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On the diameter of Schrijver graphs

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Abstract

For $k \geq 1$ and $n \geq 2k$, the well known Kneser graph $\mathrm{KG}(n,k)$ has all k-element subsets of an n-element set as vertices; two such subsets are adjacent if they are disjoint. Schrijver constructed a vertex-critical subgraph $\mathrm{SG}(n,k)$ of $\mathrm{KG}(n,k)$ with the same chromatic number. In this paper, we compute the diameter of the graph $\mathrm{SG}(2k+r,k)$ with $r \geq 1$. We obtain that the diameter of $\mathrm{SG}(2k+r,k)$ is equal to 2 if $r \geq 2k-2$; 3 if $k-2 \leq r \leq 2k-3$; k if $r=1$; and for $2 \leq r \leq k-3$, we obtain that the diameter of $\mathrm{SG}(2k+r,k)$ is at most equal to $k-r+1$.

1. Introduction

Let G be a connected graph. Given two vertices $a, b \in G$, $\mathrm{dist}(a,b)$, the *distance* between a and b, is defined as the length of the shortest path in G joining a to b. The *diameter* of G, that we denote by $D(G)$, is defined as the maximum distance between any pair of vertices in G.

Let $[n]$ denote the set $\{1,\ldots,n\}$. For positive integers $n \geq 2k$, the Kneser graph $\mathrm{KG}(n,k)$ has as vertices the k-subsets of $[n]$ and two vertices are connected by an edge if they have empty intersection. In a famous paper, Lovász [5] showed that its chromatic number $\chi(\mathrm{KG}(n,k))$ is equal to $n-2k+2$. After this result, Schrijver [7] proved that the chromatic number remains the same when we consider the subgraph $\mathrm{KG}(n,k)_{2-\mathrm{stab}}$ of $\mathrm{KG}(n,k)$ obtained by restricting the vertex set to the k-subsets that are 2-*stable*, that is, that do not contain two consecutive elements of $[n]$ (where 1 and n are considered also to be consecutive). Schrijver [7] also proved that the 2-stable Kneser graphs are *vertex critical* (or χ-*critical*), i.e. the chromatic number of any proper subgraph of $\mathrm{KG}(n,k)_{2-\mathrm{stab}}$ is strictly less than $n-2k+2$; for

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this reason, the 2-stable Kneser graphs are also known as the Schrijver graphs. From now on we will use throughout this paper the notation SG(n,k) to refer to the graph KG$(n,k)_{2-\text{stab}}$.

After these general advances, a lot of work has been done concerning properties of Kneser graphs and stable Kneser graphs (see [1, 2, 3, 4, 6, 8, 9, 10, 11] and references therein). Concerning Kneser graphs, its diameter was computed in [11]. Moreover, it is known that the distance between two vertices in Kneser graphs KG(n,k) only depends on the cardinality of their intersection [11]. However, in the case of Schrijver graphs SG(n,k) this does not work in the same way. For example, note that in SG$(10,4)$ the vertices $\{1,3,5,7\}$ and $\{1,3,6,8\}$ are at distance 3, while $\{1,3,6,8\}$ and $\{1,4,6,9\}$ are at distance 2. In this paper, we are interested in computing the diameter of Schrijver graphs. As far as we know this parameter has not been studied for such graphs. The main result of this paper is the following theorem:

Theorem 1.1. *Let n,k,r be positive integers such that $n = 2k+r$. Then, the diameter of the Schrijver graph $SG(2k+r,k)$ verifies*

$$D(SG(2k+r,k))\begin{cases} = 2 & ;\ \textit{if } r \geq 2k-2 \\ = 3 & ;\ \textit{if } k-2 \leq r \leq 2k-3 \\ \leq k-r+1 & ;\ \textit{if } 2 \leq r \leq k-3 \\ = k & ;\ \textit{if } r = 1 \end{cases}$$

The proof of Theorem 1.1 will follow from Observation 1 and Theorems 2.2, 2.12 and 2.15 given in the next section.

2. Main results

A subset $S \subseteq [n]$ is *s-stable* if any two of its elements are at least "at distance s apart" on the n-cycle, that is, if $s \leq |i-j| \leq n-s$ for distinct $i,j \in S$. For $s,k \geq 2$ and $n \geq ks$, the s-stable Kneser graph KG$(n,k)_{s-\text{stab}}$ is the subgraph of KG(n,k) obtained by restricting the vertex set of KG(n,k) to the s-stable k-subsets of $[n]$.

In [10] it was shown (see Proposition 4.3 in [10]) that KG$(ks+1,k)_{s-\text{stab}}$ is isomorphic to the complement graph of the $(k-1)$th power of a cycle C_{ks+1}. Therefore, SG$(2k+1,k)$ is isomorphic to a cycle graph C_{2k+1} and so, we have the following straightforward observation.

Observation 1. D(SG$(2k+1,k)) = k$.

From now on, we assume that $n \geq 2k+2$. We denote $[n]_2$ to the family of 2-stable subsets of $[n]$ and $[n]_2^k$ to the family of 2-stable k-subset of $[n]$, i.e. $[n]_2^k = V(\text{SG}(n,k))$. We will always assume w.l.o.g. that any vertex $v = \{v_1, v_2, \ldots, v_k\}$ in SG(n,k) is such that $v_1 < v_2 < \ldots < v_k$. Arithmetic operations will be supposed modulo n (being $0 \equiv n$).

2.1. Distances between vertices

Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ be two vertices in SG(n,k) such that $|A \cap B| = 1$. W.l.o.g. we assume that $A \cap B = \{1\}$ and $a_2 < b_2$. Note that $b_2 \geq 4$. Let $X = \{2, b_2, b_3, \ldots, b_k\}$ and $Y = X+1 = \{3, b_2+1, b_3+1, \ldots, b_k+1\}$. It is not hard to see that the set of vertices $\{A, X, Y, B\}$ induce a P_4 or a paw.

Assume now that $|A \cap B| = k-1$. W.l.o.g. let $a_i = b_i$ for $i \in [k-1]$ and $a_k < b_k$. Then $B+1$ is adjacent to A and B. So, we have:

Observation 2. *Let $A, B \in [n]_2^k$. If $|A \cap B| = k-1$ then $dist(A,B) = 2$ and if $|A \cap B| = 1$ then $dist(A,B) \in \{2,3\}$.*

Let $A, B \in [n]_2^k$. In order to compute the distance between vertices A and B, we consider the subsets of vertices in the cycle C_n with vertex set $[n]$ induced by the elements in $A \cup B$.

Let $X = A \cup B \subseteq [n]$. We denote $\mathcal{X}$ to the family of connected components of the graph induced by X in the n-cycle C_n. In the same way, $\overline{\mathcal{X}}$ is the family of connected components of the graph induced by $[n] \setminus X$ in C_n. Let $\mathcal{P} = \{C \in \overline{\mathcal{X}} : |C| \text{ is even}\}$ and $\mathcal{I} = \{C \in \overline{\mathcal{X}} : |C| \text{ is odd}\}$. From these definitions, we have the following simple observations and Lemma 2.1.

Observation 3. *Let $A, B \in [n]_2^k$ with $A \cap B \neq \emptyset$. Let $X = A \cup B$. Then,*

1. $|\mathcal{X}| = \left|\overline{\mathcal{X}}\right|$.
2. *If* $\left|\overline{\mathcal{X}}\right| \geq k$ *then,* $dist(A, B) = 2$.
3. *If* $|C| \leq 2$ *for every* $C \in \mathcal{X}$ *then,* $dist(A, B) = 2$.

Lemma 2.1. *Let* $A, B \in [n]^k_2$ *with* $A \cap B \neq \emptyset$ *and* $X = A \cup B$. *Then,* $dist(A, B) = 2$ *if and only if* $\frac{1}{2}|\mathcal{I}| + \frac{1}{2}\sum_{C\in\overline{\mathcal{X}}} |V(C)| \geq k$.

Notice that if $2k + 2 \leq n \leq 4k - 3$ (i.e. if $2 \leq r \leq 2k - 3$ and $n = 2k + r$) then $k \geq 3$ and $D(SG(n, k)) \geq 3$ since vertices $A = \{1, 4, 6, \ldots, 2k\}$ and $B = \{1, 5, 7, \ldots, 2k + 1\}$ are at distance 3 in $SG(n, k)$. In fact, observe that $[n] \setminus (A \cup B) = \{2, 3\} \cup \{2k + 2, \ldots, n\}$ and then there is no 2-stable k-subset in $[n] \setminus (A \cup B)$, i.e. there is no vertex of $SG(n, k)$, adjacent to A and B. Finally, notice that the vertices A, $\{3, 5, 7 \ldots, 2k + 1\}$, $\{2, 4, 6, \ldots, 2k\}$ and B induce a P_4 in $SG(n, k)$.
On the other hand, if $n \geq 4k - 2$, then Lemma 2.1 is enough to assure that $D(SG(n, k)) = 2$.

Theorem 2.2. *Let* n, k *and* r *be positive integers, with* $k \geq 2$ *and* $n = 2k + r$. $D(SG(n, k)) = 2$ *if and only if* $r \geq 2k - 2$.

Proof. By the preceding discussion, it only remains to show that if $r \geq k - 2$, then $D(SG(n, k)) = 2$. Let $A, B \in [n]^k_2$ such that $A \cup B \neq \emptyset$. If $r \geq 2k - 2$ then $n \geq 4k - 2$ and thus, $|[n] \setminus X| \geq 2k + 2k - 2 - (2k - 1) = 2k - 1$. Hence, $\frac{1}{2}|\mathcal{I}| + \frac{1}{2}\sum_{C\in\overline{\mathcal{X}}} |V(C)| \geq k$. Therefore, by Lemma 2.1 the result holds. □

Let $A, B \in [n]^k_2$, with $A \neq B$ and $A \cap B \neq \emptyset$, and let $X = A \cup B$. In what follows, we will study the structure of $\mathcal{X}$ and $\overline{\mathcal{X}}$ in order to construct a path between vertices A and B of length as short as possible.

Notice that a connected component C in $\mathcal{X}$ is either a single element in $A \cap B$, or it alternates between vertices of A and B. Furthermore, if $A \neq B$ and $A \cap B \neq \emptyset$, then there is at least one component in $\mathcal{X}$ made from a single element in $A \cap B$, and at least one component not containing elements in $A \cap B$. Let consider the following example:

Example 2.3. *Let* $A, B \in [20]^7_2$, *where* $A = \{2, 8, 10, 12, 15, 18, 20\}$ *and* $B = \{1, 6, 8, 10, 12, 14, 17\}$. *Thus, we have that* $A \cap B = \{8, 10, 12\}$; $X = A \cup B = \{1, 2, 6, 8, 10, 12, 14, 15, 17, 18, 20\}$; $A \setminus B = \{2, 15, 18, 20\}$; $B \setminus A = \{1, 6, 14, 17\}$; $\mathcal{X} = \{\{20, 1, 2\}, \{6\}, \{8\}, \{10\}, \{12\}, \{14, 15\}, \{17, 18\}\}$, *and* $\overline{\mathcal{X}} = \{\{3, 4, 5\}, \{7\}, \{9\}, \{11\}, \{13\}, \{16\}, \{19\}\}$.

Now, we want to construct two sets $A^*, B^* \in [n]_2$ such that $A^* \subset \overline{A}$, $|A^*| \geq k$, $B \setminus A \subset A^*$, and such that $B^* \subset \overline{B}$, $|B^*| \geq k$, $A \setminus B \subset B^*$. Once sets A^* and B^* constructed, we want to find two subsets $A' \subseteq A^*$ and $B' \subseteq B^*$ such that $A \cap A' = B \cap B' = \emptyset$ and $|A'| = |B'| = k$. Furthermore, we want $A' \cap B' = \emptyset$ or, if this cannot be achieved, we want the intersection to be as small as possible.

Looking at Example 2.3, we start with $\{1, 6, 14, 17\} \subset A^*$ and $\{2, 15, 18, 20\} \subset B^*$. Notice that $5, 7, 13$ and 16 cannot be in A^*, as we want it to be in $[n]_2$. This happens because 6, 12 and 17 are end-vertices in component of $\mathcal{X}$. Similarly $3, 16$, and 19 cannot be in B^*. With that in mind, we can have $3 \in A^*$ and $4 \in B^*$, but then 5 cannot be in either; 7 and 13 can only be in B^*; 16 cannot be in either; 19 can be in A^*. Thus far we have $\{1, 3, 6, 14, 17, 19\} \subset A^*$, and $\{2, 4, 7, 13, 15, 18, 20\} \subset B^*$. As the elements in $A \cap B$ are in neither A^* nor B^*, we have no restrictions for 9 and 11. This means that we can have them in A^* or in B^* (even in both, if it was necessary for both of them to have at least k elements). As right now B^* already has $k = 7$ elements, we can have 9 and 11 in A^*. Now we have $\{1, 3, 6, 9, 11, 14, 17, 19\} \subset A^*$, and $\{2, 4, 7, 13, 15, 18, 20\} \subset B^*$. As both A^* and B^* are in $[n]_2$ and each of them has at least 7 elements, we stop adding elements to them. Finally, we can obtain A' by eliminating any element from A^*, say $A' = \{1, 3, 6, 9, 14, 17, 19\}$, and we can have $B' = B^*$. Because of how we build them, the vertices A, A', B', and B induce a P_4 in $SG(20, 7)$, which means that $dist(A, B) \leq 3$.

Construction of sets A^* *and* B^* *and an upper bound for* $dist(A, B)$

In order to construct sets A^* and B^* corresponding to Example 2.3, we care particularly about the length of the connected components in $\overline{\mathcal{X}}$ and their relation with the end-vertices of components of $\mathcal{X}$. In particular, being able to use all the elements in a connected component $C \in \overline{\mathcal{X}}$ depends on the parity of $|C|$ and on the end-vertices of C. From now on, by Theorem 2.2, we assume that $n = 2k + r$ with $2 \leq r \leq 2k - 3$, that is, $2k + 2 \leq n \leq 4k - 3$ and we assume that $\text{dist}(A, B) \geq 3$.

We say that $\ell \in [n]$ is an *end* if $\ell \in X$ and $|\{\ell - 1, \ell + 1\} \cap \overline{X}| \geq 1$. This is, ℓ is an endpoint if it is in X and at least one of its neighbors in the cycle is in $\overline{X}$. Furthermore, we say that ℓ is an *A-end* if $\ell \in A \setminus B$, a *B-end* if $\ell \in B \setminus A$ and an *H-end* if $\ell \in A \cap B$. Finally, let $e(A)$, $e(B)$ and $e(H)$ be the sets of A-ends, B-ends and H-ends respectively. Notice that $e(H) = A \cap B$, as every vertex in $A \cap B$ must be an end. Let $h = |e(H)|$. Finally, notice that if $A \neq B$ and $A \cap B \neq \emptyset$ then $e(H) \neq \emptyset$ and $e(A) \cup e(B) \neq \emptyset$ (actually, neither $e(A)$ nor $e(B)$ are empty). In Example 2.3, $e(A) = \{2, 15, 18, 20\}$, $e(B) = \{6, 14, 17\}$, and $e(H) = \{8, 10, 12\}$. Notice that $|e(A)| = 4$ and $|e(B)| = 3$, which means that, in general, $e(A)$ and $e(B)$ are not necessarily equal. To obtain the necessary relation between $e(A)$ and $e(B)$, it is helpful to study the structure of the components in $\mathcal{X}$.

We say that a connected component $C \in \mathcal{X}$ is an A-component if $|C \cap e(A)| \geq 1$ and $|C \cap e(B)| = 0$. Notice that this can happen in two different ways, either $|C| \geq 3$ and $|e(A) \cap C| = 2$ or $|C| = 1$ and $e(A) \cap C = C$. Similarly, we say that a connected component $C \in \mathcal{X}$ is a B-component if $|C \cap e(B)| \geq 1$ and $|C \cap e(A)| = 0$. If C is neither an A-component nor a B-component, we say that C is an H-component. Notice that if C is an H-component, then $|C| = 1$ if and only if $C \subset A \cap B$. In this case, we say that C is an H'-component, otherwise, we say that C is an H''-component. In Example 2.3, $\{1, 2, 20\}$ is an A-component, $\{6\}$ is a B-component, and $\{8\}$, $\{10\}$, $\{12\}$, $\{14, 15\}$, and $\{17, 18\}$ are H-components, where $\{8\}$, $\{10\}$ and $\{12\}$ are H'-components, and $\{14, 15\}$ and $\{17, 18\}$ are H''-components. By $n(A)$, $n(B)$, $n(H)$, $n(H')$ and $n(H'')$ we denote the number of A-components, B-components H-components, H'-components and H''-components respectively. Notice that in Example 2.3, $n(A) = n(B)$, which is actually true in general as it is shown in Lemma 2.4.

Lemma 2.4. *The number of A-components equals the number of B-components.*

Next, we obtain a formula relating $e(A)$ and $e(B)$. Notice that $|e(A)|$ is equal to twice the number of A-components of size at least 3, plus the number of A-components of size 1, plus the number of H''-components. We partition $e(A)$ into $e'(A)$ and $e''(A)$, where $e''(A)$ are the elements in $e(A)$ that are in connected components with exactly one element, and $e'(A)$ are the rest. In the same way, partition $e(B)$ into $e'(B)$ and $e''(B)$.

Lemma 2.5. *If* $A, B \in [n]^k_2$ *then,* $|e'(A)| + 2|e''(A)| = |e'(B)| + 2|e''(B)|$.

Next we turn our focus on the connected components in $\overline{\mathcal{X}}$. Now, we call *block* to each element of $\overline{\mathcal{X}}$. Consider the following classifications of a block $[i, j]$:

- *Type I*: if $\{i - 1, j + 1\} \subseteq e(H)$.
- *Type II(A)*: if $i - 1 \in e(H)$ and $j + 1 \in e(A)$.
- *Type II(B)*: if $i - 1 \in e(H)$ and $j + 1 \in e(B)$.
- *Type III(A)*: if $i - 1 \in e(A)$ and $j + 1 \in e(H)$.
- *Type III(B)*: if $i - 1 \in e(B)$ and $j + 1 \in e(H)$.
- *Type IV(A)*: if $\{i - 1, j + 1\} \subset e(A)$.
- *Type IV(B)*: if $\{i - 1, j + 1\} \subset e(B)$.
- *Type IV(H)*: if $[i, j]$ is not of the types above, i.e. if $i - 1 \in e(A)$ and $j + 1 \in e(B)$ or vice versa.

Note that every block is of exactly one type of the types above. Let $\mathcal{T} = \{I, II(A), II(B), III(A), III(B), IV(A), IV(B), IV(H)\}$. We define $n(T)$ as the amount of blocks of type T, for $T \in \mathcal{T}$. In Example 2.3, $\{9\}$ and $\{11\}$ are blocks of type I, there are no blocks of type $II(A)$, $\{13\}$ is a block of type $II(B)$, there are no blocks of type $III(A)$, $\{7\}$ is a block of type $III(B)$, $\{19\}$ is a block of type $IV(A)$, there are no blocks of type $IV(B)$, and $\{3, 4, 5\}$ and $\{16\}$ are blocks of type $IV(H)$.

Notice that if $[i, j]$ is a block, then $i - 1$ and $j + 1$ are ends in some components of $\mathcal{X}$. In such a case, we say that $i - 1$ and $j + 1$ are *connected* to $[i, j]$.

There is an important difference between type IV and the rest of the types. If we are trying to form the sets A^* and B^* as was done for the Example 2.3, and $[i, j]$ is a block of type $T \in \{I, II(A), II(B), III(A), III(B)\}$, then we can use every element in $[i, j]$ because we have restrictions in at most one of $\{i, j\}$ for the sets A^* and B^*. If $[i, j]$ is a block of type $IV(A)$ (or $IV(B)$) of even length, then all but one of the elements of $[i, j]$ can be used, because neither $i + 1$ nor $j - 1$ can be in B^* (A^* resp.). If $[i, j]$ is a block of type $IV(H)$ of odd length, then all but one of the elements of $[i, j]$ can be used, as $i + 1$ and $j - 1$ cannot both be in A^* nor both be in B^*. To reflect the fact that for a block $[i, j]$ of type $T \in \{IV(A), IV(B), IV(H)\}$ we can only assure the use of $\|[i, j]\| - 1$ elements, we define

$$m([i, j]) = \begin{cases} \|[i, j]\| & \text{if } [i, j] \text{ is a block of type } T \in \{I, II(A), II(B), III(A), III(B)\}; \\ \\ \|[i, j]\| - 1 & \text{if } [i, j] \text{ is a block of type } T \in \{IV(A), IV(B), IV(H)\}. \end{cases}$$

Lemma 2.6. $\sum_{[i,j]\in\overline{X}} m([i, j]) \geq n - 3k + 2h + 2$.

Lemma 2.7. $n(II(A)) + n(III(A)) + 2\,n(IV(A)) = n(II(B)) + n(III(B)) + 2\,n(IV(B))$.

We are ready now to construct the sets A^* and B^*. We want these sets to satisfy:

(C1) $A^*, B^* \in [n]_2$;
(C2) $A \setminus B \subseteq B^*$, $B \setminus A \subseteq A^*$;
(C3) $A \cap A^* = \emptyset$, $B \cap B^* = \emptyset$;
(C4) if $[i, j]$ is a block of type $I, II(A), II(B), III(A)$ or $III(B)$, then $[i, j] \subseteq A^* \cup B^*$;
(C5) if $[i, j]$ is a block is of type $IV(A), IV(B), IV(H)$, every element of $[i, j]$ except at most one belongs to $A^* \cup B^*$.

Let us note the links between the last two items and the definition of $m([i, j])$. For each block $[i, j]$ at least $m([i, j])$ elements belong to $A^* \cup B^*$.

We define $Z(i, j)$ the set of vertices in $[i, j]$ at odd distance of $i - 1$ in the n-cycle in clockwise direction and $Y(i, j)$ the set of vertices in $[i, j]$ at even distance of $i - 1$ in the n-cycle in clockwise direction. Besides, let $Z'(i, j)$ be the set of vertices in $[i, j]$ at odd distance of $j + 1$ in the n-cycle in counterclockwise direction and $Y'(i, j)$ the set of vertices in $[i, j]$ at even distance of $j + 1$ in the n-cycle in counterclockwise direction. For instance, if $[4, 10]$ is a block, then $Z([4, 10]) = \{4, 6, 8, 10\} = Z'([4, 10])$ and $Y([4, 10]) = \{5, 7, 9\} = Y'([4, 10])$. On the other hand, if $[4, 9]$ is a block, then $Z([4, 9]) = \{4, 6, 8\} = Y'([4, 9])$, $Y([4, 9]) = \{5, 7, 9\} = Z'([4, 9])$.

Note that $Z(i, j)$ and $Z'(i, j)$ are not empty. Besides, these sets are 2-stable, i.e. they belong to $[n]_2$. Let us assign elements from the blocks to the sets A^* and B^* by the following rules.

$R1$ If $[i, j]$ is of type I with at least two elements, include $Z(i, j)$ in A^* and $Y(i, j)$ in B^*.
$R2$ If $[i, j]$ is of type II(A), include $Z'(i, j)$ in A^* and $Y'(i, j)$ in B^*.
$R3$ If $[i, j]$ is of type II(B), include $Z'(i, j)$ in B^* and $Y'(i, j)$ in A^*.
$R4$ If $[i, j]$ is of type III(A), include $Z(i, j)$ in A^* and $Y(i, j)$ in B^*.
$R5$ If $[i, j]$ is of type III(B), include $Z(i, j)$ in B^* and $Y(i, j)$ in A^*.
$R6$ If $[i, j]$ is of type IV(A), include $Z(i, j)$ in A^* and $Y(i, j) \setminus \{j\}$ in $B*$.
$R7$ If $[i, j]$ is of type IV(B), include $Z(i, j)$ in B^* and $Y(i, j) \setminus \{j\}$ in A^*.
$R8$ If $[i, j]$ is of type IV(H), with $i - 1 \in A \setminus B$ and $j + 1 \in B \setminus A$ include $Z(i, j) \setminus \{j\}$ in A^* and $Y(i, j)$ in B^*. If $i - 1 \in B \setminus A$ and $j + 1 \in A \setminus B$ include $Z(i, j) \setminus \{j\}$ in B^* and $Y(i, j)$ in A^*.

Notice that in rules $R1$-$R8$ the elements in blocks of the form $[i, i]$ of type I are not assigned. These elements play a key role that we will mention further. Hence, we define the set I' as the set of such elements, i.e. $I' = \{i \,|\, [i, i] \text{ is a block of type I}\}$.

Consider the sets A^* and B^* constructed following the rules above, and also including to A^* the elements in $B \setminus A$ and assigning to B^* the elements in $A \setminus B$. It is not hard to see that A^* and B^* satisfy:

- $A^*, B^* \in [n]_2$;
- $A \cap A^* = A^* \cap B^* = B \cap B^* = \emptyset$;
- $|A^* \cap B| = |B^* \cap A| = k - h$;
- for every block of type $T \in \{II(A), III(A), IV(A)\}$, $|A^* \cap T| \geq 1$;
- for every block of type $T \in \{II(B), III(B), IV(B)\}$, $|B^* \cap T| \geq 1$; and

- for every block $[i, j]$ of type I with at least two elements, $|A^* \cap I| \geq 1$ and $|B^* \cap I| \geq 1$.

From sets A^* et B^* we can construct two vertices $A' \subset A^*$ and $B' \subset B^*$ in $[n]_2^k$ as follows.

Lemma 2.8. *Let $A, B \in [n]_2^k$ with $|A \cap B| = h$. Then, there exist $A', B' \in [n]_2^k$ such that $|A' \cap B'| \leq h - 1$ and $A \cap A' = B \cap B' = \emptyset$.*

Proof. In order to obtain vertices $A' \subset A^*$ and $B' \subset B^*$ such that $|A'| = |B'| = k$, $A \cap A' = B \cap B' = \emptyset$, and $|A' \cap B'| \leq h - 1$, we will use the elements of I'.

First, notice that for every element $i \in A \cap B$ the block of the form $[i + 1, j]$ is a block of type $T \in \{I, II(A), II(B)\}$, thus $h = n(I) + n(II(A)) + n(II(B))$. Similarly, for every element $j \in A \cap B$ the block of the form $[i, j - 1]$ is a block of type $T \in \{I, III(A), III(B)\}$, thus $h = n(I) + n(III(A)) + n(III(B))$. Then

$$2h = 2n(I) + n(II(A)) + n(II(B)) + n(III(A)) + n(III(B)).$$

It follows that $n(II(A)) + n(III(A)) + n(II(B)) + n(III(B))$ is even, so let $s \in \mathbb{N}$ such that $2s = n(II(A)) + n(III(A)) + n(II(B)) + n(III(B))$. Then, $2h = 2n(I) + 2s$, which means $n(I) = h - s$. Furthermore, assume w.l.o.g. that $n(II(A)) + n(III(A)) = s + t \geq n(II(B)) + n(III(B)) = s - t$, for some $t \in \mathbb{N}$. By Lemma 2.7,

$$2n(IV(B)) \geq n(II(A)) + n(III(A)) - n(II(B)) - n(III(B)) = s + t - (s - t) = 2t.$$

Thus, $IV(B) \geq t$. Hence, $n(II(A)) + n(III(A)) + n(IV(A)) \geq s$ and $n(II(B)) + n(III(B)) + n(IV(B)) \geq s$. Therefore, from Rules $R2 - R7$, we have A^* has at least s elements from blocks of types $II(A), III(A)$ and $IV(A)$, and B^* has at least s elements from blocks of types $II(B), III(B)$ and $IV(B)$.

Let r be the amount of blocks $[i, j]$ of type I with at least two elements. Then from previous remarks and rule $R1$, both A^* and B^* have at least r elements from these blocks. Furthermore, $|I'| = h - s - r$.

Counting again the size of A^*, we have

- A^* has $k - h$ elements from B;
- A^* has at least s elements from blocks of types $II(A), III(A)$ and $IV(A)$;
- A^* has at least r elements from blocks of types I with at least two elements.

Thus, $|A^*| \geq k - h + s + r = k - (h - s - r)$. This means that if we assign every element in I' to A^*, then $|A^*| \geq k$. Similarly, if we assign every element in I' to B^*, then $|B^*| \geq k$. This also yields $|A^* \cap B^*| \leq h - s - r$.

Notice that there must exist at least one block of type T for some $T \in \{II(A), II(B), III(A), III(B)\}$, as otherwise only blocks of type I would exists, which implies $A = B$. Hence $s \geq 1$, and $h - s - r \leq h - 1$. Therefore, taking $A' \subset A^*$ and $B' \subset B^*$, with $|A'| = |B'| = k$, we have that $A \cap A' = B \cap B' = \emptyset$ and $|A' \cap B'| \leq |A^* \cap B^*| \leq h - s - r \leq h - 1$ which proves the result. □

The following result derives directly from the proof of Lemma 2.8.

Corollary 2.9. *If there are no blocks $[i, i]$ of type I, then $dist(A, B) \leq 3$.*

Lemma 2.10. *Let $A, B \in [n]_2^k$ with $|A \cap B| = h$. Then, $dist(A, B) \leq 1 + 2h$.*

Proof. By applying Lemma 2.8 h times, we obtain two vertices $A^{(h)}, B^{(h)} \in [n]_2^k$, with $A^{(h)} \cap B^{(h)} = \emptyset$. Hence, $dist(A, B) \leq 1 + 2h$. □

Corollary 2.11. *Let $A, B, Y \in [n]_2^k$ with $|A \cap Y| = h'$, $|Y \cap B| = h''$, and let $h^* = h' + h''$ with $h^* \geq 2$. Then, $dist(A, B) \leq 2 + 2h^*$.*

Proof. The result follows by applying Lemma 2.10 to bound $dist(A, Y)$ and $dist(Y, B)$. □

Lemmas 2.6 and 2.10 and Corollaries 2.9 and 2.11 will be used to compute the diameter of SG(n, k) when $2k + 2 \leq n \leq 4k + 3$ in the next subsections.

2.2. Case $3k-2 \le n \le 4k-3$

Let $n = 2k + r$ with $k > 2$ and $k-2 \le r \le 2k-3$. Let us consider the construction of sets A^* and B^* given in Section 2.1. Let remark that in the proof of Lemma 2.8, we may not need to assign every element in I' to both A^* and B^*. Note that following the rules $R2$-$R8$, from each block $[i, j]$ of type II, III or IV, we have included at least $m([i, j])$ elements in $A^* \cup B^*$ such that $A^* \cap B^* = \emptyset$ and $A^*, B^* \in [n]_2$. If we do not assign the $h-s-r$ elements in I', by Lemma 2.6, we have assigned at least $n-3k+2h+2-(h-s-r)$ elements from blocks to $A^* \cup B^*$, $k-h$ elements from A and $k-h$ elements from B. This means that before assigning the $h-s-r$ elements in I', we have assigned $n-k+2-(h-s-r)$ elements to $A^* \cup B^*$. Hence, if $n-k+2$ is large enough, we may be able to assign the elements in such blocks maintaining $A^* \cap B^* = \emptyset$.

Assume $n \ge 3k-2$. Then, before assigning the elements in I', we have assigned at least $n-k+2-(h-s-r) \ge 2k-h+s+r$ to $A^* \cup B^*$, at least $k-h+s+r$ elements to A^*, and at least $k-h+s+r$ elements to B^*. Let $0 \le a \le h-s-r$ and assume that we have assigned $k-h+s+r+a$ elements to A^*. This means that we assigned at least $2k-h+s+r-(k-h+s+r+a) = k-a$ elements to B^*. If $h-s-r-a > 0$, assign that many elements from I' to A^* and the remaining a elements to B^*, otherwise, if $h-s-r-a = 0$, assign every element in I' to B^*. Then $|A^*| \ge k$, $|B^*| \ge k$, and $A^* \cap B^* = \emptyset$. Let $A' \subset A^*$ and $B' \subset B^*$, such that $|A'| = |B'| = k$. Then we have $A \cap A' = A' \cap B' = B' \cap B = \emptyset$, which means that $dist(A, B) \le 3$. As by hypothesis, $n < 4k-2$ then, by Theorem 2.2, we deduce that $\text{dist}(A, B) \ge 3$.

Theorem 2.12. *Let $n = 2k + r$ with $k > 2$ and $k-2 \le r \le 2k-3$. Then,* $D(SG(n, k)) = 3$.

Notice that in Rules $R1-R8$ we have not assigned elements in blocks of type $IV(H)$ of the form $[i, i]$.

Observation 4. *Let $k > 2$ and $3k-2 \le n \le 4k-3$. If two vertices A, B are at distance 3, there exist two vertices A', B' constructed following the rules $R1$-$R8$ such that $\{A, A', B', B\}$ induce a P_4 in* $SG(n, k)$*. Besides, if $[i, i]$ is a block of type $IV(H)$, $i \notin A' \cup B'$.*

2.3. Case $2k+2 \le n \le 3k-3$

In this section, we show that $D(SG(2k + r, k)) \le k - r + 1$ when $2 \le r \le k-3$, or equivalently, $D(SG(3k-2-m, k)) \le 3 + m$ for $1 \le m \le k-4$. To do this, given two vertices $A, B \in [n]^k_2$, we use two different operations that yield sets $\tilde{A}$ and $\tilde{B}$ in $[n+1]^k_2$. We apply the operations successively until we obtain two vertices $A^p, B^p \in [n+p]^k_2$ with $dist(A^p, B^p) \le 3$ in $SG(n+p, k)$. If $dist(A^p, B^p) = 2$ in $SG(n+p, k)$, we obtain a vertex $Y \in [n]^k_2$ such that $dist(A, Y) + dist(B, Y) \le m+3$. If $dist(A^p, B^p) = 3$ in $SG(n+p, k)$, we obtain two vertices $A', B' \in [n]^k_2$ using Rules $R1$-$R8$ such that $A \cap A' = B \cap B' = \emptyset$, and $dist(A', B') \le 1+m$.

Now we can begin describing the operations that yield sets in $[n+1]^k_2$. The first operation will work by adding an element in a component $C \in \mathcal{X}$, with $|C| \ge 3$; the second operation will work by adding an element to a block $[t, t]$ of type I. Because we are going to be talking about distances in Schrijver graphs with different values of n, we denote $dist_n(A, B)$ the distance between A and B in $SG(n, k)$.

Let $A, B \in [n]^k_2$ such that $dist_n(A, B) \ge 3$ and $X = A \cup B$. From item (iii) in Observation 3, there exist $C \in \mathcal{X}$ such that $|C| \ge 3$. Consider $C = a_i\, b_j\, a_{i+1} \ldots$ (first case) or $C = b_j\, a_i\, b_{j+1} \ldots$ (second case). To obtain sets in $[n+1]^k_2$ we will add an extra element between the second and third elements in C, i.e. between b_j and a_{i+1} in the first case, and between a_i and b_{j+1} in the second case. Hence, we assign to A and B (in any case) the following sets in $[n+1]$, $A^+ = \{a_1, \ldots, a_i, a_{i+1}+1, \ldots, a_k+1\}$ and $B^+ = \{b_1, \ldots, b_j, b_{j+1}+1, \ldots, b_k+1\}$. Notice that $|A^+| = |B^+| = k$, as we did not increase the amount of elements. Furthermore, the sets are 2-stable, because A and B are 2-stable (a_1 and a_k+1 cannot be consecutive, because that would imply that $a_1 = 1$ and $a_k+1 = n+1$, and $a_k = n$). Therefore, $A^+, B^+ \in [n+1]^k_2$.

By adding this new element, we formed a new block. Observe that if $C = a_i\, b_j\, a_{i+1} \ldots$, then $a_{i+1} \notin A^+ \cup B^+$, and if $C = b_j\, a_i\, b_{j+1}$, then $b_{j+1} \notin A^+ \cup B^+$. Thus, in the first case, $[a_{i+1}, a_{i+1}]$ is a block of type $IV(H)$ in $\overline{\mathcal{X}}$ with $X = A^+ \cup B^+$. Analogously, in the second case, $[b_{j+1}, b_{j+1}]$ is a block of type $IV(H)$ in $\overline{\mathcal{X}}$.

Concerning the inverse operation of operation +, for $Y \in [n+1]_2^k$, we define a set $Y^- = \{y_1^-, \ldots, y_k^-\} \in [n]^k$ by deleting the element that we added. In other words, considering $u = a_{i+1}$ if we are in the first case and $u = b_{j+1}$ if we are in the second case, we have that $y_r^- = y_r$ if $y_r < u$ and $y_r^- = y_r - 1$ if $y_r \geq u$.

The following remark is straightforward from the previous definitions.

Observation 5. *If $Y \cap A^+ \cap B^+ = \emptyset$ then $Y^- \cap A \cap B = \emptyset$.*

Notice that Y^- is not 2-stable if and only if $u-1, u+1 \in Y$ or if $u-2, u \in Y$. But if $X = A^+ \cup B^+$, then $\{u-2, u-1\}$ is a connected component of $\mathcal{X}$, and $u+1$ is the first element (in clockwise direction) in a connected component in $\mathcal{X}$.

Observation 6. *Let $Y \in [n+1]_2^k$ and suppose that for every element $v \in Y \cap (A^+ \cup B^+)$, v is not the first element of a connected component $C \in \mathcal{X}$. Then $\{u-2, u+1\} \cap Y = \emptyset$ and $Y^- \in [n]_2^k$.*

Observation 6 is stated in such a convoluted way to make easier the proof of the main theorem in this section, which uses successive applications of the operation +, together with a second operation which is defined later in this section.

As Observation 6 assures that $\{u-2, u+1\} \cap Y = \emptyset$, we can study the relation between $Y^- \cap A$ and $Y \cap A^+$, and similarly with B, to obtain the following.

Observation 7. *Let $Y \in [n+1]_2^k$ such that $u-2$ and $u+1$ do not belong to Y. If Y^- is defined as above, then $Y^- \in [n]_2^k$ and $|Y \cap A^+| + |Y \cap B^+| = |Y^- \cap A| + |Y^- \cap B|$ if $u \notin Y$, or $|Y \cap A^+| + |Y \cap B^+| + 1 = |Y^- \cap A| + |Y^- \cap B|$ if $u \in Y$. Furthermore, if no element $v \in Y \cap (A^+ \cup B^+)$ is the first element in a connected component C of $A^+ \cup B^+$, then no element $v \in Y^- \cap (A \cup B)$ is the first element in a connected component C of $A \cup B$.*

Notice that if $u-2, u \in A$, and $Y \cap A^+ = \emptyset$ then $u-2, u+1 \notin Y$ and $Y^- \cap A = \emptyset$. On the other hand, if $u-1 \in A$, $Y \cap A^+ = \emptyset$ and $u \notin Y$, then $u-1, u \notin Y$ and $Y^- \cap A = \emptyset$. This yields the following.

Observation 8. *Let $Y \in [n+1]_2^k$. If $Y \cap A^+ = \emptyset$ and $u \notin Y$, then $Y^- \in [n]_2^k$ and $A \cap Y^- = \emptyset$.*

Finally, let $Y_1, Y_2 \in [n+1]_2^k$ and consider $Y_1^- \cap Y_2^-$. If $\{u-1, u\} \not\subset Y_1 \cup Y_2$, then $y \in Y_1 \cap Y_2$ if and only if $y^- \in Y_1^- \cap Y_2^-$ (where y^- is as in the definition of Y^-). Hence, we have the following

Observation 9. *Let $Y_1, Y_2 \in [n+1]_2^k$. If $u \notin Y_1 \cup Y_2$, then $|Y_1^- \cap Y_2^-| = |Y_1 \cap Y_2|$.*

We are ready to introduce the second operation. Notice that if $dist_n(A, B) \geq 4$, Corollary 2.9 implies that there exists a block $[t, t]$ of type I. The operation will work by adding an extra element at position $t+1$, thus increasing the size of the block. To be more precise, we define an operation on A and B, denoted $\uparrow$, by assigning the following two sets $A^\uparrow = \{a_1^\uparrow, \ldots, a_k^\uparrow\}$ and $B^\uparrow = \{b_1^\uparrow, \ldots, b_k^\uparrow\}$ in $[n+1]_2^k$ as follows:

$$a_i^\uparrow = \begin{cases} a_i & \text{if } a_i \leq t-1; \\ a_i + 1 & \text{if } a_i \geq t+1. \end{cases} \quad \text{and} \quad b_i^\uparrow = \begin{cases} b_i & \text{if } b_i \leq t-1; \\ b_i + 1 & \text{if } b_i \geq t+1. \end{cases}$$

Note that, if $X = A^\uparrow \cup B^\uparrow$ ($X \subseteq [n+1]$), then $[t, t+1]$ is a block of type I in $\overline{\mathcal{X}}$.

We define now the inverse operation of operation $\uparrow$. Given $Y \in [n+1]_2^k$, we define a set $Y^\downarrow = \{y_1^\downarrow, \ldots, y_k^\downarrow\} \in [n]^k$ as follows:

$$y_r^\downarrow = \begin{cases} y_r & \text{if } y_r \leq t; \\ y_r - 1 & \text{if } y_r \geq t+1. \end{cases}$$

As with the operation +, we care about when $Y^\downarrow$ is 2-stable, and about the relation between the intersections of Y with $A^\uparrow$ and $B^\uparrow$, and the intersection of $Y^\downarrow$ with A and B. Notice that if $Y \cap [A^\uparrow \cup B^\uparrow] = \emptyset$, $Y^\downarrow \in [n]_2^k$. Moreover, we have the following result.

Lemma 2.13. *Let $A, B \in [n]^k_2$ such that there exists a block $[t, t]$ of type I in $\mathcal{X}$ with $X = A \cup B$. Consider $A^{\uparrow}$ and $B^{\uparrow}$ defined as above. Let $Y \in [n+1]^k_2$ such that $Y \cap A^{\uparrow} \cap B^{\uparrow} = \emptyset$ and $Y^{\downarrow}$ defined as above. Then $Y^{\downarrow} \in [n]^k_2$, $Y^{\downarrow} \cap A \cap B = \emptyset$ and $|Y \cap A^{\uparrow}| + |Y \cap B^{\uparrow}| = |Y^{\downarrow} \cap A| + |Y^{\downarrow} \cap B|$.*

Proof. Consider $[t, t+1]$ the block of type I defined as above. Since $Y \in [n+1]^k_2$, $|Y \cap \{t, t+1\}| \leq 1$. In addition, from the fact that $Y \cap A^{\uparrow} \cap B^{\uparrow} = \emptyset$, $t-1$ and $t+2$ do not belong to Y. Thus $t-1$ and $t+1$ do not belong to $Y^{\downarrow}$. Therefore, $Y^{\downarrow}$ belongs to $[n]^k_2$, $Y^{\downarrow} \cap A \cap B = \emptyset$ and $|Y \cap A^{\uparrow}| + |Y \cap B^{\uparrow}| = |Y^{\downarrow} \cap A| + |Y^{\downarrow} \cap B|$. □

In particular, the following result follows immediately from Lemma 2.13 if $Y \cap (A^{\uparrow} \cup B^{\uparrow}) = \emptyset$.

Corollary 2.14. *Let $[t, t]$ be a block of type I in $\mathcal{X}$ with $X = A \cup B$. If $dist_{n+1}(A^{\uparrow}, B^{\uparrow}) = 2$ then $dist_n(A, B) = 2$.*

Let $Y_1, Y_2 \in [n+1]^k_2$ and consider $Y_1^{\downarrow} \cap Y_2^{\downarrow}$. If $\{t, t+1\} \not\subset Y_1 \cup Y_2$, then $y \in Y_1 \cap Y_2$ if and only if $y^{\downarrow} \in Y_1^{\downarrow} \cap Y_2^{\downarrow}$ (where $y^{\downarrow}$ is as in the definition of $Y^{\downarrow}$). If $\{t, t+1\} \subset Y_1 \cup Y_2$, then $t \in Y_1^{\downarrow} \cap Y_2^{\downarrow}$. Hence we have the following.

Observation 10. *If $Y_1, Y_2 \in [n+1]^k_2$, then $|Y_1^{\downarrow} \cap Y_2^{\downarrow}| \leq |Y_1 \cap Y_2| + 1$.*

Starting from $A^0 = A$ and $B^0 = B$ and by applying repeatedly the operations $+$ and $\uparrow$, we are able to construct two vertices $A^p, B^p \in [n+p]^k_2$, with $p \leq m$, such that $\text{dist}_{n+p}(A^p, B^p) \leq 3$. Then, by applying repeatedly the operations $-$ and $\downarrow$, we obtain the following result.

Theorem 2.15. *Let $n = 3k - 2 - m$ with $1 \leq m \leq k - 4$. Then,* $\text{D}(\text{SG}(n, k)) \leq m + 3$.

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