Contraction analysis of nonlinear systems and its application

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The research described in this dissertation has been carried out at the Faculty of Science and Engineering (FSE), University of Groningen, The Netherlands, within the Engineering and Technology institute Groningen.

The research described in this dissertation is part of the research program of the Dutch Institute of Systems and Control (DISC). The author has successfully completed the educational program of the Graduate School DISC.

Contraction analysis of nonlinear systems and its application

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus Prof. Jacquelien Scherpen
and in accordance with
the decision by the College of Deans.

This thesis will be defended in public on

Monday 08 January 2024 at 9.00 hours

by

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To my family.

Acknowledgment

I am deeply grateful to everyone who has played a significant role in the completion of this PhD thesis. Their support, encouragement, and contributions have been invaluable.

First and foremost, I would like to express my sincere gratitude to my supervisor, Prof. dr. Bayu Jayawardhana, and Prof. dr. Stephan Trenn, for their unwavering guidance, patience, and expertise throughout this entire research endeavor. Their insightful feedback and constant encouragement have been instrumental in shaping the direction of this work. I would like to extend special thanks to Prof. dr. Jacquelien Scherpen for her valuable insights and continuous support.

I am indebted to the faculty members of the DTPA at University of Groningen, who not only imparted their knowledge but also challenged and inspired me to pursue this research path.

I extend my thanks to the staff and personnel of the University of Groningen, who have tirelessly assisted me with administrative matters, technical support, and resources throughout my academic tenure.

I would also like to acknowledge the financial support provided by China Scholarship Council (CSC), which made it possible for me to pursue my PhD and focus on my research.

To my friends and family, thank you for your unwavering support, understanding, and encouragement during the highs and lows of this journey. Your belief in me has been my driving force.

Finally, I wish to express my heartfelt dedication of this thesis to my esteemed friends and colleagues - Arijit Sarkar, Amin Maghami, Bo Jin, Bahadir Saltik, Feng Huang, Jin Chen, Matthijs de Jong, Ningbo Li, Rodolfo Reyes-Báez, Santiago Sanchez-Escalonilla Plaza, Sutrisno Sutrisno, and Xiupeng Chen. Their unwavering commitment to fostering a delightful and supportive office ambiance, as well as their invaluable contributions throughout our collaborative brainstorming sessions, have been instrumental in shaping this research.

In conclusion, this thesis would not have been possible without the collec-

tive contributions of the aforementioned individuals and many others who have touched my life in one way or another. I am profoundly grateful for all the help and encouragement I have received.

<div align="right">
Hao Yin
Groningen
December 12, 2023
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Contents

1

Introduction

Contraction analysis, as a mathematical framework, is utilized for investigating the convergence behavior of dynamical systems. It uses tools from linear systems analysis to understand the asymptotic behavior of nonlinear systems. Contrasting with the traditional Lyapunov stability analysis, contraction analysis places emphasis on the convergence properties of trajectories, rather than solely focusing on a particular equilibrium point. Nowadays, contraction analysis has major progresses in ordinary differential equation (ODE) systems, and these progresses have found wide applications, such as, network synchronization [20], learning-based control [83], and convex optimization [14]. However, there remain opportunities for analysis the contraction properties of specific systems, including switched systems and differential-algebraic equation (DAE) systems. This thesis aims to address these gaps by investigating the contractivity of switched systems and DAE systems. Additionally, we apply our proposed approach to various areas, such as, stability analysis, observer design, and synchronization problems. In this chapter, we briefly revisit the concept of contraction analysis for ODE systems, and provide a sketch of the main contributions. Some notations used throughout the thesis are also presented.

1.1 Background

Lyapunov stability is a fundamental concept in the analysis of dynamical systems and plays a crucial role in determining the behavior and equilibrium points of these systems. The concept is used to analyze the stability of a system's equilibrium points over time. In the context of Ordinary Differential Equations (ODEs) or difference equations, an equilibrium point (often denoted as x_e) of a dynamical system is considered Lyapunov stable if, for any small positive number ε, there exists a corresponding positive number δ such that if the initial condition of the system is within a distance δ from the equilibrium point (i.e., $\|x(0) - x_e\| < \delta$), then the trajectory of the system will always remain within a distance ε of the equilib-

rium point for all future time (i.e., $\|x(t) - x_e\| < \varepsilon, \forall t \geqslant 0$). An equilibrium point is said to be *exponentially stable* if, in addition to being Lyapunov stable, the distance between the solutions and the equilibrium point decreases exponentially as time progresses. In other words, the rate of convergence towards the equilibrium is exponential. The exponential stability characteristic holds significant appeal in numerous real-world applications due to its ability to ensure rapid convergence of a system towards its equilibrium point. This asymptotic behavior plays a vital role in guaranteeing stability and resilience across diverse engineering and scientific domains, including control systems, robotics, economics, and ecology. Neverthe-less, in specific scenarios like time-varying optimization [69] and time-varying Nash equilibrium seeking [97], the system's state tends to approach a common trajectory instead of a fixed equilibrium point. Therefore, it becomes essential to study the relationship between any pair of solutions of the system.

As a generalized concept of exponentially stable, contraction (also know as ex-ponential incremental stability) states that for a given nonlinear system, regardless of the initial conditions, all trajectories converge exponentially towards a single trajectory [55]. Consider the following time-varying nonlinear system:

$$\dot{x}(t) = f(t, x(t)) \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the vector field. Consider a smooth function $f(t, x(t))$ with an initial condition $x(0) = x_0$. The function's smoothness guarantees the presence of a locally unique solution to equation (1.1) [44]. We define system (1.1) as a *contractive system* if there exist positive constants c and α, such that for any solutions $x_1(t)$ and $x_2(t)$ of (1.1), the following condition holds:

$$\|x_1(t) - x_2(t)\| \leqslant ce^{-\alpha t} \|x_1(0) - x_2(0)\|, \quad \forall t \geqslant 0. \tag{1.2}$$

The study of contraction analysis can be dated back to 1940s, contraction mappings in dynamical systems have been studied extensively in [48]. In [55], Lohmiller and Slotine bring contraction analysis back into focus by applying it to control theory. For the past two decades, contraction analysis has been intensively studied in numerous publications. Forni and Sepulchre [27] provide an analog theorem for contraction analysis by lifting the Lyapunov function to the tangent bundle. The approach proposed in [6], known as the incremental Lyapunov approach, establishes incremental stability by verifying a pointwise geometric condition within the product of the state space. In addition to the Lyapunov approach, Sontag [80] employs matrix measure, also known as logarithmic norm, to directly characterize the contraction properties. In [4], the authors studied transverse exponential stability, which can be considered as a generalized concept

of incremental exponential stability by using nonlinear Rieammanian metrics.
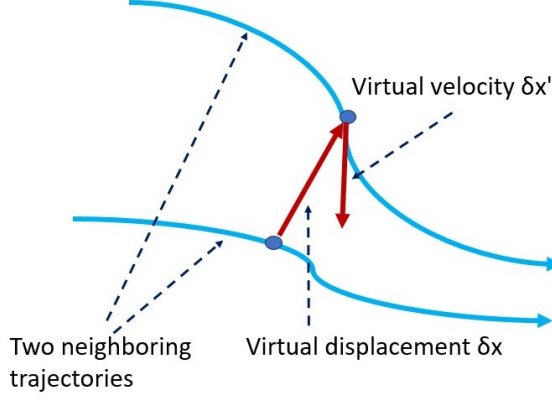
Figure 1.1: The plot of two neighboring trajectories.

One of the fundamental aspects of contraction theory is the utilization of the concept of virtual displacements [55], denoted as δx, which represent infinitesimal state displacements at a fixed time. The virtual displacement δx between two neighboring trajectories is illustrated in Fig. 1.1. Subsequently, the virtual system is introduced

$$\delta \dot{x} = \frac{\partial f}{\partial x}(t, x(t))\delta x, \tag{1.3}$$

where $x(t)$ is the solution of (1.1). The concept of infinitesimal state displacements δx, can be regarded as the differential of the trajectory $x(t) := \phi(x_0, t)$ with respect to the initial condition x_0, i.e. $\delta x := \frac{\mathrm{d}\phi(x_0,t)}{\mathrm{d}x_0}$. In this context, equation (1.3) represents the dynamics of $\frac{\mathrm{d}\phi(x_0,t)}{\mathrm{d}x_0}$, illustrating how variations in the initial condition x_0 influence the trajectory $x(t)$.

Defining $\delta x^\top \delta x$ as the squared distance between the two trajectories, we can derive its rate of change from (1.3) as:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x^\top \delta x) = 2\delta x^\top \delta \dot{x} = 2\delta x^\top \frac{\partial f}{\partial x}\delta x. \tag{1.4}$$

Let $\lambda_{\max}(t, x)$ represent the largest eigenvalue of the symmetric part of the Jacobian $\frac{\partial f}{\partial x}(t, x(t))$ i.e., the largest eigenvalue of $\frac{1}{2}(\frac{\partial f}{\partial x}^\top + \frac{\partial f}{\partial x})$. Consequently, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x^\top \delta x) \leqslant 2\lambda_{\max}\delta x^\top \delta x, \tag{1.5}$$

and hence,

$$\|\delta x\| \leqslant \|\delta x_0\| \, e^{\int_0^t \lambda_{\max}(t,x)\mathrm{d}t}. \tag{1.6}$$

Let's assume that $\lambda_{\max}(t,x)$ is uniformly and strictly negative, i.e., $\exists \beta > 0, \forall x, \forall t > 0, \lambda_{\max}(t,x) \leqslant -\beta < 0$. In this case, according to equation (1.6), any infinitesimal length $\|\delta x\|$ exponentially converges to zero. Through path integration, we can immediately infer that the length of any finite path between $x_1(t)$ and $x_2(t)$ also exponentially converges to zero. By introducing a virtual displacement δx and a virtual system (1.3), Forni et al. [27] employed the Finsler-Lyapunov function to examine the exponential incremental stability of the system. Sufficient condition is obtained.

Lemma 1.1. *[Theorem 2.1 in [27]] If there exists a Finsler-Lyapunov function $V(x, \delta x)$ such that*

$$c_1 \left\| \delta x \right\|_x^p \leqslant V(x, \delta x) \leqslant c_2 \left\| \delta x \right\|_x^p$$

and

$$\frac{\partial V(x, \delta x)}{\partial x} f(t, x) + \frac{\partial V(x, \delta x)}{\partial \delta x} \frac{\partial f(t, x)}{\partial x} \delta x \leqslant -\lambda V(x, \delta x) \tag{1.7}$$

hold for all $(x, \delta x) \in T\mathcal{M}$, where $F : T\mathcal{M} \to \mathbb{R}_{\geqslant 0}$ is a Finsler structure. Then, the system (1.1) *is exponentially incrementally stable.*

One additional fundamental aspect of contraction theory involves transferring the distance between two trajectories to the Euclidean point-to-set distance [6], i.e., $|x|_{\mathcal{A}} = \inf_{z \in \mathcal{A}} |x - z|$. The primary concept involves examining the set stability of a combined system, referred to as the auxiliary system, which consists of the original system and a copy of itself. In [6], it is demonstrated that the distance between each pair of trajectories of

$$\dot{x} = f(d, x) \tag{1.8}$$

is equivalent to the distance of the auxiliary system

$$\begin{cases} \dot{x}_1 = f(d, x_1) \\ \dot{x}_2 = f(d, x_2) \end{cases} \tag{1.9}$$

to the set

$$\Delta = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{2n} : \exists x \in \mathbb{R}^n : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix} \right\}, \tag{1.10}$$

where d is the input. In other words, the incremental stability of (1.8) and the stability of the auxiliary system (1.9) to the set (1.10) are essentially the same. The existence of a continuous function $V(x_1, x_2)$, $\mathcal{K}_\infty$ function α_1, α_2, and *positive definite* function α (the definition of which will be introduced subsequently), satisfying

$$\alpha_1(|x_1 - x_2|) \leqslant V(x_1, x_2) \leqslant \alpha_2(|x_1 - x_2|),$$

and

$$\frac{\partial V(x_1, x_2)}{\partial x_1} f(d, x_1) + \frac{\partial V(x_1, x_2)}{\partial \delta x_2} f(d, x_2) \leqslant -\alpha(|x_1 - x_2|)$$

provides both sufficient and necessary conditions for ensuring incremental global asymptotic stability.

Apart from the Lyapunov approach, Sontag [80] studies infinitesimal contraction of a vector field by means of matrix measure

$$\mu_p(A) := \lim_{h \to 0^+} \frac{||I + hA||_p - 1}{h}.$$

In particular, if there exists a strictly positive real number $\lambda > 0$, such that

$$\mu_p(\frac{\partial f(t, x)}{\partial x}) \leqslant -\lambda, \tag{1.11}$$

then the system (1.1) exhibits infinitesimal contractivity. Matrix measures enable us to analyze the contractivity of the system using non-Euclidean Norms, i.e., 1-norm or ∞-norm.

Based on the previous observations, it is clear that the contractivity of the system is highly correlated with the characteristics of the Jacobian matrix associated with the system's vector field (e.g., (1.3), (1.7), (1.11)). The Jacobian matrix of an ODE system typically takes the form of a square matrix, which represents a relatively straightforward structure. However, in certain cases, the Jacobian matrix can exhibit a more complex structure. For instance, in a DAE system, the Jacobian matrix is a rectangular matrix. It consists of rows equal to the total number of differential equations and columns equal to the total number of variables, encompassing both differential and algebraic quantities. In the context of a switched system, the size and structure of the Jacobian matrix rely on the number of state variables and the specific dynamics associated with each mode or subsystem. Since the system switches between different modes, the Jacobian matrix may have different values and structures depending on the current active mode. Due to these factors, analyzing the characteristics of the Jacobian matrix associated with DAE systems and switched systems is more complex than ODE systems. In other words, contraction analysis of DAE systems and switched systems remain a challenging task.

In the past few years, contraction theory has gained extensive application in addressing various control problems. A new approach to design globally convergent reduced-order observers for nonlinear control systems via contraction analysis and convex optimization was proposed in [98]. In [86], the authors employed virtual and horizontal contraction methodologies to reformulate the Immersion and Invariance Stabilization approach. In [20], the authors expand the

application of contraction theory from smooth dynamical systems to a broader class of piecewise smooth dynamical systems, offering a tool for analyzing network synchronization problem. The fundamental concept behind these applications involves constructing a contractive system that includes the desired trajectory within its set of trajectories. From the analysis performed before, it is obvious that the application of contraction theory to address control problems involving DAE systems or switched systems is not a straightforward task due to the inherent complexities in constructing contractive DAE systems or contractive switched systems.

1.2   Problem statements

In the context of this thesis, our primary attention will be analyzing the stability and contractivity aspects of both switched systems and DAE systems. Additionally, a significant emphasis will be placed on the applications of contraction theory. Within the framework of switched systems, recent literature such as [20, 26, 57, 71, 94] has introduced contraction analysis. Notably, all the aforementioned works assume the contractivity of all subsystems. However, delving into the analysis of contractivity for switched systems where some subsystems are non-contracting presents a challenge. Such scenarios lead to trajectories diverging from each other within each dwell time interval. Building upon the understanding that a switched system featuring entirely unstable modes can attain asymptotic stability through appropriate switching signals, the first research problem in this thesis is formulated as follows:

Problem 1.1. *How to establish the contractivity of a particular class of switched systems, which exhibit a combination of both contracting and non-contracting modes, by using appropriate switching signals?*

Regarding DAE systems, recent work on contraction analysis has been presented in [62]. In this paper, the authors demonstrated that under certain sufficient conditions on algebraic equations, the contractivity of time-invariant DAE systems can be established by performing exponential stability analysis on the associated reduced variational ODE systems. However, this methodology might have limitations and may not be applicable to time-varying DAE systems. This is primarily due to the fact that the boundedness requirement for the algebraic equation, as presented in [62], is not met by time-varying DAE systems. Therefore, another question is the following.

Problem 1.2. *Can a sufficient condition be offered to ensure the contractivity of time-varying DAE systems, particularly when there is a lack of prior information about the specific DAE system?*

Contraction theory finds its application in analyzing the stability of systems possessing a single equilibrium point, as this equilibrium lies within the trajectory set of the system. However, when dealing with switched systems that lack a common equilibrium point, the contraction theory cannot be applied because it is not possible to achieve contractivity in such systems. In these systems, it has been demonstrated that trajectories tend to approach a set as opposed to a precise equilibrium point. The characteristic of converging sets has been examined and approximated in [2, 24, 84] under the condition that all subsystems are stable. Nevertheless, the presence of unstable systems within the system could potentially disrupt this convergence property. Hence, the last question is as follows:

Problem 1.3. *Is it possible to introduce theoretical approaches that tackle set stability within switched systems, encompassing both stable and unstable subsystems?*

1.3   Outline and contributions of the thesis

This section explains how this thesis is structured and state its specific contributions.

In Chapter 2, we present our main contributions in the context of Problem 1.1. We focus on achieving incremental stability in switched systems featuring non-contracting subsystems. Furthermore, we present various strategies for designing time-dependent switching laws that ensure the system attains contractivity. By introducing the concept of switched virtual systems, we establish a sufficient and necessary condition for switched systems to be contractive. Subsequently, we derive a novel class of switching control signals specifically designed for switched virtual systems comprising both stable and unstable modes. Our approach encompasses two key facets: i) we propose a sufficient condition for designing a switching law based on mode-dependent dwell/leave time to stabilize the switched virtual systems. ii) We develop a set of Linear Matrix Inequality (LMI) conditions applicable to switched systems with sector-bound nonlinearities. The material in this chapter is based on the journal paper [100]

In Chapters 3, we outline our primary contributions within the framework of Problem 1.2. We investigate the contractivity of time-varying DAE systems. Initially, we establish the equivalence between the contractivity of the DAE system and the uniform global exponential stability (UGES) of its corresponding variational DAE system. Our methodology further evolves as we employ a matrix measure approach to tackle a higher-order auxiliary ODE system. This system's trajectory set encompasses the trajectory of the variational DAE system. Within this established structure, we formulate a sufficient condition that ensures the attainment of UGES for the variational DAE system. Lastly, we apply our approach in some specific control challenges: i) Our methodologies are utilized to design

an observer for a time-varying ODE system by treating the system's output as an algebraic constraint. ii) We stabilize a time-invariant DAE system by constructing a contractive DAE system whose trajectory set encompasses the equilibrium. The material in this chapter is based on the journal paper [101].

Chapter 4 is dedicated to investigating the pinning synchronization problem for heterogeneous multi-agent systems (MAS) on directed graphs, utilizing the principles of contraction theory. Our focus lies in designing a control strategy of the pinned agents base on the information of the exosystem and putting forward a distributed control law based only on relative local state measurement for the remaining agents. By employing the standard regulator equation, we provide both necessary and sufficient conditions for the solvability of the pinning synchronization problem. These conditions enable the transformation of the pinning synchronization problem into a contraction analysis problem. The control laws are designed to ensure the contractility of each agent's dynamics, while simultaneously ensuring that the synchronized state trajectory remains a admissible trajectory for the agent's dynamics. The material in this chapter is based on the journal paper [99].

In Chapter 5, we present our key contributions focused on addressing Problem 1.3. Our primary investigation revolves around the issue of set convergence within switched systems, even when they involve unstable subsystems. In pursuing this objective, we introduce a novel class of switching control signals that extend the current findings related to the challenge of achieving set convergence within switched systems characterized by exclusively stable subsystems. Sufficient conditions are proposed for the design of switching laws based on mode-dependent dwell/leave times to ensure the set convergence of the switched systems. Consequently, we utilize our approach on switched affine systems and formulate a collection of LMI conditions. These conditions enable the practical stability of the systems to be numerically validated. The material in this chapter is based on the journal paper [102].

Finally, in Chapter 6 we formulate our conclusions and provide some suggesstions for future work.

1.4   List of publication

Journal articles:

- H. Yin, B. Jayawardhana and S. Trenn, "On Contraction Analysis of Switched Systems with Mixed Contracting-Noncontracting Modes Via Mode-Dependent Average Dwell Time," in IEEE Transactions on Automatic Control, doi: 10.1109/TAC.2023.3237492. (Chapter 2)

- H. Yin, B. Jayawardhana and R. Reyes-Báez, "Pinning Synchronization of Heterogeneous Multi-Agent Nonlinear Systems via Contraction Analysis," in IEEE Control Systems Letters, vol. 6, pp. 157-162, 2022, doi: 10.1109/LC-SYS.2021.3053493. (Chapter 4)

- H. Yin, B. Jayawardhana and S. Trenn, "Stability of switched systems with multiple equilibria: a mixed stable-unstable subsystem case," in System & control letters, doi: 10.1016/j.sysconle.2023.105622. (Chapter 5)

- H. Yin, B. Jayawardhana and S. Trenn, "Contraction analysis of time-varying DAE systems via auxiliary ODE systems," submitted. (Chapter 3)

- H. Yin, B. Jayawardhana and S. Trenn, "Output contraction analysis of nonlinear systems," submitted.

- E. Nuño, I. Sarras, H. Yin and B. Jayawardhana, "Robust Leaderless Consensus of Euler-Lagrange Systems with Interconnection Delays," submitted.

Conference papers:

- E. Nuño, I. Sarras, H. Yin and B. Jayawardhana, "Robust Leaderless Consensus of Euler-Lagrange Systems with Interconnection Delays," 2023 American Control Conference (ACC), San Diego, CA, USA, 2023, pp. 1547-1552, doi: 10.23919/ACC55779.2023.10156387.

- S. Sutrisno, H. Yin, S. Trenn and B. Jayawardhana, "Nonlinear singular switched systems in discrete-time: Solution theory and (incremental) stability under fixed switching signals," accepted by CDC2023.

1.5 Notation

Throughout this thesis, standard notation will be used. The most commonly used definitions and notation will be listed here, while specific notions and notation can be found in each of the chapters.

General notation

The symbols $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, $\mathbb{N}$ denote the set of real, non-negative real, natural numbers, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The identity matrix with appropriate dimension is denoted by I. Given a matrix A, $A^\top$ refers to the transpose of A. For a square matrix A, $\lambda(A)$ refers to the set of eigenvalues of A. The matrix $\mathbf{diag}(M_i)$ is the the block diagonal matrix with entries of square matrices $M_1, \cdots, M_i, \cdots, M_N$. For symmetric metrics B and C, $B > 0$ ($B \geqslant 0$) indicates that B is positive definite (positive semidefinite) and $B < 0$ ($B \leqslant 0$)

indicates that B is negative definite (negative semidefinite), $B < C$ ($B \leqslant C$) means $B - C < 0$ ($B - C \leqslant 0$). $\overline{\tau}$, $\underline{\tau}$ represent the upper bound, and the lower bound of τ. For a vector or a matrix, $\| \cdot \|$ denotes the Euclidean vector norm or the induced matrix norm, respectively. The sign $\otimes$ represents matrix Kronecker product. For vector valued functions $F : x \mapsto F(x)$ with $x \in \mathbb{R}^n$, and $F_p : x \mapsto F_p(x)$ with $x \in \mathbb{R}^n$, we define the Jacobian matrix $\triangledown_x F : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ by $\triangledown_x F(x) := \frac{\partial F(x)}{\partial x}$, and $\triangledown_x F_p : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ by $\triangledown_x F_p(x) := \frac{\partial F_p(x)}{\partial x}$, respectively. Finally, whenever it is clear from the context, the symbol "$*$" inside a matrix stands for the symmetric elements in a symmetric matrix.

Sets

- We denote the *tangent bundle* of $\mathcal{M}$ by $T\mathcal{M} = \cup_{x \in \mathcal{M}} \{x\} \times T_x \mathcal{M}$, where $T_x \mathcal{M}$ is the *tangent space* of $\mathcal{M}$ at $x \in \mathcal{M}$.

- For a given set N, the sets ∂N and $\overline{N}$ denote the boundary of N and the complement of N, respectively.

Functions

- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^k$ (for some $k \in \mathbb{N}$) if it is k-times differentiable, and all the k partial derivatives are continuous functions.

- A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is *positive definite* if it verifies the identity $\{x \in \mathbb{R} | \alpha(x) = 0\} = \{0\}$;

- A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of *class-$\mathcal{K}$* if it is strictly increasing and $\alpha(0) = 0$;

- A function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of *class-$\mathcal{K}_\infty$* if it is of *class-$\mathcal{K}$* and $\lim_{s \to 0} \alpha(s) = +\infty$;

- A function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of *class-$\mathcal{KL}$* if, for each fixed s, $\beta(r, s)$ is of *class-$\mathcal{K}$* with respect to r, and for each fixed r, $\beta(r, s)$ is decreasing with respect to s, and $\lim_{s \to 0} \beta(r, s) = 0$.

- A function $F : T\mathcal{M} \to \mathbb{R}_{\geqslant 0}$ defined on the tangent bundle $T\mathcal{M}$ is a Finsler structure, if F satisfies the following conditions:

    i) F is a $\mathcal{C}^1$ function for each $(x, \delta x) \in T\mathcal{M}$ such that $\delta x \neq 0$;

    ii) $F(x, \delta x) > 0$ for each $(x, \delta x) \in T\mathcal{M}$ such that $\delta x \neq 0$;

    iii) $F(x, \lambda \delta x) = \lambda F(x, \delta x)$ for each $\lambda > 0$, and each $(x, \delta x) \in T\mathcal{M}$ such that $\delta x \neq 0$ (homogeneity);

    iv) $F(x, \delta x_1 + \delta x_2) < F(x, \delta x_1) + F(x, \delta x_2)$ for each $(x, \delta x_1), (x, \delta x_2) \in T\mathcal{M}$ such that $\delta x_1 \neq \lambda \delta x_2$ for any given $\lambda \in \mathbb{R}$ (subadditivity).

2

On contraction analysis of switched systems via mode-dependent dwell time

This chapter studies contraction analysis of switched systems that are composed of a mixture of contracting and noncontracting modes. The first result pertains to the equivalence of the contraction of a switched system and the uniform global exponential stability of its variational system. Based on this equivalence property, sufficient conditions for a mode-dependent average dwell/leave-time based switching law to be contractive are established. Correspondingly, LMI conditions are derived that allow for numerical validation of contraction property of nonlinear switched systems, which include those with all non-contracting modes.

2.1 Introduction

For the past two decades, analysis and control of switched systems (as an important and special class of hybrid systems) have been well studied due to their relevance in representing numerous modern engineering systems where an abrupt change of parameters can occur or a jump in systems dynamics can happen as a response to the sudden change in their environment. Some well-known examples of such engineering systems are the dynamics of aircraft [25], and of power electronics [59]. Typically, switched systems are described by a family of subsystems, which can either be continuous-time or discrete-time dynamics, and a switching signal $\sigma(t)$ with switching times $\{t_1, t_2, \ldots\}$ that determines which subsystem is active over each time interval $[t_i, t_{i+1})$ for all $i \geqslant 0$. Such switching times can depend on particular state events [26], or time events [35, 61, 104]. In the time-dependent switching sequence, the dwell time (DT) [61] and average dwell time (ADT) notions [35] are two basic and important concepts in switched systems, both of which refer to the time interval or the average time interval, respectively, between consecutive switching times being lower bounded by a certain positive constant. A more general and flexible switching sequence, so-called mode dependent average dwell time (MDADT), was introduced in [104], which allows each

mode to have its own ADT.

The stability of switched systems has been widely investigated in the literature [13, 35, 47, 49, 50, 53, 61, 92, 104] with a large body of works concern with switched systems comprising of stable subsystems. The common Lyapunov function technique [50] and multiple Lyapunov function technique [13] are commonly used to analyze the stability of these systems. In recent years, analysis of switched systems has also covered those with both stable and unstable subsystems [47, 53]. The main idea of these studies is to check whether the dwell-time of the stable subsystems is sufficiently large to offset the diverging trajectories caused by the unstable subsystems that are dwelt for a sufficiently short time. This approach of having a trade-off between stable and unstable subsystems is no longer applicable when all subsystems are unstable. In [49, 92], a discretized Lyapunov function technique is presented that can be used to analyze the stability of switched systems with all unstable subsystems. In this paper, we present another approach using contraction analysis to analyze the stability of switched systems which encompass all cases including those with all unstable modes.

As one of the stability analysis methods that has received a growing interest lately, contraction analysis is concerned with the relative trajectories of a systems than to a particular attractor equilibrium point in standard Lyapunov stability analysis. There are many different methods to analyze the contractivity of non-switched systems in literature, such as [4, 6, 7, 12, 27, 30, 40, 55, 65, 74] among many others. In [55], the contraction property can be guaranteed if the largest eigenvalue of the symmetric part of the associated variational systems matrix (which is loosely termed as the Jacobian) is uniformly strictly negative. Finsler–Lyapunov functions were introduced in [27] to analyze the incremental stability of the system. A hierarchical approach to study convergence using matrix norm was discussed in [74]. In the context of switched systems, the contraction analysis thereof has recently been presented in [20, 26, 57, 71, 94]. Using contraction analysis method in [55], sufficient conditions for the convergence behavior of reset control systems have been studied in [71]. The extension of matrix norm-based contraction analysis [74] to piecewise smooth continuous systems is formalized in [20]. In [26], the singular perturbation theory and matrix norm are used to study the contraction property of switched Filippov systems, which include piecewise smooth systems.

In all of above mentioned results on contraction analysis for switched systems, it is assumed that all subsystems are contracting. It remains non-trivial to analyze contractivity of switch systems with all non-contracting subsystems, where in each dwell time interval the trajectories diverge from each other. Following the fact that a switched system with all unstable modes can be made asymptotically stable by an appropriate switching signal, we study in this chapter whether the contraction of these systems, as a particular class of switched systems with mixed contracting-noncontracting modes, can be established by using the right switching

signals.

As our first main result in this chapter, we present contraction analysis for switched systems with mixed contracting-noncontracting modes. We establish that the stability of the corresponding variational dynamics is a sufficient and necessary condition to the contraction of the original switched systems. Subsequently, as our second contribution, we provide sufficient conditions on the time-varying Lyapunov function and on the mode dependent average dwell-time for switched nonlinear systems such that they are contracting. In general, these conditions ensure that the growth of time-varying Lyapunov function due to the noncontracting modes can be compensated by the switching behavior and the decaying Lyapunov function due to the contracting modes. In addition, we also consider all noncontracting subsystems case, where the increment can only be compensated by the switching behavior. Based on these conditions, as our third contribution, we propose a time-varying quadratic Lyapunov function that can be used to establish the contraction of switched systems via LMI conditions. Our result is more general and less conservative than the discretized Lyapunov function technique as proposed and used in [49, 92]. This result implies also that we can establish the stability of switched linear systems with all unstable modes.

This chapter is organized as follows. In Section 2.2, we present preliminaries and problem formulation. Necessary and sufficient conditions for the contractivity of nonlinear switched systems are presented in Sections 2.3. The switching law design strategy is provided in Section 2.4. The numerical simulations are provided in Section 2.5. The conclusions are given in Section 2.6.

2.2 Preliminaries and problem formulation

Consider switched systems in the form of

$$\dot{x}(t) = f_{\sigma(t)}(x(t), t), \quad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in \mathscr{X} \subseteq \mathbb{R}^n$ is the state vector, $t_0 \in \mathbb{R}$ is the initial time and $x_0 \in \mathscr{X}$ is the initial value. Define an index set $\mathcal{M} := \{1, 2, \cdots, N\}$, where N is the number of modes. The signal $\sigma : [t_0, \infty) \to \mathcal{M}$ denotes the switching signal, which is assumed to be a piece-wise constant function continuous from the right. The vector field $f_i : \mathscr{X} \times [t_0, \infty) \to \mathbb{R}^n$, $(x, t) \mapsto f_i(x, t), i \in \mathcal{M}$ is continuous in t and continuously differentiable in x. The switching instants are expressed by a monotonically increasing sequence $\mathscr{S} := \{t_1, t_2, \cdots, t_k, \cdots\}$, where t_k denotes the k-th switching instant. The length between successive switching instants is commonly referred to as the mode duration and given by $\tau_k = t_{k+1} - t_k, k = 0, 1, 2, \cdots$. We assume that (2.1) is forward complete, which means for each $x_0 \in \mathscr{X}$ there exists a unique

solution of (2.1) and no jump occurs in the state at a switching time.

Definition 2.1. A switched system given by (2.1) with a given switching signal $\sigma(t)$, is called

(i) incrementally uniformly globally asymptotically stable (iUGAS) if there exists a *class-$\mathcal{KL}$* function β, such that for all solutions $x_1(t)$, $x_2(t)$ of (2.1) in $t \in [t_0, +\infty)$ we have

$$\|x_1(t) - x_2(t)\| \leqslant \beta(\|x_1(t_0) - x_2(t_0)\|, t), \qquad (2.2)$$

(ii) uniformly contracting if there exists positive numbers c and α such that for all solutions $x_1(t)$, $x_2(t)$ of (2.1) we have

$$\|x_1(t) - x_2(t)\| \leqslant ce^{-\alpha t}\|x_1(t_0) - x_2(t_0)\|. \qquad (2.3)$$

In order to study contractivity of the switched systems (2.1), as usual, we will analyse the (uniform) stability of the corresponding variational systems, in which case, the following definition is relevant (note that by assumption for each time $t \geqslant t_0$ the map $x \mapsto f_{\sigma(t)}(x, t)$ is continuously differentiable at all $x \in \mathcal{X}$).

Definition 2.2. The family of (time-varying) linear switched system

$$\dot{\xi}(t) = F_{\sigma(t)}(x(t), t)\xi(t), \quad \xi(t_0) = \xi_0 \in \mathbb{R}^n \qquad (2.4)$$

with $F_p(x(t), t) = \nabla_x f_p(x(t), t)$ and $x(\cdot) : [t_0, +\infty) \to \mathbb{R}^n$ be any given solution trajectory of (2.1) is called

(i) uniformly globally asymptotically stable (UGAS), if there exist a *class-$\mathcal{KL}$* function β, (independently of the chosen solution $x(\cdot)$) such that for every solution $\xi(t) \in \mathbb{R}^n$ of (2.4) the following inequality holds,

$$\|\xi(t)\| \leqslant \beta(\|\xi(t_0)\|, t), \quad \forall t \geqslant t_0, \qquad (2.5)$$

(ii) uniformly globally exponentially stable (UGES), if there exist positive numbers c, α (independently of the chosen solution $x(\cdot)$) such that for every solution $\xi(t) \in \mathbb{R}^n$ of (2.4) the following inequality holds,

$$\|\xi(t)\| \leqslant ce^{-\alpha t}\|\xi(t_0)\|, \quad \forall t \geqslant t_0. \qquad (2.6)$$

The contraction analysis problem for switched systems with all contracting modes has attracted considerable attentions. For example, in [71, 94], a common

contraction region is required between each subsystem. Then, contracting can be achieved by activating the subsystems for a sufficient long time. However, for noncontracting subsystems, you can not find such common contraction region, to be precise, you can not find any contraction region for a noncontracting subsystem. Then, the results in [71, 94] cannot be applied. The objective of this paper is to propose a sufficient condition that guarantees the switched system (2.1) is contracting with respect to switching law $\sigma(t)$ when not all modes of (2.1) are contracting, including the case where none of the modes is contracting.

2.3 A necessary and sufficient condition for the contraction of switched systems

Since switched systems with fixed switching signal can be considered as time-varying systems, tools for time-varying systems can be used to analyse of such switched systems. In this section, inspired by contraction analysis of time-varying systems as presented in [7, 55], we have the following proposition that establish the relations between (2.1) being iUGAS/contracting and (2.4) being UGAS/UGES.

Proposition 2.3. *For a given switching signal $\sigma(t)$, the following properties hold*

(i) *the system (2.1) is iUGAS if the family of systems (2.4) is UGAS,*

(ii) *the system (2.1) is uniformly contracting if, and only if, the family of systems (2.4) is UGES.*

Proof. We first establish a relationship between the solutions of (2.1) and (2.4). Let $x(t) = \varphi(t, x_0)$, $\hat{x}(t) = \varphi(t, x_0 + \delta\xi_0)$ be two trajectories of (2.1) with initial conditions $x(t_0) = \varphi(t_0, x_0) = x_0 \in \mathbb{R}^n$ and $\hat{x}(t_0) = \varphi(t_0, x_0 + \delta\xi_0) = x_0 + \delta\xi_0$, respectively, where δ is a sufficiently small positive constant and ξ_0 will later be related to the initial condition of (2.4). We will now show that

$$\xi(t) := \lim_{\delta \to 0} \frac{\varphi(t, x_0 + \delta\xi_0) - \varphi(t, x_0)}{\delta} \tag{2.7}$$

is a solution of (2.4) with initial value $\xi(t_0) = \xi_0$. For any t, let $i \in \mathbb{N}$ be such that $t \in [t_i, t_{i+1})$, so that the flow $\varphi(t, x_0)$ of (2.1) satisfies

$$\varphi(t, x_0) = x_0 + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} f_{\sigma(t_k)}(\varphi(s, x_0), s)\mathrm{d}s + \int_{t_i}^{t} f_{\sigma(t_i)}(\varphi(s, x_0), s)\mathrm{d}s, \tag{2.8}$$

and similarly, the flow $\varphi(t, x_0 + \delta\xi_0)$ satisfies

$$\varphi(t, x_0 + \delta\xi_0) = x_0 + \delta\xi_0 + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} f_{\sigma(t_k)}(\varphi(s, x_0 + \delta\xi_0), s)\mathrm{d}s$$
$$+ \int_{t_i}^{t} f_{\sigma(t_i)}(\varphi(s, x_0 + \delta\xi_0), s)\mathrm{d}s. \tag{2.9}$$

Hence,

$$\xi(t) = \xi_0 + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} \lim_{\delta\to 0} \frac{f_{\sigma(t_k)}(\varphi(s, x_0 + \delta\xi_0), s) - f_{\sigma(t_k)}(\varphi(s, x_0), s)}{\delta}\mathrm{d}s$$
$$+ \int_{t_i}^{t} \lim_{\delta\to 0} \frac{f_{\sigma(t_i)}(\varphi(s, x_0 + \delta\xi_0), s) - f_{\sigma(t_i)}(\varphi(s, x_0), s)}{\delta}\mathrm{d}s. \tag{2.10}$$

Clearly, for $j \in \{0, 1, \ldots, i\}$

$$\lim_{\delta\to 0} \frac{f_{\sigma(t_j)}(\varphi(s, x_0 + \delta\xi_0), s) - f_{\sigma(t_j)}(\varphi(s, x_0), s)}{\delta}$$
$$= \frac{\partial}{\partial x_0}\left[f_{\sigma(t_j)}(\varphi(s, x_0), s)\right] \cdot \xi_0$$
$$= \left[\nabla_x f_{\sigma(t_j)}(\varphi(s, x_0), s) \cdot \nabla_{x_0}\varphi(s, x_0)\right] \cdot \xi_0.$$

Here we used the fact that the map $x_0 \mapsto \varphi(t, x_0)$ is differentiable for all $t \in [t_0, \infty)$ which is a consequence from the ability to write φ as a concatenation of the smooth solution flows $\varphi_{\sigma(t_i)}(t, t_i, x_i)$ of the (non-switched) differential equations $\dot{x} = f_{\sigma(t_i)}(x, t)$, $x(t_i) = x_i$. In fact, $\varphi(t, x_0) = \varphi_{\sigma(t_j)}(t, t_i, \varphi(t_i, x_0))$ and, recursively for $k = i - 1, \ldots, 2, 1$, we have $\varphi(t_k, x_0) = \varphi_{\sigma(t_{k-1})}(t_k, t_{k-1}, \varphi(t_{k-1}, x_0))$. Hence

$$\xi(t) = \xi_0 + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} \nabla_x f_{\sigma(t_k)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\xi_0\mathrm{d}s$$
$$+ \int_{t_i}^{t} \nabla_x f_{\sigma(t_i)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\xi_0\mathrm{d}s \tag{2.11}$$

and consequently

$$\dot{\xi}(t) = \nabla_x f_{\sigma(t_i)}(\varphi(t, x_0), t)\nabla_{x_0}\varphi(t, x_0)\xi_0$$
$$= F(t, x(t)))\nabla_{x_0}\varphi(t, x_0)\xi_0,$$

where the last equality follows from $\sigma(t_i) = \sigma(t)$ for all $t \in [t_i, t_{i+1})$. Furthermore,

from (2.8),

$$\nabla_{x_0}\varphi(t, x_0) = I + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} \nabla_x f_{\sigma(t_k)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\mathrm{d}s$$
$$+ \int_{t_i}^{t} \nabla_x f_{\sigma(t_i)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\mathrm{d}s \tag{2.12}$$

which when multiplied with ξ_0 and in view of (2.11) leads to

$$\nabla_{x_0}\varphi(t, x_0)\xi_0 = \xi(t).$$

Altogether this shows that indeed ξ given by (2.7) is a solution of (2.4). In particular, $\nabla_{x_0}\varphi(t, x_0)$ is the transition matrix for (2.4), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{x_0}\varphi(t, x_0) = \nabla_x f_{\sigma(t)}(x(t), t)\nabla_{x_0}\varphi(t, x_0). \tag{2.13}$$

Proof of (i) on *UGAS* ⇒ *iUGAS*. Let us consider two solutions $x(t) = \varphi(t, x_0)$ and $\hat{x}(t) = \varphi(t, \hat{x}_0)$ of (2.1). We already highlighted in the first part of the proof that the map $x_0 \mapsto \varphi(t, x_0)$ is differentiable for each fixed $t \in [t_0, \infty)$. Consequently, we can utilize the fundamental theorem of calculus for line integrals to obtain

$$\hat{x}(t) - x(t) = \int_{x_0}^{\hat{x}_0} \nabla_y\varphi(t, y)\mathrm{d}y. \tag{2.14}$$

From UGAS of (2.4) and (2.13) it follows that there exists a *class-$\mathcal{KL}$* function β, such that

$$\|\nabla_y\varphi(t, y)\| \leqslant \beta(\|\underbrace{\nabla_y\varphi(t_0, y)}_{=I}\|, t) = \beta(1, t), \tag{2.15}$$

for all $y \in \mathscr{X}$. Using (2.15) to get the upper bound of (2.14), we have

$$\|\hat{x}(t) - x(t)\| \leqslant \beta(1, t)\|\hat{x}_0 - x_0\| = \beta'(\|\hat{x}_0 - x_0\|, t), \tag{2.16}$$

where $\beta'(\|\hat{x}_0 - x_0\|, t)$ is a *class-$\mathcal{KL}$* function.

Proof of (ii) on *Contracting* ⇔ *UGES*. As we show $\varphi(t, x_0)$ is differentiable with respect to x_0 for each fixed $t \in [t_0, \infty)$.

We show: Contracting ⇒ *UGES*. The proof is done by contradiction, i.e. we show that the assumption that (2.1) is not UGES leads to a contradiction if (2.1) is contracting. Towards this goal, assume that there exists a solution $x(\cdot)$ of (2.1) and an initial value ξ_0 such that for the corresponding solution $\xi(\cdot)$ of (2.4) we have that for all positive c' and α', there exist $T > 0$ such that

$$\|\xi(T)\| > c'e^{-\alpha'T}\|\xi_0\|. \tag{2.17}$$

where $\xi(t)$ is a solution of (2.4). Let c and α be the constants corresponding to the contractivity condition (2.3), we will now show, that the choice $c' := \frac{3}{2}c$ and $\alpha' := \alpha$ leads to a contradiction. In fact, choose $T > 0$ such that, the following inequality is satisfied.

$$\|\xi(T)\| > \frac{3}{2}ce^{-\alpha T}\|\xi_0\| \tag{2.18}$$

Let $\hat{x}(\cdot)$ be a solution of (2.1) with initial value $\hat{x}(t_0) = x_0 + \delta\xi_0$ for a (small) parameter $\delta \in \mathbb{R}$. In the first part of the proof we have shown that $\xi(t) = \lim_{\delta \to 0} \frac{\hat{x}(t) - x(t)}{\delta}$, in particular, we have at time T that

$$\lim_{\delta \to 0} \frac{\hat{x}(T) - x(T)}{\delta} = \xi(T). \tag{2.19}$$

In particular, for a sufficiently small $\delta > 0$, we have that

$$\frac{\|\hat{x}(T) - x(T)\|}{\delta} > \frac{2}{3}\|\xi(T)\|. \tag{2.20}$$

Combining (2.18) with (2.20) we obtain

$$\begin{aligned}
\|\hat{x}(T) - x(T)\| &> \frac{2}{3}\delta\|\xi(T)\| \\
&> ce^{-\alpha T}\|\delta\xi_0\|\|\xi(t_0)\| \\
&= ce^{-\alpha T}\|\hat{x}(t_0) - x(t_0)\|.
\end{aligned} \tag{2.21}$$

By choosing $c = \frac{2}{3}c'$ and $\alpha = \alpha'$, one has

$$\|\hat{x}(T) - x(T)\| > ce^{-\alpha T}\|\hat{x}(t_0) - x(t_0)\|$$

This is in contradiction to the contractivity of (2.1), which concludes this step of the proof.

We show: UGES $\Rightarrow$ Contracting.

$$\|\nabla_y\varphi(t, y)\| \leqslant ce^{-\alpha t}\|\underbrace{\nabla_y\varphi(t_0, y)}_{=I}\| = ce^{-\alpha t}, \tag{2.22}$$

for all $y \in \mathscr{X}$. Using (2.22) to get the upper bound of (2.14), we have

$$\|\hat{x}(t) - x(t)\| \leqslant ce^{-\alpha t}\|\hat{x}_0 - x_0\|, \tag{2.23}$$

which implies that (2.4) is contracting. This completes the proof. □

In Proposition 2.3 we establish the concept of UGAS for variational system, which is not presented before in [7, 55]. Note that the variational system (2.4) being

UGAS is only a sufficient condition for system (2.1) being iUGAS. The reverse implication is not trivial to establish and it cannot follow the same line of proof as in [7]. Particularly, we can not conclude that $\delta\beta'(\|\xi(T)\|, T) \geqslant \beta'(\delta\|\xi(T)\|, T) = \beta'(\|\xi(t_0)\|, T)$ holds.

2.4 Switching law design

In general, when individual systems are contracting, the switched systems can be made contracting by activating each subsystem sufficiently long. Instead of considering this situation, in this section, we study the property of contraction of switched systems whose modes are composed of a mixture of contracting and non-contracting modes. The switched systems under study include also the worst case, where all individual systems are not contracting[1], and we provide sufficient conditions on MDADT/MDALT (whose precise definition will shortly be given below) that guarantee the contraction of the switched systems. The use of MDADT/MDALT property in this paper is in contrast to the existing results in literature that are based on common dwell time. For this purpose, we define $\mathcal{S}$ as the set of all stable modes and $\mathcal{U}$ as the set of all unstable modes. In our main result, we propose a new class of switching signals that is suited for switched systems with stable and unstable modes.

Denoting $N_{\sigma p}(t_1, t_2)$ as the number of times that the pth mode is activated in the interval $[t_1, t_2)$, and $T_p(t_1, t_2)$ as the sum of the running time of the pth mode in the interval $[t_1, t_2)$, $p \in \mathcal{M} = \{1, 2, ..., N\}$. We revisit the following definitions of mode dependent average dwell time in [104].

Definition 2.4. A constant $\tau_{ap} > 0$ is called (slow) mode dependent average dwell time (MDADT) for mode $p \in \mathcal{M}$ of a switching signal $\sigma : [t_0, \infty) \to \mathcal{M}$, if there exist a constant $N_{0p} \geqslant 0$ such that for all finite time intervals $[t_1, t_2) \subseteq [t_0, \infty)$ we have

$$N_{\sigma p}(t_1, t_2) \leqslant N_{0p} + \frac{T_p(t_1, t_2)}{\tau_{ap}}. \tag{2.24}$$

Definition 2.5. A constant $\tau_{ap} > 0$ is called mode dependent average leave time (MDALT) for mode $p \in \mathcal{M}$ of a switching signal $\sigma : [t_0, \infty) \to \mathcal{M}$, if there exist a constant $N_{0p} \geqslant 0$ such that for all finite time intervals $[t_1, t_2) \subseteq [t_0, \infty)$,

$$N_{\sigma p}(t_1, t_2) \geqslant N_{0p} + \frac{T_p(t_1, t_2)}{\tau_{ap}}. \tag{2.25}$$

---

[1]Equivalently, the corresponding variational system (2.4) is not UGES [7, 40].

Remark 2.6. In Definition 2.5 we refer to τ_{ap} as the mode dependent average *leave* time (MDALT) instead of *fast* mode dependent average *dwell* time as e.g. in [104]. We prefer the former, because τ_{ap} in Definition 2.5 is not related to how long (at least, on average) the system dwells (remains) in a certain mode, but when the system has to leave a certain mode at the latest (on average). So "leave time" seems a better naming choice for τ_{ap} than "fast dwell time".

We present now the following theorem on the contracting properties of switched systems (2.1) with MDADT and/or MDALT.

Theorem 2.7. *Consider switched nonlinear system* (2.1) *with switching signal* σ : $[0, \infty) \to \mathcal{M}$ *and corresponding switching times* $\mathscr{S} := \{t_0, t_1, \ldots, t_i, \ldots\}$. *Assume that we can classify each mode p as being either stable or unstable, i.e. assume* $\mathcal{M} = \mathcal{S} \,\dot\cup\, \mathcal{U}$ *and, correspondingly, assume the switching signal σ has a MDADT $\tau_{ap} > 0$ for each stable mode $p \in \mathcal{S}$ and a MDALT $\tau_{ap} > 0$ for each unstable mode $p \in \mathcal{U}$. Furthermore, assume that for each mode $p \in \mathcal{M}$ there exist a continuously differentiable function* $V_p : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ *with*

$$\dot{V}_p(x, \xi, t) := \nabla_{(x,\xi)} V_p(x, \xi, t) \begin{pmatrix} f_p(x,t) \\ F_p(x,t)\xi \end{pmatrix} + \nabla_t V_p(x, \xi, t)$$

such that for all $(x, \xi, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geqslant 0}$

$$\dot{V}_p(x, \xi, t) \leqslant \eta_p V_p(x, \xi, t), \qquad \forall p \in \mathcal{M}, \tag{2.26}$$

with $\eta_p \geqslant 0$ if $p \in \mathcal{U}$ or $\eta_p < 0$ otherwise. Finally, assume that for every $p \in \mathcal{M}$, there exists $\mu_p > 0$ such that

$$V_{\sigma(t_i)}(x, \xi, t_i) \leqslant \mu_{\sigma(t_i^-)} V_{\sigma(t_i^-)}(x, \xi, t_i), \quad \forall t_i \in \mathscr{S}. \tag{2.27}$$

Without loss of generality, we let $\mu_p > 1$ for $p \in \mathcal{S}$. Then, with the following switching law

$$\left.\begin{aligned} \tau_{ap} > \underline{\tau}_{ap} &:= -\frac{\ln \mu_p}{\eta_p}, \quad \forall p \in \mathcal{S}, \\ \tau_{ap} < \overline{\tau}_{ap} &:= -\frac{\ln \mu_p}{\eta_p}, \quad \forall p \in \mathcal{U}. \end{aligned}\right\} \tag{2.28}$$

the switched nonlinear system (2.1) *is*

(i) *incrementally uniformly globally asymptotically stable (iUGAS) if there exist class K_∞ functions $\underline{v}_p, \overline{v}_p$, such that $V_p(x, \xi, t)$ satisfies*

$$\underline{v}_p(\|\xi\|) \leqslant V_p(x, \xi, t) \leqslant \overline{v}_p(\|\xi\|), \quad \forall p \in \mathcal{M}, \tag{2.29}$$

(ii) *uniformly contracting if there exist $\overline{v}_p \geqslant \underline{v}_p \geqslant 0$, such that $V_p(x, \xi, t)$ satisfies*

$$\underline{v}_p \|\xi\|_2^2 \leqslant V_p(x, \xi, t) \leqslant \overline{v}_p \|\xi\|_2^2, \quad \forall p \in \mathcal{M}. \tag{2.30}$$

We note that $\tau_{ap} < \bar{\tau}_{ap}$ in (2.28) can only be satisfied if $\mu_p \in (0,1)$ for $p \in \mathcal{U}$.

Proof. Let $x(\cdot)$ be a solution of (2.1) and let $\xi(\cdot)$ be a solution of the corresponding system (2.4). We will show in the following that there exists $k > 0$ and $\lambda > 0$ (independent from $x(\cdot)$ and $\xi(\cdot)$) such that

$$V_{\sigma(t)}(x(t), \xi(t), t) \leqslant ke^{-\lambda(t-t_0)} V_{\sigma(t_0)}(x_0, \xi_0, t_0). \tag{2.31}$$

From (2.29) we can then conclude that

$$
\begin{aligned}
\|\xi(t)\| &\leqslant \underline{v}_{\sigma(t_n)}^{-1} \circ V_{\sigma(t_n)}(x(t), \xi(t), t) \\
&\leqslant \underline{v}_{\sigma(t_n)}^{-1} \big(ke^{-\lambda(t-t_0)} V_{\sigma(t_0)}(x_0, \xi_0, t_0) \big) \\
&\leqslant \underline{v}_{\sigma(t_n)}^{-1} \big(ke^{-\lambda(t-t_0)} \underline{v}_{\sigma(t_0)}^{-1}(\|\xi_0\|) \big).
\end{aligned}
\tag{2.32}
$$

It is easy to see that $\underline{v}_{\sigma(t_n)}^{-1} \big(ke^{-\lambda(t-t_0)} \underline{v}_{\sigma(t_0)}^{-1}(\|\xi_0\|) \big)$ is a class KL function.

From (2.30) we can then conclude that

$$
\begin{aligned}
\|\xi(t)\| &\leqslant \frac{1}{\sqrt{\underline{v}_{\sigma(t_n)}}} V_{\sigma(t_n)}^{\frac{1}{2}}(x(t), \xi(t), t) \\
&\leqslant \sqrt{\frac{k}{\underline{v}_{\sigma(t_n)}}} e^{-\frac{\lambda}{2}(t-t_0)} V_{\sigma(t_0)}^{\frac{1}{2}}(x_0, \xi_0, t_0) \\
&\leqslant \sqrt{k \frac{\overline{v}_{\sigma(t_0)}}{\underline{v}_{\sigma(t_n)}}} e^{-\frac{\lambda}{2}(t-t_0)} \|\xi_0\|.
\end{aligned}
\tag{2.33}
$$

According to (2.32), (2.33), Proposition 2.3 and Definition 2.2, we can then conclude that (i) system (2.4) is UGAS the system (2.1) is iUGAS, (ii) system (2.4) is UGES the system (2.1) is contracting.

Towards showing (2.31) first observe that for any $t \in [t_{i-1}, t_i)$ and $p := \sigma(t_i^-)$ we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_p(x(t), \xi(t), t) = \dot{V}_p(x(t), \xi(t), t).$$

Consequently, in view of (2.27) and (2.26),

$$
\begin{aligned}
&V_{\sigma(t_i)}(x(t_i), \xi(t_i), t_i) \\
&\leqslant \mu_{\sigma(t_i^-)} V_{\sigma(t_i^-)}(x(t_i), \xi(t_i), t_i) \\
&= \mu_{\sigma(t_{i-1})} V_{\sigma(t_{i-1})}(x(t_i), \xi(t_i), t_i) \\
&\leqslant \mu_{\sigma(t_{i-1})} e^{\eta_{\sigma(t_{i-1})}(t_i - t_{i-1})} V_{\sigma(t_{i-1})}(x(t_{i-1}), \xi(t_{i-1}), t_{i-1}).
\end{aligned}
$$

Recursively applying this inequality, we arrive at, for $t \in [t_i, t_{i+1})$,

$$V_{\sigma(t_i)}(x(t), \xi(t), t) \leqslant c_\sigma(t) V_{\sigma(t_0)}(x, \xi, t_0), \tag{2.34}$$

with

$$c_\sigma(t) = e^{\eta_{\sigma(t_i)}(t-t_i)} \prod_{k=0}^{i-1} \mu_{\sigma(t_k)} e^{\eta_{\sigma(t_k)}(t_{k+1}-t_k)}$$

$$= \prod_{p \in \mathcal{M}} \mu_p^{N_{\sigma p}(t,t_0)} e^{\eta_p T_p(t,t_0)}$$

$$= \prod_{p \in \mathcal{M}} e^{N_{\sigma p}(t,t_0) \ln \mu_p + \eta_p T_p(t,t_0)}.$$

By assumption, we have for $p \in \mathcal{S}$ that $\ln \mu_p > 0$ and hence by (2.24)

$$N_{\sigma p}(t, t_0) \ln \mu_p + \eta_p T_p(t, t_0) \leqslant N_{0p} \ln \mu_p + (\eta_p + \tfrac{\ln \mu_p}{\tau_{ap}}) T_p(t, t_0);$$

and for $p \in \mathcal{U}$ we have $\ln \mu_p < 0$ and hence by (2.25) we arrive at the same inequality as above. Let $\lambda_p := \eta_p + \frac{\ln \mu_p}{\tau_{ap}}$, then from (2.28) together with $\ln \mu_p > 0$ for $p \in \mathcal{S}$ and $\ln \mu_p < 0$ for $p \in \mathcal{U}$, we have that $\lambda_p < 0$ for all $p \in \mathcal{M}$. With $k = \prod_{p \in \mathcal{M}} \mu_p^{N_{0p}}$ and $\lambda := \min_{p \in \mathcal{M}}(-\lambda_p) > 0$ we obtain

$$c_{\sigma(t)} \leqslant k \prod_{p \in \mathcal{M}} e^{-\lambda T_p(t,t_0)} = k e^{-\lambda(t-t_0)},$$

where we used the fact that $\sum_{p \in \mathcal{M}} T_p(t, t_0) = t - t_0$. This concludes the proof. □

Different from Corollary 1 in [104], we do not need here to consider the ordering of stable and unstable subsystems. Some Lyapunov methods of incremental stability have recently appeared in the literature. Let us compare our results to these works. In this paper we do not exclude the case that the system switches from a non-contracting mode q to another non-contracting mode p and then back to mode q again (Example 2.2). In this case, according to (2.26), the variational system of each subsystem is divergent with a bounded rate η_p. Therefore, we need condition (2.27) to compensate for the divergent trajectory by having $\mu_p < 1$. This is not possible if $V_p(x, \xi, t)$ is time independent. Indeed, otherwise we have $V_p(x, \xi) < \mu_q V_q(x, \xi) < \mu_q \mu_p V_p(x, \xi) < V_p(x, \xi)$, which is a contradiction. In [27], the authors study the incremental stability of time-varying system based on the Finsler distance. A sufficient and necessary condition for incremental stability of time-invariant system with input is given in [6], which shows that the time-invariant system is incrementally stable if and only if there exists an incremental Lyapunov function with respect to the manifold $\{x_1 = x_2\}$). Neither [27] nor [6] study the stability properties of the variational systems. In addition,

the Lyapunov functions in [27], [6] are all time-independent, which cannot solve switched systems with all non-contracting subsystems. By means of Proposition 2.3 and Theorem 2.7, we can analyze the contraction of switched systems with all non-contracting subsystems by finding multiple time-dependent Lyapunov functions for its variational system. Since constructing time-dependent Lyapunov functions is much more difficult than constructing time-independent Lyapunov functions, a LMI method is established in the subsequent Theorem 2.13 to construct time-dependent Lyapunov functions for a family of nonlinear switched systems.

Remark 2.8. The results for all modes are contracting in [57, 94], can be considered as a particular case of Theorem 2.7. In particular, if we assume that $\mathcal{M} = \mathcal{S}$ in Theorem 2.7 then the switched nonlinear system (2.1) is contracting for any MDADT switching signals satisfying $\tau_{ap} > \underline{\tau}_{ap} = -\frac{\ln \mu_p}{\eta_p}, \forall p \in \mathcal{M}$, which recovers the results of Theorem 1 in [94] and Proposition 1 in [57].

For switched system (2.1), if all subsystems are non-contracting, which represents the worst case scenario, the distance increment between two trajectories will not be contracting in each mode and it can only be compensated by at the switching events. In this case, we have the following corollary from Theorem 2.7.

Corollary 2.9. *Using the notation of Theorem 2.7, assume that $\mathcal{M} = \mathcal{U}$, i.e. we assume all modes are non-contracting. Then the switched nonlinear system* (2.1) *is contracting for any MDALT switching signals satisfying*

$$\tau_{ap} \leqslant \overline{\tau}_{ap} = -\frac{\ln \mu_p}{\eta_p}, \qquad \forall p \in \mathcal{M}. \tag{2.35}$$

Although Theorem 2.7 provides a general framework to handle the contraction analysis problem, it is impractical for actual use, since it does not provide means to construct the Lyapunov functions $V_p(x, \xi, t)$ using existing computational techniques. In addition, when noncontracting subsystems are involved, we cannot find a monotonically decreasing Lyapunov function for each subsystem. Inequality (2.26) implies that the value of $V_p(x, \xi, t)$ may increase in some time interval with a bounded rate $\eta_p > 0$. The same as switched systems with all subsystems unstable, it is not easy to find a Lyapunov function and the corresponding parameter η_p satisfying (2.26). Different from [92, Thm. 1] that uses the DT to ensure asymptotic stability for all unstable mode switching systems, we consider here the use of MDALT to ensure exponentially stability of the switched systems. Based on Theorem 2.7, we will establish a sufficient condition that is easily verifiable for analysing the contraction property of switched systems.

As pursued in recent literature, the contraction analysis pertains to the stability analysis of nonlinear system using linear systems theory via its variational system

(2.4). As the variational system can be regarded as a state-dependent linear system with the state ξ, quadratic Lyapunov function can directly be used to prove the stability. Hence let us consider a time dependent Lyapunov function of the quadratic form $V_p(x, \xi, t) = \xi^\top M_p(t)\xi$ for some matrix function $M_p : [0, \infty) \to \mathbb{R}^{n \times n}$ with symmetric, positive definite values. The following lemma provides conditions on such Lyapunov functions to ensure the contracting property of switched system (2.1).

Lemma 2.10. *Consider a switched nonlinear system* (2.1) *with given switching times* $\mathscr{S} := \{t_0, t_1, \ldots, t_i, \ldots t_n, \ldots\}$ *generated by* $\sigma : [0, \infty) \to \mathcal{M}$. *Let each mode p be classified as either stable or unstable, i.e.* $\mathcal{M} = \mathcal{S} \,\dot{\cup}\, \mathcal{U}$ *and correspondingly assume that there exists* $\tau_{ap} > 0$ *such that* (2.24) *holds for the stable mode $p \in \mathcal{S}$ or* (2.25) *holds for the unstable mode $p \in \mathcal{U}$. Suppose that for each mode $p \in \mathcal{M}$ there exist* $\overline{m}_p \geqslant \underline{m}_p \geqslant 0$ *and a time dependent symmetric matrix $M_p(t)$ such that*

$$\underline{m}_p I \leqslant M_p(t) \leqslant \overline{m}_p I, \quad \forall p \in \mathcal{M}, \tag{2.36}$$

$$F_p(x,t)^\top M_p(t) + \dot{M}_p(t) + M_p(t)F_p(x,t) \leqslant \eta_p M_p(t), \forall p \in \mathcal{M}, \tag{2.37}$$

with $\eta_p \geqslant 0$ *if $p \in \mathcal{U}$ or $\eta_p < 0$ otherwise. Assume that for every $p \in \mathcal{M}$, there exists* $\mu_p > 0$, *such that*

$$M_{\sigma(t_i)}(t_i) \leqslant \mu_{\sigma(t_i^-)} M_{\sigma(t_i^-)}(t_i^-), \quad \forall t_i \in \mathscr{S}. \tag{2.38}$$

Then the switched nonlinear system (2.1) *is contracting for any MDADT/MDALT switching signals satisfying* (2.28).

Proof. By taking a Lyapunov function in the form of $V_p(x, \xi, t) = \xi_p^\top M_p(t)\xi_p$, it follows that (2.36) and (2.38) satisfy (2.30) and (2.27) in Theorem 2.7, respectively. By differentiating $V_p(x, \xi, t)$ along the trajectory of system (2.4), we have $\dot{V}_p(x, \xi, t) = \xi_p^\top \left(F_p(x,t)^\top M_p(t) + \dot{M}_p(t) + M_p(t)F_p(x,t)\right)\xi_p$. Using (2.37), it follows that $\dot{V}_p(x, \xi, t) \leqslant \eta_p V_p(x, \xi, t)$, e.g. (2.26) holds. By Theorem 2.7, it implies that (2.1) is contracting for any switching signals satisfying (2.28). $\qquad\square$

We note that the most popular quadratic Lyapunov function in contraction analysis literature is $V_p(x, \xi, t) = \xi^\top M_p\xi$, where M_p is a positive definite constant matrix [65]. In this case, $\dot{M}_p(t)$ in (2.37) is vanished. However, in the contraction analysis problem, since $F(x)$ in (2.4) is time-varying and state-dependent, the existence of such a *constant* matrix M_p is not always possible. In addition, in this paper, we allow subsystems are all non-contracting, M_p should be time-dependent. Hence, in general, allowing for time-varying matrix $M_p(t)$ in Lemma 2.10 leads to a significantly less conservative stability condition. For a general time dependent matrix $M_p(t)$, the inequality (2.38) is not trivial to solve. An-

other well-known technique to solve such a problem is the discretized Lyapunov function technique which is widely used in the stabilization of linear switched systems [49, 92]. The basic idea of the discretized Lyapunov function technique is to linearize $M_p(t)$ into the form of $\frac{t-t_i}{\tau_{dp}}P_p + (1 - \frac{t-t_i}{\tau_{dp}})Q_p$. However, it can be difficult to find such $M_p(t)$ for some simple systems, e.g. for the switched system

$$p = 1 : \begin{cases} \dot{x}_1 = -1.9x_1 + 0.6x_2, \\ \dot{x}_2 = 0.5x_1 + 0.7x_2, \end{cases} \quad p = 2 : \begin{cases} \dot{x}_1 = 0.5x_1 - 0.9x_2, \\ \dot{x}_2 = 0.1x_1 - 1.4x_2. \end{cases}$$. If we apply dis-

cretized Lyapunov function technique as presented in [92] to this switched system, the corresponding LMIs are not feasible or $\tau_{dp} > -\frac{\ln \mu_p}{\eta_p}$. We will present later in Corollary 2.15 a method to design stabilizing switching signals for this switched system.

In order to compensate the conservativity brought by the Matrix Young inequality, in the following, we propose a construction of $M_p(t)$ in a nonlinear fashion by the addition of $\phi_p(t)\big(1 - \phi_p(t)\big)G_p$ to $M_p(t)$, which is more general than the discretized Lyapunov function proposed in [49, 92]. By considering the class of switching signals with mode dependent strict dwell time $\tau_{dp} > 0$, i.e., each mode p is active at least for τ_{dp} time before switching to another mode, we can transform the inequality condition of (2.36)-(2.38) into LMI conditions in Theorem 2.13 presented below. This is achieved by introducing a time-varying Lyapunov function that interpolates two quadratic constant Lyapunov functions in a prescribed dwell time τ_{dp}. Before stating our main result, we first recall two technical lemmas on matrix algebra.

Lemma 2.11. *(Matrix Young inequality): For any $X, Y \in \mathbb{R}^{n \times m}$ and any symmetric positive-definite matrix $S \in \mathbb{R}^{n \times n}$,*

$$X^\top Y + Y^\top X \leqslant X^\top SX + Y^\top S^{-1}Y \tag{2.39}$$

holds.

Lemma 2.12. *(Lemma 2 in [49]) Consider the matrix polynomial $f : [0,1]^n \to \mathbb{R}^{n \times n}$ defined by*

$$f(\tau_1, \tau_2, \cdots, \tau_n) = \Sigma_0 + \tau_1 \Sigma_1 + \tau_1 \tau_2 \Sigma_2 + \cdots + (\prod_{k=1}^{n} \tau_k)\Sigma_n, \quad \forall \tau_k \in [0, 1], \tag{2.40}$$

for some matrices $\Sigma_0, \Sigma_1, \ldots, \Sigma_n$. If the matrices Σ_k, $k \in \mathbb{N}$, are symmetric and satisfy $\sum_{k=0}^{d} \Sigma_k < 0$ (or $\sum_{k=0}^{d} \Sigma_k > 0$) for all $d = 0, 1, \cdots, n$, then $f(\tau_1, \tau_2, \cdots, \tau_n) < 0$ (or $f(\tau_1, \tau_2, \cdots, \tau_n) > 0$).

Theorem 2.13. *Consider switched nonlinear system* (2.1) *with globally Lipschitz f_p, $p \in \mathcal{M}$ and with given switching times $\mathscr{S} := \{t_0, t_1, \ldots, t_i, \ldots t_n, \ldots\}$ generated by*

$\sigma : [0, \infty) \rightarrow \mathcal{M}$. *Assume that the modes can be classified as stable or unstable, i.e.* $\mathcal{M} = \mathcal{S} \cup \mathcal{U}$ *and assume that for every mode p there exists $\tau_{ap} > 0$ such that* (2.24) *for* $p \in \mathcal{S}$ *or* (2.25) *for $p \in \mathcal{U}$ holds. Suppose that for each mode $p \in \mathcal{M}$ there exist a minimum mode dependent dwell time $\tau_{dp} > 0$, a constant matrix A_p, a positive semi-definite matrix Γ_p, symmetric constant matrices P_p, Q_p, G_p, and positive constants $\overline{m}_p > 0, \epsilon_p \geqslant 0$ such that f_p is decomposed[2] into the following form*

$$f_p(x, t) = A_p x + g_p(x, t), \tag{2.41}$$

with

$$\nabla_x g_p(x, t)^\top \nabla_x g_p(x, t) \leqslant \Gamma_p, \qquad \forall x \in \mathbb{R}^n, t \geqslant 0, \tag{2.42}$$

and

$$0 < Q_p < \overline{m}_p I, \quad 0 < P_p < \overline{m}_p I, \quad 0 < P_p + G_p < \overline{m}_p I, \tag{2.43}$$

$$A_p^\top Q_p + Q_p A_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) + \epsilon_p^{-1}\Gamma_p + \epsilon_p \overline{m}_p Q_p \leqslant \eta_p Q_p, \tag{2.44}$$

$$A_p^\top (P_p + G_p) + (P_p + G_p)A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) + \epsilon_p^{-1}\Gamma_p$$
$$+ \epsilon_p \overline{m}_p (P_p + G_p) \leqslant \eta_p(P_p + G_p), \tag{2.45}$$

$$A_p^\top P_p + P_p A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) + \epsilon_p^{-1}\Gamma_p + \epsilon_p \overline{m}_p P_p \leqslant \eta_p P_p, \tag{2.46}$$

$$A_p^\top P_p + P_p A_p + \epsilon_p^{-1}\Gamma_p + \epsilon_p \overline{m}_p P_p \leqslant \eta_p P_p, \tag{2.47}$$

hold with $\eta_p \geqslant 0$ if $p \in \mathcal{U}$ or $\eta_p < 0$ otherwise. Assume that for every $p \in \mathcal{M}$, there exists $\mu_p > 0$ such that

$$Q_{\sigma(t_i)} \leqslant \mu_{\sigma(t_i^-)} P_{\sigma(t_i^-)}, \quad \forall t_i \in \mathscr{S}. \tag{2.48}$$

Then the switched nonlinear system (2.1) *is contracting for any MDADT/MDALT switching signals satisfying* (2.28), *and which have mode dependent dwell time $\tau_{dp} > 0$.*

Proof. Let us define $M_p(t)$ in the following form

$$M_p(t) = \begin{cases} \phi_p(t)\Big(1 - \phi_p(t)\Big)G_p + \phi_p(t)P_p + \Big(1 - \phi_p(t)\Big)Q_p, t \in [t_i, t_i + \tau_{dp}), \\ P_p, \qquad t \in [t_i + \tau_{dp}, t_{i+1}), \end{cases} \tag{2.49}$$

where $\phi_p(t) = \frac{t-t_i}{\tau_{dp}}$, so that $M_p(t_i) = Q_p$ and $M_p(t_i + \tau_{dp}) = P_p$. Note that $M_p(t)$ is positive definite according to (2.43) and Lemma 2.12. Now, let us consider $M_p(t)$

---

[2]This decomposition is well-posed since the vector field f_p is assumed to be globally Lipschitz. The matrix A_p in this decomposition can be non-Hurwitz, which is relevant for the unstable modes.

in the time interval $[t_i, t_i + \tau_{dp})$. The time derivative of $M_p(t)$ is given by

$$\dot{M}_p(t) = \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \phi_p(t)\frac{2}{\tau_{dp}}G_p. \qquad (2.50)$$

For $t \in [t_i, t_i + \tau_{dp})$, we obtain from (2.37), (2.49) and (2.50) that

$$F_p(x,t)^\top M_p(t) + \dot{M}_p(t) + M_p(t)F_p(x,t) - \eta_p M_p(t) = \Sigma_1 + \phi_p(t)\Sigma_2 + \phi_p^2(t)\Sigma_3, \qquad (2.51)$$

where

$$\Sigma_1 = F_p^\top Q_p + Q_p F_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p,$$

$$\Sigma_2 = F_p^\top (G_p + P_p - Q_p) + (G_p + P_p - Q_p)F_p - \frac{2}{\tau_{dp}}G_p - \eta_p(G_p + P_p - Q_p),$$

$$\Sigma_3 = -F_p^\top G_p - G_p F_p + \eta_p G_p. \qquad (2.52)$$

According to (2.41), and Lemma 2.11 (for $S = \epsilon_p I$), we have

$$\begin{aligned}
\Sigma_1 &= \left(A_p + \triangledown_x g_p\right)^\top Q_p + Q_p\left(A_p + \triangledown_x g_p\right) + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p \\
&\leqslant A_p^\top Q_p + Q_p A_p + \epsilon_p^{-1} \triangledown_x g_p^\top \triangledown_x g_p + \epsilon_p Q_p Q_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p \\
&\leqslant A_p^\top Q_p + Q_p A_p + \epsilon_p^{-1}\Gamma_p + \epsilon_p \overline{m}_p Q_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p,
\end{aligned} \qquad (2.53)$$

Similarly we have

$$\begin{aligned}
\Sigma_1 + \Sigma_2 &= F_p^\top(G_p + P_p) + (G_p + P_p)F_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \eta_p(G_p + P_p) \\
&\leqslant A_p^\top(G_p + P_p) + (G_p + P_p)A_p + \epsilon_p^{-1}\Gamma_p + \epsilon_p \overline{m}_p(G_p + P_p) \\
&\quad + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \eta_p(G_p + P_p),
\end{aligned} \qquad (2.54)$$

and

$$\begin{aligned}
\Sigma_1 + \Sigma_2 + \Sigma_3 &= F_p^\top P_p + P_p F_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \eta_p P_p \\
&\leqslant A_p^\top P_p + P_p A_p + \epsilon_p^{-1}\Gamma_p + \epsilon_p \overline{m}_p P_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \eta_p P_p,
\end{aligned} \qquad (2.55)$$

Using the hypotheses (2.44), (2.45), and (2.46) of the theorem, it follows that $\Sigma_1 < 0$, $\Sigma_1 + \Sigma_2 < 0$, $\Sigma_1 + \Sigma_2 + \Sigma_3 < 0$. Since $\phi_p(t) \in [0, 1]$, it follows from Lemma 2.12

that (2.51) is negative definite. Similarly, for $t \in [t_i + \tau_{dp}, t_{i+1})$, (2.51) is negative definite according to (2.47). Consequently, in combination with (2.43), (2.48), and (2.28), all hypotheses in Lemma 2.10 are satisfied and the claim of the theorem follows immediately. □

We remark that there are a number of families of systems that can be written in the form of (2.41). This includes Lipschitz systems [103], Lorentz systems, Lur'e systems, and Persidskii systems. Note that, the assumption of g_p after (2.41) is uniformly in x. This is because for a time-varying system, contracting property does not guarantee the boundedness of x (we refer to Example 2.1 later where one of the states can diverge to infinity). However, this condition is less conservative then the global Lipschitz condition presented in [103], and the references therein. For the global Lipschitz condition, one has $\nabla_x g_p(x,t)^\top \nabla_x g_p(x,t) \leqslant \gamma^2 I$, where γ is the Lipschitz constant, while in our condition, Γ_p can be much smaller than $\gamma^2 I$. To illustrate this, let us consider $g_p(x,t) = \left[\begin{smallmatrix} \sin(x_1) \\ 0 \end{smallmatrix}\right]$, where the Lipschitz constant is given by $\gamma = 1$. For this example, we have $\nabla_x g_p(x,t)^\top \nabla_x g_p(x,t) = \left[\begin{smallmatrix} \cos^2(x_1) & 0 \\ 0 & 0 \end{smallmatrix}\right] \leqslant \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Hence $\Gamma_p = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, which is less than $\gamma^2 I$. In addition, if $\nabla_x g_p(x,t)$ is a symmetric matrix, such inequality reduces to the incremental monotonic condition present in [30], or the uniformly Lipschitz smooth condition introduced in [12].

Remark 2.14. Suppose that the hypotheses in Theorem 2.13 hold with $\mathcal{M} = \mathcal{U}$, i.e. all modes are non-contracting. Then the switched nonlinear system (2.1) is contracting for any MDALT switching signals satisfying (2.35).

As an interesting particular case of our main results above, let us consider the stabilization of linear switched systems where all modes are unstable. Using results in Theorem 2.13, we can stabilize such switched unstable systems. Consider a linear switched system given by

$$\dot{x}(t) = A_{\sigma(t)} x(t), \tag{2.56}$$

where $x(t)$ and $\sigma(t)$ are as in (2.1), and A_p, $p \in \mathcal{M}$, are unstable matrices for each mode p.

Corollary 2.15. *Consider a linear switched system* (2.56) *with a given switching sequence* $\mathscr{S} := \{t_0, t_1, \cdots, t_i, \cdots t_n\}$ *generated by* $\sigma(t)$. *Assume that there exists* $\tau_{ap} > 0$ *such that* (2.25) *holds. Suppose that for each mode* $p \in \mathcal{M}$ *there exist a minimum mode dependent dwell time* $\tau_{dp} > 0$, *symmetric constant matrices* P_p, Q_p, G_p, *and scalars* $\overline{m}_p > 0$ *and* $0 < \mu_p < 1$, *such that* (2.43), (2.48), *and the following inequalities*

$$A_p^\top Q_p + Q_p A_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) \leqslant \eta_p Q_p, \ \forall p \in \mathcal{M}, \tag{2.57}$$

$$A_p^\top (P_p + G_p) + (P_p + G_p)A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) \leqslant \eta_p(P_p + G_p), \, \forall p \in \mathcal{M},$$
(2.58)

$$A_p^\top P_p + P_p A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) \leqslant \eta_p P_p, \, \forall p \in \mathcal{M},$$
(2.59)

$$A_p^\top P_p + P_p A_p \leqslant \eta_p P_p, \, \forall p \in \mathcal{M},$$
(2.60)

hold. Then the switched system (2.56) *is exponentially stable for any MDALT switching signals satisfying* (2.35) *and with mode-dependent dwell times* $\tau_{dp} > 0$.

Proof. The proof follows vis-á-vis with the proof of Theorem 2.13 adapted to the switched linear system (2.56). In this case, we have

$$A_p^\top M_p(t) + \dot{M}_p(t) + M_p(t)A_p - \eta_p M_p(t) = \Sigma_1 + \phi_p(t)\Sigma_2 + \phi_p^2(t)\Sigma_3,$$
(2.61)

where

$$\Sigma_1 = A_p^\top Q_p + Q_p A_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p,$$

$$\Sigma_2 = A_p^\top (G_p + P_p - Q_p) + (G_p + P_p - Q_p)A_p - \frac{2}{\tau_{dp}}G_p - \eta_p(G_p + P_p - Q_p),$$

$$\Sigma_3 = -A_p^\top G_p - G_p A_p + \eta_p G_p.$$
(2.62)

It follows from (2.57), (2.58), (2.59), (2.62) and Lemma 2.12 that (2.61) is negative definite. Then, following Theorem 2.13, the linear switched systems (2.56) is contracting. Since $x(t) = 0$ is one of admissible trajectories of (2.56) and it is contracting, it follows that all the trajectories will converge to $x(t) = 0$ exponentially. □

Discretized Lyapunov function technique for stabilizing switched systems with all unstable subsystems can be found in [92, Theorem 2]. The main differences with the results in Corollary 2.15 are as follows. Firstly, the construction of our Lyapunov functions is based on nonlinear interpolation that connects Q_p and P_p via G_p, as opposed to a linear interpolation used in [92]. Consequently, the derivative of $M_p(t)$ in (2.50) may be negative so that the corresponding Lyapunov function may decrease in $[t_i, t_i + \tau_{dp})$, in contrast to the non-decreasing Lyapunov function in [92]. We note that the discretized Lyapunov function technique in [92, Theorem 2] can be obtained by taking $G_p = 0$. Secondly, our approach consider MDALT condition which generalizes the DT condition assumed in [92]. For the previous linear case after Lemma 2.10, by using Corollary 2.15 we can fix $\eta_1 = \eta_2 = 1.7$, $\mu_1 = \mu_2 = 0.7$, then a periodic switching signal with mode duration τ_{di} is given by $\tau_{d1} = \tau_{d2} = 0.2$.

2.5   Simulation setup and results

In this section, three numerical examples will be presented. In the first case, we analyze the contraction of a switched system with mixed contracting-noncontracting modes by using Theorem 2.7. In the second case, we apply Theorem 2.13 to design the switching law for the system whose subsystems are all noncontracting. In the third case, we analyze the stability of a linear switched system whose subsystems are all unstable by using Corollary 2.15. In the last case, we apply our results to solve the synchronization problem of one-way identical non-autonomous systems with switching coupling.

Example 2.1. Consider a switched system (2.1) consisting of two time-varying subsystems, whose dynamics take the form

$$
\begin{aligned}
p = 1 : & \begin{cases} \dot{x}_1 = -x_1 - x_1^3 + 3x_2 \sin t, \\ \dot{x}_2 = -2x_1 \sin t - x_2 + 2 \cos t, \end{cases} \\
p = 2 : & \begin{cases} \dot{x}_1 = x_1 + x_2 + t, \\ \dot{x}_2 = -x_1 - 2x_2 + \cos x_2. \end{cases}
\end{aligned}
\tag{2.63}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector. Subsystem $p = 2$ is non-contracting. The Lyapunov function can be selected as $V_1(\xi) = 2\xi_1^2 + 3\xi_2^2$, $V_2(\xi) = \xi_1^2 + \xi_2^2$. According to Theorem 2.7, we can fix $\eta_1 = -2$, $\eta_2 = 2$, $\mu_1 = 3$, $\mu_2 = 0.5$, the switched law (2.28) is given by $\tau_{a1} \geqslant 0.55$, $\tau_{a2} \leqslant 0.35$. For the simulation shown Figure 2.1 we use a periodic switching signal with $\tau_1 = 0.65$ and $\tau_2 = 0.35$.

Example 2.2. Consider a switched system (2.1) consisting of two noncontracting subsystems, whose dynamics take the form

$$
\begin{aligned}
p = 1 : & \begin{cases} \dot{x}_1 = 0.1x_1 - 0.9x_2 - 0.2\cos(0.1x_1), \\ \dot{x}_2 = 0.1x_1 - 1.4x_2 - 0.7\cos(0.1x_2), \end{cases} \\
p = 2 : & \begin{cases} \dot{x}_1 = -1.9x_1 + 0.6x_2 + 0.7\cos(0.1x_2), \\ \dot{x}_2 = 0.6x_1 - 0.1x_2 + 0.2\cos(0.1x_2). \end{cases}
\end{aligned}
\tag{2.64}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector.

It can be checked that for each mode, there exist a positive eigenvalue of $\nabla_x f_i(x, t)$ which satisfies $\lambda_1 \geqslant 0.0130$ (for the first mode) or $\lambda_2 \geqslant 0.0948$ (for the second mode). In other words, each individual system is non-contracting. As a result, the methods used in [71, 94] are no longer applicable in this particular case.

Using Theorem 2.13, where we fix $\overline{m}_1 = \overline{m}_2 = 0.1$, $\eta_1 = \eta_2 = 0.3$, $\mu_1 = 0.65$, $\mu_2 = 0.6$, $\epsilon_1 = \epsilon_2 = 1$, $\Gamma_1 = \begin{bmatrix} 0.0004 & 0 \\ 0 & 0.005 \end{bmatrix}$, $\Gamma_2 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.0004 \end{bmatrix}$, it can be checked that

Figure 2.1: The plot of trajectories of switched system in Example 2.1 initialized at $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ for mode 1 and 2, respectively, and using a periodic switching signal with $\tau_1 = 0.65$ and $\tau_2 = 0.35$.

using the following symmetric constant matrices

$$
\begin{aligned}
P_i : \quad & \begin{bmatrix} 0.0398 & -0.0071 \\ -0.0071 & 0.0933 \end{bmatrix}, \quad \begin{bmatrix} 0.0881 & -0.0208 \\ -0.0208 & 0.0547 \end{bmatrix}, \\
Q_i : \quad & \begin{bmatrix} 0.0493 & -0.0129 \\ -0.0129 & 0.0326 \end{bmatrix}, \quad \begin{bmatrix} 0.0235 & -0.0013 \\ -0.0013 & 0.0554 \end{bmatrix}, \\
G_i : \quad & \begin{bmatrix} -0.0038 & 0.0013 \\ 0.0013 & -0.0272 \end{bmatrix}, \quad \begin{bmatrix} -0.0340 & 0.0107 \\ 0.0107 & -0.0064 \end{bmatrix},
\end{aligned}
\tag{2.65}
$$

the LMI problem given by (2.43)-(2.48) is feasible. Correspondingly, we have MDALTs as $\bar{\tau}_{a1} = 1.435$, $\bar{\tau}_{a2} = 1.702$, and the minimum dwell time for each mode as $\tau_{d1} = \tau_{d2} = 0.5$.

To illustrate the contraction property, we consider switching signals with periodic switching time (each p mode has the same dwell time). Trajectories of a periodic switching signal with mode duration: $\tau_{a1} = 0.5$, $\tau_{a2} = 1.7$ with two different initial conditions $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are shown in Figure 2.2. The switching signal satisfies hypotheses of Theorem 2.13, Figure 2.2 shows that despite each mode is noncontracting and the distance between the trajectories may increase in each mode (before the first switching, the distance are increasing), the increments are compensated by the switching behaviors, so that the trajectories converge to each other asymptotically.

Example 2.3. Let us consider a linear switched system (2.56) that is composed of two modes as follows.

$$p = 1 : \begin{cases} \dot{x}_1 = -1.9x_1 + 0.6x_2, \\ \dot{x}_2 = 0.5x_1 + 0.7x_2, \end{cases} \quad p = 2 : \begin{cases} \dot{x}_1 = 0.5x_1 - 0.9x_2, \\ \dot{x}_2 = 0.1x_1 - 1.4x_2. \end{cases} \tag{2.66}$$

It can be checked that the unstable pole for each mode is given by $\lambda_1 = 0.8107$ and $\lambda_2 = 0.4514$, respectively. We can directly apply Corollary 2.9 to this switched system, where we fix $\eta_1 = \eta_2 = 1.7$, $\mu_1 = \mu_2 = 0.7$, and $\tau_{d1} = \tau_{d2} = 0.2$. If we apply discretized Lyapunov function technique as presented in [1, 92] to this switched system, the corresponding LMIs are not feasible or $\tau_{dp} > -\frac{\ln \mu_p}{\eta_p}$. However, in our result, it can be checked that using the following symmetric constant matrices

$$P_i : \begin{bmatrix} 62.9289 & -1.6230 \\ -1.6230 & 50.5991 \end{bmatrix}, \quad \begin{bmatrix} 41.6025 & 7.7470 \\ 7.7470 & 75.9246 \end{bmatrix},$$

$$Q_i : \begin{bmatrix} 26.4695 & 5.7076 \\ 5.7076 & 51.6393 \end{bmatrix}, \quad \begin{bmatrix} 41.1033 & -0.6396 \\ -0.6396 & 34.2129 \end{bmatrix}, \tag{2.67}$$

$$G_i : \begin{bmatrix} -12.7895 & 2.7501 \\ 2.7501 & 0.3126 \end{bmatrix}, \quad \begin{bmatrix} 1.1959 & -1.5380 \\ -1.5380 & -13.3743 \end{bmatrix},$$

the LMI problem given by (2.57)-(2.60) is feasible. Correspondingly, we have $\bar{\tau}_{a1} = \bar{\tau}_{a2} = 0.21$. By considering a periodic switching signal with $\tau_{a1} = \tau_{a2} = 0.2$ for the corresponding switched system with initial state $x(0) = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, Figure 2.3 shows the resulting state trajectories which converge to zero as expected.

Example 2.4. In this numerical example, we apply our main results to the synchronization problem of one-way coupled identical oscillators in [87, 94], whose dynamics take the form
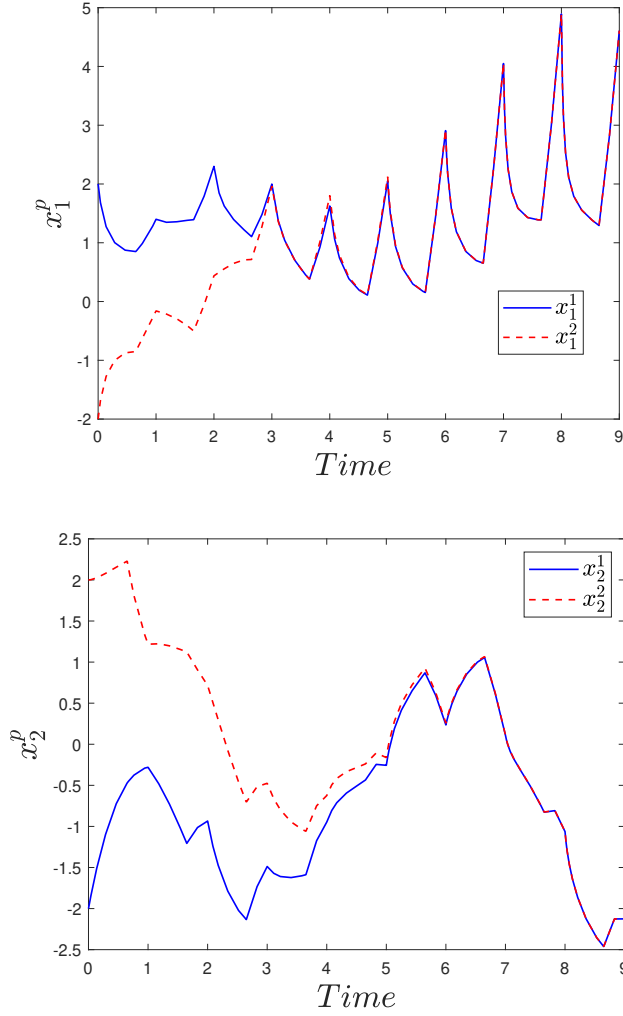
$$\dot{w} = f(w(t), t), \tag{2.68}$$
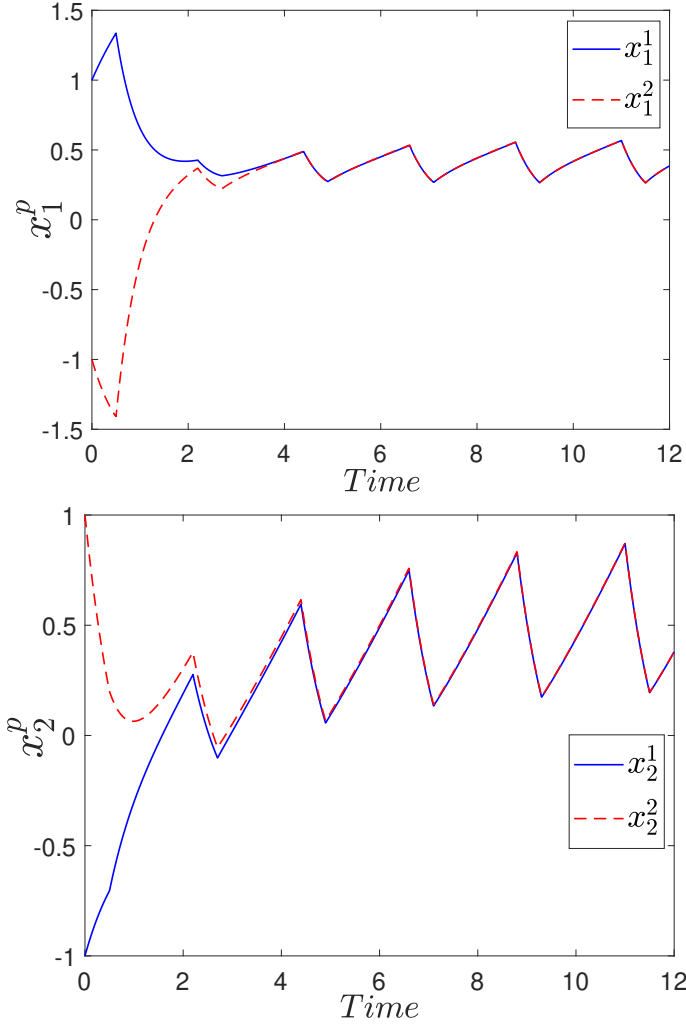
Figure 2.2: The plot of trajectories of switched system in Example 2.2 initialized at $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for mode 1 and 2, respectively, and using a periodic switching signal with $\tau_{a1} = 0.5$ and $\tau_{a2} = 1.7$.

$$\dot{x} = f(x(t), t) + u_{\sigma(t)}(w(t)) - u_{\sigma(t)}(x(t)), \qquad (2.69)$$

where $w(t), x(t) \in \mathbb{R}^n$ is the state vector, $f(w(t), t)$ is the dynamics of the uncoupled oscillators, and $u_{\sigma(t)}(w(t)) - u_{\sigma(t)}(x(t))$ is the switched coupling force. The synchronization goal is to design a switching sequence $\sigma(t)$ such that the trajectories of (2.68), (2.69) satisfy $\lim_{t \to +\infty} \|x(t) - w(t)\| = 0$. Since $w(t)$ and $x(t)$ are both the solutions of (2.69), it follows that if $\dot{x} = f(x(t), t) - u_{\sigma(t)}(x(t))$ is contracting,
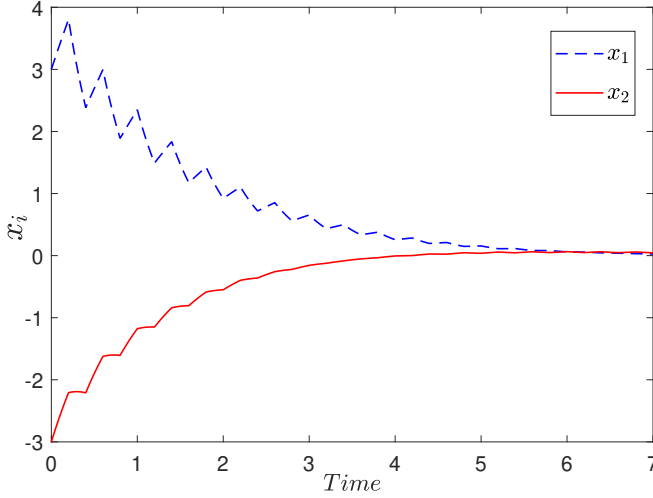
Figure 2.3: The plot of state trajectories of x_i in Example 2.3 with initial condition $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ and using periodic switching signal with $\tau_{a1} = \tau_{a2} = 0.2$.

the synchronization will be achieved.

Let us now consider the following non-autonomous system, and the coupled switched oscillators with two modes.

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -0.9w_1 + 0.6w_2 + 2t \\ 0.1w_1 + 0.4w_2 + 2t \end{bmatrix}, \tag{2.70}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.9x_1 + 0.6x_2 + 2t \\ 0.1x_1 + 0.4x_2 + 2t \end{bmatrix} + u_{\sigma(t)}(w) - u_{\sigma(t)}(x), \tag{2.71}$$

where $\sigma(t) \in \{1, 2\}$, $u_{\sigma(t)}\left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} -z_1 + 1.5z_2 + 0.2\cos(0.1z_1) \\ 1.8z_2 + 0.7\cos(0.1z_2) \end{bmatrix}$. Then, $f(x(t), t) - u_{\sigma(t)}(x(t))$ are given by

$$\begin{cases} \dot{x}_1 = 0.5x_1 - 0.9x_2 + 0.2\cos(0.1x_2) \\ \dot{x}_2 = 0.1x_1 - 1.4x_2 + 0.7\cos(0.1x_1) \end{cases} \tag{2.72}$$

$$\begin{cases} \dot{x}_1 = -1.9x_1 + 0.6x_2 + 0.7\cos(0.1x_2) \\ \dot{x}_2 = 0.5x_1 + 0.3x_2 + t + 0.2\cos(0.1x_1) \end{cases} \tag{2.73}$$

The Jacobian of $f(x(t), t) - u_p(x(t))$, $p = 1, 2$, are given by

$$\nabla_x f_1(x, t) - \nabla_x u_1(x, t) = \begin{bmatrix} -1.9 & 0.6 - 0.02\sin(0.1x_2) \\ 0.5 - 0.07\sin(0.1x_1) & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & -0.9 \\ 0.1 & -1.4 \end{bmatrix} + \begin{bmatrix} 0.02\sin(0.1x_1) & 0 \\ 0 & 0.07\sin(0.1x_2) \end{bmatrix},$$

(2.74)

and

$$\nabla_x f_2(x, t) - \nabla_x u_2(x, t) = \begin{bmatrix} 0.5 & -0.9 - 0.07\sin(0.1x_2) \\ 0.1 - 0.02\sin(0.1x_1) & -1.4 \end{bmatrix}$$

$$= \begin{bmatrix} -1.9 & 0.6 \\ 0.6 & -0.1 \end{bmatrix} + \begin{bmatrix} 0 & -0.07\sin(0.1x_2) \\ -0.02\sin(0.1x_1) & 0 \end{bmatrix}.$$

(2.75)

It can be checked that for each mode, there exist a positive eigenvalue of $\nabla_x f(x, t) - \nabla_x u_{\sigma(t)}(x, t)$ which satisfies $\lambda_1 \geqslant 0.0130$ (for the first mode) or $\lambda_2 \geqslant 0.0948$ (for the second mode). In other words, each individual system is non-contracting. As a result, the methods used in [71, 94] are no longer applicable in this particular case. We choose Γ_1 and Γ_2 in (2.42) as the following, respectively.

$$\begin{bmatrix} 0.005 & 0 \\ 0 & 0.0004 \end{bmatrix} \quad \begin{bmatrix} 0.0004 & 0 \\ 0 & 0.005 \end{bmatrix}$$

(2.76)

Using Theorem 2.13, where we fix $\overline{m}_1 = \overline{m}_2 = 0.1$, $\eta_1 = \eta_2 = 0.3$, $\mu_1 = 0.65$, $\mu_2 = 0.6$, $\epsilon_1 = \epsilon_2 = 1$, it can be checked that using the following symmetric constant matrices

$$P_i : \begin{bmatrix} 0.0398 & -0.0071 \\ -0.0071 & 0.0933 \end{bmatrix}, \quad \begin{bmatrix} 0.0881 & -0.0208 \\ -0.0208 & 0.0547 \end{bmatrix},$$

$$Q_i : \begin{bmatrix} 0.0493 & -0.0129 \\ -0.0129 & 0.0326 \end{bmatrix}, \quad \begin{bmatrix} 0.0235 & -0.0013 \\ -0.0013 & 0.0554 \end{bmatrix},$$

$$G_i : \begin{bmatrix} -0.0038 & 0.0013 \\ 0.0013 & -0.0272 \end{bmatrix}, \quad \begin{bmatrix} -0.0340 & 0.0107 \\ 0.0107 & -0.0064 \end{bmatrix},$$

(2.77)

the LMI problem given by (2.43)-(2.48) is feasible. Correspondingly, we have MDALTs as $\overline{\tau}_{a1} = 1.435$, $\overline{\tau}_{a2} = 1.702$, and the minimum dwell time for each mode as $\tau_{d1} = \tau_{d2} = 0.5$.

To illustrate the synchronization property. We consider switching signals with periodic switching time (each p mode has the same dwell time). The states with three different switching signals are initialized as: $w(0) = [10, -10]$; (i). $x_1(0) = [-30, 30]$, $\tau_{a1} = 1$, $\tau_{a2} = 1$, (ii). $x_2(0) = [30, -30]$, $\tau_{a1} = 0.5$, $\tau_{a2} = 1.7$;

and (iii). $x_3(0) = [30, 30]$, $\tau_{a1} = 1.43$, $\tau_{a2} = 0.5$. As all switching signals that satisfy hypotheses of Theorem 2.13, i.e. $\tau_{a1} \in [0.5, 1.435]$, $\tau_{a2} \in [0.5, 1.702]$, we can conclude that $\dot{x} = f(x(t), t) - u_{\sigma(t)}(x(t))$ is contracting (i.e. synchronization occurs). The resulting three different trajectories are shown in Figure 2.4.
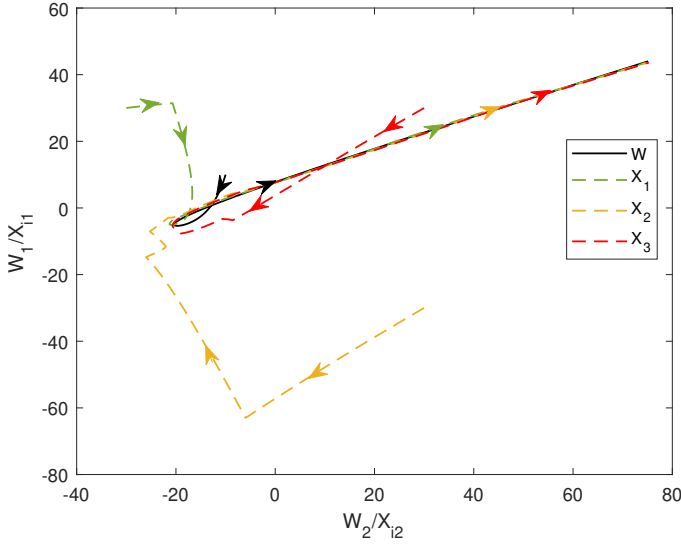
Figure 2.4: State trajectories of w/x_i. x_1: $\tau_{a1} = 1$, $\tau_{a2} = 1$, x_2: $\tau_{a1} = 0.5$, $\tau_{a2} = 1.7$, x_3: $\tau_{a1} = 1.43$, $\tau_{a2} = 0.5$.

2.6   Conclusion

In this chapter, the contraction property of switched systems with mixed contracting-noncontracting modes have been studied. It is established based on a necessary and sufficient condition that connects the contraction property of the original switched systems and the UGES of its variational systems. A time-dependent Lyapunov function and a mixed MDADT/MDALT method are introduced to study the UGES of the switched variational systems. Furthermore LMI conditions are presented that allow for numerical validation on the contraction property of switched systems with computable mode-dependent average dwell-time. Our results can be applied to stabilize linear switched systems with all unstable modes, and solve the synchronization problem of switched systems.

3

Contraction analysis of time-varying DAE systems via auxiliary ODE systems

This chapter studies the contraction property of time-varying differential-algebraic equation (DAE) systems by embedding them to higher-dimension ordinary differential equation (ODE) systems. The first result pertains to the equivalence of the contraction of a DAE system and the uniform global exponential stability (UGES) of its variational DAE system. Such equivalence inherits the well-known property of contracting ODE systems on a specific manifold. Subsequently, we construct an auxiliary ODE system from a DAE system whose trajectories encapsulate those of the corresponding variational DAE system. Using the auxiliary ODE system, a sufficient condition for contraction of the time-varying DAE system is established by using matrix measure which allows us to estimate an upper bound on the parameters of the auxiliary system. Finally, we apply the results to analyze the stability of time-invariant DAE systems, and to design observers for time-varying ODE systems.

3.1 Introduction

As a generalization of ordinary differential equation (ODE) systems, differential-algebraic equation (DAE) systems have been studied for the past decades due to their relevance in representing numerous modern engineering systems with constraints. Some well-known examples of such engineering systems are electrical networks with Kirchhoff's laws [60] and mechanical systems with rigid body constraints [68]. The DAE systems can also be used to model power systems [37] and chemical processes [46]. Typically, DAE systems consist of a set of differential equations describing the system dynamics, and a set of algebraic equations describing the constraints. Solving DAE systems is more challenging than solving ODE systems due to the implicit relationship between the differential equations and algebraic equations. DAE systems can be classified as either index-1 [37] or higher index [29] by the degree of the highest derivative of the algebraic equation.

Index-1 DAE systems are particularly important in control theory, because the algebraic restriction can be substituted directly into the dynamics. Consequently, they can numerically be solved by using standard techniques for solving ODE [81], which allows numerical simulations to be carried out straightforwardly [76].

The analysis of DAE systems has been widely studied in the literature [16, 32, 37, 58, 63, 64, 67] with a large body of works concerned with index-1 DAE systems. In [32], the authors show that any solvable DAE power systems are of index one. By this result, robust H_∞ observers for power networks are presented in [64], and a load- and renewable-following control approach for power system is proposed in [63]. There are several methods for analyzing the stability of DAE systems, including the Lyapunov method [16, 37], the energy-based methods [67], as well as, the eigenvalue analysis [58]. In [37], the DAE systems are embedded in reduced ODE systems, where Lyapunov method can be applied to get the Lyapunov stability. In [16], an incremental Lyapunov function is applied to analyze the asymptotic stability and optimal resource allocation for a network preserved microgrid model with active and reactive power loads. In [67], bifurcation theory is used to characterize the stability boundary for power systems in DAE, and an energy function method is developed to guarantee both rotor angular stability and voltage stability. A method for defining the small-signal stability of delay DAE systems based on eigenvalue analysis and an approximation of the characteristic equation at equilibrium points are proposed in [58].

In all of the above results, the properties of time-invariant DAE systems are analyzed, while the extension of these results to the time-varying DAE systems remains non-trivial. The exponential stability and robustness of linear time-varying DAE systems with index-1 are studied in [9, 11]. In [11], the Bohl exponent theory for stability analysis of ODEs is extended to DAEs. When there are perturbations in systems' matrices, a stability radius has been investigated in [9], which includes the lower bound computation of the stability radius.

As one of the stability analysis methods for time-varying systems, which has gained popularity in recent years, contraction analysis focuses on the relative trajectory of the nonlinear time-varying system rather than a specific equilibrium point. There are many methods to analyze the contractivity of nonlinear time-varying ODE systems in literature, such as, [5, 7, 27, 55, 100] among many others. An ODE system is contracting if and only if the associated variational system is uniformly globally exponentially stable (UGES) [7]. In [55], the contraction property can be guaranteed if the largest eigenvalue of the symmetric part of the variational systems is uniformly strictly negative. Finsler–Lyapunov functions are introduced in [27] to analyze the incremental exponential stability of the system. Mode-dependent Lyapunov functions for contracting switched nonlinear systems are presented in [100]. In [5], the focus is on investigating transverse exponential stability (a generalized notion of contraction) by employing a Lyapunov matrix

transversal equation. In the context of DAE systems, the contraction analysis thereof has recently been presented in [62]. In [62], the authors proved that if the algebraic equation satisfies some sufficient conditions, the contractivity of time-invariant DAE systems can be obtained through exponential stability analysis of the corresponding reduced variational ODE systems using matrix measure. In Section 3.3.2, we show that this approach can be restrictive and is not applicable to some time-varying DAE systems.

In this chapter, we analyze the contraction property of index-1 nonlinear time-varying DAE systems by using an ODE approach. As our first main result, we establish that the uniform global exponential stability (UGES) of the variational DAE dynamics is a sufficient and necessary condition to the contractivity of the original DAE systems. This condition can be viewed as a DAE counterpart of the results for ODE presented in [7, Prop. 1]. Subsequently, we construct an auxiliary ODE system whose convergence property can encapsulate the same property of the variational DAE dynamics. With this construction, we can analyze the contractivity of the DAE systems by applying conventional control theory (such as Lyapunov approach, matrix measure method) to the auxiliary ODE system. As our second contribution, we provide sufficient conditions on the contraction of nonlinear time-varying DAE system by using matrix measure method to analysis the UGES property of the auxiliary ODE system. In general, these conditions ensure the contraction of the DAE system without analysing its reduced system. As our third contribution, we employ these conditions to design observers for time-varying ODE systems by treating the output as an algebraic equation. Furthermore, we investigate the exponential stability of time-invariant DAE systems by ensuring that the DAE system is contractive and the equilibrium lies within its trajectory set.

The chapter is organized as follows. In Section 3.2, we present preliminaries and problem formulation. Our main results are proposed in Section 3.3, where we present necessary and sufficient conditions for the contractivity of time-varying DAE systems, and the ODE approach. The numerical simulations and applications are provided in Section 3.4 and the conclusions are given in Section 3.5.

3.2 Preliminaries and problem formulation

Throughout this paper, we consider the following nonlinear time-varying DAE systems

$$\begin{cases} \dot{w} = f(t, w, z), \\ 0 = g(t, w, z), \end{cases} \tag{3.1}$$

where $w(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^m$ refers to the algebraic vector, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the vector field, and $g : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$

describes the algebraic manifold. We assume that f is continuously differentiable and g is twice-continuously differentiable. In this note, we consider only the continuously differentiable solutions $\begin{cases} w(t) = \varphi(t_0, w_0, z_0) \\ z(t) = \psi(t_0, w_0, z_0) \end{cases}$ of (3.1) with admissible initial conditions $(w_0, z_0) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying the algebraic constraint

$$g(t, w_0, z_0) = 0. \tag{3.2}$$

We assume the DAE system (3.1) is of index-1, i.e. the (partial) Jacobian matrix $\frac{\partial g}{\partial z}(t, w, z) \in \mathbb{R}^{m \times m}$ is invertible for all (t, w, z). This assumption guarantees the existence and uniqueness of a local solution to (3.1) for any initial condition satisfying (3.2) (see also, [37]). Throughout the paper, we will assume that every local solution can be extended to a solution defined on the whole time domain $[0, \infty)$.

Note that the index-1 assumption allows to apply the implicit function theorem to solve the constraint (3.2) for z_0 for any given (t_0, w_0) and in the following we will therefore write $z_0(w_0)$ to denote the unique value for z_0 which satisfies (3.2) for an arbitrarily given w_0 (we omit the dependency on t_0 as we consider the initial time as fixed in the following analysis).

Definition 3.1. A time-varying DAE system (3.1) is called *contracting* if there exists positive numbers c and α such that for any pair of solutions $W_i(t) = \begin{bmatrix} w_i(t) \\ z_i(t) \end{bmatrix} \in \mathbb{R}^n$ of (3.1) with $i = 1, 2$, we have

$$\|W_1(t) - W_2(t)\| \leqslant ce^{-\alpha t}\|W_1(t_0) - W_2(t_0)\|, \quad \forall t \geqslant t_0. \tag{3.3}$$

In order to study contractivity of the DAE system (3.1), we will analyse the (uniform) stability of the corresponding variational DAE systems. The variational system of system (3.1) is given by

$$\begin{cases} \dot{\xi} = \frac{\partial f}{\partial w}(t, w(t), z(t)) \cdot \xi + \frac{\partial f}{\partial z}(t, w(t), z(t)) \cdot \nu, \\ 0 = \frac{\partial g}{\partial w}(t, w(t), z(t)) \cdot \xi + \frac{\partial g}{\partial z}(t, w(t), z(t)) \cdot \nu, \end{cases} \tag{3.4}$$

where $w(t)$ and $z(t)$ are solutions of (3.1). We omit the explicit parametrization (t, w, z) whenever it is clear from the context.

Definition 3.2. The variational DAE system (3.4) is called uniformly globally exponentially stable (UGES), if there exist positive numbers c, α (independent of the solution $W(\cdot)$) such that for every solution $\Xi(t) := \begin{bmatrix} \xi(t) \\ \nu(t) \end{bmatrix} \in \mathbb{R}^n$ of (3.4) the inequality

$$\|\Xi(t)\| \leqslant ce^{-\alpha t}\|\Xi(t_0)\|, \tag{3.5}$$

holds for all $t \geqslant t_0$.

Given the assumption of reversibility for $\frac{\partial g}{\partial z}$, we can express the reduced system of equation (6) in the following form:

$$\dot{\xi} = \Big(\frac{\partial f}{\partial w} - \frac{\partial f}{\partial z} \big(\frac{\partial g}{\partial z} \big)^{-1} \frac{\partial g}{\partial w} \Big) \xi. \tag{3.6}$$

In [62, Proposition 1], a sufficient condition is introduced to analyse the contractivity of time-invariant DAE systems by analysing its reduced systems (3.6). However, this approach may not be applicable in the time-varying case. As a simple example, consider the following time-varying DAE system

$$\begin{cases} \dot{w} = -w + e^{-3t} z, \\ \quad 0 = e^{3t} w + z, \end{cases} \tag{3.7}$$

which has the same form as its variational DAE system. Its reduced system is given by $\dot{w} = -2w$, which is a contractive system. Consequently the trajectory of the system is $\begin{cases} w = w_0 e^{-2t} \\ z = -w_0 e^t \end{cases}$, which shows the DAE system is not contracting (z is not contracting). This is due to the unboundedness of $\big(\frac{\partial g}{\partial z} \big)^{-1} \frac{\partial g}{\partial w} = e^{3t}$, which implies that the method proposed in [62] is not applicable when dealing with time-varying systems. Notice that, in this particular case, the contraction analysis problem can be effectively addressed by comparing the exponential rate of the reduced system with an exponential bound on $\left\| \big(\frac{\partial g}{\partial z} \big)^{-1} \frac{\partial g}{\partial w} \right\|$. However, it is worth acknowledging that obtaining information about the reduced system or establishing a bound for $\left\| \big(\frac{\partial g}{\partial z} \big)^{-1} \frac{\partial g}{\partial w} \right\|$ can be a challenging task for certain systems.

The objective of this paper is to provide a sufficient condition that guarantees the contractivity of time-varying DAE systems, even in situations where prior knowledge about the DAE system is unavailable.

Before stating our main results, let us recall the matrix measure $\mu_q(A)$[19]. For a matrix measure $\mu_q(A)$, it follows that when $q = 1, 2$ or ∞, we have

$$\begin{aligned} \mu_1(A) &= \max_j \Big(a_{jj} + \sum_{i \neq j} |a_{ij}| \Big), \\ \mu_2(A) &= \max_i \Big(\lambda_i \{ \frac{A + A^\top}{2} \} \Big), \text{ or} \\ \mu_\infty(A) &= \max_i \Big(a_{ii} + \sum_{j \neq i} |a_{ij}| \Big), \end{aligned} \tag{3.8}$$

respectively. In this chapter, all the norms are defined using the p-norm

3.3   Main results

In this section, we firstly establish an equivalent relationship between the contraction of a DAE system and the uniform global exponential stability (UGES) of its variational DAE system. Secondly, we construct an auxiliary ODE system that encapsulates the behaviors of the variational DAE system. Thirdly, a sufficient condition is presented that guarantees the UGES of the auxiliary ODE system and is numerically implementable.

3.3.1   A necessary and sufficient condition

Proposition 3.3. *The DAE system* (3.1) *is contracting if and only if the corresponding variational DAE system* (3.4) *is UGES.*

Proof. Let us establish a relationship between the solutions of (3.1) and those of (3.4). Let $\begin{bmatrix} w(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \varphi(t,w_0,z_0(w_0)) \\ \psi(t,w_0,z_0(w_0)) \end{bmatrix}$ and $\begin{bmatrix} \hat{w}(t) \\ \hat{z}(t) \end{bmatrix} = \begin{bmatrix} \varphi(t,w_0+\delta\xi_0,z_0(w_0+\delta\xi_0)) \\ \psi(t,w_0+\delta\xi_0,z_0(w_0+\delta\xi_0)) \end{bmatrix}$ be two trajectories of (3.1) with initial conditions $\begin{bmatrix} w_0 \\ z_0(w_0) \end{bmatrix}$ and $\begin{bmatrix} w_0+\delta\xi_0 \\ z_0(w_0+\delta\xi_0) \end{bmatrix}$, respectively, where δ is a sufficiently small positive constant and ξ_0 will be related later to the initial condition of (3.1). As they are solutions of (3.1), they satisfy

$$g(t, w(t), z(t)) = 0, \tag{3.9}$$

and

$$g(t, \hat{w}(t), \hat{z}(t)) = 0. \tag{3.10}$$

Denote the partial Jacobian matrices of φ and ψ with respect to the second or third argument evaluated at $(t, w_0, z_0(w_0))$ by $\Phi_{w_0}(t), \Phi_{z_0}(t), \Psi_{w_0}(t), \Psi_{z_0}(t)$, respectively. By differentiating both sides of (3.9) with respect to w_0, we have

$$
\begin{aligned}
0 = {} & \tfrac{\partial g}{\partial w}(t, w(t), z(t)) \cdot \left(\Phi_{w_0}(t) + \Phi_{z_0}(t) \cdot z_0'(w_0) \right) \\
& + \tfrac{\partial g}{\partial z}(t, w(t), z(t)) \cdot \left(\Psi_{w_0}(t) + \Psi_{z_0}(t) \cdot z_0'(w_0) \right).
\end{aligned}
\tag{3.11}
$$

In the following, we will show that

$$
\begin{cases}
\xi(t) := \lim\limits_{\delta \to 0} \frac{\varphi(t,w_0+\delta\xi_0,z_0(w_0+\delta\xi_0))-\varphi(t,w_0,z_0(w_0))}{\delta}, \\
\nu(t) := \lim\limits_{\delta \to 0} \frac{\psi(t,w_0+\delta\xi_0,z_0(w_0+\delta\xi_0))-\psi(t,w_0,z_0(w_0))}{\delta},
\end{cases}
\tag{3.12}
$$

are a pair of solutions of (3.4) with initial value

$$\xi(t_0) = \xi_0,$$
$$\nu(t_0) = \nu_0 := \lim_{\delta \to 0} \frac{z_0(w_0 + \delta\xi_0) - z_0(w_0)}{\delta}.$$

We can rewrite (3.12) as

$$\begin{cases} \xi(t) := \Big(\Phi_{w_0}(t) + \Phi_{z_0}(t) \cdot z_0'(w_0)\Big) \cdot \xi_0, \\ \nu(t) := \Big(\Psi_{w_0}(t) + \Psi_{z_0}(t) \cdot z_0'(w_0)\Big) \cdot \xi_0. \end{cases} \tag{3.13}$$

From (3.11) and (3.13), we know that (3.12) satisfies $0 = \frac{\partial g}{\partial w}\xi + \frac{\partial g}{\partial z}\nu$. The flow $\varphi(t, w_0, z_0(w_0))$ of (3.1) satisfies

$$\varphi(t, w_0, z_0(w_0)) = w_0 + \int_0^t f\big(\varphi(\tau, w_0, z_0(w_0)), \psi(\tau, w_0, z_0(w_0))\big)\mathrm{d}\tau, \tag{3.14}$$

and similarly, the flow $\varphi(t, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0))$ satisfies

$$\varphi(t, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0)) = w_0 + \delta\xi_0 +$$
$$\int_0^t f\big(\varphi(\tau, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0)), \psi(\tau, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0))\big)\mathrm{d}\tau. \tag{3.15}$$

Hence,

$$\xi(t) = \xi_0 +$$
$$\int_0^t \lim_{\delta \to 0} \frac{1}{\delta} \times \Big(f\big(\varphi(\tau, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0)), \psi(\tau, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0))\big) \tag{3.16}$$
$$- f\big(\varphi(\tau, w_0, z_0(w_0)), \psi(\tau, w_0, z_0(w_0))\big)\Big)\mathrm{d}\tau$$

Clearly,

$$\lim_{\delta \to 0} \frac{1}{\delta} \times \Big(f\big(\varphi(\tau, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0)), \psi(\tau, w_0 + \delta\xi_0, z_0(w_0 + \delta\xi_0))\big)$$
$$- f\big(\varphi(\tau, w_0, z_0(w_0)), \psi(\tau, w_0, z_0(w_0))\big)\Big)$$
$$= \frac{\partial f}{\partial w}(\tau, w(\tau), z(\tau)) \cdot \Big(\Phi_{w_0}(\tau) + \Phi_{z_0}(\tau) \cdot z_0'(w_0)\Big) \cdot \xi_0 \tag{3.17}$$
$$+ \frac{\partial f}{\partial z}(\tau, w(\tau), z(\tau))\Big(\Psi_{w_0}(\tau) + \Psi_{z_0}(\tau) \cdot z_0'(w_0)\Big) \cdot \xi_0$$
$$\stackrel{(3.13)}{=} \frac{\partial f}{\partial w}(\tau, w(\tau), z(\tau)) \cdot \xi(\tau) + \frac{\partial f}{\partial z}(\tau, w(\tau), z(\tau)) \cdot \nu(\tau).$$

Substituting this back to (3.16) and differentiating with respect to time gives us

$$\dot{\xi}(t) = \frac{\partial f}{\partial w}\xi(t) + \frac{\partial f}{\partial z}\nu(t). \tag{3.18}$$

Altogether this shows that indeed $\xi(t)$, $\nu(t)$ given by (3.12) is a solution of (3.4).

We can now show the sufficiency result.

Contracting $\Rightarrow$ UGES. Let c and α be the constants corresponding to the contractivity condition. Seeking a contradiction, assume the variational DAE system (3.4) is not UGES. Then there exists a solution $\begin{bmatrix} w(\cdot) \\ z(\cdot) \end{bmatrix}$ of (3.1) and an initial value $\begin{bmatrix} w_0 \\ z_0(w_0) \end{bmatrix}$ such that for the corresponding solution $\begin{bmatrix} \xi(\cdot) \\ \nu(\cdot) \end{bmatrix}$ of (3.4) we have that for $c' := \frac{3}{2}c$ and $\alpha' := \alpha$, there exists $T > 0$ such that

$$\left\| \begin{bmatrix} \xi(T) \\ \nu(T) \end{bmatrix} \right\| > c'e^{-\alpha'T} \left\| \begin{bmatrix} \xi_0 \\ \nu_0 \end{bmatrix} \right\| = \frac{3}{2}ce^{-\alpha T} \left\| \begin{bmatrix} \xi_0 \\ \nu_0 \end{bmatrix} \right\| \qquad \forall \xi_0. \tag{3.19}$$

Let $\begin{bmatrix} \hat{w}(\cdot) \\ \hat{z}(\cdot) \end{bmatrix}$ be a solution of (3.1) with initial value $\begin{bmatrix} \hat{w}(t_0) \\ \hat{z}(t_0) \end{bmatrix} = \begin{bmatrix} w_0 + \delta\xi_0 \\ z_0(w_0 + \delta\xi_0) \end{bmatrix}$ for sufficiently small $\delta \in \mathbb{R}$. In the first part of the proof we have shown that $\begin{cases} \xi(t) := \lim_{\delta \to 0} \frac{\hat{w}(t) - w(t)}{\delta} \\ \nu(t) := \lim_{\delta \to 0} \frac{\hat{z}(t) - z(t)}{\delta} \end{cases}$, then, for a sufficiently small $\delta > 0$, we have that at time T,

$$\frac{\left\| \begin{bmatrix} \hat{w}(T) \\ \hat{z}(T) \end{bmatrix} - \begin{bmatrix} w(T) \\ z(T) \end{bmatrix} \right\|}{\delta} > \frac{4}{5} \left\| \begin{bmatrix} \xi(T) \\ \nu(T) \end{bmatrix} \right\|, \tag{3.20}$$

where the lower-bound constant $\frac{4}{5}$ is chosen arbitrarily for the following computation of bounds. Similarly, since $\nu_0 := \lim_{\delta \to 0} \frac{\hat{z}(t_0) - z(t_0)}{\delta}$, for a sufficiently small $\delta > 0$, we have that

$$\|\nu_0\| > \frac{5}{6} \frac{\|\hat{z}(t_0) - z(t_0)\|}{\delta}. \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), we obtain

$$\left\| \begin{bmatrix} \hat{w}(T) \\ \hat{z}(T) \end{bmatrix} - \begin{bmatrix} w(T) \\ z(T) \end{bmatrix} \right\| \overset{(3.20)}{>} \frac{4}{5}\delta \left\| \begin{bmatrix} \xi(T) \\ \nu(T) \end{bmatrix} \right\| \overset{(3.19)}{>} \frac{6}{5}ce^{-\alpha T} \left\| \begin{bmatrix} \delta\xi_0 \\ \delta\nu_0 \end{bmatrix} \right\|$$

$$\overset{(3.21)}{>} ce^{-\alpha T} \left\| \begin{bmatrix} \frac{6}{5}\hat{w}(t_0) \\ \hat{z}(t_0) \end{bmatrix} - \begin{bmatrix} \frac{6}{5}w(t_0) \\ z(t_0) \end{bmatrix} \right\|$$

$$> ce^{-\alpha T} \left\| \begin{bmatrix} \hat{w}(t_0) \\ \hat{z}(t_0) \end{bmatrix} - \begin{bmatrix} w(t_0) \\ z(t_0) \end{bmatrix} \right\|,$$

for all ξ_0, the last inequality arises from the property of the p-norm. This is in contradiction to the contractivity of (3.1) and concludes the proof of sufficiency

part.

UGES ⇒ Contracting. Let us consider two solutions $\begin{bmatrix} w(\cdot) \\ z(\cdot) \end{bmatrix} = \begin{bmatrix} \varphi(\cdot, w_0, z_0(w_0)) \\ \psi(\cdot, w_0, z_0(w_0)) \end{bmatrix}$ and $\begin{bmatrix} \hat{w}(\cdot) \\ \hat{z}(\cdot) \end{bmatrix} = \begin{bmatrix} \varphi(\cdot, \hat{w}_0, \hat{z}_0(\hat{w}_0)) \\ \psi(\cdot, \hat{w}_0, \hat{z}_0(\hat{w}_0)) \end{bmatrix}$ of (3.1). Consequently, we can utilize the fundamental theorem of calculus for line integrals to obtain

$$\begin{cases} \hat{w}(t) - w(t) = \int_{w_0}^{\hat{w}_0} \frac{\mathrm{d}\varphi(t, \zeta, z_0(\zeta))}{\mathrm{d}\zeta} \mathrm{d}\zeta \\ \hat{z}(t) - z(t) = \int_{w_0}^{\hat{w}_0} \frac{\mathrm{d}\psi(t, \zeta, z_0(\zeta))}{\mathrm{d}\zeta} \mathrm{d}\zeta \end{cases}. \tag{3.22}$$

According to (3.12), one has

$$\begin{bmatrix} \xi(t) \\ \nu(t) \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}\varphi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} & 0 \\ \frac{\mathrm{d}\psi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} & 0 \end{bmatrix} \begin{bmatrix} \xi_0 \\ \nu_0 \end{bmatrix}. \tag{3.23}$$

Then, in (3.23), $\begin{bmatrix} \frac{\mathrm{d}\varphi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} & 0 \\ \frac{\mathrm{d}\psi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} & 0 \end{bmatrix}$ is the (singular) state transition matrix of the DAE (3.4). From the UGAS property of (3.4), it follows with similar arguments as in the necessity proof of [44, Thm. 4.11] that

$$\left\| \begin{bmatrix} \frac{\mathrm{d}\varphi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} & 0 \\ \frac{\mathrm{d}\psi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} & 0 \end{bmatrix} \right\| \leqslant ce^{-\alpha t} \tag{3.24}$$

It follows[1] from (3.24) that

$$\left\| \begin{bmatrix} \frac{\mathrm{d}\varphi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} \\ \frac{\mathrm{d}\psi(t, w_0, z_0(w_0))}{\mathrm{d}w_0} \end{bmatrix} \right\| \leqslant ce^{-\alpha t} \tag{3.25}$$

Using (3.25) to get the upper bound of (3.22), we have

$$\left\| \begin{bmatrix} \hat{w}(t) \\ \hat{z}(t) \end{bmatrix} - \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \right\| = \left\| \int_{w_0}^{\hat{w}_0} \begin{bmatrix} \frac{\mathrm{d}\varphi(t, \zeta, z_0(\zeta))}{\mathrm{d}\zeta} \\ \frac{\mathrm{d}\psi(t, \zeta, z_0(\zeta))}{\mathrm{d}\zeta} \end{bmatrix} \mathrm{d}\zeta \right\|$$
$$\leqslant ce^{-\alpha t} \|\hat{w}_0 - w_0\| \tag{3.26}$$
$$\leqslant ce^{-\alpha t} \left\| \begin{bmatrix} \hat{w}_0 \\ z_0(\hat{w}_0) \end{bmatrix} - \begin{bmatrix} w_0 \\ z_0(w_0) \end{bmatrix} \right\|,$$

which implies that (3.1) is contracting. This completes the proof. □

---

[1]In general, for any p-norm and its induced matrix norm we have $\left\| \begin{bmatrix} M & 0 \end{bmatrix} \right\| = \sup_{\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = 1} \left\| \begin{bmatrix} M & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sup_{\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = 1} \|Mx\| \geqslant \sup_{\|x\| = 1} \|Mx\| = \|Mx\|$. The inequality follows from the set inclusion $\{x| \|x\| = 1\} = \{x| \|\begin{bmatrix} x \\ 0 \end{bmatrix}\| = 1\} \subseteq \{x| \exists y : \|\begin{bmatrix} x \\ y \end{bmatrix}\| = 1\}$.

Proposition 3.3 shows that the contractivity of DAE systems inherit the property of contracting ODE systems (i.e., [7, Proposition 1]) on a corresponding manifold $\frac{\partial g}{\partial w}(t, w, z)\xi + \frac{\partial g}{\partial z}(t, w, z)\nu = 0$. From (3.23) and (2.23), we can deduce that $\|\xi(t)\| \leqslant ce^{-\alpha t}\|\xi_0\|$, which means that when the system (3.4) is exponentially stable, its reduced system (3.6) is exponentially stable. In the case of a time-invariant system, the reverse implication holds automatically. This is due to the fact that it becomes feasible to guarantee the boundedness of $\left[\frac{\partial g}{\partial z}\right]^{-1}\frac{\partial g}{\partial w}$ within a particular invariant set. As a result, we can utilize Proposition 3.3 for analyzing the stability of time-invariant DAE systems, as demonstrated in Corollary 3.8. Apart from the time-invariant system, the presence of an invariant set might not be guaranteed for a time-varying system, i.e., $\left[\frac{\partial g}{\partial z}\right]^{-1}\frac{\partial g}{\partial w}$ could potentially become unbounded. This implies that the contractivity of the time-varying DAE system cannot be derived from the contractivity of the reduced system in the absence of bounded condition on $\left[\frac{\partial g}{\partial z}\right]^{-1}\frac{\partial g}{\partial w}$. In Lemma 3.4 below we relax the condition on $\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}$ by analysing the stability of an auxiliary ODE system instead of analysing the contractivity of the reduced system. By adopting this approach, it becomes unnecessary to have prior knowledge about the reduced system (3.6), as demonstrated in Example 3.1.

3.3.2 The ODE approach

In this section, we present an ODE approach to analyze the contractivity of nonlinear time-varying DAE systems. We present a construction of auxiliary ODE whose trajectories can represent the convergence property of the variational DAE system. As a consequence, we can apply traditional control theories to the auxiliary ODE in order to analyze the UGES of (3.4).

For a given variational DAE system (3.4), replacing the algebraic equation with its relative ordinary differential equation, we construct its auxiliary ODE system by

$$
\begin{cases}
\dot{\xi}_\gamma = \frac{\partial f}{\partial w}\xi_\gamma + \frac{\partial f}{\partial z}\nu_\gamma, \\
\dot{\nu}_\gamma = -\left(\frac{\partial g}{\partial z}\right)^{-1}\left[\left(\gamma\frac{\partial g}{\partial w} + \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial g}{\partial w}) + \frac{\partial g}{\partial w}\frac{\partial f}{\partial w}\right)\xi_\gamma + \left(\gamma\frac{\partial g}{\partial z} + \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial g}{\partial z}) + \frac{\partial g}{\partial w}\frac{\partial f}{\partial z}\right)\nu_\gamma\right],
\end{cases}
\tag{3.27}
$$

where γ is a no-negative constant, $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial g}{\partial w}) = \frac{\partial^2 g}{\partial w\partial t} + \frac{\partial^2 g}{\partial^2 w}f - \frac{\partial^2 g}{\partial w\partial z}(\frac{\partial g}{\partial z})^{-1}(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial w}f)$, and $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial g}{\partial z}) = \frac{\partial^2 g}{\partial z\partial t} + \frac{\partial^2 g}{\partial w\partial z}f + \frac{\partial^2 g}{\partial^2 z}(\frac{\partial g}{\partial z})^{-1}(\frac{\partial g}{\partial t} - \frac{\partial g}{\partial w}f)$. It is worth noting that in various control problems that involve the system's output, such as state observer design and output feedback control, the output of the system, denoted as $z = h(w, t)$, can be considered as a time-varying constraint. In other words, the function $g(t, w, z)$ takes the form of $z - h(w, t)$. In such cases, the term $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial g}{\partial z})$

becomes zero, and $\frac{d}{dt}\left(\frac{\partial g}{\partial w}\right)$ can be simplified to $\frac{\partial^2 g}{\partial w \partial t} + \frac{\partial^2 g}{\partial^2 w} f$.

Lemma 3.4. *The variational DAE system (3.4) is exponentially stable if there exist $\gamma \geqslant 0$ such that its auxiliary ODE system (3.27) is exponentially stable.*

Proof. Consider an auxiliary system of (3.4) as follows, which is equivalent to (3.27),

$$\begin{cases} \dot{\xi}_\gamma = \frac{\partial f}{\partial w}\xi_\gamma + \frac{\partial f}{\partial z}\nu_\gamma, \\ \overbrace{\left(\frac{\partial g}{\partial w}\xi_\gamma + \frac{\partial g}{\partial z}\nu_\gamma\right)} = -\gamma\left(\frac{\partial g}{\partial w}\xi_\gamma + \frac{\partial g}{\partial z}\nu_\gamma\right). \end{cases} \tag{3.28}$$

From the second equation in (3.28), we have

$$\left(\frac{\partial g}{\partial w}\xi_\gamma + \frac{\partial g}{\partial z}\nu_\gamma\right) = \left(\frac{\partial g_0}{\partial w}\xi_0 + \frac{\partial g_0}{\partial z}\nu_0\right)e^{-\gamma t}. \tag{3.29}$$

where $\frac{\partial g_0}{\partial w} = \frac{\partial g}{\partial w}\big|_{t=0}$. It follows then that the solution of (3.4) is a particular case of (3.28) with initial condition $\frac{\partial g_0}{\partial w}\xi_0 + \frac{\partial g_0}{\partial z}\nu_0 = 0$. $\qquad\square$

In Lemma 3.4, we use the stability properties of (3.28) to derive the stability properties of (3.4). In this regard, the constant γ must be chosen properly as (3.28) may fail to capture the stability properties of (3.4), e.g., the system (3.28) can be unstable while correspondingly the system (3.4) is stable. For instance, let us consider the following contracting DAE system whose trajectories can be calculated explicitly

$$\begin{cases} \dot{w} = -2e^t z, \\ 0 = e^{-t}w - z, \end{cases} \tag{3.30}$$

which again has the same form as its variational system. The trajectory of the system is $\begin{cases} w = w_0 e^{-2t} \\ z = w_0 e^{-3t} \end{cases}$. If we choose $\gamma = 1$, (3.27) can be rewritten as

$$\begin{cases} \dot{w} = -2e^t z, \\ \dot{z} = -3z, \end{cases} \tag{3.31}$$

which has the trajectory of $\begin{cases} w = z_0 e^{-2t} + w_0 - z_0 \\ z = z_0 e^{-3t} \end{cases}$, i.e. it is not contracting (the value of w depends on the initial condition $w_0 - z_0$). However, by choosing $\gamma = 4$, (3.27) can be rewritten as

$$\begin{cases} \dot{w} = -2e^t z, \\ \dot{z} = 3e^{-t}w - 6z, \end{cases} \tag{3.32}$$

its trajectory is given by $\begin{cases} w = (3w_0 - 2z_0)e^{-2t} - 2(w_0 - z_0)e^{-3t} \\ z = (3w_0 - 2z_0)e^{-3t} - 3(w_0 - z_0)e^{-4t} \end{cases}$, which is con-

tracting. Based on this observation, it appears that in order to ensure the contractivity of the system, the parameter γ should be chosen appropriately.

In the following proposition, we establish a lower bound for γ based on a priori information of the DAE system.

Proposition 3.5. *For any given signals w and z, suppose that the reduced system (3.6) is uniformly exponentially upper and lower bounded by the rates $-\underline{\alpha}$, $-\overline{\alpha}$, with $\overline{\alpha} \geqslant \underline{\alpha} > 0$, and there exist positive constants $l_f(w_0, z_0)$, $k_f(w_0, z_0)$, $l_g(w_0, z_0)$, $k_g(w_0, z_0)$ such that, $\left\| \left(\frac{\partial f}{\partial z} \right) \right\| \leqslant k_f e^{l_f t}$ and $\left\| \left(\frac{\partial g}{\partial z} \right)^{-1} \right\| \leqslant k_g e^{l_g t}$ for all $t \geqslant 0$, where all gradients are evaluated on the trajectory $w(t) = \varphi(t, w_0, z_0)$ and $z(t) = \psi(t, w_0, z_0)$ of (3.1). For $\gamma > l + \overline{\alpha}$ holds in the ODE system (3.27), where $l = \max\{l_g, l_g + l_f\}$, if the system (3.4) is exponentially stable, then it can be concluded that (3.27) is asymptotically stable*

Proof. Let us define $q = \frac{\partial g}{\partial w} \xi + \frac{\partial g}{\partial z} \nu$, so that (3.28) can be rewritten as

$$\begin{cases} \dot{\xi}_\gamma = \left(\frac{\partial f}{\partial w} - \frac{\partial f}{\partial z} \left(\frac{\partial g}{\partial z} \right)^{-1} \frac{\partial g}{\partial w} \right) \xi_\gamma + \frac{\partial f}{\partial z} \left(\frac{\partial g}{\partial z} \right)^{-1} q, \\ \dot{q} = -\gamma q. \end{cases} \tag{3.33}$$

In this case, the solution of (3.33) is given by

$$\begin{cases} \xi_\gamma = \Phi(t, 0) \xi_0 + q_0 \int_0^t \Phi(t, s) \left(\frac{\partial f}{\partial z} \left(\frac{\partial g}{\partial z} \right)^{-1} e^{-\gamma s} \right) \mathrm{d}s, \\ q = q_0 e^{-\gamma t}, \end{cases} \tag{3.34}$$

where $\Phi(t, s)$ is the transition matrix of the reduced system. Since the reduced system is exponentially bounded by the rates $-\underline{\alpha}$, $-\overline{\alpha}$, we use the bound $ce^{-\overline{\alpha}(t_1 - t_0)} \leqslant \|\Phi(t_1, t_0)\| \leqslant ce^{-\underline{\alpha}(t_1 - t_0)}$ to arrive at the following estimation

$$\|\xi_\gamma\| \leqslant ce^{-\underline{\alpha}t} \|\xi_0\| + \|q_0\| \int_0^t ce^{-\underline{\alpha}(t-s)} \left\| \frac{\partial f}{\partial z} \left(\frac{\partial g}{\partial z} \right)^{-1} \right\| e^{-\gamma s} \mathrm{d}s$$

$$= ce^{-\underline{\alpha}t} \|\xi_0\| + ce^{-\underline{\alpha}t} \|q_0\| \int_0^t k_f e^{l_f s} k_g e^{l_g s} e^{-(\gamma - \underline{\alpha})s} \mathrm{d}s \tag{3.35}$$

$$\leqslant ce^{-\underline{\alpha}t} (\|\xi_0\| + k \|q_0\|).$$

The last equality is derived from $l_f + l_g - \gamma + \underline{\alpha} \leqslant l - \gamma + \overline{\alpha} < 0$ and for some suitable $k > 0$. This implies that the convergence rate of ξ is at least $\underline{\alpha}$. By the definition of q defined before (3.33), we have

$$\|\nu_\gamma\| \leqslant \|q_0\| \left\| \left(\frac{\partial g}{\partial z} \right)^{-1} \right\| e^{-\gamma t} + \left\| \left(\frac{\partial g}{\partial z} \right)^{-1} \frac{\partial g}{\partial w} \xi_\gamma \right\|$$

$$\leqslant k_g \|q_0\| e^{-\underline{\alpha}t} + \left\| \left(\frac{\partial g}{\partial z} \right)^{-1} \frac{\partial g}{\partial w} \xi_\gamma \right\|. \tag{3.36}$$

Now, let us consider $\left\|\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\xi_\gamma\right\|$, by using ξ_γ in (3.34), we have

$$
\begin{aligned}
\left\|\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\xi_\gamma\right\| &\leqslant \left\|\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\Phi(t,0)\xi_0\right\| + \\
&\quad \left\|q_0\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\int_0^t\Phi(t,s)\left(\frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial z}\right)^{-1}e^{-\gamma s}\right)\mathrm{d}s\right\| \\
&= \left\|\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\Phi(t,0)\xi_0\right\| + \\
&\quad \left\|q_0\int_0^t\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\Phi(t,0)\Phi(0,s)\left(\frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial z}\right)^{-1}e^{-\gamma s}\right)\mathrm{d}s\right\|.
\end{aligned}
\tag{3.37}
$$

Since the system (3.4) is exponentially stable, and

$$
\nu = \left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\xi = \left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\Phi(t,0)\xi_0,
$$

we have

$$
\left\|\begin{bmatrix}\Phi(t,0) & 0 \\ \left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\Phi(t,0) & 0\end{bmatrix}\right\| \leqslant c'e^{-\alpha' t},
$$

for some positive c' and $\alpha' \geqslant \underline{\alpha}$. Then, we have

$$
\left\|\begin{bmatrix}\Phi(t,0) \\ \left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\Phi(t,0)\end{bmatrix}\right\| \leqslant c'e^{-\alpha' t},
\tag{3.38}
$$

It follows [2] from (3.38) that

$$
\left\|\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\Phi(t,0)\right\| \leqslant c'e^{-\alpha' t},
\tag{3.39}
$$

By using (3.39), we can rewrite (3.37) as

$$
\left\|\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial w}\xi_\gamma\right\| \leqslant c'\|\xi_0\|e^{-\alpha' t} + c'\|q_0\|\left\|\int_0^t e^{\overline{\alpha}s}\left(\frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial z}\right)^{-1}e^{-\gamma s}\right)\mathrm{d}s\right\|e^{-\alpha' t}.
\tag{3.40}
$$

---

[2]In general, for any p-norm and its induced matrix norm we have $\left\|\begin{bmatrix}M \\ N\end{bmatrix}\right\| = \sup_{\|x\|=1}\left\|\begin{bmatrix}M \\ N\end{bmatrix}x\right\| = \sup_{\|x\|=1}\left\|\begin{bmatrix}Mx \\ Nx\end{bmatrix}\right\| \geqslant \sup_{\|x\|=1}\|Nx\| = \|N\|.$

Combine (3.37), (3.40) and $l + \overline{\alpha} - \gamma < 0$, we arrive at

$$\|\nu_\gamma\| \leqslant k_g \|q_0\| e^{-\underline{\alpha} t} + c'(\|\xi_0\| + k_g k_f \|q_0\|)e^{-\alpha' t}. \tag{3.41}$$

By (3.35) and (3.41), the system (3.27) is asymptotically stable.   □

In Proposition 3.5, we have a mild assumption where the time varying functions $\frac{\partial f}{\partial z}$ and $\left(\frac{\partial g}{\partial z}\right)^{-1}$ are bounded by ke^{lt} instead of by constants. Let us recall again the previous example in (3.30), where the reduced system is $\dot{w} = -2w$, we have $\left\|\frac{\partial f}{\partial z}\right\| = 2e^t$, $\left\|\left(\frac{\partial g}{\partial z}\right)^{-1}\right\| = 1$. Thus the hypotheses in Proposition 3.5 hold with $\alpha = 2$, $k = l = 1$ and $\gamma = 4 > l + \alpha = 3$. Then, the ODE system (3.32) has the same exponential convergence rate as the corresponding DAE system (3.31), namely $\alpha' = 2$. By considering Proposition 3.5, it becomes evident that selecting $\gamma > l + \overline{\alpha}$ allows us to conclude that if (3.27) is unstable, then (3.4) is also unstable. This proposition helps prevent any potential "misjudgment" similar to the previous example (3.31). By examining (3.36), it shows that as γ increases, the estimation performance of (3.28) w.r.t. (3.4) improves. If obtaining prior information about the DAE system (3.3) is difficult, one can select a sufficiently large value for γ to apply Proposition 3.5 and determine the contracting rate of the DAE system.

By utilizing the concept of matrix measure, we will examine the contractivity of the time-varying DAE system (3.1) by focusing on the generalized Jacobian matrix of the corresponding auxiliary ODE system (3.27). In order to simplify the notation, we rewrite (3.27) into

$$\begin{cases} \dot{\xi} = A(t)\xi + B(t)\nu, \\ \dot{\nu} = -F^{-1}(t)C(t)\xi - F^{-1}(t)D(t)\nu, \end{cases} \tag{3.42}$$

where $A(t) = \frac{\partial f}{\partial w}$, $B(t) = \frac{\partial f}{\partial z}$, $F(t) = \frac{\partial g}{\partial z}$, $C(t) = \gamma\frac{\partial g}{\partial w} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial g}{\partial w}\right) + \frac{\partial g}{\partial w}\frac{\partial f}{\partial w}$, and $D(t) = \gamma\frac{\partial g}{\partial z} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial g}{\partial z}\right) + \frac{\partial g}{\partial w}\frac{\partial f}{\partial z}$. For a invertible matrix $M(w, z, t)$, let us define

$$J(t) = \dot{M}M^{-1} + M \begin{bmatrix} A & B \\ -F^{-1}C & -F^{-1}D \end{bmatrix} M^{-1}. \tag{3.43}$$

Theorem 3.6. *If there exist a invertible matrix $M(w, z, t)$ satisfying the condition that $\|M^{-1}\| \|M\|$ is bounded, and let β be a positive constant such that $\mu_q\left(J(t)\right) \leqslant -\beta$, where $J(t)$ is given by (3.43) then the DAE system (3.1) is contracting.*

Proof. For any $M(w, z, t)$, we define differential coordinate transformations $\begin{bmatrix} p \\ r \end{bmatrix} =$

$M \begin{bmatrix} \xi_\gamma \\ \nu_\gamma \end{bmatrix}$, so that (3.42) can be rewritten as

$$\overbrace{\begin{bmatrix} p \\ r \end{bmatrix}}^{\cdot} = \underbrace{\left(\dot{M}M^{-1} + M \begin{bmatrix} A & B \\ -F^{-1}C & -F^{-1}D \end{bmatrix} M^{-1} \right)}_{J(t)} \begin{bmatrix} p \\ r \end{bmatrix}. \tag{3.44}$$

Its upper right Dini derivative satisfies

$$
\begin{aligned}
D_t^+ \left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\| &= \lim_{h \to 0^+} \frac{\left\| \begin{bmatrix} p(t+h) \\ r(t+h) \end{bmatrix} \right\| - \left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\|}{h} \\
&= \lim_{h \to 0^+} \frac{\left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} + h \overbrace{\begin{bmatrix} p(t) \\ r(t) \end{bmatrix}}^{\cdot} \right\| - \left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\|}{h} \\
&\overset{(3.44)}{=} \lim_{h \to 0^+} \frac{\left\| \left(I + hJ(t) \right) \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\| - \left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\|}{h} \\
&\leqslant \lim_{h \to 0^+} \frac{\| I + hJ(t) \| - 1}{h} \left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\| = \mu_q\left(J(t)\right) \left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\| \\
&\leqslant -\beta \left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\|.
\end{aligned}
\tag{3.45}
$$

By the comparison lemma, the inequality

$$\left\| \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\| \leqslant e^{-\beta t} \left\| \begin{bmatrix} p(0) \\ r(0) \end{bmatrix} \right\|, \tag{3.46}$$

holds, from which it follows[3] that

$$\left\| \begin{bmatrix} \xi_\gamma(t) \\ \nu_\gamma(t) \end{bmatrix} \right\| = \left\| M^{-1} \begin{bmatrix} p(t) \\ r(t) \end{bmatrix} \right\| \overset{(3.46)}{\leqslant} e^{-\beta t} \left\| M^{-1} \right\| \| M \| \left\| \begin{bmatrix} \xi_\gamma(0) \\ \nu_\gamma(0) \end{bmatrix} \right\| \leqslant c e^{-\beta t} \left\| \begin{bmatrix} \xi_\gamma(0) \\ \nu_\gamma(0) \end{bmatrix} \right\|, \tag{3.47}$$

i.e., the system (3.42) is exponentially stable, the last inequality arises from the boundedness of $\left\| M^{-1} \right\| \| M \|$. By Lemma 1.1 and Proposition 3.3, we can conclude that the DAE system (3.1) is contracting. $\qquad \square$

---

[3]In general, for any induced matrix norm, we have $\| Mx \| = \| x \| \left\| M \frac{x}{\|x\|} \right\| \leqslant \| x \| \sup_{\left\| \frac{x}{\|x\|} \right\| = 1} \left\| M \frac{x}{\|x\|} \right\| = \| M \| \| x \|.$

In Theorem 3.6, the condition that $\|M^{-1}\| \|M\|$ is bounded is less conservative compared to the Lyapunov transformation [21, Def. 2.4]. Specifically, instead of requiring both $\|M^{-1}\|$ and $\|M\|$ to be bounded, it only demands their product to be bounded. As illustrated in Example 3.3, this less conservative condition allows for cases where $M = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$, which has an unbounded inverse. Nevertheless, $\left\| \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \right\| \left\| \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \right\| = 1$, satisfying the boundedness condition.

Remark 3.7. As a powerful tool to analyze the stability of ODE systems, the Lyapunov method has been widely used in the analysis and control design of nonlinear systems. By employing the 2-norm matrix measure, the hypotheses in Theorem 3.6, i.e., $\mu_2\big(J(t)\big) \leqslant -\beta$ can be expressed into solving the subsequent Riccati inequality condition

$$\begin{bmatrix} A^{\top}(t) & C^{\top}(t)F^{-\top}(t) \\ B^{\top}(t) & D^{\top}(t)F^{-\top}(t) \end{bmatrix} P(t) + P(t) \begin{bmatrix} A(t) & B(t) \\ F^{-1}(t)C(t) & F^{-1}(t)D(t) \end{bmatrix} + \dot{P}(t) \tag{3.48}$$
$$\leqslant -\beta P(t),$$

where $P(t) = M^{\top}M$ for some $M(w, z, t)$. Afterwards, Theorem 3.6 can be simplified to [55, Thm. 2]. As will be shown later in Example 3.3, such Riccati inequality will be used for the design of an observer. Nevertheless, in certain scenarios, it can be challenging to find a solution $M^{\top}M$ that satisfies (3.48). By employing the 1-norm (∞-norm) as matrix measure, we can arrive at a simple numerical test to each column (or row) of the matrix $J(t)$. We demonstrate this approach via Example 3.1 and Example 3.2 later.

3.4   Simulation Setup and Applications

This section presents three numerical examples to illustrate different applications of the proposed methods. In the first example, we examine the contraction property of a nonlinear time-varying DAE system using Theorem 3.6. The second example illustrates the practical application of Proposition 3.3 in stabilizing a time-invariant DAE system, as demonstrated by Corollary 3.8. Subsequently, we employ Corollary 3.8 to design a state feedback controller for a power source system with an inverter interface, which can be modeled as a time-invariant DAE system. Finally, the third illustration involves the application of Theorem 3.6 to devise an observer for a time-varying ODE system, namely, Corollary 3.9. Subsequently, we employ Corollary 3.9 to create an observer for an unstable time-varying ODE system.

3.4.1 Contractivity of nonlinear time-varying DAE systems

Example 3.1. Consider a nonlinear time-varying DAE system

$$\begin{cases} \dot{w}_1 = -4w_1 - 0.5\cos z, \\ \dot{w}_2 = \frac{4}{3+\sin t}w_1 - \frac{3+\cos t}{3+\sin t}w_2 - \frac{4}{3+\sin t}, \\ 0 = 4z + 0.5\sin z + w_1 + (3+\sin t)w_2. \end{cases} \tag{3.49}$$

where $w(t) \in \mathbb{R}^2$ is the state vector and $z(t) \in \mathbb{R}$ refers to the algebraic vector. Its variational system is

$$\begin{cases} \dot{\xi}_1 = -4\xi_1 + (0.5\sin z)\nu, \\ \dot{\xi}_2 = \frac{4}{3+\sin t}\xi_1 - \frac{3+\cos t}{3+\sin t}\xi_2, \\ 0 = \xi_1 + (3+\sin t)\xi_2 + (4+0.5\cos z)\nu. \end{cases} \tag{3.50}$$

In the given example (3.30), obtaining information about the system is a straightforward process. However, in this specific case, the existence of a time-varying nonlinear constraint creates difficulties in obtaining information about (3.50), we are unable to utilize Proposition 3.5 to select γ. Nevertheless, the simplest and most direct option for γ in this instance is $\gamma = 0$ (if this choice proves ineffective, a larger value of γ should be considered). As a result, the auxiliary ODE system corresponding to (3.50) can be expressed as follows:

$$\begin{cases} \dot{\xi}_1 = -4\xi_1 + (0.5\sin z)\nu, \\ \dot{\xi}_2 = \frac{4}{3+\sin t}\xi_1 - \frac{3+\cos t}{3+\sin t}\xi_2, \\ \dot{\nu} = -\frac{4}{3+0.5\cos z}\nu. \end{cases} \tag{3.51}$$

It follows that

$$J(t) = \begin{bmatrix} -4 & 0 & 0.5\sin z \\ \frac{4}{3+\sin t} & -\frac{3+\cos t}{3+\sin t} & 0 \\ 0 & 0 & -\frac{4}{3+0.5\cos z} \end{bmatrix}. \tag{3.52}$$

Applying the 1-norm, we find that $\mu_1\big(J(t)\big) < -0.5$ holds, which implies that the DAE system (3.53) is contracting according to Theorem 3.6. Figure 3.1 illustrates the trajectories of the system with two distinct initial conditions. Hence, we can conclude that the time-varying DAE system is contracting.

3.4.2 Stability of inverter-interfaced power source systems

It is well known that for a contracting time-invariant system, all trajectories converge to an equilibrium exponentially. As an interesting particular case of our main results above, we can prove stability of the time-invariant DAE systems
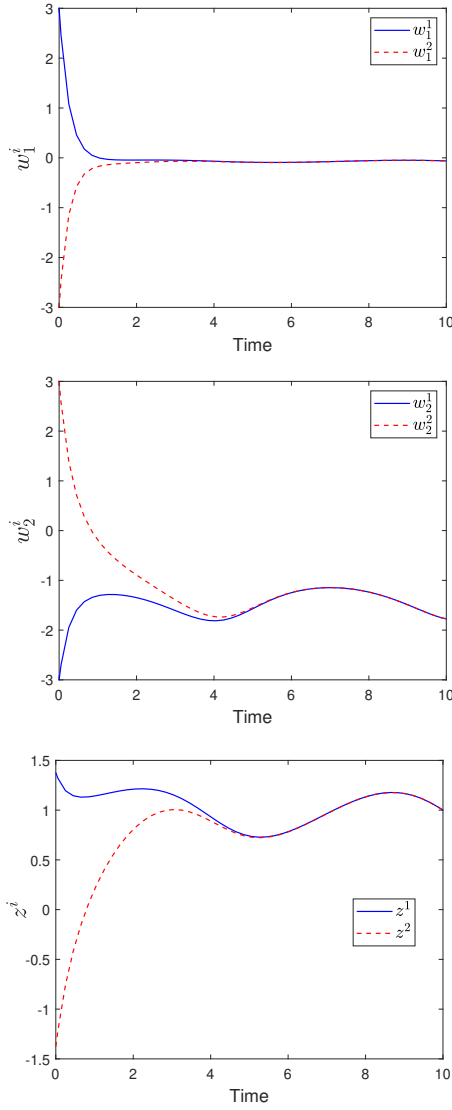
Figure 3.1: The plot of trajectories of time-varying DAE system in Example 2.2 initialized at $\begin{bmatrix} 3 \\ -3 \\ 1.38 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 3 \\ -1.38 \end{bmatrix}$.

by using Proposition 3.3. The basic idea involves guaranteeing the contravtivity of the DAE system while ensuring that its equilibrium point resides within its

trajectory set. Consider a time-invariant DAE system given by

$$\begin{cases} \dot{w} = f(w, z), \\ 0 = g(w, z), \end{cases} \tag{3.53}$$

where $f(0,0) = 0$, and $g(0,0) = 0$. The variational system of (3.53) is

$$\begin{cases} \dot{\xi} = \frac{\partial f}{\partial w}(w(t), z(t)) \cdot \xi + \frac{\partial f}{\partial z}(w(t), z(t)) \cdot \nu, \\ 0 = \frac{\partial g}{\partial w}(w(t), z(t)) \cdot \xi + \frac{\partial g}{\partial z}(w(t), z(t)) \cdot \nu. \end{cases} \tag{3.54}$$

To ensure safety in engineering systems, it is imperative to establish bounds on the states of (3.53). In the context of a power system, for example, the terminal complex voltage $V \angle \theta$ is restricted within the range $[0.95, 1.05] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ to ensure operational stability and prevent undesirable conditions. In this section, we investigate the time-invariant DAE system (3.53) within the set $[W \times Z]$, i.e., $w(t) \in W \subseteq \mathbb{R}^n$ and $z(t) \in Z \subseteq \mathbb{R}^m$. Moreover, we assume that $\left(\frac{\partial g}{\partial z}\right)^{-1} \frac{\partial g}{\partial w}$ is bounded in $[W \times Z]$. In the subsequent corollary, we analyze the local exponential stability of (3.54) through its reduced system $\dot{\xi} = \left(\frac{\partial f}{\partial w} - \frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial z}\right)^{-1} \frac{\partial g}{\partial w}\right)\xi$.

Corollary 3.8. *The time-invariant DAE system* (3.53) *is locally exponentially stable in* $[W \times Z]$ *if there exist an invertible metric* $\rho(w, z)$, *such that* $\mu_q\left(J(w, z)\right) \leqslant -\beta$ *for some positive constant* β, *where* $J(w, z)$ *is given by*

$$J(w, z) = \rho\left(\frac{\partial f}{\partial w} - \frac{\partial f}{\partial z}\left(\frac{\partial g}{\partial z}\right)^{-1} \frac{\partial g}{\partial w}\right)\rho^{-1}. \tag{3.55}$$

Proof. If condition (3.55) is satisfied, the reduced system is exponentially stable, i.e., $\|\xi\| \leqslant c \|\xi_0\| e^{-\alpha t}$. By using the boundedness of $\left[\frac{\partial g}{\partial z}\right]^{-1} \frac{\partial g}{\partial w}$ in $[W \times Z]$, we can derive $\|[\begin{smallmatrix} \xi \\ \nu \end{smallmatrix}]\| \leqslant c' \|\xi_0\| e^{-\alpha t} \leqslant c' \|[\begin{smallmatrix} \xi_0 \\ \nu_0 \end{smallmatrix}]\| e^{-\alpha t}$. Consequently, the variational system (3.54) is exponentially stable. Applying Proposition 3.3, we can conclude that the DAE system (3.53) is contracting. As $\left[\begin{smallmatrix} w(t) \\ z(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ represents one of admissible trajectories of (3.53) and it is contracting, it follows that all the trajectories will exponentially converge to $\left[\begin{smallmatrix} w(t) \\ z(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$. $\square$

Example 3.2. Consider an inverter-interfaced power source connected to the infinite bus via a transmission line [96], as shown in Fig.3.2. The system dynamics is given by the following

$$\begin{cases} \dot{P} = \frac{1}{\tau_1}\left(-P + P^{ref} - d_1(\theta - \theta^{ref})\right), \\ \dot{Q} = \frac{1}{\tau_2}\left(-Q + Q^{ref} - d_2(V - V^{ref})\right) + u. \end{cases} \tag{3.56}$$
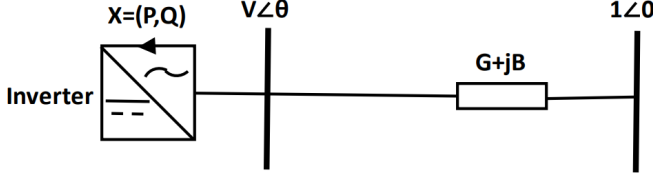
Figure 3.2: An inverter-interfaced power source connecting to the infinite bus.

where P, Q, and u are the terminal output active, reactive power, and control input, respectively. The variable $V\angle\theta$ is the terminal complex voltage, $1\angle 0$ is the desired complex voltage. The symbols P^{ref}, Q^{ref}, θ^{ref}, and V^{ref} are pre-specified constant reference values. The constants $\tau_1 > 0$ and $\tau_2 > 0$ are time constants while $d_1 > 0$ and $d_2 > 0$ are droop coefficients. For this particular case, the algebraic equations are given by

$$\begin{cases} P - GV\cos\theta - BV\sin\theta = 0, \\ Q - GV\sin\theta + BV\cos\theta = 0. \end{cases} \tag{3.57}$$

In this example, a state feedback controller $u = k_1 P + k_2 Q$ will be designed based on Corollary 2.9. For numerical purposes, the parameters are given by $\tau_1 = \tau_2 = \frac{1}{3}$, $d_1 = d_2 = \frac{1}{3}$, $P^{ref} = 1$, $Q^{ref} = -1$, $\theta^{ref} = 0$, $V^{ref} = 1$, $G = B = 1$. In this case, the whole system can be rewritten as

$$\begin{cases} \dot{P} = -3P - \theta + 3, \\ \dot{Q} = -3Q - V - 2 + k_1 P + k_2 Q, \end{cases} \tag{3.58}$$

$$\begin{cases} P - V\cos\theta - V\sin\theta = 0, \\ Q - V\sin\theta + V\cos\theta = 0, \end{cases} \tag{3.59}$$

where $W(t) = \begin{bmatrix} P \\ Q \end{bmatrix}$, $Z(t) = \begin{bmatrix} \theta \\ V \end{bmatrix}$. Its corresponding variational system is

$$\begin{cases} \delta\dot{P} = -3\delta P - \delta\theta, \\ \delta\dot{Q} = -3\delta Q - \delta V + k_1\delta P + k_2\delta Q, \\ \delta P - (\sin\theta + \cos\theta)\delta V - V(\cos\theta - \sin\theta)\delta\theta = 0, \\ \delta Q - (\sin\theta - \cos\theta)\delta V - V(\cos\theta + \sin\theta)\delta\theta = 0. \end{cases} \tag{3.60}$$

The reduced system of (3.60) is

$$\begin{cases} \delta\dot{P} = (-3 + \frac{\sin\theta - \cos\theta}{2V})\delta P - \frac{\sin\theta + \cos\theta}{2V}\delta\theta, \\ \delta\dot{Q} = (k_1 - \frac{\sin\theta + \cos\theta}{2})\delta P + (-3 + k_2 - \frac{\sin\theta - \cos\theta}{2})\delta Q. \end{cases} \tag{3.61}$$

In power system, we have $V \in [0.95, 1.05]$, and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, $\left\| \left(\frac{\partial g}{\partial z} \right)^{-1} \frac{\partial g}{\partial w} \right\| = \left\| \left[\begin{smallmatrix} \frac{\sin\theta - \cos\theta}{2V} & -\frac{\sin\theta + \cos\theta}{2V} \\ -\frac{\sin\theta + \cos\theta}{2} & -\frac{\sin\theta - \cos\theta}{2} \end{smallmatrix} \right] \right\|$ is bounded. So we can analysis the stability of (3.60) by (3.61). The matrix measure of (3.61) is

$$J(V, \theta) = \begin{bmatrix} -3 + \frac{\sin\theta - \cos\theta}{2V} & -\frac{\sin\theta + \cos\theta}{2V} \\ k_1 - \frac{\sin\theta + \cos\theta}{2} & -3 + k_2 - \frac{\sin\theta - \cos\theta}{2} \end{bmatrix}. \tag{3.62}$$

We can determine the range of the elements in $J(V, \theta)$ within $[0.95, 1.05] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. By setting $k_1 = k_2 = 0.5$, and using 1-norm, we find that $\mu_1 \left(J(V, \theta) \right) < -1.1$. According to Corollary 2.9, the DAE system (3.58), (3.59) is exponentially stable with respect to the equilibrium $[P^*, Q^*]^\top = [1, -1]^\top$. The trajectories of inverter-interfaced power source systems are depicted in Fig. 3.3. With the state feedback controller, the terminal complex voltage $V \angle \theta$ converges to the desired complex voltage $1 \angle 0$.
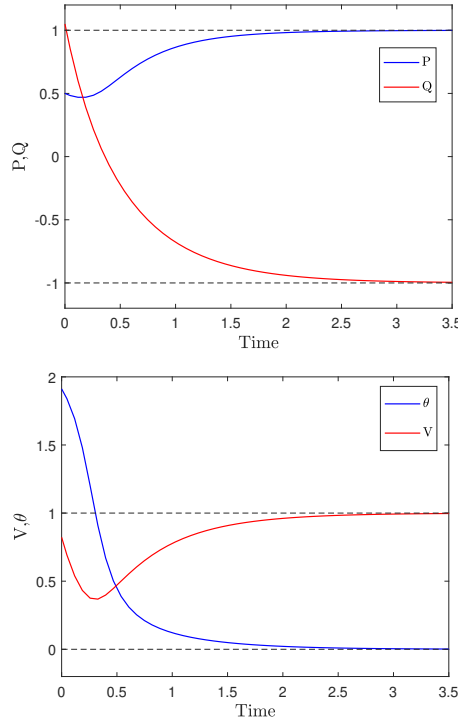

Figure 3.3: The plot of solutions of P, Q, θ, and V in Example 2.3 initialized at $\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.05 \end{bmatrix}$.

3.4.3   Observer design of time-varying ODE systems

By treating the output of the systems as an algebraic constraint, our approaches can be effectively employed in various classical control problems, including output feedback design, output regulation, and state observer design. In the following, we deploy our methodologies to develop an observer for a time-varying ODE system. Consider a time-varying ODE system

$$\begin{cases} \dot{w} = k(t, w), \\ z = h(t, w), \end{cases} \tag{3.63}$$

where w is the state and z is the output. The design problem of interest is to determine an observer of the form

$$\begin{cases} \dot{\hat{w}} = k(t, \hat{w}) + l(t, \hat{z}, z), \\ \hat{z} = h(t, \hat{w}), \end{cases} \tag{3.64}$$

with $l(t, z, z) = 0$, such that

$$\|\hat{w} - w\| \leqslant c \|\hat{w}_0 - w_0\| e^{-\alpha t}, \tag{3.65}$$

holds for all $t \geqslant 0$ with some positive constants c and α. In literature, a commonly used method for designing $l(t, \hat{z}, z)$ is $l(t, \hat{z}, z) = k(t)(\hat{z} - z)$. As a result, the observer (3.64) can be simplified to the well-known Luenberger observer [72].

The time-varying observer (3.63) can be regarded as a time-varying DAE system in the form of (3.1) with $f(t, \hat{w}, \hat{z}) = k(t, \hat{w}) + l(t, \hat{z}, z)$, and $g(t, \hat{w}, \hat{z}) = h(t, \hat{w}) - \hat{z}$. It is evident that w and z in (3.63) represent a solution of (3.64). If (3.64) is contracting, both w and $\hat{w}$ will satisfy (3.65). By applying Theorem 3.6 to (3.64), we obtain the subsequent corollary.

Corollary 3.9. *Given a system described by (3.63), the system (3.64) is an observer of (3.63) if there exist invertible metrics $\theta(t)$ and $\vartheta(t)$, such that $\mu_q\big(J(t)\big) \leqslant -\beta$ for some positive constant β, where $J(t)$ is in the form of (3.42), with $A(t) = \frac{\partial k}{\partial \hat{w}}$, $B(t) = \frac{\partial l}{\partial \hat{z}}$, $F(t) = -I$, $C(t) = \gamma \frac{\partial h}{\partial \hat{w}} + \frac{d}{dt}(\frac{\partial h}{\partial \hat{w}}) + \frac{\partial h}{\partial \hat{w}}\frac{\partial k}{\partial \hat{w}}$, and $D(t) = -\gamma I + \frac{\partial h}{\partial \hat{w}}\frac{\partial l}{\partial \hat{z}}$.*

Example 3.3. Consider an unstable time-varying ODE system as presented in [44, Ex. 4.22]

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5\sin t \cos t & -1 + 1.5\sin^2 t \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{3.66}$$

with the output $z = w_1$. For simplifying the design process, we assume $l(t, \hat{z} - z)$
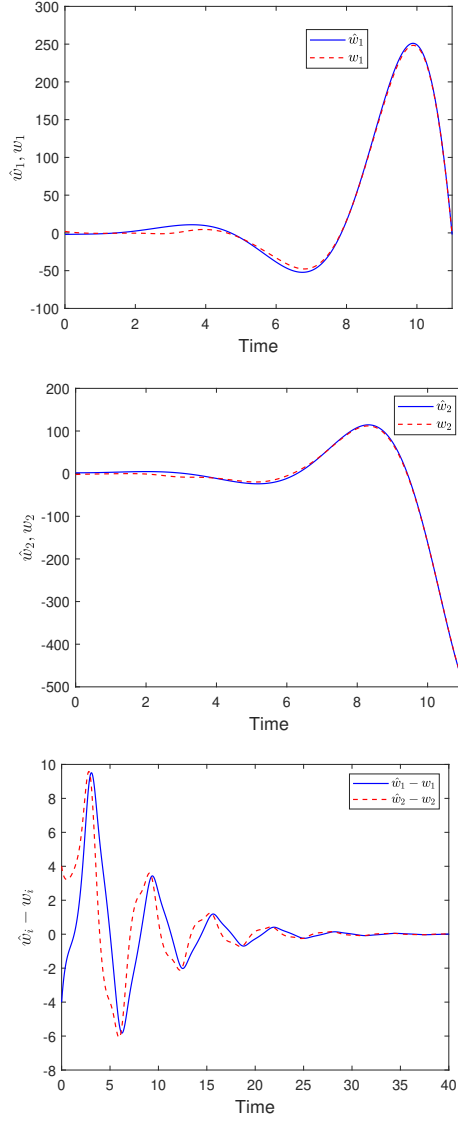
Figure 3.4: The plot of errors of $\hat{w}_i - w_i$ in Example 3.3 initialized at $\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

in (3.64) takes the form of $\begin{bmatrix} k_1(t)(\hat{z}-z) \\ k_2(t)(\hat{z}-z) \end{bmatrix}$. The Luenberger observer is given by

$$\begin{bmatrix} \dot{\hat{w}}_1 \\ \dot{\hat{w}}_2 \end{bmatrix} = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t \cos t \\ -1 - 1.5\sin t \cos t & -1 + 1.5\sin^2 t \end{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} + \begin{bmatrix} k_1(t)(\hat{z} - z) \\ k_2(t)(\hat{z} - z) \end{bmatrix}, \qquad (3.67)$$

where the output $\hat{z} = \hat{w}_1$. By selecting $\gamma = 1$, the auxiliary ODE system (3.27) of (3.67) is given by

$$\begin{cases} \dot{\xi}_1 = (-1 + 1.5\cos^2 t)\xi_1 + (1 - 1.5\sin t \cos t)\xi_2 + k_1(t)\nu \\ \dot{\xi}_2 = (-1 - 1.5\sin t \cos t)\xi_1 + (-1 + 1.5\sin^2 t)\xi_2 + k_2(t)\nu \\ \dot{\nu} = 1.5\xi_1\cos^2 t + (1 - 1.5\sin t \cos t)\xi_2 + (k_1(t) - 1)\nu \end{cases} \quad (3.68)$$

By choosing $k_1 = -1.5\cos^2 t$, $k_2 = -1 + 1.5\sin t \cos t$ and by using the 2-norm, i.e., inequality (3.48) holds with $M = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$, we have

$$J(t) = \begin{bmatrix} -4 + 3\cos^2 t & -3\sin t \cos t & 0 \\ -3\sin t \cos t & -4 + 3\sin^2 t & 0 \\ 0 & 0 & -3\cos^2 t - 4 \end{bmatrix} e^{-2t} \leqslant -I. \quad (3.69)$$

The plots of error of $\hat{w}_i - w_i$ are shown in Fig. 3.4, where the tracking property of the observer is ensured by the proposed methodologies.

3.5   Conclusion

In this chaper, the contraction property of time-varying DAE systems have been studied by an ODE approach. It is established based on a necessary and sufficient condition that connects the contraction property of the original DAE systems and the UGES of its variational DAE systems. Subsequently, the variational DAE systems are lifted to an higher dimension auxiliary ODE systems, and the trajectories of these systems exhibit the same convergence property. The concept of matrix measure is introduced to study the UGES of the auxiliary ODE system. Moreover, the results obtained in this study can be applied to stabilize time-invariant DAE systems, and to observer design for time-varying ODE systems.

4

Pinning synchronization of heterogeneous multi-agent nonlinear systems

This chapter revisit the pinning synchronization problem in nonlinear multi-agent systems (MAS) by using recent results on the incremental stability analysis via contraction and on internal model principle. We provide sufficient and necessary conditions for both the pinned agents as well as the rest of the agents to guarantee the state synchronization. For the non-pinned agents, we present a distributed control framework based only on the relative local state measurement and we give sufficient conditions for the contractivity of the individual virtual systems in order to achieve pinning synchronization. Numerical simulation is given to illustrate the main results.

4.1   Introduction

Distributed consensus control problem has been one of the most well-studied control problems for multi-agent systems (MAS) for the past two decades. It is due to its broad applications in engineering that involves interconnected autonomous systems, such as, sensor networks [105], collaborative robots [82] and unmanned aerial vehicles [22]. The basic problem setup is to design a distributed control algorithm [91], which is implemented locally in each agent and is based only on local information from the neighboring agents, such that the state of all agents converges to each other. While many literature studies deal with the state convergence to a common point (that depends on the initial state of all agents), we are interested in this paper in the convergence of the agents' state to a periodic trajectory of an external oscillator or exosystem which is Poisson stable[1]. The synchronization problem of a single agent to an oscillator is commonly known as entrainment control problem [56] or pinning synchronization control problem in MAS setting [79]. It is motivated by natural phenomena in biology, such as, circadian rhythm

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[1]We refer to [38] for the definition of Poisson stable systems.

[41] and central pattern generators [28], as well as, by engineering application, for example, in power network systems [23]. The pinning control problem refers to a distributed control problem of MAS where a subset of agents, referred to later as the *pinned agents*, is assigned to synchronize to an exosystem (known as the *pinner agent*) and the rest of the agents must synchronize themselves via the network to these pinned nodes. We refer interested readers to the work of Song and Cao in [79] for the pinning synchronization problem of linear MAS. Recent results on pinning synchronization are, to name a few, [66], [85], [88] and [51]. In [66], the controllability property of the pinning network for linear MAS is studied that is relevant to determining important pinning nodes in the network for achieving the synchronization. In recent years, the generalization of pinning synchronization to the nonlinear MAS has been presented in [51, 85, 88].

In this chapter, a contraction-based control scheme is developed for solving the pinning synchronization for heterogeneous MAS that incorporates both linear and non-linear systems. In the literature of distributed control for MAS, contraction analyses and contraction-based methods have been applied to homogeneous MAS [73]-[34]. These synchronization approaches for homogeneous MAS pose non-trivial challenges to achieve the synchronization of heterogeneous MAS due to the heterogeneity among all agents. There are only a few research works that focus on heterogeneous MAS, see for example, [3]. As typically considered in the pinning synchronization approaches, our proposed control design is comprised of two design steps. The first one pertains to the tracking control design for the pinning nodes where the agents are connected to the external oscillator or virtual leader. By employing standard regulator equation [38], some sufficient and necessary conditions are given to guarantee the solvability of the synchronization problem for the pinned agents. By employing these conditions, the pinning synchronization problem of heterogeneous nonlinear MAS can be recast as a contraction problem of virtual systems. The control law for the pinned agents follows a contraction approach [27] applied to the virtual systems. Subsequently, for the rest of the agents, we put forward a distributed control law based only on relative local state measurement in order to ensure the contractivity of the each agent's virtual systems where the synchronized state trajectory is also an admissible trajectory of the virtual systems.

The chapter is organized as follows. In Section 4.2 we present preliminaries and problem formulation. Our main results are presented in Sections 4.3. The numerical simulation is provided in Section 4.4 and the conclusions are given in Section 4.5.

4.2 Preliminaries and problem formulation

4.2.1 Communication Graph

For the MAS, we denote $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ as the set of N nodes in the network and the associated set of K edges is denoted by $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. We consider a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ where the communication direction is embedded in $\mathcal{E}$. Correspondingly, we define the adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$, where the element $a_{ij} = 1$ for all $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The set of *neighbors* of node v_i, denoted by N_i, is the set of nodes $v_j \in \mathcal{V}$ such that $(v_j, v_i) \in \mathcal{E}$. The *Laplacian matrix* is defined as $L = D - A$, where $D = \mathbf{diag}(d_i)$ is called the *in-degree matrix*, with $d_i = \sum_{j \in N_i} a_{ij}$ as the in-degree of node v_j. A directed graph contains a directed spanning tree if there exists a node called the *root* such that there exists a directed path (a sequence of nodes connected by directed edges, where each edge has a direction from one node to another) from this node to every other node. For defining the pinning synchronization problem later, we define $\mathcal{V}_{\text{pin}} \subset \mathcal{V}$ as the set of pinned nodes which are all directly connected (two nodes that are linked or connected by an edge) to a virtual node v_0 representing the exosystem or external oscillator. Correspondingly, we can define the communication graph $\mathcal{G}_0 = (\mathcal{V}_{\text{pin}} \cup \{v_0\}, \mathcal{E}_0)$ of the pinned nodes and the virtual node where $\mathcal{E}_0 = \{(v_0, v_i) | v_i \in \mathcal{V}_{\text{pin}}\}$. All other nodes in $\mathcal{V} \backslash \mathcal{V}_{\text{pin}}$ are assumed to be accessible from $\mathcal{V}_{\text{pin}}$ by the following assumption.

Assumption 4.1. The graph $\mathcal{G} \cup \mathcal{G}_0 = (\mathcal{V} \cup \{v_0\}, \mathcal{E} \cup \mathcal{E}_0)$ contains a spanning tree where v_0 is pinned to the root node.

4.2.2 Agent dynamics, dynamic distributed controller and exosystem

Consider again the nodes of the graph $\mathcal{G} \cup \mathcal{G}_0$ as defined before. The oscillator dynamics at node v_0 is given by

$$\dot{w} = S(w), \tag{4.1}$$

where $w(t) \in \mathbb{R}^n$ is the oscillator's state and S is a smooth function with $S(0) = 0$ that generates a Poisson stable exosystem [38]. We recall from [38] that in this case the eigenvalues of $S_w(0)$ are all on the imaginary axis. For all nodes in $\mathcal{V}$, we consider (non-identical) agents described by

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \quad \forall i \in \mathcal{V}, \tag{4.2}$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^{m_i}$ are the state and control input variables, respectively, and f_i, g_i are smooth functions with $f_i(0) = 0$ and $g_i(x_i) \neq 0$ are full-rank for all x_i. For each node in $\mathcal{V}$, we will assign a distributed dynamic controller (according to the communication graph $\mathcal{G} \cup \mathcal{G}_0$) as follows. For all pinned nodes $i \in \mathcal{V}_{\text{pin}}$, the dynamic controller for i-th agent is given by

$$\begin{cases} \dot{\xi}_i & = C_i(\xi_i, x_i - w) \\ u_i & = D_i(\xi_i) \end{cases} \tag{4.3}$$

where $\xi_i \in \mathbb{R}^p$ is the controller state, C_i, D_i are continuously differentiable with $C_i(0,0) = 0$ and $D_i(0) = 0$. On the other hand, for the rest of the agents, e.g. for all $i \in \mathcal{V} \backslash \mathcal{V}_{\text{pin}}$, we consider the following dynamic controller

$$\begin{cases} \dot{\xi}_i & = \hat{C}_i(\xi_i, e_i) \\ u_i & = \hat{D}_i(\xi_i) \end{cases} \tag{4.4}$$

where $\xi_i \in \mathbb{R}^p$ is the controller state, $e_i = \sum_{j \in N_i} a_{ij}(x_i - x_j)$ is the local error state and $\hat{C}_i, \hat{D}_i$ are continuously differentiable with $\hat{C}_i(0,0) = 0$ and $\hat{D}_i(0) = 0$.

4.2.3 Problem formulation

Based on the previous systems' description, we can formulate the pinning synchronization problem as follows.

Pinning synchronization problem: For a given exosystem and agents' dynamics given by (4.1) and (4.2), respectively, design the controllers (4.3) for the pinned nodes $\mathcal{V}_{\text{pin}}$ and the controller (4.4) for the rest of the nodes $\mathcal{V} \backslash \mathcal{V}_{\text{pin}}$ such that

(O1). The origin of the closed-loop systems is locally exponentially stable; and

(O2). For all initial conditions $w(0), x_i(0), \xi_i(0)$ in $W_0 \times X_{i0} \times \Xi_0 \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ which contains the origin, the trajectories of the closed-loop systems are bounded and satisfy

$$\lim_{t \to \infty} (x_i(t) - w(t)) = 0, \qquad \forall i \in \mathcal{V}. \tag{4.5}$$

4.3 Main result

For solving the pinning synchronization problem via contraction approach, we separate the distributed control design problem into two parts. The first part is associated to the agents in the pinned nodes $\mathcal{V}_{\text{pin}}$ and the second one is designed for the rest of the agents. On the one hand, as the pinned agents are directly connected

to the exosystem, the error manifold can be defined directly as $e_i = x_i - w$. On the other hand, for the rest of agents $\mathcal{V} \backslash \mathcal{V}_{\mathrm{pin}}$, the error manifold is defined by $e_i = \sum_{j \in N_i} (x_j - x_i)$.

4.3.1 Tracking control law for $\mathcal{V}_{\mathbf{pin}}$

In order to motivate the control design approach, we present firstly the control law for the pinned agents $\mathcal{V}_{\mathrm{pin}}$, which are described as linear systems case, and subsequently we propose the control design for the nonlinear ones. As our design approach is based on making the corresponding virtual systems contractive, the analysis on linear systems facilitates the discussion of contraction-based control law in the subsequent sub-section.

Linear systems case

For all pinned agents $i \in \mathcal{V}_{\mathrm{pin}}$, let us represent it as a linear system as follows

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{4.6}$$

where A_i, B_i are of appropriate dimension. Correspondingly, we also assume a linear external oscillator as the pinner agent, which is given by

$$\dot{w} = Sw, \tag{4.7}$$

where S has simple eigenvalues on the imaginary axis and is a generator of sinusoidal and constant signals. Following the form in (4.3) with $\hat{C}_i = K_i \xi_i + H_i(x_i - w)$, $\hat{D}_i = L_i \xi_i$, we have Proposition 4.2 where K_i, H_i, L_i are matrices to be designed.

Proposition 4.2. *Consider the network of pinned agents $\mathcal{V}_{\mathrm{pin}}$ and pinner agent v_0 with graph $\mathcal{G}_0$ which are represented by (4.6) and (4.7), respectively. Assume that the pair (A_i, B_i) is stabilizable for all $i \in \mathcal{V}_{\mathrm{pin}}$. Then the pinning synchronization problem for $\mathcal{V}_{\mathrm{pin}}$ agents is solvable by the distributed controller (4.3) (i.e. there exists a controller (4.3) such that the closed-loop system (4.3), (4.6) with the exosystem (4.7) satisfies (O1) and (O2)) if and only if, for all $i \in \mathcal{V}_{\mathrm{pin}}$ there exist K_i, L_i and Σ_i satisfying*

$$\begin{cases} A_i + B_i L_i \Sigma_i = S \\ \quad K_i \Sigma_i = \Sigma_i S. \end{cases} \tag{4.8}$$

Proof. As each agent in $\mathcal{V}_{\mathrm{pin}}$ is connected directly to v_0 and they do not communicate with each other, we prove the proposition by analyzing the state synchronization of an arbitrary agent $i \in \mathcal{V}_{\mathrm{pin}}$ to the pinner agent and the arguments hold *mutatis mutandis* for the other agents. Correspondingly, the claim of the

proposition follows from classical results on linear output regulation via internal model principle, e.g., [38] and [89]. □

Let us now relate the above result to the exponential incremental stability as before, which will be instrumental later for our contraction-based control approach. It can be observed that if (4.8) is satisfied, $\left[\begin{smallmatrix} x_i \\ \xi_i \end{smallmatrix} \right] = \left[\begin{smallmatrix} w \\ \Sigma_i w \end{smallmatrix} \right]$ is a particular solution of the following virtual system

$$\begin{pmatrix} \dot{q}_i \\ \dot{\xi}_i \end{pmatrix} = \begin{pmatrix} A_i & B_i L_i \\ H_i & K_i \end{pmatrix} \begin{pmatrix} q_i \\ \xi_i \end{pmatrix} - \begin{pmatrix} 0 \\ H_i \end{pmatrix} w, \tag{4.9}$$

Nonlinear systems case

We will now consider the nonlinear system (4.2) with the controller (4.3), in which case, the closed-loop system is given by

$$\begin{pmatrix} \dot{x}_i \\ \dot{\xi}_i \end{pmatrix} = \begin{pmatrix} f_i(x_i) + g_i(x_i)D_i(\xi_i) \\ C_i(\xi_i, x_i - w) \end{pmatrix} \tag{4.10}$$

Let us define $\tilde{A}_i = f^i_{x_i}(0)$, $\tilde{B}_i = g_i(0)$, $\tilde{S} = S_w(0)$, $\tilde{L}_i = D^i_{\xi_i}(0)$.

Proposition 4.3. *Consider the network of pinned controlled agents $\mathcal{V}_{\text{pin}}$ and pinner agent v_0 with graph $\mathcal{G}_0$ which are represented by (4.10) and (4.1), respectively. Assume that for all $i \in \mathcal{V}_{\text{pin}}$, the pair (f_i, g_i) has a stabilizable linear approximation at 0. Then the pinning synchronization problem for $\mathcal{V}_{\text{pin}}$ agents is solvable if and only if, for all $i \in \mathcal{V}_{\text{pin}}$ there exist mappings $x_i = w$, $\xi_i = \sigma_i(w)$ with $\sigma_i(0) = 0$ satisfying*

$$\left. \begin{array}{rl} S(w) &= f_i(w) + g_i(w)D_i(\sigma_i(w)) \\ \sigma^i_w S(w) &= C_i(\sigma_i(w), 0), \end{array} \right\} \tag{4.11}$$

where $\sigma^i_w = \frac{\partial \sigma^i(w)}{\partial w}$. We remark that (4.11) is the standard Byrnes-Isidori regulator equation [38] restricted to the state regulation case.

Proof. By Taylor expansion around the origin, the system (4.1), (4.2) and (4.3) can be rewritten as
$$\left. \begin{array}{rl} \dot{\tilde{x}}_{ci} &= \tilde{A}_{ci}\tilde{x}_{ci} + \tilde{B}_{ci}w + \lambda_i(\xi_i, x_i, w) \\ \dot{w} &= \tilde{S}w + \omega(w) \end{array} \right\} \tag{4.12}$$
where $\tilde{x}_{ci} = \begin{pmatrix} x_i \\ \xi_i \end{pmatrix}$, $\tilde{A}_{ci} = \begin{pmatrix} \tilde{A}_i & \tilde{B}_i \tilde{L}_i \\ \tilde{H}_i & \tilde{K}_i \end{pmatrix}$, $\tilde{B}_{ci} = \begin{pmatrix} 0 \\ -\tilde{H}_i \end{pmatrix}$ and with $\lambda_i(\xi_i, x_i, w)$ and $\omega(w)$ are the remainder terms that vanish at the origin. The rest of the proofs can be

found in [38] Theorem 2 by selecting the controller as

$$
\begin{cases}
\dot{\xi}_{i1} = f_i(\xi_{i1}) + g_i(\xi_{i1})(D_i(\xi_{i2}) + H_i(\xi_{i1} - \xi_{i2})) - G_{i1}(\xi_{i1} - \xi_{i2} - e_i) \\
\dot{\xi}_{i2} = S(\xi_{i2}) + G_{i2}\xi_{i2} - (G_{i2} - G_{i3})(\xi_{i1} - e_i) \\
u = D_i(\xi_{i2}) + H_i(\xi_{i1} - \xi_{i2})
\end{cases}
\tag{4.13}
$$

where $e_i = x_i - w$ and $K_i, H_i, G_{i1}, G_{i2}, G_{i3}$ are matrices to be chosen appropriately such that $\begin{pmatrix} \tilde{A}_i + \tilde{B}_i H_i & \tilde{B}_i H_i & \tilde{B}_i(L_i - H_i) \\ 0 & \tilde{A}_i - G_{i1} & G_{i1} \\ 0 & -G_{i2} + G_{i3} & \tilde{S} + G_{i2} \end{pmatrix}$ is Hurwitz. Since the manifold $x_i = \pi_i(w)$ in [38] is locally attractive and invariant, then the error $e_i = x_i - w = \pi_i(w) - w$ converges to 0 only if $x_i = \pi_i(w) = w$. $\qquad\square$

Equations (4.8) and (4.11) correspond to the necessary and sufficient conditions for output synchronization of linear systems in [89]. In contrast to the existing results for nonlinear systems case, such as the ones presented in [15, 39] where there are information exchanges among the exosystems and the internal model part of the controller, our controller uses only the relative measurement of state variable with its neighbors and not that of the controller state variable. Instead of designing the controller based on the linearization at the origin as in the proof of Proposition 4.3 above, we will consider below the use of a virtual system that can enlarge the region of attraction of the synchronization manifold. If the pinning synchronization problem for $\mathcal{V}_{\text{pin}}$ is solvable then $\begin{bmatrix} x_i \\ \xi_i \end{bmatrix} = \begin{bmatrix} w \\ \sigma_i(w) \end{bmatrix}$ is a particular solution of the following virtual system

$$
\begin{pmatrix} \dot{q}_i \\ \dot{\xi}_i \end{pmatrix} = \begin{pmatrix} f_i(q_i) + g_i(q_i)D_i(\xi_i) \\ C_i(\xi_i, q_i - w) \end{pmatrix}.
\tag{4.14}
$$

Its variational system is given by

$$
\begin{pmatrix} \delta\dot{q}_i \\ \delta\dot{\xi}_i \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial(f_i(q_i) + g_i(q_i)D_i(\xi_i))}{\partial q_i} & g_i(q_i)D_{\xi_i}^i \\ C_{e_i}^i & C_{\xi_i}^i \end{pmatrix}}_{\tilde{A}_i} \begin{pmatrix} \delta q_i \\ \delta\xi_i \end{pmatrix}.
\tag{4.15}
$$

With the following proposition, we can ensure the zero error invariant manifold is attractive.

Proposition 4.4. *Consider the virtual system* (4.14) *with its associated variational system* (4.15) *satisfying* (4.11). *If there exists a symmetric matrix $P_i(q_i, \xi_i)$ such that $c_{1i}I \leqslant P_i(q_i, \xi_i) \leqslant c_{2i}I$ and*

$$
\tilde{A}_i^T P_i + \dot{P}_i + P_i \tilde{A}_i \leqslant -\lambda P_i
\tag{4.16}
$$

hold for some $\lambda > 0$ and for all $(q_i, \xi_i) \in Q_0 \times \Xi_0$ with $W_0 \cup X_{i0} \subset Q_0$, $i \in \mathcal{V}_{pin}$ then,

for all $i \in \mathcal{V}_{pin}$, the closed-loop systems (4.1), (4.2) and (4.3) satisfy **(O1)** and **(O2)**.

Proof. By defining the Finsler-Lyapunov function

$$V_i(\delta q_i, \delta \xi_i) = \begin{pmatrix} \delta q_i & \delta \xi_i \end{pmatrix} P_i \begin{pmatrix} \delta q_i \\ \delta \xi_i \end{pmatrix}$$

we have $c_{1i} \left\| \begin{bmatrix} \delta q_i \\ \delta \xi_i \end{bmatrix} \right\|^2 \leqslant V_i(\delta q_i, \delta \xi_i) \leqslant c_{2i} \left\| \begin{bmatrix} \delta q_i \\ \delta \xi_i \end{bmatrix} \right\|^2$.

Its time derivative along (4.15) is given by

$$\dot{V}_i = \begin{pmatrix} \delta q_i & \delta \xi_i \end{pmatrix} \left(\tilde{A}_i(q_i, \xi_i)^T P_i + \dot{P}_i + P_i \tilde{A}_i(q_i, \xi_i) \right) \begin{pmatrix} \delta q_i \\ \delta \xi_i \end{pmatrix}$$

$$\leqslant -\lambda_i \begin{pmatrix} \delta q_i & \delta \xi_i \end{pmatrix} P_i \begin{pmatrix} \delta q_i \\ \delta \xi_i \end{pmatrix} = -\lambda_i V_i.$$

Based on Lemma 1.1, this inequality implies that the system (4.14) is exponentially incrementally stable. In other words, we have

$$\left\| x_i(t) - w(t) \right\| \leqslant k e^{-\lambda t} \left\| x_i(0) - w(0) \right\|$$

for all $(x_i(0), w(0)) \in X_{i0} \times W_0$. $\qquad\qquad\qquad\square$

We note that the closed-loop system (4.2)-(4.3) satisfying all hypotheses in Proposition 4.4 can be written as $\dot{x}_r = f(x_r, t)$, where $x_r = \begin{bmatrix} x_i \\ \xi_i \end{bmatrix}$ and t represents its dependence to external signal $w(t)$. As shown in the proof above, it is contractive with a contraction rate λ. Let us consider a "perturbed" system $\dot{x}_p = f(x_p, t) + d(x_p, t)$ where $\left| d(x_p, t) \right| \leqslant d$. In this case any trajectory of the perturbed system satisfies $\left| x_d(t) - x_r(t) \right| \leqslant (\left| x_d(0) - x_r(0) \right| - \frac{d}{\lambda}) e^{-\lambda t} + \frac{d}{\lambda}$ (for generalisation, we refer to Section 3.7 in [55].

4.3.2   Distributed control law for $\mathcal{V} \backslash \mathcal{V}_{\mathbf{pin}}$

After we have designed the control law for the pinned nodes in the previous sub-section, we can present now the control design for the rest of the agents.

Proposition 4.5. *Under Assumption 4.1 the synchronization problem is solvable if and only if for all $i \in \mathcal{V} \backslash \mathcal{V}_{pin}$ there exist mappings $x_i = w$, $\xi_i = \hat{\sigma}_i(w)$ with $\hat{\sigma}_i(0) = 0$ satisfying*

$$\left. \begin{aligned} S(w) &= f_i(w) + g_i(w) \hat{D}_i(\hat{\sigma}_i(w)) \\ \hat{\sigma}_w^i S(w) &= \hat{C}_i(\hat{\sigma}_i(w), 0), \end{aligned} \right\} \tag{4.17}$$

where $\hat{\sigma}_w^i = \frac{\partial \hat{\sigma}^i(w)}{\partial w}$.

Proof. By Proposition 4.3, we have established the necessary and sufficient conditions of pinning synchronization for the pinned nodes. In the following, we will extend the analysis to the rest of the agents based on their connectivity to these pinned agents.

(*If part*): By Assumption 4.1, the graph $\mathcal{G} \cup \mathcal{G}_0$ contains a spanning tree with v_0 be at the root node. For the nodes $\mathcal{V}_{\text{pin}}$, we have established before that we can design a control law such that $x_i - w \to 0$ for all $i \in \mathcal{V}_{\text{pin}}$. Thus we can now consider the rest of the agents. Suppose now, for all agents $i \in \mathcal{V} \backslash V_{\text{pin}}$, there exist mappings $x_i = w$ and $\xi_i = \hat{\sigma}_i(w)$ satisfying (4.17). Since $\hat{C}_i(\hat{\sigma}_i(w), 0)$, $\hat{D}_i(\hat{\sigma}_i(w))$ in (4.17) and $C_i(\sigma_i(w), 0)$, $D_i(\sigma_i(w))$ in (4.11) (for the pinned agents) are all functions of w, we can take $\hat{D}_i(\hat{\sigma}_i(w)) = D_i(\sigma_i(w))$, $\hat{C}_i(\hat{\sigma}_i(w), 0) = C_i(\sigma_i(w), 0)$. Following similar arguments as in the if part of Proportion 4.3, we can define the controller for agent i as in (4.13) with $e = \sum_{j \in N_i} (x_i - x_j)$ such that the linearization at the origin can be made Hurwitz and the mappings $x_i = w$ and $\xi_i = \hat{\sigma}_i(w)$ are attractive center manifold.

(*Only if part*): Let us define $\hat{A}_i = f_{x_i}^i(0)$, $\hat{B}_i = g_i(0)$, $\hat{K}_i = \hat{C}_{\xi_i}^i(0,0)$, $\hat{H}_i = \hat{C}_{x_i}^i(0,0)$, and $\hat{G}_i = \hat{D}_{\xi_i}^i(0)$, $\hat{w} = \begin{bmatrix} w & 0 \end{bmatrix}^T$. Then the composite system with the corresponding dynamic controller can be linearized as follows

$$
\begin{pmatrix} \dot{\hat{x}}_{\text{pin}}^i \\ \dot{\hat{x}}_{\text{rest}}^i \end{pmatrix} = \begin{pmatrix} \hat{A}_{\text{pin}}^i & 0 \\ 0 & \mathbf{diag}(\hat{A}_{\text{rest}}^i) \end{pmatrix} \begin{pmatrix} \hat{x}_{\text{pin}}^i \\ \hat{x}_{\text{rest}}^i \end{pmatrix}
$$
$$
+ \begin{pmatrix} L_{\text{pin}} \otimes I_n^{\text{pin}} & 0 \\ L' \otimes I_n^{\text{pin}} & L_{\text{rest}} \otimes I_n^{\text{rest}} \end{pmatrix} \begin{pmatrix} \mathbf{diag}(\hat{H}_{\text{pin}}^i) & 0 \\ 0 & \mathbf{diag}(\hat{H}_{\text{rest}}^i) \end{pmatrix} \begin{pmatrix} \hat{x}_{\text{pin}}^i \\ \hat{x}_{\text{rest}}^i \end{pmatrix}
$$

where $\hat{x}_{\text{pin}}^i = \begin{pmatrix} \hat{w}^T & x_{\text{pin}}^{i \ T} & \xi_{\text{pin}}^{i \ T} \end{pmatrix}^T$, $\hat{x}_{\text{rest}}^i = \begin{pmatrix} x_{\text{rest}}^{i \ T} & \xi_{\text{rest}}^{i \ T} \end{pmatrix}^T$,
$\hat{A}_{\text{pin}}^i = \mathbf{diag}(\hat{S}, \{\mathbf{diag}(A_{\text{pin}}^i)\})$, $\hat{H}_{\text{pin}}^i$ or $\hat{H}_{\text{rest}}^i = \begin{pmatrix} 0 & 0 \\ \hat{H}_i & 0 \end{pmatrix}$, $\hat{A}_{\text{pin}}^i$ or $\hat{A}_{\text{rest}}^i = \begin{pmatrix} \hat{A}_i & \hat{B}_i \hat{G}_i \\ 0 & \hat{K}_i \end{pmatrix}$,
the *Laplacian matrix* $L = \begin{pmatrix} L_{\text{pin}} & 0 \\ L' & L_{\text{rest}} \end{pmatrix}$. According to Assumption 4.1, the *Laplacian matrix* L has exactly one zero eigenvalue and the rest eigenvalues are all have positive real parts [70]. Since the first row of L_{pin} are all zero, L_{rest} has eigenvalues with positive real parts. The rest agents can be linearized at the origin as follows

$$
\dot{\hat{x}}_{\text{rest}}^i = \mathbf{diag}(\hat{A}_{\text{rest}}^i)\hat{x}_{\text{rest}}^i + (L' \otimes I_n^{\text{pin}})\mathbf{diag}(\hat{H}_{\text{pin}}^i)\hat{x}_{\text{pin}}^i + (L_{\text{rest}} \otimes I_n^{\text{rest}})\mathbf{diag}(\hat{H}_{\text{rest}}^i)\hat{x}_{\text{rest}}^i
$$

Since the pinned agents are stabilizable and synchronized to w, we can consider the state of pinned agents as bounded input for the rest of the agents. Then the whole system is stable if and only if the system

$$
\dot{\hat{x}}_{\text{rest}}^i = \mathbf{diag}(\hat{A}_{\text{rest}}^i)\hat{x}_{\text{rest}}^i + (L_{\text{rest}} \otimes I_n^{\text{rest}})\mathbf{diag}(\hat{H}_{\text{rest}}^i)\hat{x}_{\text{rest}}^i + (L' \otimes I_n^{\text{pin}})\mathbf{diag}(\hat{H}_{\text{pin}}^i)u
$$

is input-to-state stable (see Lemma 5.6 in [44]). We recall that for a linear system $\dot{x} = Ax + Bu$, it is input-to-state stable if and only if $\dot{x} = Ax$ is stable. Thus it follows then that

$$\dot{\hat{x}}^i_{\text{rest}} = \mathbf{diag}(\hat{A}^i_{\text{rest}})\hat{x}^i_{\text{rest}} + (L_{\text{rest}} \otimes I^{\text{rest}}_n)\mathbf{diag}(\hat{H}^i_{\text{rest}})\hat{x}^i_{\text{rest}} \tag{4.18}$$

is stable. The system (4.18) can compactly be written as

$$\begin{pmatrix} \dot{X} \\ \dot{\Xi} \end{pmatrix} = \begin{pmatrix} \hat{A} & \hat{B}\hat{G} \\ (\hat{L}_{\text{rest}} \otimes I_n)\hat{H} & \hat{K} \end{pmatrix} \begin{pmatrix} X \\ \Xi \end{pmatrix}$$

where $X = (x^i_{\text{rest}}{}^T)^T$, $\Xi = (\xi^i_{\text{rest}}{}^T)^T$, $\hat{A} = \mathbf{diag}(\hat{A}_i)$, $\hat{B} = \mathbf{diag}(\hat{B}_i)$, $\hat{K} = \mathbf{diag}(\hat{K}_i)$, $\hat{G} = \mathbf{diag}(\hat{G}_i)$, $\hat{H} = \mathbf{diag}(\hat{H}_i)$, and $\hat{L}_{\text{rest}}$ has the same eigenvalues as L_{rest}. The latter means that there exists an $\hat{H}$ such that $\hat{A} - (\hat{L}_{\text{rest}} \otimes I_n)\hat{H}$ is Hurwitz. If pinning synchronization problem is solvable then there exist controllers u_i for all $i \in \mathcal{V}$ that locally exponentially stabilize the multi-agent systems (4.2). Since $\hat{A}_c = \begin{pmatrix} \hat{A} & \hat{B}\hat{G} \\ (\hat{L}_{\text{rest}} \otimes I_n)\hat{H} & \hat{K} \end{pmatrix}$ is similar to $\begin{pmatrix} \hat{A} + \hat{B}\hat{G} & \hat{B}\hat{G} \\ (\hat{L}_{\text{rest}} \otimes I_n)\hat{H} + \hat{K} - \hat{A} - \hat{B}\hat{G} & \hat{K} - \hat{B}\hat{G} \end{pmatrix}$, then $\hat{A}_c$ can be made Hurwitz by choosing appropriate $\hat{G}_i, \hat{K}_i, \hat{H}_i$ such that $(\hat{L}_{\text{rest}} \otimes I_n)\hat{H} + \hat{K} - \hat{A} - \hat{B}\hat{G} = 0$, $\hat{A} + \hat{B}\hat{G}$ is Hurwitz and $\hat{K} - \hat{B}\hat{G} = \hat{A} - (\hat{L}_{\text{rest}} \otimes I_n)\hat{H}$ is Hurwitz. This, in combination with the dynamics of the exosystem (4.1) and by *Center Manifold Theorem*[44], implies that there exists a center manifold which is the graph of $\begin{pmatrix} X \\ \Xi \end{pmatrix} = \begin{pmatrix} \hat{\pi}(w) \\ \hat{\sigma}(w) \end{pmatrix}$, where $\hat{\sigma}(w) = \mathbf{diag}(\hat{\sigma}_i(w))$, $\hat{\pi}(w) = \mathbf{diag}(\hat{\pi}_i(w))$. Since the manifold $X = \hat{\pi}(w)$ is locally attractive and invariant, then the error converges to 0 only if $\hat{\pi}(w) = w$. The graph of (X, Ξ) in the manifold is given by $\begin{pmatrix} X \\ \Xi \end{pmatrix} = \begin{pmatrix} W \\ \hat{\sigma}(W) \end{pmatrix}$ satisfying (4.17). □

If pinning synchronization problem is solvable, the dynamics of each agent in $\mathcal{V}\backslash\mathcal{V}_{\text{pin}}$ can be rewritten as

$$\begin{cases} \dot{x}_i = S(x_i) + g_i(x_i)(\hat{D}_i(\xi_i) - \hat{D}_i(\hat{\sigma}_i(x_i))) \\ \quad \dot{\xi}_i = \hat{C}_i(\xi_i, \sum a_{ij}(x_i - x_j)) \end{cases} \tag{4.19}$$

Consequently the following virtual system has a particular solution $q_i = x_i = x_1 = \cdots = x_N = w$

$$\begin{cases} \dot{q}_i = f_i(q_i) + g_i(q_i)\hat{D}_i(\xi_i) \\ \dot{\xi}_i = \hat{C}_i(\xi_i, \sum a_{ij}(q_i - x_j)) \end{cases} \tag{4.20}$$

The variational system of (4.20) is

$$
\begin{pmatrix} \delta \dot{q}_i \\ \delta \dot{\xi}_i \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial (f_i(q_i) + g_i(q_i)\hat{D}_i(\xi_i))}{\partial q_i} & g_i(q_i)\hat{D}_{\xi_i}^i \\ \hat{C}_{e_i}^i \sum a_{ij} & \hat{C}_{\xi_i}^i \end{pmatrix}}_{\hat{A}_i} \begin{pmatrix} \delta q_i \\ \delta \xi_i \end{pmatrix}
\tag{4.21}
$$

According to Proposition 4.5, by solving (4.17) we have

$$
\begin{cases}
\hat{D}_i(\xi_i)\big|_{\xi_i = \hat{\sigma}_i(w)} = (g_i^T(w)g_i(w))^{-1}g_i^T(w)(S(w) - f_i(w)) \\
\hat{C}_i(\xi_i, 0)\big|_{\xi_i = \hat{\sigma}_i(w)} = \hat{\sigma}_w^i S(w)
\end{cases}
$$

We note again that in our distributed controllers (4.3) and (4.4), there is no information exchange of the controller state variable among the agents. Each local controller uses only the relative plant state measurement with its neighbors as opposed to the one considered in [15, 39] which assume information exchanges among the local nonlinear oscillators. The local reference generator in [39] can be regarded as a particular case of (4.19) with a linear input term.

Proposition 4.6. *Assume that for the pinned agents $i \in \mathcal{V}_{pin}$, the hypotheses in Proposition 4.4 hold. For the rest of the agents, consider the virtual system (4.20) with its variational system (4.21) satisfying (4.17) and Assumption 4.1. If there exists a symmetric matrix $P_i(q_i, \xi_i)$ s.t. $c_1 I \leqslant P_i(q_i, \xi_i) \leqslant c_2 I$, and*

$$
\hat{A}_i^T P_i + \dot{P}_i + P_i \hat{A}_i \leqslant -\lambda P_i
\tag{4.22}
$$

*or (4.16) holds for some $\lambda > 0$ and for all $(q_i, \xi_i) \in Q_0 \times \Xi_0$ with $W_0 \cup X_{i0} \subset Q_0$, $i \in \mathcal{V}$ then the closed-loop systems (4.1), (4.2), (4.3) and (4.4) satisfy **(O1)** and **(O2)**.*

Proof. The system (4.20) is exponentially IS if condition (4.22) is satisfied. Since $q_i = x_i = x_1 = \cdots = x_N = w$ is a particular solution of the virtual system (4.20), then **(O2)** is satisfied and the zero-error invariant manifold is locally attractive. Since $(0, 0)$ is a particular solution of (4.20) with $x_j = 0$, all trajectories converge to $(0, 0)$ asymptotically, i.e. **(O1)** is satisfied. $\qquad\square$

In Proposition 4.6 above, we have presented sufficient conditions on each node that allow us to enlarge the region of attraction to $W_0 \times X_{i0}$ via the contraction analysis on the virtual systems.

4.4   Simulation setup and results

For the simulation setup, we consider four agents with two pinned nodes, e.g., $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{V}_{\text{pin}} = \{v_1, v_2\}$. The nonlinear exosystem is given by
$\begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = -w_1 + \cos(w_1). \end{cases}$ which is Poisson stable whose orbit may not revolve
around the origin. We assume the following dynamics for each node:

$$v_1 : \begin{cases} \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = -3x_{11} - 2x_{12} + u_1 \end{cases}, \qquad v_2 : \begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = -5x_{21} - x_{22} + \sin(x_{21}) + u_2 \end{cases}$$

$$v_3 : \begin{cases} \dot{x}_{31} = x_{32} \\ \dot{x}_{32} = -3x_{31} - x_{32} - x_{32}^3 + u_3 \end{cases}, \qquad v_4 : \begin{cases} \dot{x}_{41} = x_{42} \\ \dot{x}_{42} = -3x_{41} - x_{42} + 2\cos(x_{41}) + u_4 \end{cases}$$

The nodes v_1 and v_2 are connected to the pinner agent v_0, while the agents v_3 and v_4 are connected to all other agents (except v_0). Correspondingly, we consider the following controllers:

$$v_1 : \begin{cases} \dot{\xi}_{11} = \xi_{12} \\ \dot{\xi}_{12} = -\xi_{11} + 2\cos(0.5\xi_{11}) - 3e_{11} - 3e_{12} \\ u_1 = \xi_{11} + \xi_{12} + \cos(0.5\xi_{11}) \end{cases} \tag{4.23}$$

$$v_2 : \begin{cases} \dot{\xi}_{21} = \xi_{22} - 5e_{22} \\ \dot{\xi}_{22} = -\xi_{21} + 2\cos(0.5\xi_{21}) - 5e_{21} - 5e_{22} \\ u_2 = 2\xi_{21} + 0.5\xi_{22} - \sin(0.5\xi_{21}) + \cos(0.5\xi_{21}) \end{cases} \tag{4.24}$$

$$v_3 : \begin{cases} \dot{\xi}_{31} = \xi_{32} \\ \dot{\xi}_{32} = -\xi_{31} + 2\cos(0.5\xi_{31}) - 0.167e_{31} - 0.267e_{32} \\ u_3 = \xi_{31} + 0.5\xi_{32} + 0.125\xi_{32}^3 + \cos(0.5\xi_{31}) \end{cases} \tag{4.25}$$

$$v_4 : \begin{cases} \dot{\xi}_{41} = \xi_{42} - 0.5e_{42} \\ \dot{\xi}_{42} = -\xi_{41} + 2\cos(0.5\xi_{41}) - 0.5e_{41} - 0.5e_{42} \\ u_4 = \xi_{41} + 0.5\xi_{42} - \cos(0.5\xi_{41}) \end{cases} \tag{4.26}$$

where $e_{i,j} = x_{i,j} - w_i$, for $i = 1, 2$ and $j = 1, 2$ and $e_{3j} = 3x_{3j} - x_{1j} - x_{2j} - x_{4j}$, $e_{4j} = 3x_{4j} - x_{1j} - x_{2j} - x_{3j}$ for $j = 1, 2$. The conditions (4.11) and (4.17) in Propositions 4.3 and 4.5 hold with $\hat{\sigma}_i(w) = 2w$ for all $i = 1, 2, 3, 4$. It can be checked that using the following positive definite constant matrices for P_1, P_2, P_3

and P_4, respectively:

$$
\begin{bmatrix}
4.16 & 1.83 & -1.33 & -1.66 \\
1.83 & 3.16 & 0.33 & -1.33 \\
-1.33 & 0.33 & 2.5 & 0.83 \\
-1.66 & -1.33 & 0.83 & 1.5
\end{bmatrix},
\begin{bmatrix}
40.25 & -1.36 & 3.45 & 1.27 \\
-1.36 & 4.75 & -0.65 & -0.47 \\
3.45 & -0.65 & 2.62 & -0.26 \\
1.27 & -0.47 & -0.26 & 0.96
\end{bmatrix},
$$

$$
\begin{bmatrix}
5.54 & 0.25 & -3.89 & -0.52 \\
0.25 & 2.20 & 0.28 & -1.81 \\
-3.89 & 0.28 & 6.68 & 0.40 \\
-0.52 & -1.81 & 0.40 & 3.34
\end{bmatrix},
\begin{bmatrix}
11.87 & -0.58 & 1.95 & 2.04 \\
-0.58 & 1.56 & -0.83 & -0.26 \\
1.95 & -0.83 & 3.91 & -0.36 \\
2.04 & -0.26 & -0.36 & 1.87
\end{bmatrix},
$$

the conditions (4.16) and (4.22) hold with $Q_0 = [0.4, 0.9] \times [-0.7, 0.7]$ and $\Xi_0 = [1.2, 1.9] \times [-0.7, 0.7]$. By taking initial conditions within $Q_0 \times \Xi_0$, we will have exponential incremental stability property and the agents' trajectories converge to w according to Proposition 4.6. For numerical simulation, we take $w(0) = \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}$ whose orbit is shown in Figure 4.1 (shown in solid black). Using initial conditions $x_1(0) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, $x_2(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$, $x_3(0) = \begin{bmatrix} 0.8 \\ 0.1 \end{bmatrix}$, $x_4 = \begin{bmatrix} 0.6 \\ -0.1 \end{bmatrix}$ and $\xi_i(0) = \begin{bmatrix} 1.4 \\ 0 \end{bmatrix}$ for all $i = 1, 2, 3, 4$, the phase plot of each agent's trajectories is shown in Figure 4.1(a) and (b) where pinning synchronization is achieved as expected.

4.5 Conclusion

In this chaper, the pinning synchronization problem of heterogeneous multi-agent systems is studied. When only the relative state measurement is available, we present sufficient and necessary conditions for the solvability of the problem. Subsequently, sufficient conditions are given to guarantee pinning synchronization based on establishing contraction of each agent's virtual systems.
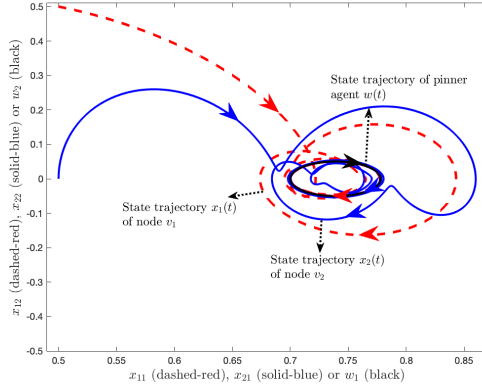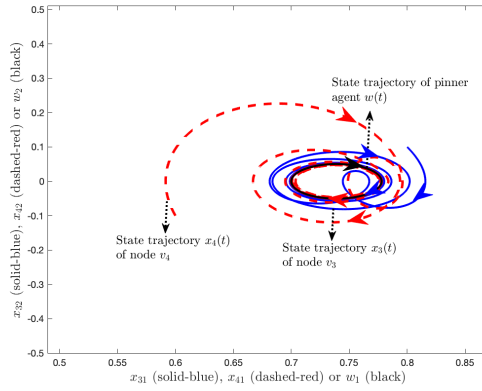
(a) Synchronization of pinned agents x_1 and x_2 to w

(b) Pinning synchronization of x_3 and x_4 to w

Figure 4.1: The phase plot of pinning agents' state x_1, x_2, x_3, x_4 and of the pinner agent w. The simulation results are based on the controllers in (4.23)-(4.26).

5

Stability of switched systems with multiple equilibria

This chapter studies the stability of switched systems that are composed of a mixture of stable and unstable modes with multiple equilibria. The main results of this paper include some sufficient conditions concerning set convergence of switched nonlinear systems. We show that under suitable dwell-time and leave-time switching laws, trajectories converge to an initial set and then stay in a convergent set. Based on these conditions, Linear Matrix Inequality (LMI) conditions are derived that allow for numerical validation of the practical stability of switched affine systems, which include those with all unstable modes. Two examples are provided to verify the theoretical results.

5.1 Introduction

Many complex engineering systems operate as finite-state machines with different modes of operations and functions. These modes can correspond to the multitude of tasks designed for these systems and to the adaptability of these systems in dealing with the dynamic environment. In this regard, these systems can be modeled as switched systems, which have received much attention in the past few decades. Some well-known examples of engineering systems described by switched systems are aircraft systems [25], power electronics [59], and electrical circuits [77].

Typically, a switched system is described by a finite set of continuous-time or discrete-time dynamic subsystems/modes and a switching law/signal that determines which subsystem/mode is active at any given moment of time. Such switching laws can depend on particular state values, time events, or an external state as a memory.

In the time-dependent switching signal, the dwell-time (DT) as studied in [61] provides an important notion that gives us the minimal time where the switched systems must remain in a subsystem before switching to another one. Correspondingly, a significant amount of literature has been directed towards

the stability of switched systems [1, 13, 49, 50, 61, 92, 95, 104]. In [13, 50] the common Lyapunov function and multiple Lyapunov function techniques are used to analyze the stability of switched systems with all stable subsystems. In recent years, some results have also been reported on switched systems with both stable and unstable subsystems [95, 104]. The main idea of these studies is to make the dwell-time of the stable subsystems long enough while shortening the dwell time of the unstable subsystems to offset the divergent trajectory of the unstable subsystem. This approach of having a trade-off between stable and unstable subsystems is no longer applicable when all subsystems are unstable. In [1, 49, 92], a discretized Lyapunov function technique is presented that can be used to analyze the stability of switched systems with all unstable subsystems. The switched systems considered in these papers all share a common equilibrium point and they provide analysis on the convergence of the trajectories to the common equilibrium.

On the contrary, in some engineering applications, there may not be a common equilibrium among subsystems. Some well-known examples are neural networks [106] and bipedal walking robots [31]. In these systems, it has been shown that the trajectories converge to a set rather than to a specific equilibrium point. The property of convergent sets has been studied and estimated in [2, 24, 84]. When all subsystems are stable, dwell-time criteria was investigated in [2] to guarantee that the trajectories converge globally to a superset and remain in such a superset. This work was extended to switched systems satisfying the input-to-state stability property with bounded disturbance in [84], and to switched discrete systems in [84]. Another extension of [2] was presented in [24], which allows each subsystem to have multiple stable equilibria. The minimal invariant convergent superset for switched affine systems is studied in [18]. However, the studies in [2, 18, 24, 84] have not yet considered the case where the switched systems can contain unstable subsystems.

Inspired by the previous study in [2], we study in this paper the set convergence property of switched systems with distinct equilibria in a more general case. The switched systems can contain both stable and unstable subsystems. Such situations can be found, for instance, in aeroengine systems [78] or in RLC circuit systems [52], where a component failure or external disturbance can render a subsystem to be unstable. In power systems [54], DoS attacks are aimed at creating a power blackout through cascading failure by inducing instability in a subsystem. In the game theoretic setting [8, 10, 93], each game's Nash equilibria may be different and unstable.

This chapter provides theoretical tools relevant to ensuring set stability for switched systems with stable and unstable subsystems. Multiple Lyapunov functions techniques are applied to obtain dwell/leave time conditions. With these conditions, the trajectories of switched systems are ultimately bounded by a com-

pact set. Instead of finding multiple Lyapunov functions for the entire state space as studied recently in [2], we only need to find multiple Lyapunov-like functions in the state space outside some compact sets, which is less conservative than the former in [2]. Consequently, we have enlarged the superset of the equilibria as reported in [2]. The generalization allows us to consider switched systems with all unstable subsystems as well.

Related work on the study of switched systems with multiple equilibria is the practical stability analysis of switched affine systems in [17, 18, 36, 42, 43, 45, 75]. In [17, 18, 45, 75], time dependent switching laws are given to guarantee the stability with respect to a set. In these studies, they analyze switched affine systems with all stable subsystems [45] or with stable switching condition among subsystems [17, 75]. In [36, 42, 43], a stabilization problem is studied for the switched systems with unstable mode. Quadratic and non-quadratic Lyapunov functions are used to develop the state dependent switching laws to compute the domain-of-attraction. However, as pointed out in [18], such computation of domain-of-attraction does not exist when time dependent switching laws are applied to such systems. Related to this, we present the practical stability analysis for mixed stable-unstable switched affine systems with time dependent switching laws. Based on the obtained sufficient conditions for set convergence, we present a numerical construction of such multiple Lyapunov functions using time-dependent multiple quadratic Lyapunov functions. It leads to Linear Matrix Inequality (LMI) conditions that can be numerically implemented.

The chapter is organized as follows. In Section 5.2, we present preliminaries and problem formulation. The construction of the convergent set and some sufficient conditions for the set convergence property of switched systems are presented in Sections 5.3 along with an example. Application of such sufficient conditions to the practical stability analysis of switched affine systems that include examples with all unstable subsystems are provided in Section 5.4. Finally, we present the conclusions in Section 5.5.

5.2 Preliminaries and problem formulation

Consider switched systems in the form of

$$\dot{x}(t) = f_{\sigma(t)}(x(t), t), \quad x(t_0) = x_0, \tag{5.1}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, $t_0 \in \mathbb{R}$ is the initial time and $x_0 \in \mathcal{X}$ is the initial value. Define an index set $\mathcal{Q} := \{1, 2, \cdots, M\}$, where M is the number of modes. The signal $\sigma : [t_0, \infty) \to \mathcal{Q}$ denotes the switching signal, which is assumed to be a piecewise constant function and continuous from the right. The vector field

$f_i : \mathcal{X} \times [t_0, \infty) \to \mathbb{R}^n$, $i \in \mathcal{Q}$, is continuous in t and continuously differentiable in x. The switching instants are expressed by a monotonically increasing sequence $\mathscr{S} := \{t_1, t_2, \cdots, t_k, \cdots\}$, where t_k denotes the k-th switching instance. We assume that (5.1) is forward complete, which means for each $x_0 \in \mathcal{X}$ there exists a unique solution of (5.1) on $[t_0, \infty)$ and no jump occurs in the state at a switching time.

In this paper, we don't assume that there is common equilibria for the switched systems (5.1). In addition, we allow each subsystem has multiple equilibria. Since the equilibria are different, trajectories will converge to a set rather than a specific point.

The set convergence problem for switched systems with all stable modes has attracted considerable attentions. For example, in [2, 24, 84], a convergent set is constructed by the level sets of multiple Lyapunov functions. Then, convergence can be achieved by activating the stable subsystems for a sufficient long time. In contrast to previous works, reference [33] permits arbitrary switching between subsystems without imposing any dwell-time constraints. However, for unstable subsystems, we can not find such multiple Lyapunov functions that limit the application of results in [2, 24, 33, 84]. Correspondingly, the main objective of this paper is to propose a sufficient condition that guarantees the switched system (5.1) is set convergent with respect to any switching law $\sigma(t)$ satisfying the dwell/leave time constraints, which includes the case when not all modes of (5.1) are stable and when none of the modes are stable.

5.3   Main result

In this section, the sets construction is introduced and some sufficient conditions are given to guarantee the set convergence of the switched system (5.1).

We denote the subset of modes in $\mathcal{Q}$ that compose of unstable sub-systems by $\mathcal{U}$ and its complement (i.e., the stable ones) by $\mathcal{S}$. Hence, $\mathcal{Q} = \mathcal{U} \dot{\cup} \mathcal{S}$. Consider the switched system (5.1) under a certain switching signal $\sigma(t)$. Suppose that there exists a compact set K such that for each mode $q \in \mathcal{Q}$ there exists a continuously differentiable function $V_q : \mathcal{X} \setminus K \times [0, \tau_{q,\max}) \to \mathbb{R}_{\geqslant 0}$, where $\tau_{q,\max} \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ represents the maximal local time, such that the following inequality holds for all $\xi \in \mathcal{X} \setminus K$ and $\tau \in [0, \tau_{q,\max})$

$$\dot{V}_q(\xi, \tau) := \frac{\partial V_q(\xi, \tau)}{\partial \xi} f_q(\xi, \tau) + \frac{\partial V_q(\xi, \tau)}{\partial \tau} \leqslant \eta_q V_q(\xi, \tau), \qquad (5.2)$$

with $\eta_q \geqslant 0$ if $q \in \mathcal{U}$ or $\eta_q < 0$ otherwise. This mode-dependent locally time-varying Lyapunov function provides us with a means to describe the stability of the compact set K in a local time-interval whenever mode q is activated. For facilitating the numerical computation later via LMI conditions, we will use an

explicit relation of the compact set K with the Lyapunov function V_q through a parametrized compact set $N(k)$, where the parameter $k > 0$ gives us the degree-of-freedom to check the Lyapunov condition.

We introduce this locally time-varying Lyapunov function in order to relax the requirement of finding a common time-invariant Lyapunov function for switched systems, which may be hard to find. The maximal time of definition $\tau_{q,\max}$ can be ∞ and we do not exclude the usual time-invariant V_q in this definition by taking $V_q(\xi, \tau)$ to be time-invariant for all $\tau \in [0, \tau_{q,\max})$ with arbitrary $\tau_{q,\max} > 0$. As will be clear later, such maximal time $\tau_{q,\max}$ must necessarily be greater than the usual required dwell-time condition. In our previous work [100], we have shown the applicability of such locally time-varying Lyapunov functions in order to set up verifiable LMI conditions for establishing stability of switched systems comprising (un)stable modes. The function constructed in [100] is based on time interpolation of two time-invariant quadratic Lyapunov function.

In the following, we will define $N(k)$, $N^\alpha(k)$, $L(k)$, which are a subset of $\mathcal{X}$ and parametrized by positive constant $k > 0$. These sets will be used in our main result to define the attractive invariant set of the switched systems. For a given positive constant $k > 0$ and for any given mode $q \in \mathcal{Q}$, we define $N_q(k)$ as a level set of $V_q(\xi, \tau)$ given by

$$N_q(k) := \{\xi \in \mathcal{X} : V_q(\xi, \tau) \leqslant k, \forall \tau \in [0, \tau_{q,\max})\}. \tag{5.3}$$

The superset $N(k)$ is defined by the union of $N_q(k)$ over all modes $q \in \mathcal{Q}$ as follows

$$N(k) := \bigcup_{q \in \mathcal{Q}} N_q(k). \tag{5.4}$$

Since $N(k)$ is generally larger than any of the individual $N_q(k)$, let us define the maximum range of V_q in $N(k)$ by

$$\alpha_q(k) := \max_{\substack{\xi \in N(k) \\ \tau \in [0, \tau_{\max})}} V_q(\xi, \tau), \tag{5.5}$$

and

$$\alpha(k) := \max_{q \in \mathcal{Q}} \alpha_q(k). \tag{5.6}$$

For every $q \in \mathcal{Q}$, we define a level set $N_q^\alpha(k)$ by

$$N_q^\alpha(k) := N_q(\alpha(k)), \tag{5.7}$$

where $N_q(\cdot)$ is given by (5.3). And we define $N^\alpha(k)$ by

$$N^\alpha(k) := \bigcap_{q \in \mathcal{Q}} N_q^\alpha(k). \tag{5.8}$$

Note that $N^\alpha(k) \neq N(\alpha(k))$ because the former is the intersection, while the latter is the union of all $N_q(\alpha(k))$.

Now, using the above notions of V_q and the sets $\mathcal{U}, \mathcal{S}$, and $N(k)$, we will consider the following locally time-varying Lyapunov characterisation for establishing the set stability of (5.1). For each mode $q \in \mathcal{Q}$, we assume that (5.2) holds for $K = N(k)$, i.e.

$$\dot{V}_q(\xi, \tau) \leqslant \eta_q V_q(\xi, \tau), \forall \xi \in \mathcal{X} \setminus N(k), \forall \tau \in [0, \tau_{q,\max}) \tag{5.9}$$

with $\eta_q > 0$ if $q \in \mathcal{U}$ or $\eta_q < 0$ otherwise, and with $\tau_{q,\max} > 0$. Additionally, we assume that the mode-dependent functions V_q are bounded by each other as follows: there exists $0 < \mu_q < 1$ if $q \in \mathcal{U}$ or $\mu_q > 1$ otherwise, such that

$$V_p(\xi, 0) \leqslant \mu_q V_q(\xi, \tau), \forall \xi \in \mathcal{X} \setminus N(k) \forall p, q \in \mathcal{Q}, \forall \tau \in [\tau_{q,\min}, \tau_{q,\max}), \tag{5.10}$$

with $\tau_{q,\min} > 0$.

Notice that, for a stable subsystem, there exists η_q that satisfies (5.9) globally. For an unstable subsystem, inequality (5.9) implies that the value of $V_q(\xi, \tau)$ may increase in some time interval with a bounded rate $\eta_q > 0$. In such case, the divergence can be compensated by the switched event according to (5.10) with $0 < \mu_q < 1$.

Finally, let us introduce the set $L(k)$, in which the trajectories will eventually remain in. Firstly, we denote for every $q \in \mathcal{Q}$

$$\beta_q(k) := \alpha(k) \cdot \max\left\{\frac{1}{\mu_q}, 1\right\}, \tag{5.11}$$

$$M_q(k) := \{x \in \mathcal{X} : V_q(\xi, \tau) \leqslant \beta_q(k),$$
$$\forall \tau \in [0, \tau_{q,\max})\}, \tag{5.12}$$

Accordingly, we define $L(k)$ by

$$L(k) := \bigcup_{q \in \mathcal{Q}} M_q(k). \tag{5.13}$$

For an unstable sub-system, it follows from (5.11) that $\beta_q(k) = \frac{1}{\mu_q}\alpha(k) \geqslant \alpha(k)$ which implies that $N_q^\alpha(k) \subseteq M_q(k)$.

In the following, the relations among the above defined sets are discussed and

illustrated. For every $\xi \in N(k)$, according to (5.3)-(5.6), we have $V_q(\xi, \tau) \leqslant \alpha(k)$, for every $q \in \mathcal{Q}$. Then, by the definition of $N_q^\alpha(k)$ in (5.7), for every $q \in \mathcal{Q}$, we have $\xi \in N_q^\alpha(k)$, i.e., $\xi \in \bigcap_{q \in \mathcal{Q}} N_q^\alpha(k)$. Combining this with the definition of $N^\alpha(k)$ in (5.8), we can conclude that $N(k) \subseteq N^\alpha(k)$. In addition, for every $\xi \in N^\alpha(k)$, i.e., $V_q(\xi, \tau) \leqslant \alpha(k)$, according to (5.11), (5.12), we have $V_q(\xi, \tau) \leqslant \beta_q(k)$, i.e., $\xi \in \bigcup_{q \in \mathcal{Q}} M_q(k)$. In combination with the definition of $L(k)$ in (5.13), we conclude that $N^\alpha(k) \subseteq L(k)$. Hence, we have that $N(k) \subseteq N^\alpha(k) \subseteq L(k)$. An illustration of this construction for two modes can be seen in Fig.5.1. In this illustration, $N_1(k)$ and $N_2(k)$ are disconnected; however, they can also be connected as shown later in Example 3.
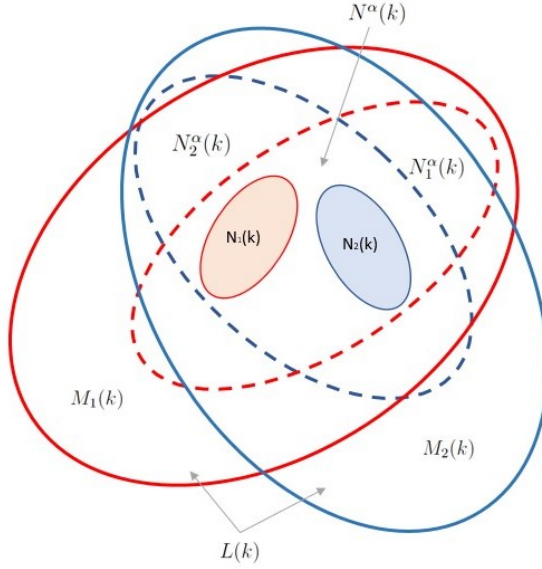

Figure 5.1: An illustration of the set constructions for two modes.

Remark 5.1. There are two main differences between our results and those in [2, 84]. Firstly, the results in this paper can include unstable subsystems; moreover, we do not exclude the case of all unstable subsystems. To cater for the presence of unstable subsystems, we use piecewise time-varying Lyapunov functions instead of time-invariant Lyapunov functions as used in [2, 84] with the restriction of (5.10). Secondly, the time-varying Lyapunov characterization of the sub-systems are applied outside a compact set $N(k)$ instead of the whole state space $\mathcal{X}$ as assumed in [2, 84]. It will be shown later in Example 1 that checking these Lyapunov conditions outside a compact set in (5.9) is easier that checking the counterparts in the whole state space $\mathcal{X}$.

Before we present our main result, we introduce the following definition of the mode dependent dwell (leave) time.

Definition 5.2. A constant $\tau_p > 0$ is called mode dependent dwell (leave) time for stable (unstable) mode $p \in \mathcal{Q}$ of a switching signal $\sigma : [t_0, \infty) \to \mathcal{Q}$ if the time interval between two consecutive switches or jumps being no smaller (or larger) than τ_p.

We present now the main result of this section for the set convergence of switched systems (5.1).

Theorem 5.3. *Suppose that for every $q \in \mathcal{Q}$ there exists $V_q : \mathcal{X} \times [0, \tau_{q,\max}) \to \mathbb{R}_+$ satisfying (5.9) and (5.10) with a given η_q, μ_q and $k > 0$. Then, for every switching signal $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ satisfying the following dwell and leave time condition*

$$
\left.
\begin{aligned}
\tau_q &> \max\left\{ -\tfrac{\ln \mu_q}{\eta_q}, \tau_{q,\min} \right\} \quad \forall q \in \mathcal{S},\ and \\
\tau_q &< \min\left\{ -\tfrac{\ln \mu_q}{\eta_q}, \tau_{q,\max} \right\}, \quad \forall q \in \mathcal{U}.
\end{aligned}
\right\}
\tag{5.14}
$$

the following properties hold for the state trajectory of the switched system (5.1)*:*

(i) *there exists $T = T(x_0) > 0$ such that $x(T) \in N(k)$;*

(ii) *for any time $t \in [T, +\infty)$, the trajectory will stay in $L(k)$, i.e. $x(t) \in L(k)$;*

(iii) *for all starting points $x_0 \in N^\alpha(k)$, the trajectory of switched system (5.1) remains in the set $L(k)$, i.e. $x(t) \in L(k)$.*

Proof. Let us consider a given switching signal σ satisfying the hypotheses of the theorem with the switching time $\{t_0, t_1, t_2, \ldots\}$. For such switching signal σ, we can construct a piecewise time-varying Lyapunov function, by piecing all V_q together, as follows

$$
V(x(t), t) = V_q(x(t), t - t_i),
$$

where t_i is the latest switching moment before time t and q is the current mode.

   Proof of part (i):   Trivially, if $x_0 \in N(k)$ then $T = 0$. Let us now consider $x_0 \in \mathcal{X} \setminus N(k)$. In the following, we will show that under the condition (5.14), the function $V(x(t), t)$ will converge to an arbitrarily small constant. This implies that there exists a time $T > 0$ s.t. at which the trajectory enters $N(k)$, i.e. $V(x(T), T) \leqslant k$.

Firstly, for any $t \in [t_i, t_{i+1})$ and $q := \sigma(t_i)$, according to (5.2), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), t) &= \frac{\mathrm{d}}{\mathrm{d}t} V_q(x(t), t - t_i) \qquad t \in [t_i, t_{i+1}) \\
&= \frac{\partial V_q(x(t), \tau)}{\partial x} f_q(x(t)) + \frac{\partial V_q(x(t), \tau)}{\partial \tau} \qquad \tau \in [0, t_{i+1} - t_i) \\
&\leqslant \eta_q V_q(x(t), \tau)
\end{aligned}
$$

where we have introduced a time transformation of $\tau = t - t_i$ in the second equality. The comparison lemma implies

$$
V(x(t), t) \leqslant e^{\eta_q(t - t_i)} V(x(t_i), t_i) \tag{5.15}
$$

for all $t \in [t_i, t_{i+1})$. Using (5.9) and (5.10), and by denoting now $p := \sigma(t_i)$ and $q := \sigma(t_{i-1})$,

$$
\begin{aligned}
V(x(t_i), t_i) &= V_p(x(t_i), 0) \\
&\leqslant \mu_q V_q(x(t), t - t_{i-1}) \qquad t \in [t_{i-1} + \tau_{p,\min}, t_i) \\
&\leqslant \mu_q e^{\eta_q(t_i - t_{i-1})} V_q(x(t_{i-1}), 0),
\end{aligned}
$$

where the last inequality follows a similar line as in (5.15). By recursively computing the inequality bound down to t_0, for $t = t_{i+1}$ we arrive at the following inequality

$$
\begin{aligned}
V(x(t_{i+1}), t_{i+1}) &= V_{\sigma(t_{i+1})}(x(t_{i+1}), 0) \\
&\leqslant V_{\sigma(t_0)}(x(t_0), 0) \prod_{j=0}^{i} \mu_{\sigma(t_j)} \exp\left(\eta_{\sigma(t_j)}(t_{j+1} - t_j)\right) \\
&= V_{\sigma(t_0)}(x(t_0), 0) \prod_{j=0}^{i} \exp\left(\eta_{\sigma(t_j)}(t_{j+1} - t_j) + \ln \mu_{\sigma(t_j)}\right).
\end{aligned} \tag{5.16}
$$

It follows from the dwell and leave time condition (5.14) that for all $q \in \mathcal{S} \cup \mathcal{U}$, the inequality $\ln \mu_q + \eta_q \tau_q =: d_q < 0$ holds. This implies immediately for $d := \max_q d_q < 0$ that $\exp\left(\eta_{\sigma(t_j)}(t_{j+1} - t_j) + \ln \mu_{\sigma(t_j)}\right) \leqslant e^d < 1$. If σ has infinitely many switches, it therefore follows from (5.16) that $V(x(t_i), t_i)$ converges to zero for $i \to \infty$. Together with (5.15), we conclude that for $t \in [t_i, t_{i+1})$ either $V(x(t), t) \leqslant V(x(t_i), t_i))$ if $\sigma(t_i) \in \mathcal{S}$ or $V(x(t), t) \leqslant e^{\eta_{\max} \tau_{\max}} V(x(t_i), t_i)$ if $\sigma(t_i) \in \mathcal{U}$ and where $\eta_{\max} := \max_{q \in \mathcal{U}} \eta_q$, $\tau_{\max} := \max_{q \in \mathcal{U}} \tau_{q,\max}$. Consequently, $t \mapsto V(x(t), t)$ converges also to zero as $t \to \infty$. In the case that σ only has finitely many switches, the last mode most be a stable mode (because each unstable mode has a maximal leave time by assumption), hence (5.15) considered for the last

(stable) mode also implies that $t \mapsto V(x(t), t)$ converges to zero.

Particularly, for any given $k > 0$, there exists $T > 0$ and q such that $V(x(T), T) = V_q(x(T), T) \leqslant k$.

Proof of part (ii): The proof is decomposed in two steps. In the first step, we show that once the trajectory enters $N(k)$, i.e. $V(x(T), T) \leqslant k$ with the switch time $t_i \leqslant T$, it stays in $L(k)$ before the next switch at t_{i+1}, i.e. $V(x(t), t) \leqslant \beta(k)$ for all $T \leqslant t < t_{i+1}$. Thereafter it stays in $N^\alpha(k)$ after the next switch time t_{i+1}, i.e. $V(x(t_{i+1}), t_{i+1}) \leqslant \alpha(k)$. In the second step, we show that when the trajectory starts in $N^\alpha(k)$, it stays in $L(k)$ for all forward time.

First step: Let us consider the time interval $[t_i, t_{i+1})$, and $T \in [t_i, t_{i+1})$, i.e. the trajectory enters $N(k)$ in $[t_i, t_{i+1})$. Let us first show that during the subsequent switch time t_{i+1}, we have $V(x(t_{i+1}), t_{i+1}) \leqslant \alpha(k)$. It follows from (5.5) and (5.6) that when the trajectory enters $N(k)$ at time T, we have $V(x(T), T) \leqslant \alpha_{\sigma(t_i)}(k) \leqslant \alpha(k)$.

We first show for a stable mode, by means of contradiction, that once the trajectory enters $N(k)$ at time T, it will stay in $N(k)$ in the time interval $[T, t_{i+1})$. Let us assume there exists $T'' > T$ such that $x(T'') \notin N(k)$. According to the continuity of the trajectory, there exists $T' > T$ such that $T < T' < T''$ and $x(T') \in \partial N(k)$. According to (5.9), outside $N(k)$, we have $V(x(T''), T'') = V_q(x(T''), T'' - t_i) \leqslant e^{\eta_q(T'' - T')} V_q(x(T'), T' - t_i)$. Using (5.14) and $T'' - T' \leqslant \tau_q$, we have $e^{\eta_q(T'' - T')} V_q(x(T'), T' - t_i) \leqslant e^{\eta_q \tau_q} V_q(x(T'), T' - t_i) \leqslant e^{\ln \frac{1}{\mu_q}} V_q(x(T'), T' - t_i) = \frac{1}{\mu_q} V_q(x(T'), T' - t_i)$. Since we are in a stable mode with $\mu_q > 1$, it follows that $V_q(x(T''), T'' - t_i) < V_q(x(T'), T' - t_i)$. In other words, $x(T'') \in N(k)$, which is a contradiction. Since $x(t) \in N(k)$, for all $t \in [T, t_{i+1})$, it follows from (5.5) and (5.6) that in the subsequent switch time t_{i+1}, we have $V(x(t_{i+1}), t_{i+1}) = V_{\sigma(t_{i+1})}(x(t_{i+1}), 0) \leqslant \alpha_{\sigma(t_{i+1})}(k) \leqslant \alpha(k)$. This means that the trajectory stays in $N^\alpha(k)$ at the subsequent switch time t_{i+1}.

Now let us consider the other case when an unstable mode is active during the time interval $[T, t_{i+1})$. For this situation, there are two further possible cases: $x(t_{i+1}) \in N(k)$ and $x(t_{i+1}) \notin N(k)$.

For the first case, with $x(t_{i+1}) \in N(k)$, we will show that $x(t) \in L(k)$ for all $t \in [T, t_{i+1})$ and $x(t_{i+1}) \in N^\alpha(k)$. Since it is an unstable mode, in the time interval $[T, t_{i+1})$, there can be a moment $T' > T$ such that $x(T') \notin N(k)$. It follows from (5.9), (5.14), and $T' - T < \tau_q$ that $V(x(T'), T') = V_q(x(T'), T' - t_i) \leqslant e^{\eta_q(T' - T)} V_q(x(T), T - t_i) < e^{\eta_q \tau_q} V_q(x(T), T - t_i) < \frac{1}{\mu_q} V_q(x(T), T - t_i) < \frac{1}{\mu_q} \alpha_q(k) \leqslant \beta_q(k)$. This inequality implies that $x(T') \in L(k), \forall T' \in [T, t_{i+1})$. Since $x(t_{i+1}) \in N(k)$, according to (5.5) and (5.6), after the switching at t_{i+1} we have $V(x(t_{i+1}), t_{i+1}) = V_{\sigma(t_{i+1})}(x(t_{i+1}), 0) \leqslant \alpha_q(k) \leqslant \alpha(k)$.

For the second case, with $x(t_{i+1}) \notin N(k)$, following the same arguments as in the first case, we have that $x(t) \in L(k)$, for all $t \in [T, t_{i+1})$. Accordingly, at t_{i+1},

with $x(t_{i+1}) \notin N(k)$, $p := \sigma(t_{i+1})$ and $q := \sigma(t_i)$, we can apply (5.9)-(5.10), which gives us

$$\begin{aligned}
V(x(t_{i+1}), t_{i+1}) &= V_p(x(t_{i+1}), 0) \\
&\leqslant V_q(x(t_{i+1}), t_{i+1} - t_i) \\
&\leqslant e^{\ln \mu_q + \eta_q(t_{i+1} - T)} V_q(x(T), T - t_i).
\end{aligned}$$

Using (5.14), it follows that $\ln \mu_q < -\eta_q(t_{i+1} - t_i)$. Hence the above inequality can be further upper-bounded by

$$\begin{aligned}
V(x(t_{i+1}), t_{i+1}) &< e^{-\eta_q(t_{i+1} - t_i) + \eta_q(t_{i+1} - T)} V_q(x(T), T - t_i) \\
&= e^{\eta_q(t_i - T)} V_q(x(T), T - t_i).
\end{aligned}$$

Since $t_i \leqslant T$ and $\eta_q > 0$, we have $V(x(t_{i+1}), t_{i+1}) < V_q(x(T), T - t_i) \leqslant \alpha_q(k) \leqslant \alpha(k)$. This implies that the trajectory remains in $N^\alpha(k)$.

In summary, for all $t \in [T, t_{i+1})$, the trajectory always stays in $L(k)$, and at t_{i+1}, the trajectory remains in $N^\alpha(k)$.

Second step: Let us now consider the subsequent time interval $[t_{i+1}, t_{i+2})$. Following the previous step, we have established that $V(x(t_{i+1}), t_{i+1}) \leqslant \alpha(k)$. We will now show that also $x(t) \in L(k)$, for all $t \in [t_{i+1}, t_{i+2})$. On the one hand, if the active mode in $[t_{i+1}, t_{i+2})$ is a stable one, the maximal value of $V(x(t), t)$ occurs at t_{i+1} since $V(x(t), t)$ is non-increasing in $[t_{i+1}, t_{i+2})$. In this case, we have $V(x(t), t) \leqslant V(x(t_{i+1}), t_{i+1}) \leqslant \alpha(k)$.

On the other hand, if the active mode in $[t_{i+1}, t_{i+2})$ is an unstable one then the upper bound of $V(x(t), t)$ occurs at t_{i+2}. By denoting $q := \sigma(t_{i+1})$ then for all $t \in [t_{i+1}, t_{i+2})$

$$\begin{aligned}
V(x(t), t) &= V_q(x(t), t - t_{i+1}) \\
&\leqslant V_q(x(t_{i+2}), t_{i+2} - t_{i+1}) e^{\eta_q(t - t_{i+1})} \\
&\leqslant \frac{1}{\mu_q} V_q(x(t_{i+2}), t_{i+2} - t_{i+1}) \\
&\leqslant \frac{1}{\mu_q} \alpha(k) \leqslant \beta_q(k),
\end{aligned}$$

where we have used (5.14) in the second inequality above to establish that $e^{\eta_q(t - t_{i+1})} < e^{\eta_q \frac{-\ln \mu_q}{\eta_q}} = \frac{1}{\mu_q}$ for all $t \leqslant t_{i+2} < t_{i+1} + \tau_q$.

It follows from (5.12) and (5.13) that $x(t) \in L(k), \forall t \in [t_{i+1}, t_{i+2})$.

Finally, let us consider the trajectory at the switch-time t_{i+2}. When $x(t_{i+2}) \in N(k)$, it immediately holds that $V(x(t_{i+2}), t_{i+2}) \leqslant \alpha(k)$. Otherwise, using (5.9),

(5.10) and (5.14) and by denoting $p := \sigma(t_{i+2})$, we have

$$V(x(t_{i+2}), t_{i+2}) = V_p(x(t_{i+2}), 0)$$
$$\leqslant \frac{1}{\mu_q} V_q(x(t_{i+2}), t_{i+2} - t_{i+1})$$
$$\leqslant e^{-\ln \mu_q + \eta_q(t_{i+2} - t_{i+1})} V_q(x(t_{i+1}), 0)$$
$$\leqslant e^{-\ln \mu_q + \eta_q \tau_q} V_q(x(t_{i+1}), 0)$$
$$< V_q(x(t_{i+1}), 0) \leqslant \alpha(k).$$

Thus the trajectory $x(t)$ remains in $N^\alpha(k)$ at t_{i+2}.

By computing recursively for the subsequent time intervals, we can conclude that the trajectory $x(t)$ remains in $L(k)$ for all $t \geqslant T$.

Proof of part (iii): The proof of part (iii) follows directly from the second step of the proof of part (ii). □

Remark 5.4. The results presented in [2], which deals with all stable modes, can be considered as a particular case of our results. In particular, if we assume that the subsystems in Theorem 5.3 are all stable, i.e. $\mathcal{Q} = \mathcal{S}$, the trajectory of switched nonlinear system (5.1) is in $L(k)$ after time T for any switching signals satisfying $\tau_q > -\frac{\ln \mu_q}{\eta_q}$. In this regards, part (i) and (ii) of Theorem 5.3 coincide with [2, Theorem 1] with a common $\mu = \max \mu_q$ and a common $\eta = \max \eta_q$. For part (iii) of the theorem, we established that for all trajectories starting in $N^\alpha(k)$, which is larger than $N(k)$ used in [2, Corollary 2], will stay in the same level set $L(k)$. This shows that our result is less conservative.

For switched system (5.1), if all subsystems are unstable, which represents the worst case scenario, the trajectories will not converge to any of the modes and the divergence can only be compensated by the switching events as shown in the following corollary.

Corollary 5.5. *Assume that $\mathcal{Q} = \mathcal{U}$ (i.e., all modes are unstable). Suppose that for every $q \in \mathcal{Q}$ there exists $V_q(\xi, \tau) : \mathcal{X} \times [0, \tau_{q,\max}) \to \mathbb{R}_+$ satisfying (5.9) and (5.10) with a given η_q and μ_q. Then for any trajectory of switched nonlinear system (5.1) with switching signals σ satisfying*

$$\tau_q < \min \left\{ -\frac{\ln \mu_q}{\eta_q}, \tau_{q,\max} \right\}, \forall q \in \mathcal{U}, \tag{5.17}$$

there exists $T > 0$ such that $x(t)$ remains in $L(k)$ for all $t \geqslant T$.

From (5.17), the term $-\frac{\ln \mu_q}{\eta_q}$ gets closer to 0 as η_q gets larger, in which case the unstable mode must switch sufficiently fast. Alternatively, it can be compensated

by having a small μ_q, which increases the difficulty of designing the Lyapunov function. From this point of view, it is desirable to have a small η_q.

Example 1: Let us consider a switched system (5.1) composed of two scalar subsystems as follows

$$q = 1: \quad \dot{x} = x + 4,$$
$$q = 2: \quad \dot{x} = -x(1 + x)^2. \tag{5.18}$$

The mode $q=1$ is an unstable system and the mode $q=2$ is a stable system with multiple equilibria. Both systems do not have common equilibria. For these sub-systems, we can define V_1 and V_2 that satisfy (5.9) and (5.10). Indeed, by using $V_1(x(t), t) = 2x^2$ and $V_2(x(t), t) = \frac{1}{2}x^2$, we have $\mu_1 = \frac{V_2(x(t),t)}{V_1(x(t),t)} = \frac{1}{4}$ and $\mu_2 = \frac{V_1(x(t),t)}{V_2(x(t),t)} = 4$; thus (5.10) is satisfied globally.

Let us fix $k = 2$ in (5.3) so that $N_1(2) = (-1, 1)$, $N_2(2) = (-2, 2)$, $N(2) = (-2, 2)$, and $\mathcal{X} \setminus N(2) = (-\infty, -2] \bigcup [2, +\infty)$. In $\mathcal{X} \setminus N(2)$, one can obtain that (5.9) holds with $\eta_1 = 6$ and $\eta_2 = -2$. Following the computation in (5.5), we have $\alpha(2) = 8$ in $N(2)$. Subsequently, using (5.11), we can obtain that $\beta_1 = 32$ and $\beta_2 = 8$. Consequently, $M_1(2) = [-4, 4]$, $M_2(2) = [-4, 4]$, so that $L(2) = [-4, 4]$. Thus $N(\alpha) = N(k) = [-2, 2]$, $L(k) = [-4, 4]$. According to the main dwell-time condition (5.14) in Theorem 3.6, the dwell-time for each subsystem is given by $\tau_1 \leqslant 0.231$, $\tau_2 \geqslant 0.693$.

For numerical simulation, we consider $\tau_1 = 0.231$ and $\tau_2 = 0.693$, and the switched system is initialized at two different position: $x(0) = -6$, which is outside $L(2)$, and $x(0) = 2$, which is on the boundary of $N^\alpha(k)$. Figure 5.2 shows the resulting trajectories where the blue line gives the trajectory initialized at -6 while the red one is the trajectory initialized at 2. According to part (i) and part (ii) in Theorem 3.6, there exists $T > 0$ such that the trajectory will enter $N(k)$ and remains in $L(k)$ for all $t \geqslant T$. As shown in Figure 5.2, the trajectory that starts at -6 enters $N(k)$ at $T = 0.52s$, and remains in $L(k)$ afterwards. On the other hand, when the state is initialized at 2, which is in $N^\alpha(k)$, the trajectory will remain in $L(k)$ for all $t \geqslant 0$.

The analysis tools provided by Theorem 5.3 only require us to get Lyapunov characterization for each sub-system outside a given compact set. For instance, the Lyapunov inequality (5.9) does not need to be fulfilled in the neighborhood of the equilibria. In the example above, one can check for the second subsystem that by using the given Lyapunov function $V_2(x(t), t) = \frac{1}{2}x^2$, we have $\dot{V}_2(x(t), t) = -x^2(1 + x)^2 \leqslant 0$. However, it is not possible to fulfill the inequality (5.9) for all $\mathbb{R}$. By letting $k = 2$, $-2(1 + x)^2 \leqslant -2$ holds for all x in $(-\infty, -2) \bigcup (2, +\infty)$ (which is a state domain outside the compact interval $[-2, 2]$). Thus, in this domain, we have $\dot{V}_2(x(t), t) \leqslant -2V_2(x(t), t)$ fulfilling (5.9) with the dissipation rate $\eta_2 = -2$.
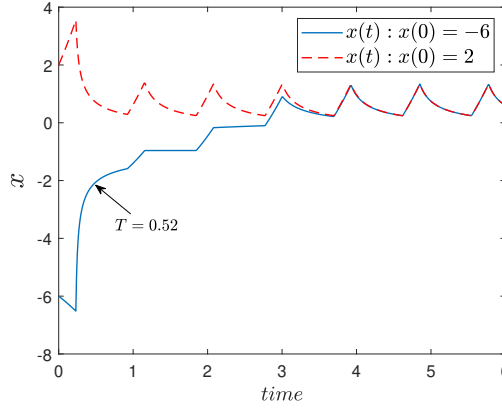
Figure 5.2: The plot of trajectories of switched system in Example 1 initialized at $x(0) = -6$, $x(0) = 2$, and using a periodic switching signal with $\tau_1 = 0.231$ and $\tau_2 = 0.693$, $x(t) : x(0) = -6$ enters $N(k)$ at 0.52s.

5.4   Practical stability for the switched affine systems

In this section, we focus on the application of Theorem 5.3 in the practical stability analysis of switched affine systems with mixed stable-unstable subsystems. Let us consider a switched affine system in the form of

$$\dot{x}(t) = A_q x(t) + B_q, \quad \forall q \in \mathcal{Q}, \tag{5.19}$$

where $x(t)$ and $\sigma(t)$ are as in (5.1). Here we do not restrict A_q to be stable matrices, nor they have stable matrix combination as pursued in [17, 75].

Following [45], the switched affine system (5.19) is said to be *practically stable with respect to the sets* $\Omega_1 \subseteq \mathcal{X}$ and $\Omega_2 \subseteq \mathcal{X}$ ($\Omega_1 \subseteq \Omega_2$) for any switching signal $\sigma(t)$ from the given class, if the implication $x(t_0) \in \Omega_1 \Rightarrow x(t) \in \Omega_2$ holds for all $t \geqslant 0$.

In Theorem 5.3, it is assumed that there exist multiple Lyapunov functions $V_q(\xi, \tau)$ in $\mathcal{X} \setminus N(k) \times [0, \tau_{q,\max})$ satisfying (5.9) and (5.10). In general, checking the existence of such Lyapunov functions is not trivial. In the following lemma, we present a sufficient condition that can simplify the construction of such Lyapunov functions.

Lemma 5.6. *Suppose that for each mode $q \in \mathcal{Q}$ there exists a continuously differentiable function $V_q : \mathcal{X} \times [0, \tau_{q,\max}) \to \mathbb{R}_{\geqslant 0}$ such that the following inequalities*

$$\dot{V}_q(\xi, \tau) \leqslant \eta_q V_q(\xi, \tau) + \gamma_q \Big(k - V_q(\xi, \tau) \Big), \quad \forall \xi \in \mathcal{X}, \forall \tau \in [0, \tau_{q,\max}) \tag{5.20}$$

$$V_p(\xi, 0) \leqslant \mu_q V_q(\xi, \tau) + \gamma_q'\Big(k - V_q(\xi, \tau)\Big), \quad \forall \xi \in \mathcal{X}, \forall p, q \in \mathcal{Q}, \forall \tau \in [\tau_{q,\min}, \tau_{q,\max}), \tag{5.21}$$

hold with $0 < \tau_{q,\min} < \tau_{q,\max}$, *where* $\gamma_q \geqslant 0$, $\gamma_q' \geqslant 0$, *and the constant* k *is as used before in* (5.3). *Then the statements of Theorem 3.6(i), (ii) and (iii) hold for any switching signals* σ *satisfying* (5.14).

Proof. It follows from (5.20) and (5.21) that $V_q(\xi, \tau) > k \Rightarrow \dot{V}_q(\xi, \tau) \leqslant \eta_q V_q(\xi, \tau)$ and $V_p(\xi, 0) \leqslant \mu_q V_q(\xi, \tau)$. This implies that $V_q(\xi, \tau)$ and $V_p(\xi, 0)$ as given in (5.20) and (5.21) satisfy (5.9) and (5.10) outside the compact set $N(k)$, i.e. in the set $\bigcap_{q \in \mathcal{Q}} \{\xi \in \mathcal{X} \mid V_q(\xi) \geqslant k\}$. In this case, all hypotheses of Theorem 5.3 are satisfied and hence the claim of the lemma follows immediately. Moreover, it holds globally if $k = 0$. □

For switched linear systems, it is common to use a quadratic Lyapunov function $V_q(\xi, \tau) = \xi^\top R_q \xi$, where R_q is a positive definite matrix. The use of such quadratic form may not be suitable, particularly when the systems switch consecutively between unstable modes. For instance, when the system switches from an unstable mode q to another unstable mode p and then back to mode q again, for constant matrices R_q and R_p, (5.10) becomes $R_p \leqslant \mu_q R_q \leqslant \mu_q \mu_p R_p$ with $0 < \mu_q < 1$, $0 < \mu_p < 1$, which cannot be satisfied. This shows that the matrix R_q can not be a constant matrix when switching between unstable modes are admitted, such as the switched systems considered in our main result above. In order to compensate the conservativity brought by the affine term B_q, we construct a shifted time-varying quadratic Lyapunov function given by

$$V_q(\xi, \tau) = (\xi - x^\star)^\top R_q(\tau)(\xi - x^\star), \quad \forall q \in \mathcal{Q}, \tag{5.22}$$

where $x^\star \in \mathbb{R}^n$ is the centroid of the level set V_q. By defining $\tilde{x}(t) = x(t) - x^\star$, we can rewrite (5.19) as

$$\dot{\tilde{x}}(t) = A_q \tilde{x}(t) + L_q, \quad \forall q \in \mathcal{Q}, \tag{5.23}$$

where $L_q = A_q x^\star + B_q$. Note that for estimating the domain-of-attraction, it is desirable to have $||L_q||$ as small as possible. Otherwise, the LMI conditions, which we will present later in Lemma 5.7, may not be feasible, i.e., the determinant $\Xi_{q,i} + \gamma_q k P_q L_q L_q^T P_q$ in (5.24)-(5.25) may not be negative for large L_q. By tuning $x^\star$, we can obtain a minimal value of L_q by minimizing the cost function $\sum_{p,q \in \mathcal{Q}} ||L_q||$. In other words, $x^\star$ can be determined as $x^\star := \underset{x \in \mathcal{X}, p, q \in \mathcal{Q}}{\arg\min} \sum ||A_q x + B_q||$.

For the time-varying matrix $R_q(\tau)$, the inequality (5.10) is not trivial to solve. A well-known technique to solve such problem is the discretized Lyapunov function technique, which is widely used in the stabilization of linear switched systems [1, 49, 92]. The basic idea of the discretized Lyapunov function technique is to linearize $R_q(\tau)$ into the form of $\frac{\tau}{\tau_{q,\min}} P_q + (1 - \frac{\tau}{\tau_{q,\min}}) Q_q$ for all $\tau \in [0, \tau_{q,\min})$,

and $R_q(\tau) = P_q$ elsewhere. In the following, we transform Lemma 5.6 into LMI conditions by using the discretized Lyapunov function technique and coordinate transformation.

Lemma 5.7. *Suppose that for each mode $q \in \mathcal{Q}$ there exist positive symmetric matrices P_q, Q_q, and constants $\mu_q > 0$, $\gamma_q > 0$, $\gamma_q' > 0$, $\tau_{q,\min} > 0$, and $\eta_q \neq 0$, such that the following inequalities*

$$\begin{bmatrix} \Xi_{q,1} & P_q L_q \\ * & -\gamma_q k \end{bmatrix} \leqslant 0, \quad \forall q \in \mathcal{Q},$$

(5.24)

$$\begin{bmatrix} \Xi_{q,2} & Q_q L_q \\ * & -\gamma_q k \end{bmatrix} \leqslant 0, \quad \forall q \in \mathcal{Q},$$

$$\begin{bmatrix} \Xi_{q,3} & P_q L_q \\ * & -\gamma_q k \end{bmatrix} \leqslant 0, \quad \forall q \in \mathcal{Q},$$

(5.25)

$$Q_p \leqslant (\mu_q - \gamma_q') P_q, \quad \forall q \in \mathcal{Q}$$

(5.26)

hold, where $\Xi_{q,1} = A_q^\top P_q + P_q A_q + \frac{1}{\tau_{q,\min}}(P_q - Q_q) + (\gamma_q - \eta_q)P_q$, $\Xi_{q,2} = A_q^\top Q_q + Q_q A_q - \frac{1}{\tau_{q,\min}}(P_q - Q_q) + (\gamma_q - \eta_q)Q_q$, and $\Xi_{q,3} = A_q^\top P_q + P_q A_q + (\gamma_q - \eta_q)P_q$. Then the statements of Theorem 3.6(i), (ii) and (iii) hold for any switching signals σ satisfying (5.14).

Proof. We will prove the lemma by constructing the matrix $R_q(\tau)$ used in (5.22) such that $V_q(\xi, \tau)$ in (5.22) satisfies the hypotheses in Theorem 3.6. Let us define $R_q(\tau)$ by

$$R_q(\tau) = \begin{cases} \frac{\tau}{\tau_{q,\min}} P_q + (1 - \frac{\tau}{\tau_{q,\min}})Q_q & \forall \tau \in [0, \tau_{q,\min}) \\ P_q & \text{otherwise,} \end{cases},$$

(5.27)

so that $R_q(0) = Q_q$ and $R_q(\tau_{q,\min}) = P_q$. For $\tau \geqslant \tau_{q,\min}$, (5.20) is guaranteed according to (5.25). Now, let us consider $R_q(\tau)$ in the interval $[0, \tau_{q,\min})$ where the time-derivative of $R_q(\tau)$ satisfies

$$\frac{\mathrm{d}R_q(\tau)}{\mathrm{d}\tau} = \frac{1}{\tau_{q,\min}}(P_q - Q_q).$$

(5.28)

Correspondingly, using (5.23), (5.20), (5.27) and (5.28) we can compute the time-derivative of V_q in (5.22) on $[0, \tau_{q,\min}]$ as follows

$$\dot{V}_q(x, \tau) - \eta_q V_q(x, \tau) - \gamma_q\Big(k - V_q(x, \tau)\Big) = \frac{\tau}{\tau_{q,\min}}\Big[\tilde{x}^\top \Xi_{q,1}\tilde{x} + \tilde{x}^\top P_q L_q + L_q^\top P_q \tilde{x} -$$

$$\gamma_q k\Big] + \Big(1 - \frac{\tau}{\tau_{q,\min}}\Big)\Big[\tilde{x}^\top \Xi_{q,2}\tilde{x} + \tilde{x}^\top Q_q L_q + L_q^\top Q_q \tilde{x} - \gamma_q k\Big],$$

(5.29)

where we have used the relation $\tilde{x} = x - x^\star$ in $V_q(x, \tau)$ above. The right-hand side of (5.29) can be written compactly as

$$\frac{\tau}{\tau_{q,\min}} \begin{bmatrix} \tilde{x}^\top & 1 \end{bmatrix} \begin{bmatrix} \Xi_{q,1} & P_q L_q \\ * & -\gamma_q k \end{bmatrix} \begin{bmatrix} \tilde{x} \\ 1 \end{bmatrix} + \left(1 - \frac{\tau}{\tau_{q,\min}}\right) \begin{bmatrix} \tilde{x}^\top & 1 \end{bmatrix} \begin{bmatrix} \Xi_{q,2} & Q_q L_q \\ * & -\gamma_q k \end{bmatrix} \begin{bmatrix} \tilde{x} \\ 1 \end{bmatrix}. \tag{5.30}$$

Correspondingly, using (5.24) and (5.30), it follows that

$$\dot{V}_q(x, \tau) - \eta_q V_q(x, \tau) - \gamma_q \left(k - V_q(x, \tau)\right) \leqslant 0, \tag{5.31}$$

for all $\tau \in [0, \tau_{q,\min})$. According to (5.26), it implies that

$$V_q(x, \tau) \leqslant (\mu_q - \gamma_q') V_q(x, \tau) \leqslant \mu_q V_q(x, \tau) + \gamma_q' \left(k - V_q(x, \tau)\right). \tag{5.32}$$

Similarly, for all $\tau \geqslant \tau_{q,\min}$, $\dot{V}_q(x, \tau) - \eta_q V_q(x, \tau) - \gamma_q(k - V_q(x, \tau))$ is negative definite according to (5.25). Consequently, in combination with (5.26) and (5.14), all hypotheses in Theorem 5.3 are satisfied and the claim of the lemma follows immediately. □

In general switched affine systems, L_q in (5.23) is not equal to zero and may not be identical among the different modes q when each mode has a different equilibrium point. The possibility of admitting a different equilibrium point for every mode makes it impossible to find a global quadratic common Lyapunov function given by (5.22).

Remark 5.8. Since R_q is a convex combination of P_q and Q_q, then for any given $k > 0$, $\alpha(k)$ in (5.6) can be upperbounded by

$$\alpha(k) \leqslant \overline{\alpha}(k) := \frac{\lambda_{max}}{\lambda_{\min}} k, \tag{5.33}$$

where $\lambda_{max} = \max\{\lambda(P_q), \lambda(Q_q)\}$, and $\lambda_{\min} = \min\{\lambda(P_q), \lambda(Q_q)\}$, $\forall q \in \mathcal{Q}$.

Equipped with the LMI conditions in Lemma 5.7, the following theorem provides sufficient conditions for practical stability of the switched affine system (5.19).

Theorem 5.9. *(Practical stability) Let the sets $\Omega_1 = N^\alpha(k)$ and $\Omega_2 = L(k)$ ($\Omega_1 \subset \Omega_2$). Suppose that the hypotheses in Lemma 5.7 hold. Then for all initial states in Ω_1, i.e. $x(t_0) \in N^\alpha(k)$, the trajectories of switched system (5.19) remain in the set Ω_2, i.e. $x(t) \in L(k)$, for every switching signals $\sigma(t)$ satisfying (5.14).*

Similar to Corollary 5.5, if all subsystems of (5.19) are unstable, the results in Lemma 2.12 can be used to establish the following remark.

Remark 5.10. Suppose that the hypotheses in Lemma 5.7 hold with $\mathcal{Q} = \mathcal{U}$. Then the trajectories of switched affine system (5.23) will remain in $L(k)$ after time $T > 0$ for any switching signals satisfying (5.17). In addition, if $P_q > Q_q$ then Ω_1 and Ω_2 can be estimated by $\bigcap_{q \in \mathcal{Q}} \{\tilde{x}_q \mid \tilde{x}_q^\top Q_q \tilde{x}_q \leqslant \alpha(k)\}$ and $\bigcup_{q \in \mathcal{Q}} \{\tilde{x}_q \mid \tilde{x}_q^\top Q_q \tilde{x}_q \leqslant \beta_q(k)\}$, respectively. Since the LMI may have multiple solutions, there may be different Ω_1 and Ω_2 with the same parameter setting. To reduce the region of Ω_1 and Ω_2, we can use some positive matrices $c_i I$ to bound Q_i and P_i, thus enlarging the terms in Q_i.

Let us illustrate the applicability of the LMI conditions in Lemma 5.7. By a direct application of Lemma 5.7, we establish the stability of a switched system with stable and unstable subsystems in Example 2 below, and it is followed by the stability of a switched system with all unstable subsystems in Example 3.

Example 2: Let us consider the switched system (5.19) composed of both unstable ($q = 1$) and stable ($q = 2$) subsystems as follows

$$q = 1: \quad \dot{x} = \begin{bmatrix} -2 & 0.5 \\ 0.5 & 0 \end{bmatrix} x + \begin{bmatrix} 1.4 \\ -0.4 \end{bmatrix},$$

$$q = 2: \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ -0.5 & -2 \end{bmatrix} x + \begin{bmatrix} -0.9 \\ 2.4 \end{bmatrix}, \tag{5.34}$$

and we set the parameter $k = 2$. Then by applying Lemma 2.12 to this switched affine system, where we fix $x^\star = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $\gamma_1 = \gamma_2 = 0.05$, $\eta_1 = 0.34$, $\eta_2 = -0.24$, $\mu_1 = 0.5$, $\mu_2 = 2$, $\gamma_1' = \gamma_2' = 0$, it can be checked that using the following symmetric constant matrices

$$P_i : \begin{bmatrix} 0.9160 & -0.0841 \\ -0.0841 & 0.3847 \end{bmatrix}, \begin{bmatrix} 0.0788 & 0.0296 \\ 0.0296 & 0.1767 \end{bmatrix},$$

$$Q_i : \begin{bmatrix} 0.1350 & 0.0624 \\ 0.0624 & 0.3511 \end{bmatrix}, \begin{bmatrix} 0.1596 & 0.0186 \\ 0.0186 & 0.1789 \end{bmatrix}, \tag{5.35}$$

the LMI problem given by (2.43)-(2.48) is feasible. Correspondingly, we have $\tau_{1,\min} = 2$, $\tau_{1,\max} = 2.04$, $\tau_{2,\min} = 2.89$. An upper bound of $\alpha(2)$ is given by (5.33) as $\overline{\alpha}(2) = 26.3546$ and $\frac{\lambda_{max}}{\lambda_{\min}} = 13.1773$. According to (5.11), $\beta_1 = 52.7092$, $\beta_2 = 26.3546$. Then, we have,

$$N(2) = \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 2 \right\} \cup \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top P_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 2 \right\},$$

$$N^\alpha(2) \subseteq N^{\overline{\alpha}}(2) = \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 26.3546 \right\} \cap \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top \right.$$

$$P_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 26.3546 \Big\};$$

$$L(2) = \Big\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 52.7092 \Big\} \cup \Big\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top P_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}$$

$$\leqslant 26.3546 \Big\}.$$

Figure 5.3 shows the trajectories of switched system (5.34) with a periodic switching signal σ satisfying $\tau_1 = 2$, $\tau_2 = 3$ and with four different initial conditions $\begin{bmatrix} 2.584 \\ -7.86 \end{bmatrix}$, $\begin{bmatrix} -0.056 \\ 9.848 \end{bmatrix}$, $\begin{bmatrix} -13.432 \\ 4.92 \end{bmatrix}$, $\begin{bmatrix} 15.608 \\ -2.36 \end{bmatrix}$.
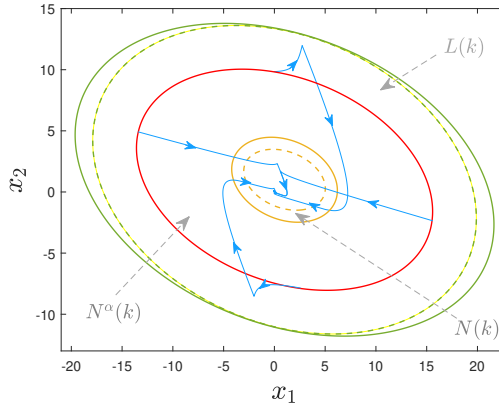

Figure 5.3: The plot of trajectories of switched system in Example 3 initialized at $\begin{bmatrix} 2.584 \\ -7.86 \end{bmatrix}$, $\begin{bmatrix} -0.056 \\ 9.848 \end{bmatrix}$, $\begin{bmatrix} -13.432 \\ 4.92 \end{bmatrix}$, $\begin{bmatrix} 15.608 \\ -2.36 \end{bmatrix}$, and using a periodic switching signal with $\tau_1 = 2$, $\tau_2 = 3$, the green solid line is $L(k)$, the red solid line is $N^\alpha(k)$, the orange line is $N(k)$.

Example 3: Let us consider the switched system (5.19) composed of two unstable subsystems as follows

$$\begin{aligned} q = 1: \quad \dot{x} &= \begin{bmatrix} -1.9 & 0.6 \\ 0.6 & -0.1 \end{bmatrix} x + \begin{bmatrix} 1.4 \\ -0.6 \end{bmatrix}, \\ q = 2: \quad \dot{x} &= \begin{bmatrix} 0.1 & -0.9 \\ 0.1 & -1.4 \end{bmatrix} x + \begin{bmatrix} 0.7 \\ 1.4 \end{bmatrix}, \end{aligned} \tag{5.36}$$

and let us set $k = 5$. Then by applying Lemma 2.12 to this switched affine system, where we fix $x^\star = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $\gamma_1 = \gamma_2 = 0.06$, $\eta_1 = \eta_2 = 0.34$, $\mu_1 = \mu_2 = 0.5$, $\gamma_1' = \gamma_2' = 0$, it can be checked that using the following symmetric constant
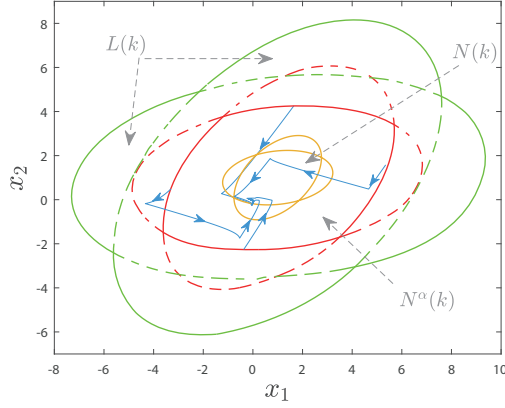
Figure 5.4: The plot of trajectories of switched system in Example 3 initialized at $\begin{bmatrix} -3.3 \\ 0.49 \end{bmatrix}$, $\begin{bmatrix} 5.35 \\ 1.58 \end{bmatrix}$, $\begin{bmatrix} -0.37 \\ -2.26 \end{bmatrix}$, $\begin{bmatrix} 1.65 \\ 4.26 \end{bmatrix}$, and using a periodic switching signal with $\tau_1 = \tau_2 = 2$, the green solid line is $L(k)$, the red solid line is $N^\alpha(k)$, the orange line is $N(k)$.

matrices

$$P_i : \begin{bmatrix} 6.6543 & -1.0418 \\ -1.0418 & 3.7555 \end{bmatrix}, \begin{bmatrix} 2.0998 & -0.6941 \\ -0.6941 & 6.8937 \end{bmatrix},$$

$$Q_i : \begin{bmatrix} 1.0475 & -0.3351 \\ -0.3351 & 3.3797 \end{bmatrix}, \begin{bmatrix} 2.0716 & -0.9015 \\ -0.9015 & 1.7611 \end{bmatrix}, \tag{5.37}$$

the LMI problem given by (2.43)-(2.48) is feasible. Correspondingly, we have $\tau_{1,\min} = \tau_{2,\min} = 2$, $\tau_{1,max} = \tau_{2,max} = 2.04$. An upper bound for $\alpha(5)$ is given by (5.33) as $\overline{\alpha}(5) = 34.95$ and $\frac{\lambda_{max}}{\lambda_{\min}} = 6.99$. According to (5.11), $\beta_1 = \beta_2 = 69.9$. According to Remark 5.10, $N(5) = \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 5 \right\} \cup \Big\{ x_1, x_2 :$

$\begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 5 \Big\}$;

$N^\alpha(5) \subseteq N^{\overline{\alpha}}(5) = \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 34.95 \right\} \cap \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top \right.$

$Q_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 34.95 \Big\}$;

$L(5) = \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \leqslant 69.9 \right\} \cup \left\{ x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^\top Q_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \right.$

$\leqslant 69.9 \Big\}$. Figure 5.4 shows the plot of trajectories of the switched system with periodic switching signal σ satisfying $\tau_1 = \tau_2 = 2$ and with four different initial

conditions $\begin{bmatrix} -3.3 \\ 0.49 \end{bmatrix}$, $\begin{bmatrix} 5.35 \\ 1.58 \end{bmatrix}$, $\begin{bmatrix} -0.37 \\ -2.26 \end{bmatrix}$, $\begin{bmatrix} 1.65 \\ 4.26 \end{bmatrix}$. The figure shows that when the trajectory starts on the boundary of $N(\alpha)$, the trajectory stays in $L(k)$ for all time. We note that the first switching moment in the trajectory, which starts from $\begin{bmatrix} 1.65 \\ 4.26 \end{bmatrix}$, illustrates the second case of step one in the proof of Theorem 3.6, i.e. for an unstable system, the trajectory can go into $N(k)$ and later escape from $N(k)$ on the next switching moment.

5.5 Conclusion

In this chaper, the set convergence properties of switched systems with mixed stable-unstable modes have been studied. Based on the dwell-time and leave-time property of the switching signals, multiple Lyapunov functions are defined and used to characterise the set of initial conditions that admits an attractive set, to which all trajectories will converge to. Based on these sufficient conditions, LMI conditions are presented that allow for numerical validation on the practical stability of switched affine systems with computable dwell-time.

6   Conclusions and Outlook

In this thesis, we undertake an investigation into the contractility and stability of both switched systems and differential-algebraic equation (DAE) systems. Subsequently, we apply these analyses to address a range of control challenges, including state observer, pinning synchronization, and practical stability. In the present chapter we will discuss the main results presented in Chapters 2- 5. We will also provide directions for possible future research.

6.1   Conclusions

Throughout Chapter 2 to Chapter 4, our main focus is on contraction theory and its practical applications. Specifically, in Chapter 2, we conduct an analysis of the contractivity of switched systems that contain noncontracting subsystems. This analysis involves a thorough examination of the associated switched variational systems. We establish that the switched systems are contractive if and only if the switched virtual systems are uniformly globally exponentially stable (UGES). Building upon this discovery, we proceed to design a time-dependent switching law that effectively stabilizes the virtual system. A noteworthy conclusion from our research is that, even in scenarios where all subsystems are noncontractive, it is possible to achieve contractivity in the switched systems by appropriately setting an upper bound for the active time. In addition, our approach can be transformed into Linear Matrix Inequality (LMI) conditions, facilitating numerical validation, for specific types of switched systems such as switched Lipschitz systems and switched Lur'e systems. This conversion into LMI conditions enhances the feasibility of practical implementation and verification of our method.

In Chapter 3, we analyze the contractivity of time-varying differential-algebraic equation (DAE) system. Through the analysis of the corresponding virtual DAE system, we reached the conclusion that the DAE system exhibits contractivity if and only if its associated virtual DAE system is UGES. To achieve UGES, we lift the variational DAE system to a higher-dimension variational ODE system.

This higher-dimension system encompasses the trajectory set that includes the trajectory of the variational DAE system. Based on this analysis, we can conclude that if the variational DAE system is UGES, the DAE system is contractive. A significant application of this approach lies in effectively utilizing our methods in various classical control problems by considering the system's output as an algebraic constraint. These control problems include output feedback design, output regulation, and state observer design. We employ our approach to design a time-varying observer.

In Chapter 4, we apply contraction theory to solve the state synchronization problem of heterogeneous time-invariant multi-agent systems. Our conclusion is that the synchronization problem is solvable if and only if there is a mapping that allows the embedding of the agents' dynamics into the exosystem. By employing these conditions, we apply contraction theory to design dynamic controllers for the agents, guaranteeing the contractivity of each agent's system and ensuring that the desired trajectory is included among the controlled dynamics' trajectories. Furthermore, for the pinned agents, we establish a control law only depend on the exosystems' information. On the other hand, for the remaining agents, we propose a distributed control law that exploits relative local state measurements.

In Chapter 2 and Chapter 3, we also address the stabilization problem by designing control laws that render contractivity in the systems. We apply our approaches to stabilize the linear switch systems with all unstable subsystem in Chapter 2. Additionally, in Chapter 3, we focus on stabilizing time-invariant DAE systems. This is feasible due to the fact that the equilibrium represents one of the trajectories of the contractive system. However, in cases where multiple equilibrium exist within the systems, achieving contractivity becomes challenging. In order to rectify this gap, in Chapter 5, we focuses on the investigation of switched systems without common equilibrium. We establish dwell/leave time conditions to ensuring set stability for switched systems comprising both stable and unstable subsystems. Based on the obtained sufficient conditions for set convergence, we propose a method utilizing time-dependent multiple quadratic Lyapunov functions to establish practical stability for switched affine systems. This approach results in LMI conditions that can be easily implemented through numerical computations.

6.2   Future research

Future studies will focus on the following aspects.

- In Chapter 2, we conducted an analysis of the contractivity of time event dependent switched systems. However, there exist various types of state event dependent switched systems, which require a different approach. The

contraction analysis of state event dependent switched systems, where all subsystems are contractive, was explored by Fiore et al. in [26]. When a noncontractive subsystem is introduced into the system, it disrupts the overall contractivity of the switched systems, presenting a challenging analysis task. Consequently, a potential avenue for further study would be to analyze state event dependent switched systems that incorporate noncontractive subsystems.

- In Chapter 3, the contractivity of index-1 DAE system is analyzed, where the DAE system can be viewed as an ODE system subject to an equality constraint. There are two potential avenues for future investigation. Firstly, it is possible to convert the equality constraint into an inequality constraint. In this case, it is not a DAE system anymore but an underdetermined ODE system. Secondly, an analysis of the contraction properties of the DAE system in a more general from can be pursued, specifically considering the form $E(x)\dot{x} = f(x)$.

- In Chapter 4, the focus lies on the application of the contraction theory to address the state synchronization problem. However, a more complicate challenge arises in the form of the output synchronization problem. One potential approach to address this issue is to advance an output contraction theory, which aims to attain convergence of the outputs rather than the states. By establishing such a theory, we can subsequently apply this novel approach to effectively tackle the output synchronization problem.

- In Chapter 5, we study the set convergence of switched systems that exhibit multiple equilibrium. The utilization of contraction theory to address this challenge becomes intricate, as achieving contractivity in such scenarios is not feasible. However, an alternative approach called k-contraction has been introduced in [90] as a generalized concept of contraction, specifically designed to analyze the contractivity of ODE systems with multiple equilibrium. Exploring the application of k-contraction theory to analyze switched systems with multiple equilibrium could be a potential avenue for future research.

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Summary

The convergence properties of nonlinear dynamics are deeply influenced by the characteristics of the specific system or model under investigation, resulting in significant variations. Many attempts have been made to understand these properties, which have also inspired a lot of applications. The thesis addresses various issues concerning the convergence properties of switched systems and differential-algebraic equation (DAE) systems. Specifically, we focus on contraction analysis problem, as well as tackling problems related to stabilization and synchronization.

We first consider the contraction analysis of switched systems and DAE systems. To address this, a transformation is employed to convert the contraction analysis problem into a stabilization analysis problem. This transformation involves the introduction of virtual systems, which exhibit a strong connection with the Jacobian matrix of the vector field. Analyzing these systems poses a significant challenge due to the distinctive structure of their Jacobian matrices. Regarding the switched systems, a time-dependent switching law is established to guarantee uniform global exponential stability (UGES). As for the DAE system, we begin by embedding it into an ODE system. Subsequently, the UGES property is ensured by analyzing its matrix measure. As our first application, we utilize our approach to stabilize time-invariant switched systems and time-invariant DAE systems, respectively. This involves designing control laws to achieve system contractivity, thereby ensuring that the trajectory set encompasses the equilibrium point. In our second application, we propose the design of a time-varying observer by treating the system's output as an algebraic equation of the DAE system.

In our study on synchronization problems, we investigate two types of synchronization issues: the trajectory tracking of switched oscillators and the pinning state synchronization. In the case of switched oscillators, we devise a time-dependent switching law to ensure that these oscillators effectively follow the trajectory of a time-varying system. As for the pinning synchronization problem, we define solvable conditions and, building upon these conditions, we utilize contraction theory to design dynamic controllers that guarantee synchronization is achieved among the agents.

Finally, we study the set convergence of a particular class of switched systems

that lack a common equilibrium point. Applying contraction theory to this category of systems proves impractical. To overcome this challenge, we develop switching laws based on mode-dependent dwell/leave times, effectively ensuring the set convergence of the switched systems. Moreover, we extend this approach to establish a condition, employing Linear Matrix Inequalities (LMIs), for achieving practical stability in switched affine systems.

Samenvatting

De convergentie-eigenschappen van niet-lineaire dynamica worden diepgaand
beïnvloed door de kenmerken van het specifieke systeem of model dat wordt
onderzocht, wat leidt tot aanzienlijke variaties. Er zijn veel pogingen gedaan om
deze eigenschappen te begrijpen, die ook veel toepassingen hebben geïnspireerd.
De scriptie behandelt verschillende kwesties met betrekking tot de convergentie-
eigenschappen van geschakelde systemen en differentiaal-algebraïsche vergelij-
king (DAE) systemen. Specifiek richten we ons op het probleem van contractie-
analyse, evenals het aanpakken van problemen met betrekking tot stabilisatie en
synchronisatie.

Als eerste overwegen we de contractieanalyse van geschakelde systemen en
DAE-systemen. Om dit aan te pakken, wordt een transformatie gebruikt om
het contractieanalyseprobleem om te zetten in een stabilisatieanalyseprobleem.
Deze transformatie houdt in dat virtuele systemen worden geïntroduceerd, die
een sterke verbinding vertonen met de Jacobiaanse matrix van het vectorveld.
Het analyseren van deze systemen vormt een aanzienlijke uitdaging vanwege de
kenmerkende structuur van hun Jacobiaanse matrices. Wat betreft de geschakelde
systemen wordt een tijdsafhankelijke schakelwet vastgesteld om uniforme we-
reldwijde exponentiële stabiliteit (UGES) te garanderen. Wat het DAE-systeem
betreft, beginnen we ermee door het in te bedden in een ODE-systeem. Vervolgens
wordt de UGES-eigenschap gewaarborgd door de matrixmaat te analyseren. Als
onze eerste toepassing gebruiken we onze aanpak om respectievelijk tijdinvariante
geschakelde systemen en tijdinvariante DAE-systemen te stabiliseren. Dit omvat
het ontwerpen van regels om systeemcontractiviteit te bereiken, waardoor ervoor
wordt gezorgd dat de trajectenset het evenwichtspunt omvat. In onze tweede
toepassing stellen we het ontwerp van een tijdvariërende waarnemer voor door
de uitvoer van het systeem te behandelen als een algebraïsche vergelijking van
het DAE-systeem.

In onze studie naar synchronisatieproblemen onderzoeken we twee typen
synchronisatiekwesties: het trajectvolgen van geschakelde oscillatoren en het
vastzetten van toestandssynchronisatie. In het geval van geschakelde oscilla-
toren ontwerpen we een tijdafhankelijke schakelwet om ervoor te zorgen dat

deze oscillatoren effectief het traject van een variërend systeem volgen. Wat betreft het probleem van het vastzetten van synchronisatie, definiëren we oplosbare voorwaarden en, op basis van deze voorwaarden, maken we gebruik van contractietheorie om dynamische controllers te ontwerpen die garanderen dat synchronisatie wordt bereikt tussen de agenten.

Uiteindelijk bestuderen we de verzamelconvergentie van een specifieke klasse van geschakelde systemen die geen gemeenschappelijk evenwichtspunt hebben. Het toepassen van contractietheorie op deze categorie van systemen blijkt onpraktisch. Om deze uitdaging te overwinnen, ontwikkelen we schakelwetten op basis van modusafhankelijke verblijfs/vertrektijden, wat effectief zorgt voor de verzamelconvergentie van de geschakelde systemen. Bovendien breiden we deze aanpak uit om een voorwaarde te stellen, met behulp van lineaire matrixongelijkheden (LMIs), voor het bereiken van praktische stabiliteit in geschakelde affiene systemen.