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About Convergence and Order of Convergence of Some Fractional Derivatives

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Abstract: In this paper we obtain some convergence results for Riemann-Liouville, Caputo, and Caputo–Fabrizio fractional operators when the order of differentiation approaches one. We consider the errors given by $\left|\left|D^{1-\alpha}f-f'\right|\right|_p$ for p=1 and $p=\infty$ and we prove that for bothm the Caputo and Caputo Fabrizio operators, the order of convergence is a positive real $r\in(0,1)$. Finally, we compare the speed of convergence between Caputo and Caputo–Fabrizio operators obtaining that they are related by the Digamma function.

Keywords: Caputo–Fabrizio derivative, Caputo derivative, orders of convergence.

1 Introduction

Several definitions of fractional operators were emerged in the last years. As a consequence, many interesting discussions have been taken relevance, as for example which properties define a fractional operator (see for example, [1,2,3,4,5,6]). Between the different answers given for this question, we stand out the classification criteria given in [7], where a list of properties that an operator must verify to be considered fractional derivative is proposed.

The suggested criteria given by different authors may vary. But they all agree regarding on the fact that a fractional derivative must be a linear operator converging to an ordinary derivative when the order of differentiation approaches a positive integer in an appropriated space of functions. That is, if an operator D^α is considered a fractional derivative, then it is linear and $\lim\limits_{\alpha\to n}||D^\alpha f-f^{(n)}||=0$, for each f belonging to some normed space of functions, $(X,||\cdot||)$.

Many fractional operators were analyzed in this direction. In particular, the well known fractional derivatives of Riemann-Liouville (RL) and Caputo (C), that were widely studied in [8,9], were considered in [**?**]. Another integrodifferential operator is the Caputo-Fabrizio (CF) derivative defined in [10]. In addition, the CF derivative is defined through a kernel without singularity, whereas that the RL and C derivatives are defined through integrodifferential operators with singular kernels. The use of fractional derivatives in the field of applications is in continuous expansion. For example, the applications to the theory of viscoelasticity or subdiffusion processes for C and RL operators where studied in [11,12,13,14], whereas a model for a biological epidemic involving a CF operator is presented.

In this article we will study some topics related to the convergence of the fractional derivatives C, CF and RL to the ordinary derivative when $\alpha\nearrow 1$. In section 2 some basic definitions and results on the convergence of the fractional operators mentioned above are established. In section 3, we will use the L^p norm, for $p\in[1,\infty)$, to analyze the order of convergence of each operator when the fractional order of differentiation approaches 1. In particular, we obtain that the order of convergence for the C and CF derivatives are both less than 1, while it is not possible to analize the order of convergence for RL derivative when working with these norms. Finally, we compare the speed of convergence between C and CF derivatives in the L^1 norm for some particular cases. We obtain a close formula for the speed of convergence in the L^1 norm for power functions, in terms of the Digamma function.

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2 Preliminaries

In the next definitions, $(a,b) \subset \mathbb{R}$ is a bounded interval (that is $-\infty < a < b < \infty$) and $W^{1,1}(a,b) = \left\{ f \in L^1(a,b) \,/\, \exists g \in L^1(a,b) \text{ such that } \int_a^b f\varphi' = -\int_a^b g\varphi, \forall \varphi \in C_c^1(a,b) \right\}$.

Definition 1 *Let $\alpha \in (0,1)$.*

1. If $f \in L^1(a,b)$, the fractional Riemann–Liouville integral of order α is defined by

$$ {}_aI^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} \mathrm{d}\tau. $$

2. If $f \in W^{1,1}(a,b)$, the fractional Riemann–Liouville derivative of order α is defined by

$$ {}_a^{RL}D^{\alpha} f(t) = \left[\frac{\mathrm{d}}{\mathrm{d}t} {}_aI^{1-\alpha} f \right](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(\tau)(t-\tau)^{-\alpha} \mathrm{d}\tau. $$

3. If $f \in W^{1,1}(a,b)$, the fractional Caputo derivative of order α is defined by

$$ {}_a^{C}D^{\alpha} f(t) = \left[{}_aI^{1-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t} f \right) \right](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t f'(\tau)(t-\tau)^{-\alpha} \mathrm{d}\tau $$

Proposition 1 *[9] If $0 < \alpha < 1$ and $f \in W^{1,1}(a,b)$ then*

$$ {}_a^{RL}D^{\alpha} f(t) = \frac{f(a)}{\Gamma(1-\alpha)} (t-a)^{-\alpha} + {}_a^{C}D^{\alpha} f(t). $$

Definition 2 *Let f be a function in $W^{1,1}(a,b)$. The fractional Caputo-Fabrizio derivative of order α is defined by*

$$ {}_a^{CF}D^{\alpha} f(t) = \frac{1}{1-\alpha} \int_a^t f'(t) e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau. \tag{1} $$

Proposition 2 *Let $f(t) = (t-a)^{\gamma}$ defined in $[a,b]$ ($\gamma > 0$) and $\alpha \in (0,1)$. Then*

a) ${}_a^{C}D^{\alpha} f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} (t-a)^{\gamma-\alpha}$.

b) ${}_a^{CF}D^{\alpha} f(t) = \frac{\gamma}{\alpha}(t-a)^{\gamma-1} \left[1 - \Gamma(\gamma) \mathscr{E}_{1,\gamma} \left(-\frac{\alpha}{1-\alpha}(t-a) \right) \right]$, where $\mathscr{E}_{\rho,\omega}(\cdot)$ is the Mittag–Leffler function defined for every $t \in \mathbb{R}$ by $\mathscr{E}_{\rho,\omega}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k+\omega)}$.

Proof. See [13] for the proof of a) and [16, Prop 4] for b).

Hereafter if f is defined in an interval (a,b), its extension by zero to $\mathbb{R}$ will be considered if the context requires a whole definition.

In order to give in the following section some general results, we now define a general linear operator that coincides, according to certain hypotheses, with the fractional derivatives defined above.

Definition 3 *Let $(a,b) \subset \mathbb{R}$ and let $h : \mathbb{R}^+ \times (0,1) \to \mathbb{R}$ be a function such that $h(\cdot,\beta) \in W^{1,1}(\mathbb{R}^+)$ and $h(\cdot,\beta) \in L^1(\mathbb{R}^+)$ uniformly in $(0,\beta_0], \beta_0 \in (0,1)$. We define the operator ${}_a^{h}D^{1-\beta} : W^{1,1}(a,b) \to W^{1,1}(a,b)$ by*

$$ {}_a^{h}D^{1-\beta} g(t) := (g' * h(\cdot,\beta))(t) \qquad a.e. \, in \ (a,b). \tag{2} $$

Remark. Note that if we the kernels $h_C(t,\alpha) = \frac{1}{\Gamma(\alpha)} t^{-(1-\alpha)}$ and $h_{CF}(t,\alpha) = \frac{e^{-\frac{1-\alpha}{\alpha}t}}{\alpha}$ defined for $t > 0$ are considered, then the C and CF derivatives of order $1-\alpha$ given in Definition 1-3 and Definition 2 are recovered.

Next, we recall some classical results about convergence almost everywhere for C and RL derivatives when the the order of differentiation tends to 1.

Proposition 3 *The following limits hold.*

a)If $f \in W^{1,1}(a,b)$, then

$$\lim_{\alpha \nearrow 1} {}^{RL}_{a}D^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t), \quad a.e. \ in \ (a,b).$$

b)If $f \in C^1[a,b]$, then

$$\lim_{\alpha \nearrow 1} {}^{RL}_{a}D^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) \qquad for\ every\ t \in (a,b].$$

The proof of $a)$ follows from [17, Theorem 2.6] and Proposition 1. For $b)$ see [8, Theorem 2.20]. The next proposition is a direct consequence of Propositions 1 and 3.

Proposition 4 *The following limits hold.*

a)If $f \in W^{1,1}(a,b)$, then

$$\lim_{\alpha \nearrow 1} {}^{C}_{a}D^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t), \quad a.e. \ in \ (a,b).$$

b)If $f \in C^1[a,b]$, then

$$\lim_{\alpha \nearrow 1} {}^{C}_{a}D^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) \qquad for\ every\ t \in (a,b].$$

For the CF derivative we present the next result, which is a generalization of the one obtained in [16].

Proposition 5 *The following limits hold.*

a)If $f \in W^{1,1}(a,b)$, then

$$\lim_{\alpha \nearrow 1} {}^{CF}_{a}D^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) \qquad a.e. \ in \ (a,b).$$

b)If $f \in C^2[a,b]$, then

$$\lim_{\alpha \nearrow 1} {}^{CF}_{a}D^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t) \qquad for\ every\ t \in (a,b].$$

Proof. a) Let $f \in W^{1,1}(a,b)$ be. Then $f' \in L^1(a,b)$ and by classical density results in $L^1(a,b)$ (see e.g. [18]) there exists a simple function, which will be called g'_ε by an abuse of language, such that $g'_\varepsilon(t) = \sum_{i=1}^{n} q_i \chi_{[a_i,b_i]}(t)$, where $b_i \leq a_{i+1}$ for every $i = 1,\dots,n-1$ and

$$\|f' - g'_\varepsilon\|_{L^1(a,b)} < \frac{\varepsilon}{2}. \tag{3}$$

Now for every $t \in [a,b]$, let $g_\varepsilon(t) = \int_a^t g'_\varepsilon$ be. Then if $t \in [a_k,b_k]$, for any k given it follows that

$${}^{CF}_{a}D^{\alpha} g_\varepsilon(t) = \sum_{i=1}^{k-1} q_i \frac{1}{\alpha}\left(e^{-\frac{\alpha}{1-\alpha}(t-b_i)} - e^{-\frac{\alpha}{1-\alpha}(t-a_i)}\right) + q_k \frac{1}{\alpha}\left(1 - e^{-\frac{\alpha}{1-\alpha}(t-a_k)}\right). \tag{4}$$

Taking the limit when $\alpha \nearrow 1$ in (4), we have that

$$\lim_{\alpha \nearrow 1} {}^{CF}_{a}D^{\alpha} g_\varepsilon(t) = q_k = g'_\varepsilon(t).$$

For every $t \in (a,b)$ and $\alpha \geq \frac{1}{2}$ the following estimations hold

$$|g'_\varepsilon(t)| \leq \sum_{i=i}^{n} q_i \quad \text{and} \quad \left|{}^{CF}_{a}D^{\alpha} g_\varepsilon(t)\right| \leq \sum_{i=i}^{k} 2q_i,$$

thus the Lebesgue Convergence Theorem can be applied to compute the next limit

$$\lim_{\alpha \nearrow 1} \left\|{}^{CF}_{a}D^{\alpha} g_\varepsilon - g'_\varepsilon\right\|_{L^1(a,b)} = \lim_{\alpha \nearrow 1} \int_a^b \left|{}^{CF}_{a}D^{\alpha} g_\varepsilon(t) - g'_\varepsilon(t)\right| \mathrm{d}t = 0. \tag{5}$$

By the other side,

$$||{}^{CF}_{a}D^{\alpha}f - {}^{CF}_{a}D^{\alpha}g_{\varepsilon}||_{L^1(a,b)} \leq \int_a^b \left|f'(\tau) - g'_{\varepsilon}(\tau)\right| \frac{1-e^{-\frac{\alpha}{1-\alpha}(b-\tau)}}{\alpha} \mathrm{d}\tau \leq \frac{1}{\alpha}||f' - g'_{\varepsilon}||_{L^1(a,b)}. \tag{6}$$

where Fubini's Theorem has been applied due to the fact that $f' - g'_{\varepsilon} \in L^1(a,b)$. From (3) and (6) we have that

$$||{}^{CF}_{a}D^{\alpha}f - {}^{CF}_{a}D^{\alpha}g_{\varepsilon}||_{L^1(a,b)} < \frac{\varepsilon}{2\alpha}. \tag{7}$$

Finally, from (3) and (7) it holds that

$$||{}^{CF}_{a}D^{\alpha}f - f'||_{L^1(a,b)} \leq \frac{\varepsilon}{2\alpha} + ||{}^{CF}_{a}D^{\alpha}g_{\varepsilon} - g'_{\varepsilon}||_{L^1(a,b)} + \frac{\varepsilon}{2}. \tag{8}$$

Taking the limit when $\alpha \nearrow 1$, in (8) we conclude that

$$\lim_{\alpha \nearrow 1} ||{}^{CF}_{a}D^{\alpha}f - f'||_{L^1(a,b)} \leq \varepsilon, \qquad \text{for every } \varepsilon > 0,$$

and in consequence item $a)$ is proved. The proof of $b)$ is given in [16].

As mentioned before, the purpose of this paper is to analyze the convergence and the "speed" of convergence of the mentioned fractional derivatives in different norms. Are they strongly different taking into account that C and RL derivatives are defined in terms of singular kernels while CF derivative is defined for a no singular kernel? We will see in the next section that the answer is no.

In Figures 1 and 2, functions $f(t) = t + 1$ and $g(t) = \cos t$ are compared, and it can be seen how the fractional RL, CF and C derivatives converge pointwise in $(0,1)$ to f' and g' respectively.

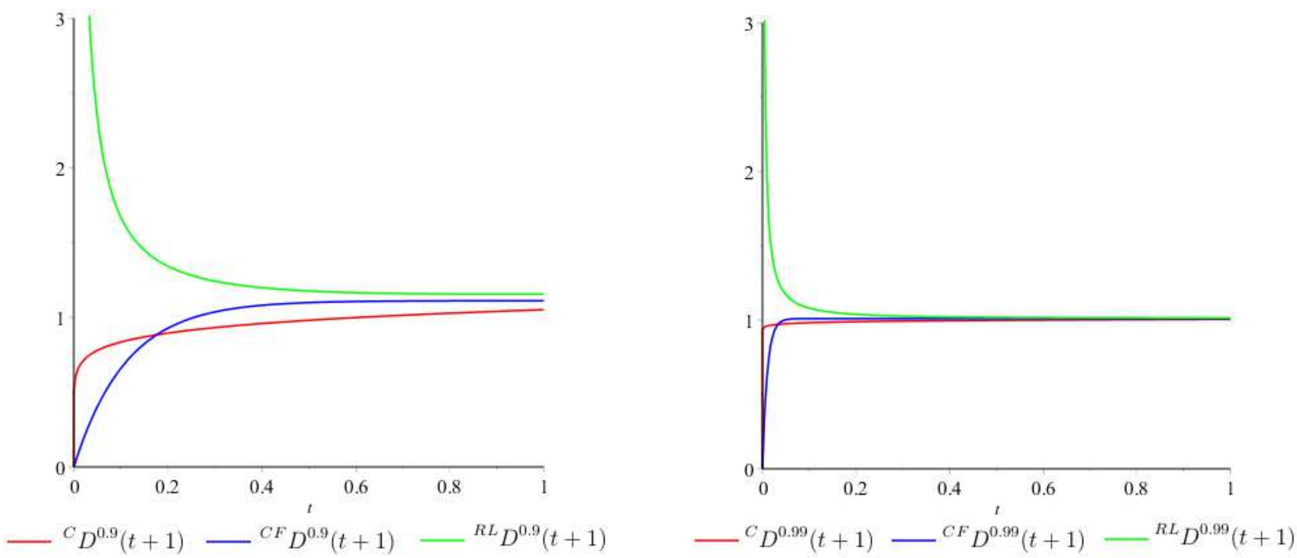

Fig. 1: Some fractional derivatives of $f(t) = t + 1$

3 Caputo and Caputo-Fabrizio convergence

3.1 General estimates

In this section we study the order of convergence respect on the parameter related to the order of differentiation, for different L^p norms. In this sense, we define the following general error expression associated with a general fractional operator.

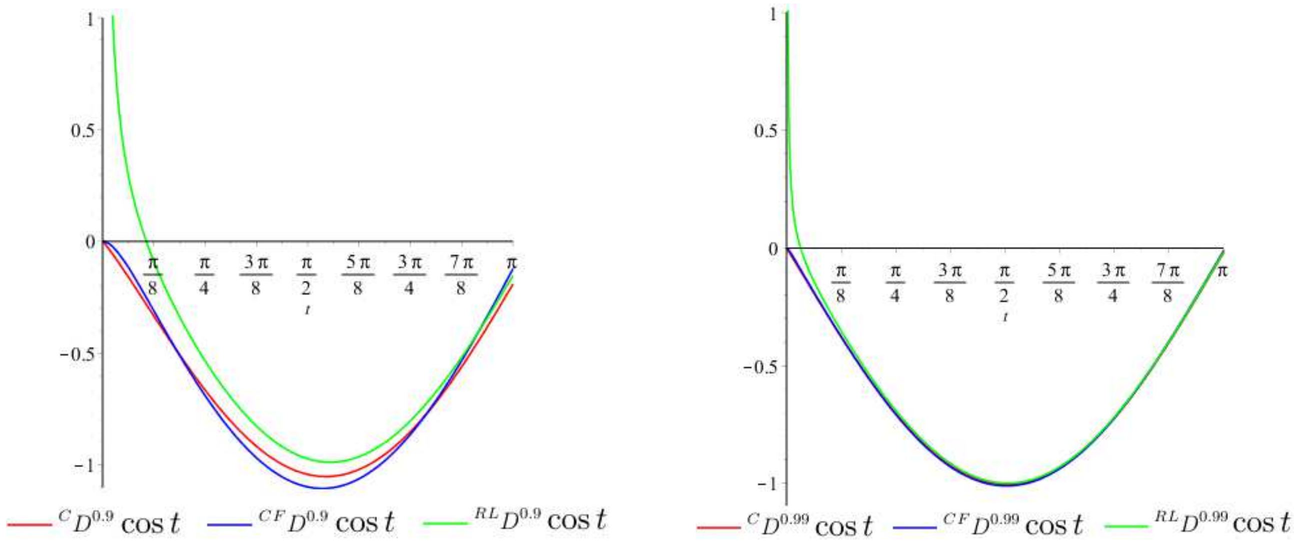

Fig. 2: Some fractional derivatives of $f(t) = \cos(t)$

Definition 4*Let $(a,b) \subset \mathbb{R}$, $f \in W^{1,p}(a,b)$, $1 \leq p \leq \infty$ and ${}_aD^{1-\beta}$ the fractional operator given in (2). We define the error estimate in the L^p norm associated to the fractional derivative as*

$$\begin{aligned} E_{f,p} \,:\, (0,1) &\to \mathbb{R}_0^+ \\ \beta &\to E_{f,p}(\beta) = ||{}_aD^{1-\beta} f - f'||_{L^p(a,b)}. \end{aligned} \tag{9}$$

From now on we will denote by ${}_aD^{1-\beta}$ to refer to the fractional derivative of C or CF type, where the superscript may be omitted if it is beyond doubt).

*Note 1.*Note that the error estimate for the RL operator is not well possed. En fact, if we consider $f(t) \equiv 1$ and $a = 0$, Propositions 1 and 2 yields that

$${}^{RL}D^{1-\beta} f(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}. \tag{10}$$

By replacing (10) in (9) for $\beta = 1/2$ and we compute the norm in L^2 we have that

$$E_{f,2}(\beta) = \left|\left| {}^{RL}D^{1/2} f - f' \right|\right|_{L^2(0,b)} = \left[\int_0^b \left(\frac{t^{-1/2}}{\Gamma(1/2)} \right)^2 \mathrm{d}t \right]^{1/2}$$

which clearly diverges.

The next Lemma states that, if we can estimate the rates $\frac{E_{g,1}(\beta)}{\beta^r}$ for every g such that $g' \in D$ (where D is a dense set contained in $L^1(a,b)$), then we can estimate the rate $\frac{E_{f,1}(\beta)}{\beta^r}$ for any function $f \in W^{1,1}(a,b)$.

Lemma 1.*Let $(a,b) \subset \mathbb{R}$ and ${}_aD^{1-\beta}$ a fractional operator defined in (2). Suppose that there exist a dense subset $D \subset L^1(a,b)$ and a fix number $r > 0$ such that*

$$E_{g,1}(\beta) = O(\beta^r) \quad (resp.\, E_{g,1}(\beta) = o(\beta^r)), \quad \beta \to 0^+ \quad \forall g \in W^{1,1}(a,b)\, such\, that\, g' \in D. \tag{11}$$

Then,

$$E_{f,1}(\beta) = O(\beta^r) \quad (resp.\, E_{f,1}(\beta) = o(\beta^r)), \quad \beta \to 0^+ \quad \forall f \in W^{1,1}(a,b). \tag{12}$$

*Proof.*Let $f \in W^{1,1}(a,b)$ and $\varepsilon > 0$ be. Then $f' \in L^1(a,b)$ and there exist a function g'_ε in D (we are making an abuse of language again by using the apostrophe), such that

$$||g'_\varepsilon - f'||_{L^1(a,b)} < \varepsilon. \tag{13}$$

Thus, if we set $g_\varepsilon(t) = \int_a^t g'_\varepsilon$ for every $t \in (a,b)$ it holds that

$$E_{f,1}(\beta) \leq \left|\left|{}_aD_t^{1-\beta}(f - g_\varepsilon)\right|\right|_{L^1(a,b)} + E_{g_\varepsilon,1}(\beta) + ||g'_\varepsilon - f'||_{L^1(a,b)}. \tag{14}$$

By applying definition (2) and Young's inequality we get

$$\left|\left|{}_aD_t^{1-\beta}(f - g_\varepsilon)\right|\right|_{L^1(a,b)} = ||(f - g_\varepsilon)' * h(\cdot,\beta)||_{L^1(a,b)} \leq ||f' - g'_\varepsilon||_{L^1(a,b)} ||h(\cdot,\beta)||_{L^1(a,b)}. \tag{15}$$

And using the uniformly boundedness of h and (13) gives that

$$\left|\left|{}_aD_t^{1-\beta}(f - g_\varepsilon)\right|\right|_{L^1(a,b)} \leq K\varepsilon. \tag{16}$$

By applying inequalities (11) (13) and (16), to (14), it yields that

$$E_{f,1}(\beta) \leq (K+1)\varepsilon + C\beta^r \qquad \beta \to 0. \tag{17}$$

From the arbitrary choice of ε, the thesis holds.

*Remark.*An analogous result to the given in Lemma 1 can be obtained by replacing the $||\cdot||_{L^1(a,b)}$ norm by the $||\cdot||_p$ norm, for $1 \leq p \leq \infty$, due to the validity of Young's inequality.

3.2 Order the convergence for the CF derivative

Theorem 1.*Let $f \in W^{1,1}(a,b)$ and ${}_aD_t^{1-\beta} = {}^{CF}_{\ a}D^{1-\beta}$. Then,*

$$E_{f,1}(\beta) = o(\beta^r), \quad \beta \to 0^+, \ \forall r \in (0,1), \tag{18}$$

and

$$E_{f,1}(\beta) = O(\beta), \quad \beta \to 0^+. \tag{19}$$

*Proof.*Let $h : \mathbb{R}^+ \times (0,1) \to \mathbb{R}$ be defined by $h(t,\beta) = \frac{e^{-\frac{1-\beta}{\beta}t}}{\beta}$. Note that ${}^h_aD_t^{1-\beta} = {}^{CF}_{\ a}D^{1-\beta}$ because h is an admissible kernel in Definition 3.

Now, let $\varepsilon > 0$ and $f \in W^{1,1}(a,b)$. In order to apply Lemma 1, we consider the set D of simple functions which is dense subset in $L^1(a,b)$. By making again an abuse of language by using the apostrophe, let the function $g'_\varepsilon(t) = \sum_{i=1}^{n} q_i \chi_{[a_i,b_i]}(t) \in D$ be such that

$$||f' - g'_\varepsilon||_{L^1(a,b)} < \varepsilon. \tag{20}$$

Note that

$$||g'_\varepsilon||_{L^1(a,b)} \leq ||g'_\varepsilon - f'||_{L^1(a,b)} + ||f'||_{L^1(a,b)} < \varepsilon + ||f'||_{L^1(a,b)}. \tag{21}$$

Then if we set $g_\varepsilon(t) = \int_a^t g'_\varepsilon$ and integrate by parts, it holds that the error estimate in the interval (a_k,b_k) verifies that

$$\begin{aligned} E_{g_\varepsilon \chi_{[a_k,b_k]},1}(\beta) &= \int_{a_k}^{b_k} \left| \sum_{i=1}^{k-1} \frac{q_i}{1-\beta}\left(e^{-\frac{1-\beta}{\beta}(t-b_i)} - e^{-\frac{1-\beta}{\beta}(t-a_i)}\right) + \frac{q_k}{1-\beta}\left(1 - e^{-\frac{1-\beta}{\beta}(t-a_k)}\right) - q_k \right| \mathrm{d}t \\ &\leq \beta^r \left[\sum_{i=1}^{k-1} \frac{|q_i|\beta^{1-r}}{(1-\beta)^2}\left(e^{-\frac{1-\beta}{\beta}(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}(b_k-b_i)} + e^{-\frac{1-\beta}{\beta}(a_k-a_i)} - e^{-\frac{1-\beta}{\beta}(b_k-a_i)}\right) \right. \\ &\quad \left. + |q_k|(b_k-a_k)\frac{\beta^{1-r}}{1-\beta} + |q_k|\frac{\beta^{1-r}}{(1-\beta)^2}\left(1 - e^{-\frac{1-\beta}{\beta}(b_k-a_k)}\right) \right]. \end{aligned} \tag{22}$$

Clearly, the last inequality in (22) tends to 0 when $\beta \searrow 0$ if $r \in (0,1)$. Being $E_{g_\varepsilon,1}(\beta) = \sum_{k=1}^{n} E_{g\chi_{[a_k,b_k]},1}(\beta)$, we conclude that $E_{g_\varepsilon,1}(\beta) = o(\beta^r), \quad \forall r \in (0,1)$ and (18) holds from Lemma 1.

Consider now the case $r = 1$ where (22) becomes

$$\begin{aligned} E_{g\varepsilon\chi_{[a_k,b_k]},1}(\beta) \le \beta \Bigg[&\sum_{i=1}^{k-1} \frac{|q_i|}{(1-\beta)^2} \left(e^{-\frac{1-\beta}{\beta}(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}(b_k-b_i)} + e^{-\frac{1-\beta}{\beta}(a_k-a_i)} - e^{-\frac{1-\beta}{\beta}(b_k-a_i)} \right) \\ &+ (b_k-a_k)\frac{|q_k|}{1-\beta} + \frac{|q_k|}{(1-\beta)^2}\left(1 - e^{-\frac{1-\beta}{\beta}(b_k-a_k)}\right) \Bigg]. \end{aligned} \tag{23}$$

We define

$$A_k = \sum_{i=1}^{k-1} \frac{|q_i|}{(1-\beta)^2} \left(e^{-\frac{1-\beta}{\beta}(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}(b_k-b_i)} + e^{-\frac{1-\beta}{\beta}(a_k-a_i)} - e^{-\frac{1-\beta}{\beta}(b_k-a_i)} \right), \quad k = 1,\ldots,n.$$

By applying the mean value theorem to the last term in brackets in (23), we get

$$E_{g\varepsilon\chi_{[a_k,b_k]},1}(\beta) = \beta \left[A_k + (b_k-a_k)\frac{|q_k|}{1-\beta}\left(1 + \frac{e^{-\frac{1-\beta}{\beta}(b_k-c_k)}}{\beta}\right)\right], \tag{24}$$

where $a_k < c_k < b_k$. Being $\lim_{\beta\to 0^+} A_k = 0$ and $\lim_{\beta\to 0^+} \frac{e^{-\frac{1-\beta}{\beta}(b_k-c_k)}}{\beta} = 0, \forall k = 1,2,\cdots,n$, we conclude that

$$E_{g_\varepsilon,1}(\beta) = \sum_{k=1}^{n} E_{g\varepsilon\chi_{[a_k,b_k]},1}(\beta) \le \beta \left[K_1 + K_2 \sum_{k=1}^{n} |q_k|(b_k-a_k) \right] \le \beta K, \tag{25}$$

where the last inequality comes form

$$\sum_{k=1}^{n} |q_k|(b_k-a_k) = ||g_\varepsilon||_{L^1(a,b)} \le \varepsilon + ||f'||_{L^1(a,b)} \le 1 + ||f'||_{L^1(a,b)}, \quad \forall \varepsilon \in (0,1).$$

Finally, (19) is obtained from (25) and Lemma 1.

Reasoning in a similar way, the next theorem follows.

Theorem 2. *Let $f \in W^{1,p}(a,b)$ and ${}_aD_t^{1-\beta} = {}^{CF}_{\ a}D^{1-\beta}$. Then, for $1 < p < \infty$*

$$E_{f,p}(\beta) = o(\beta^r), \quad \beta \to 0^+, \forall r \in \left(0, \frac{1}{p}\right), \tag{26}$$

and

$$E_{f,p}(\beta) = O\left(\beta^{\frac{1}{p}}\right), \quad \beta \to 0^+. \tag{27}$$

Proof. The proof is analogous to the given in Theorem 1. The difference relies in the estimation (22). For this case, we apply Minkowsky inequality, and integrate by parts in order to obtain an error estimate in the interval (a_k,b_k).

$$\begin{aligned} E_{g\varepsilon\chi_{[a_k,b_k]},1}(\beta) &= \left(\int_{a_k}^{b_k} \left| \sum_{i=1}^{k-1} q_i \frac{1}{1-\beta}\left(e^{-\frac{1-\beta}{\beta}(t-b_i)} - e^{-\frac{1-\beta}{\beta}(t-a_i)}\right) + q_k\frac{1}{1-\beta}\left(1 - e^{-\frac{1-\beta}{\beta}(t-a_k)}\right) - q_k \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \\ &\le \beta^r \Bigg\{ \sum_{i=1}^{k-1} |q_i| \frac{\beta^{\frac{1}{p}-r}}{(1-\beta)^{\frac{p+1}{p}} p^{\frac{1}{p}}} \left[\left(e^{-\frac{1-\beta}{\beta}p(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}p(b_k-b_i)} \right)^{\frac{1}{p}} + \left(e^{-\frac{1-\beta}{\beta}p(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}p(b_k-b_i)} \right)^{\frac{1}{p}} \right] \\ &\quad + (b_k-a_k)^{\frac{1}{p}} |q_k| \frac{\beta^{1-r}}{1-\beta} + |q_k| \frac{\beta^{\frac{1}{p}-r}}{(1-\beta)^{\frac{p+1}{p}} p^{\frac{1}{p}}} \left(1 - e^{-\frac{1-\beta}{\beta}p(b_k-a_k)}\right)^{\frac{1}{p}} \Bigg\} \end{aligned} \tag{28}$$

Clearly, the last inequality in (28) tends to 0 when $\beta \searrow 0$ if $r \in \left(0, \frac{1}{p}\right)$. Being $E_{g_\varepsilon,p}(\beta) = \sum_{k=1}^{n} E_{g\chi_{[a_k,b_k]},1}(\beta)$, we conclude that $E_{g_\varepsilon,1}(\beta) = o(\beta^r)$, $\quad \forall r \in \left(0, \frac{1}{p}\right)$ and (26) holds from Lemma 1.

Consider now the case $r = \frac{1}{p}$ where (28) becomes

$$E_{g\varepsilon\chi_{[a_k,b_k]},\frac{1}{p}}(\beta) \leq \beta^{\frac{1}{p}} \left\{ \sum_{i=1}^{k-1} \frac{|q_i|}{(1-\beta)^{\frac{p+1}{p}} p^{\frac{1}{p}}} \left[\left(e^{-\frac{1-\beta}{\beta}p(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}p(b_k-b_i)} \right)^{\frac{1}{p}} + \left(e^{-\frac{1-\beta}{\beta}p(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}p(b_k-b_i)} \right)^{\frac{1}{p}} \right] \right.$$
$$\left. + (b_k-a_k)^{\frac{1}{p}} |q_k| \frac{\beta^{\frac{p-1}{p}}}{1-\beta} + \frac{|q_k|}{(1-\beta)^{\frac{p+1}{p}} p^{\frac{1}{p}}} \left(1 - e^{-\frac{1-\beta}{\beta}p(b_k-a_k)} \right)^{\frac{1}{p}} \right\} \tag{29}$$

As before, we define $A_k = \sum_{i=1}^{k-1} \frac{|q_i|}{(1-\beta)^{\frac{p+1}{p}} p^{\frac{1}{p}}} \left[\left(e^{-\frac{1-\beta}{\beta}p(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}p(b_k-b_i)} \right)^{\frac{1}{p}} + \left(e^{-\frac{1-\beta}{\beta}p(a_k-b_i)} - e^{-\frac{1-\beta}{\beta}p(b_k-b_i)} \right)^{\frac{1}{p}} \right]$ for every $k = 1, \ldots, n$. Then, by applying the mean value theorem to the last term in brackets in (29) first and noting that $\lim_{\beta \to 0^+} A_k = 0$ and $\lim_{\beta \to 0^+} \frac{e^{-\frac{1-\beta}{\beta}p(b_k-c_k)}}{\beta^{\frac{2}{p}}} = 0$, $\forall k = 1, 2, \cdots, n$, we conclude that

$$E_{g_\varepsilon,p}(\beta) = \sum_{k=1}^{n} E_{g\varepsilon\chi_{[a_k,b_k]},1}(\beta) \leq \beta^{\frac{1}{p}} \left[K_1 + K_2 \sum_{k=1}^{n} |q_k| (b_k - a_k) \right] \leq \beta^{\frac{1}{p}} K. \tag{30}$$

Finally, (27) is obtained from (30) and Lemma 1.

*Remark.*Note that the estimate (26) does not hold for $r = \frac{1}{p}$. In fact, let $f_1(t) = t$ defined in $[0, b]$. It is easy to see that

$$E_{f_1,p}(\beta) = \left|\left| {}^{CF}D^{1-\beta} f_1 - f_1' \right|\right|_{L^p(0,b)} = \frac{\beta^{\frac{1}{p}}}{(1-\beta)^{\frac{1}{p}}} \left(\int_0^b \left| 1 - \frac{e^{-\frac{1-\beta}{\beta}t}}{\beta} \right|^p \mathrm{d}t \right)^{\frac{1}{p}}. \tag{31}$$

Taking into account that the limit $\frac{e^{-\frac{1-\beta}{\beta}t}}{\beta} \to 0$ when $\beta \to 0$ holds, we can make the integrand in (31) grater than $1/2$ for small values of β obtaining that $\left(\int_0^b \left| 1 - \frac{e^{-\frac{1-\beta}{\beta}t}}{\beta} \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \geq \frac{b^{\frac{1}{p}}}{2}$. Then, we conclude that

$$\lim_{\beta \to 0^+} \frac{E_{f_1,p}(\beta)}{\beta^{\frac{1}{p}}} \neq 0.$$

Besides,if we consider the function $f_2(t) = e^t$ defined in $t \in [0, b]$ we see that it is an $O(\beta)$ but it is not an $o(\beta)$ because $E_{f_2,1}(\beta) = \left|\left| {}^{CF}D^{1-\beta} f_2 - f_2' \right|\right|_{L^1(0,b)} = \frac{\beta}{1-\beta} \left(1 - e^{-\frac{1-\beta}{\beta}b} \right)$, from where we deduce that

$$\lim_{\beta \to 0^+} \frac{E_{g,1}(\beta)}{\beta} = 1 \neq 0.$$

3.3 Order the convergence for the C derivative

Theorem 3.*Let $f \in W^{1,1}(a,b)$ and ${}_aD_t^{1-\beta} = {}_a^C D^{1-\beta}$. Then,*

$$E_{f,1}(\beta) = o(\beta^r), \quad \beta \to 0^+, \forall r \in (0,1),$$

and

$$E_{f,1}(\beta) = O(\beta), \quad \beta \to 0^+.$$

*Proof.*Let $h : \mathbb{R}^+ \times (0,1) \to \mathbb{R}$ defined by $h(t,\beta) = \frac{t^{-(1-\beta)}}{\Gamma(\beta)}$. Note that h is an admissible kernel in Definition 3, moreover, ${}^h_a D_t^{1-\beta} = {}^C_a D^{1-\beta}$.

Let $\varepsilon > 0$ and $f \in W^{1,1}(a,b)$ be. Let D be a dense subset in $L^1(a,b)$ described in Lemma 1. Reasoning like in Theorem 1, consider the function $g'_\varepsilon(t) = \sum_{i=1}^{n} q_i \chi_{[a_i,b_i]}(t) \in D$ such that

$$||f' - g'_\varepsilon||_{L^1(a,b)} < \varepsilon. \tag{32}$$

Then, (21) holds. Let us estimate the error in the interval (a_k,b_k). Setting $g_\varepsilon(t) = \int_a^t g'_\varepsilon$ and applying Proposition 2 it holds that

$$E_{g_\varepsilon \chi_{[a_k,b_k]},1}(\beta) = \beta^r \frac{|q_k|}{\Gamma(1+\beta)} \int_{a_k}^{b_k} \left| \frac{(t-a_k)^\beta - \Gamma(1+\beta)}{\beta^r} \right| \mathrm{d}t. \tag{33}$$

And,

$$\lim_{\beta \to 0^+} \int_{a_k}^{b_k} \left| \frac{(t-a_k)^\beta - \Gamma(1+\beta)}{\beta^r} \right| \mathrm{d}t = 0, \text{ if } r \in (0,1).$$

Then, $\lim_{\beta\to 0^+} = E_{g_\varepsilon \chi_{[a_k,b_k]},1}(\beta) = 0$. Being $E_{g_\varepsilon,1}(\beta) = \sum_{k=1}^{n} E_{g\chi_{[a_k,b_k]},1}(\beta)$, we conclude that $E_{g_\varepsilon,1}(\beta) = o(\beta^r), \quad \forall r \in (0,1)$. In consequence, Lemma 1 gives that $E_{f,1}(\beta) = o(\beta^r), \quad \forall r \in (0,1)$.
Consider the case $r = 1$. From (33) we have

$$E_{g_\varepsilon \chi_{[a_k,b_k]},1}(\beta) = \beta \frac{|q_k|}{\Gamma(1+\beta)} \int_{a_k}^{b_k} \left| \frac{(t-a_k)^\beta - \Gamma(1+\beta)}{\beta} \right| \mathrm{d}t. \tag{34}$$

Now,

$$\lim_{\beta \to 0^+} \frac{(t-a_k)^\beta - \Gamma(\beta+1)}{\beta} = \ln(t-a_k) - \Gamma'(1).$$

Thus, by Lebesgue convergence Theorem, we have that

$$\lim_{\beta \to 0^+} \int_{a_k}^{b_k} \left| \frac{(t-a_k)^\beta - \Gamma(1+\beta)}{\beta} \right| \mathrm{d}t = \int_{a_k}^{b_k} |\ln(t-a_k) - \Gamma'(1)| < \infty,$$

hence $E_{g_\varepsilon,1}(\beta) = O(\beta)$ for every g such that $g' \in D$. By applying Lemma 1 the thesis holds.

Reasoning in a similar way, the next theorem follows.

Theorem 4.*Let $f \in W^{1,p}(a,b)$ and ${}_a D_t^{1-\beta} = {}^C_a D^{1-\beta}$. Then,*

$$E_{f,p}(\beta) = o(\beta^r), \quad \beta \to 0^+, \forall r \in (0,1),$$
$$E_{f,p}(\beta) = O(\beta), \quad \beta \to 0^+.$$

*Remark.*It is easy to see that $E_{f,1}(\beta)$ is not in general an $o(\beta)$ when $\beta \to 0^+$. In fact, let $g(t) = |t-1|$ be defined in $t \in [0,2]$ and compute it's C derivative

$${}^C D^{1-\beta} g(t) = \begin{cases} -\frac{t^\beta}{\Gamma(\beta+1)} & \text{si } t \in [0,1], \\ \frac{2(t-1)^\beta - t^\beta}{\Gamma(\beta+1)} & \text{si } t \in (1,2] \end{cases}.$$

We have

$$E_{g,1}(\beta) = \left|\left| {}^C D^{1-\beta} g - g' \right|\right|_{L^1(0,2)} = \int_0^1 \left| -\frac{t^\beta}{\Gamma(\beta+1)} + 1 \right| \mathrm{d}t + \int_1^2 \left| \frac{2(t-1)^\beta - t^\beta}{\Gamma(\beta+1)} - 1 \right| \mathrm{d}t$$
$$\geq \int_1^2 1 - \frac{2(t-1)^\beta - t^\beta}{\Gamma(\beta+1)} - 1 \mathrm{d}t = \frac{\Gamma(\beta+2) - 3 + 2^{\beta+1}}{\Gamma(\beta+2)} \geq 2(\Gamma(\beta+2) - 3 + 2^{\beta+1}).$$

Being $\beta > 0$,

$$\lim_{\beta\to 0^+} \frac{E_{g,1}(\beta)}{\beta} \geq \lim_{\beta\to 0^+} 2\frac{\Gamma(\beta+2)-3+2^{\beta+1}}{\beta} = 2(\Gamma'(2)+\ln 2) \neq 0.$$

Moreover, by using Hölder's inequality we have that

$$E_{g,1}(\beta) = \left|\left|{}^{C}D^{1-\beta}g - g'\right|\right|_{L^1(0,2)} \leq \left|\left|{}^{C}D^{1-\beta}g - g'\right|\right|_{L^p(0,2)} ||1||_{L^q(0,2)} = 2^{\frac{1}{q}}E_{g,p}(\beta),$$

where $\frac{1}{p}+\frac{1}{q}=1$, which leads to

$$\lim_{\beta\to 0^+} \frac{E_{g,p}(\beta)}{\beta} \geq 2^{1-\frac{1}{q}}(\Gamma'(2)+1) \neq 0.$$

*Remark.*It is worth noting that Theorems 1 and 3 give us a similar order of convergence for CF and C derivatives respectively. It is interesting that the difference between the kernels (the first one non-singular and the second one singular!) has not been relevant in the convergence result.

3.4 Some comments about the speed of convergence

Based on the results obtained in the preceding section, it is natural to ask: Does the C derivative converges to the ordinary derivative faster than the CF derivative? Or conversely, does the CF derivative converges to the ordinary derivative faster than the C derivative?

Let us compare the speed of convergence in the L^1 norm. Consider the function

$$\begin{aligned} g\colon [0,T] &\to \mathbb{R} \\ t &\to g(t) = t^m, \qquad m \in \mathbb{N}. \end{aligned} \tag{35}$$

From Proposition 2 we have that

$$\left|\left|{}^{CF}D^{1-\beta}g - g'\right|\right|_{L^1(0,T)} = \int_0^T \frac{m}{1-\beta}t^{m-1}\left|\beta - \Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right)\right|\mathrm{d}t. \tag{36}$$

and

$$\left|\left|{}^{C}D^{1-\beta}g - g'\right|\right|_{L^1(0,T)} = \int_0^T mt^{m-1}\left|1 - t^{\beta}\frac{\Gamma(m)}{\Gamma(m+\beta)}\right|\mathrm{d}t. \tag{37}$$

With the aim to compute the integrals (36) and (37) we present the next Lemma.

Lemma 2.*Let $m \in \mathbb{N}-\{1\}$, and let $t^*\colon (0,1) \to \mathbb{R}^+$ and $s^*\colon (0,1) \to \mathbb{R}^+$ be the functions defined as*

$$t^*(\beta) = v \quad \text{if} \quad \Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}v\right) = \beta \tag{38}$$

and

$$s^*(\beta) = w \quad \text{if} \quad w^{\beta}\frac{\Gamma(m)}{\Gamma(m+\beta)} = 1 \tag{39}$$

respectively. Then we have that

a)$m-1 \leq t^*(\beta)$ for every $\beta \in (0,1)$.
b)$m-1 \leq s^*(\beta)$ for every $\beta \in (0,1)$.

Proof. a)Let $m \in \mathbb{N}$. From [5, Example 4.1 - (d)] it follows that $\lim_{t\to\infty} \mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right) = 0$. We also know that the Mittag-Leffler function $\mathscr{E}_{\alpha,\beta}(-t)$ is complete monotonic[1] for every $\alpha \in [0,1]$ and $\beta \geq \alpha$ (see [19, Ch. 4]) which lead us to conclude that $\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right)$ is a decreasing function on $(0,\infty)$.

Therefore, if $\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}(m-1)\right) \geq \beta$, there exists a unique $t^* \geq m-1 > 0$ such that $\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t^*\right) = \beta$.

We will prove that $\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}(m-1)\right) \geq \beta$.

From [13, p. 18] we know that

$$\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right) = \frac{(m-1)!}{\left(-\frac{1-\beta}{\beta}t\right)^{m-1}}\left(e^{-\frac{1-\beta}{\beta}t} - \sum_{k=0}^{m-2}\frac{\left(-\frac{1-\beta}{\beta}t\right)^k}{k!}\right) = \frac{(m-1)!}{\left(-\frac{1-\beta}{\beta}\right)^{m-1}t^{m-1}}R_{m-2}\left(e^{-\frac{1-\beta}{\beta}t},0\right), \tag{40}$$

where $R_{m-2}\left(e^{-\frac{1-\beta}{\beta}t},0\right)$ is the Taylor error of $e^{-\frac{1-\beta}{\beta}t}$ centered at 0 which can be expressed as

$$R_{m-2}\left(e^{-\frac{1-\beta}{\beta}t},0\right) = \frac{\left(-\frac{1-\beta}{\beta}\right)^{m-1}}{(m-2)!}\int_0^t (t-x)^{m-2}e^{-\frac{1-\beta}{\beta}x}dx. \tag{41}$$

From (40) and (41) we have

$$\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right) = \frac{m-1}{t^{m-1}}\int_0^t (t-x)^{m-2}e^{-\frac{1-\beta}{\beta}x}dx.$$

Applying integration by parts and the Mean Value Theorem, we obtain

$$\begin{aligned}\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right) &= \frac{m-1}{t^{m-1}}\left[t^{m-2}\frac{\beta}{1-\beta} - (m-2)\frac{\beta}{1-\beta}\int_0^t (t-x)^{m-3}e^{-\frac{1-\beta}{\beta}x}dx\right]\\ &= \frac{m-1}{t}\frac{\beta}{1-\beta} - \frac{m-1}{t}\frac{\beta}{1-\beta}e^{-\frac{1-\beta}{\beta}\xi}, \quad \xi \in [0,t].\end{aligned}$$

Hence, using that $x \leq e^x, \forall x \in \mathbb{R}$, for $t = m-1$ we have

$$\begin{aligned}\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}(m-1)\right) &= \frac{\beta}{1-\beta}\left(1-e^{-\frac{1-\beta}{\beta}\xi}\right)\\ &\geq \frac{\beta}{1-\beta}\left(1-e^{-\frac{1-\beta}{\beta}(m-1)}\right)\\ &\geq \frac{\beta}{1-\beta}\left(1-\frac{\beta}{(1-\beta)(m-1)}\right) \geq \frac{\beta}{1-\beta}\frac{(1-\beta)(m-1)-\beta}{(1-\beta)(m-1)}.\end{aligned} \tag{42}$$

On the other hand

$$1-\frac{1}{m-1} \geq \beta \Leftrightarrow \frac{\beta}{1-\beta}\frac{(1-\beta)(m-1)-\beta}{(1-\beta)(m-1)} \geq \beta, \tag{43}$$

and form (42) and (43) it yields that $\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}(m-1)\right) \geq \beta$. Now, note that if $m \geq 3$ then $1-\frac{1}{m-1} \geq \frac{1}{2}$. Thus, for $\beta \in \left(0,\frac{1}{2}\right)$ and $m \geq 3$, $\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right) \geq \beta$. In addition, taking $m = 2$ in (42), we have

$$\Gamma(2)\mathscr{E}_{1,2}\left(-\frac{1-\beta}{\beta}t\right) \geq \frac{1}{t}\frac{\beta}{1-\beta}\left(1-e^{-\frac{1-\beta}{\beta}t}\right).$$

[1] Recall that a function $f : (',\infty) \to \mathbb{R}$ is completely monotonic if f is differentiable for every natural order k and $(-1)^k f^{(k)}(x) \geq 0$ for every $k \in \mathbb{N}_0$ and x

Finally, for $t = m-1 = 1$,

$$\Gamma(2)\mathscr{E}_{1,2}\left(-\frac{1-\beta}{\beta}\right) \geq \frac{\beta}{1-\beta}\left(1-e^{-\frac{1-\beta}{\beta}}\right).$$

Define the function $g(\beta) = \beta - e^{-\frac{1-\beta}{\beta}}$, which is continuous and $\lim_{\beta\to 0^+} g(\beta) = 0$, $g'(\beta) = 1+\frac{1-\beta}{\beta}e^{-\frac{1-\beta}{\beta}} > 0$, from where $g(\beta) \geq 0, \forall \beta \in (0,1)$. Then $-e^{-\frac{1-\beta}{\beta}} \geq -\beta$, and

$$\Gamma(2)\mathscr{E}_{1,2}\left(-\frac{1-\beta}{\beta}\right) \geq \frac{\beta}{1-\beta}(1-\beta) = \beta.$$

Then, $\forall m \geq 2$, there exists $t^* \geq m-1$ such that $\Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t^*\right) = \beta$.

b)Let $m \in \mathbb{N}$. Considere the function $H(\beta) = \left(\frac{\Gamma(m+\beta)}{\Gamma(m)}\right)^{\frac{1}{\beta}}$. By the Gautschi's inequality [15],

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}, \quad \forall x > 0, \forall s \in (0,1).$$

Then

$$(m+\beta-1)^{1-(1-\beta)} < \frac{\Gamma((m+\beta-1)+1)}{\Gamma((m+\beta-1)+(1-\beta))} < ((m+\beta-1)+1)^{1-(1-\beta)}$$

$$(m+\beta-1)^{\beta} < \frac{\Gamma(m+\beta)}{\Gamma(m)} < (m+\beta)^{\beta}$$

$$m-1 < m+\beta-1 < \left(\frac{\Gamma(m+\beta)}{\Gamma(m)}\right)^{\frac{1}{\beta}} < m+\beta,$$

so $m-1 \leq s^*(\beta)$ for every $\beta \in (0,1)$.

Finally, we present the next proposition where the speed of convergence for natural power functions is obtained.

Proposition 1.*Let g be the function defined in (35), $m \geq 2$ a fixed natural, and $T \in [0, m-1]$. Then*

$$\lim_{\beta\to 0^+} \frac{||{}^{CF}D^{1-\beta}g - g'||_{L^1(0,T)}}{||{}^{C}D^{1-\beta}g - g'||_{L^1(0,T)}} = \frac{m-T}{T}\frac{1}{\Psi(m+1)-\ln T},$$

where $\Psi(\cdot)$ is the Psi function (or Digamma function), defined as $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for $x \in \mathbb{R}-\mathbb{Z}_0^-$.

*Proof.*Let $m \geq 2$ a fixed natural and $T \in [0, m-1]$. Observe that $\beta - \Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}\cdot 0\right) < 0$. Being $T \leq m-1$, by Lemma 2 we deduce that

$$\beta - \Gamma(m)\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right) < 0.$$

Then, from (36) and (37) we have

$$\begin{aligned}\left|\left|{}^{CF}D^{1-\beta}g - g'\right|\right|_{L^1(0,T)} &= \frac{\Gamma(m+1)}{1-\beta}\int_0^T t^{m-1}\mathscr{E}_{1,m}\left(-\frac{1-\beta}{\beta}t\right)dt - \frac{\beta}{1-\beta}T^m \\ &= \frac{T^m}{1-\beta}\left(\Gamma(m+1)\mathscr{E}_{1,m+1}\left(-\frac{1-\beta}{\beta}T\right)-\beta\right),\end{aligned}$$

and

$$\left|\left|{}^{C}D^{1-\beta}g - g'\right|\right|_{L^1(0,T)} = \frac{T^m}{\Gamma(m+\beta+1)}\left(\Gamma(m+\beta+1)-\Gamma(m+1)T^{\beta}\right).$$

Therefore,

$$\frac{||{}^{CF}D^{1-\beta}g-g'||_{L^1(0,T)}}{||{}^{C}D^{1-\beta}g-g'||_{L^1(0,T)}}=\frac{\Gamma(m+\beta+1)}{1-\beta}\frac{\frac{\Gamma(m+1)}{\beta}\mathscr{E}_{1,m+1}\left(-\frac{1-\beta}{\beta}T\right)-1}{\frac{\Gamma(m+\beta+1)-\Gamma(m+1)T^{\beta}}{\beta}}.$$

Now, from [5, Example 4.1]

$$\mathscr{E}_{1,m+1}\left(-\frac{1-\beta}{\beta}T\right)=\frac{1}{\left(-\frac{1-\beta}{\beta}T\right)^m}\left(e^{-\frac{1-\beta}{\beta}T}-\sum_{k=0}^{m-1}\frac{\left(-\frac{1-\beta}{\beta}T\right)^k}{k!}\right),$$

from where we conclude that

$$\lim_{\beta\to 0^+}\frac{\Gamma(m+1)}{\beta}\mathscr{E}_{1,m+1}\left(-\frac{1-\beta}{\beta}T\right)=\frac{m}{T}.$$

By the other side

$$\lim_{\beta\to 0^+}\frac{\Gamma(m+\beta+1)-\Gamma(m+1)T^{\beta}}{\beta}=\Gamma'(m+1)-\Gamma(m+1)\ln T.$$

Then,

$$\lim_{\beta\to 0^+}\frac{||{}^{CF}D^{1-\beta}g-g'||_{L^1(0,T)}}{||{}^{C}D^{1-\beta}g-g'||_{L^1(0,T)}}=\frac{m-T}{T}\frac{1}{\Psi(m+1)-\ln T},$$

and the thesis holds.

Proposition 1 affirms that the speed of convergence for t^m vary depending on the power m and the interval length T for the L^1 norm. This is in concordance with the graphics in Figures 1 and 2 where it can be seen that the fractional derivatives is more unestable for short times.

Another interesting thing to remark is that the speed of convergence of these two operators (when computed to power functions) is comparable. This is in contrasts to what we expected, because the exponential no-singular kernel in the CF derivative does not make it faster than the C derivative.

Let us see some examples in the next table where we have taken the values $T=1$ and $T=m-1$.

Table 1: Different speed of convergence

m	$\lim_{\beta\to 0^+}\frac{\|\|{}^{CF}D^{1-\beta}g-g'\|\|_{L^1(0,1)}}{\|\|{}^{C}D^{1-\beta}g-g'\|\|_{L^1(0,1)}}$	$\lim_{\beta\to 0^+}\frac{\|\|{}^{CF}D^{1-\beta}g-g'\|\|_{L^1(0,m-1)}}{\|\|{}^{C}D^{1-\beta}g-g'\|\|_{L^1(0,m-1)}}$
3	1.592207522	0.8881460240
4	1.991876242	0.8179851126
5	2.344504178	0.7816816178
6	2.669821563	0.7594559202

4 Conclusion

We have analyzed the order of convergence of different fractional differential operators to the ordinary derivative, when the order of derivation tends to one, for L^1 and L^p norms, $p>1$. We proved that the derivatives have a similar order of convergence for both norms (in fact the order is a number r in the interval $(0,1)$). As expected, the error estimate for the RL operator is not well defined. Finally, we studied the speed of convergence of the C and CF derivatives for power functions, concluding that, in general, neither of them is faster than the other.

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