \\ * & 1-a & * \\ * & 1-a & *\end{pmatrix}, \tag{3.9}$$

$$A_\chi(2,3)=\begin{pmatrix} * & * & * \\ * & a & * \\ * & * & *\end{pmatrix},\quad A_\chi(3,2)=\begin{pmatrix} * & * & * \\ * & 1-a & * \\ * & * & *\end{pmatrix}. \tag{3.10}$$

So, by equality (1.2) and [1, Remark 5.13],

$$B_\chi(1,1)=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0\end{pmatrix},\quad B_\chi(2,2)=\begin{pmatrix} 0 & a & 1-a \\ 0 & a & 1-a \\ 0 & a & 1-a\end{pmatrix},\quad B_\chi(3,3)=\begin{pmatrix} 0 & 1-a & a \\ 0 & 1-a & a \\ 0 & 1-a & a\end{pmatrix}.$$

Hence, again by equality (1.2) and [1, Remark 5.13],

$$A_\chi(2,2)=\begin{pmatrix} 0 & a & 1-a \\ 0 & a & 1-a \\ 0 & a & 1-a\end{pmatrix} \quad\text{and}\quad A_\chi(3,3)=\begin{pmatrix} 0 & 1-a & a \\ 0 & 1-a & a \\ 0 & 1-a & a\end{pmatrix}. \tag{3.11}$$

Using equality (1.2), that $B_\chi(3,2)1=B_\chi(2,3)1=0$ and that $\mathrm{rk}(B_\chi(3,2))=\mathrm{rk}(B_\chi(2,3))=1$, we obtain

$$B_\chi(3,2)=\begin{pmatrix}0 & z & -z\\ 0 & 1-a & a-1\\ 0 & -a & a\end{pmatrix}\quad\text{and}\quad B_\chi(2,3)=\begin{pmatrix}0 & \frac{a(1-a)}{z} & \frac{a(a-1)}{z}\\ 0 & a & -a\\ 0 & a-1 & 1-a\end{pmatrix}.$$

Consequently, since $\mathrm{id}=\sum_i B_\chi(i,j)$, we have

$$B_\chi(1,2)=\begin{pmatrix}1 & -a-z & a+z-1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\quad\text{and}\quad B_\chi(1,3)=\begin{pmatrix}1 & \frac{(a-1)(a+z)}{z} & \frac{a(1-a-z)}{z}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Using now equality (1.2) and that $A_\chi(3,2)1=A_\chi(2,3)1=0$, we obtain that there exits $x,y\in K$, such that

$$A_\chi(3,2)=\begin{pmatrix}0 & x & -x\\ 0 & 1-a & a-1\\ 0 & -a & a\end{pmatrix}\quad\text{and}\quad A_\chi(2,3)=\begin{pmatrix}0 & -y & y\\ 0 & a & -a\\ 0 & a-1 & 1-a\end{pmatrix}. \tag{3.12}$$

Consequently, since $\mathrm{id}=\sum_i A_\chi(i,j)$, we have

$$A_\chi(1,2)=\begin{pmatrix}1 & -a-x & a+x-1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\quad\text{and}\quad A_\chi(1,3)=\begin{pmatrix}1 & a+y-1 & -a-y\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{3.13}$$

Finally, again by equality (1.2),

$$B_\chi(2,1)=\begin{pmatrix}0 & 0 & 0\\ -a-x & a & x\\ a+y-1 & -y & 1-a\end{pmatrix}\quad\text{and}\quad B_\chi(1,3)=\begin{pmatrix}0 & 0 & 0\\ a+x-1 & 1-a & -x\\ -a-y & y & a\end{pmatrix}.$$

Since $\mathrm{rk}(B_\chi(2,1))=1$ we have $xy=a(1-a)$. So $x,y\in K^\times$ and $y=\frac{a(1-a)}{x}$. Reciprocally a direct computation shows that if $\mathcal{A}=(A_\chi(i,l))_{i,l\in N_3^*}$, where the $A(i,\ l)$'s are the matrices of (3.9), (3.10), (3.13), (3.14) and (3.15), where $a\in K\setminus\{0,1\}$ and $x,y,z\in K^\times$ with $y=\frac{a(1-a)}{x}$, then $\mathcal{A}$ satisfies the conditions in [1, Proposition 3.11]. □

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