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An intuitive introduction to the evolution of physical systems

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Summary. — We outline a unified introduction to the general problem of dynamics intended for a high-school students audience. The attempt consists in circumventing the lack of mathematical knowledge with the use of 1) geometric diagrams, 2) a discretized version of the equations of motion and 3) a simplified form of computation and analysis of their solutions. The aim is to allow students to approach theoretical features as well as computational aspects of the evolution equations through the use of spreadsheets, a work environment students are usually familiar with and an ideal tool for an intuitive approach to recursive algorithms. The proposal was presented to an audience composed of students of University courses of Physics teaching and to high-school Science teachers.

1. – Introduction

In the last years, the Italian Ministry of Education has been recommending that modern physics should be part of any high-school physics curriculum [1]. As well known, however, there are many peculiar obstacles to overcome to comply with those recommendations, namely the lack of mathematical knowledge that is necessary to treat ordinary and partial differential equations in a rigorous way. Also, it has been often remarked that high-school physics reduces to an arid enumeration of laws, a list of fragmentary pieces of information related to each other only by the chronological order of their "discovery". To make an example, in the volume "Innovations in Science and Technology Education",

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edited by UNESCO, Svein Sjøberg analyzes the status of science and technology education and the way science and the scientists' work are perceived by young people [2]. In a section devoted to the possible reasons for the "disenchantment with science and technology", the author criticizes many actual curricula and textbooks which, in his opinion, "are overloaded with facts and information at the expense of concentration on few big ideas and key principles" often leading to "rote learning without any deeper understanding." We think that a natural way to exhibit the deeper meaning of physical theories and their extraordinary power to account for observable phenomena would be to perform a qualitative and/or quantitative analysis of the solutions of the dynamical equations describing the evolution of actual systems. However, If the skills for performing such an analysis are lacking, students are deprived of the astonishment and motivation resulting from the verification of the effectiveness of the theory. In many contributions to the previously cited book, the focus is also on the role of technological applications as a resource to outline the "scientific method" as being "not just about knowing but also about doing and making things work". Both new technologies and conventional software (*i.e.*, databases, spreadsheets, graphical programs, etc.) are presented as key elements to develop teaching and learning together with modelling, visualization and simulation of processes. Recent proposals in response to students' difficulties are in fact heading in this direction. Other approaches along the same lines demonstrate how computational physics, beyond being an effective way to find approximate solutions to specific problems, can help students understand basic concepts in physics [3-5].

The main aim of this paper is to propose a unified methodology that tries to bypass technical difficulties involved in the rigorous treatment of differential equations. We have worked on developing an approach that can foster the connection between phenomena that very often appear in a fragmented way. We propose an approach to the study of dynamical systems starting from classical dynamics up to stochastic and quantum dynamics which try to circumvent this problem. A forthcoming paper, currently in preparation, will be devoted to statistical and quantum dynamics. In the following, we introduce the general aspects of our proposal and illustrate them with some phenomena of classical dynamics, *e.g.*, planetary motions, linear and non-linear oscillations, and field equations. In this work, the study of the dynamics of physical systems is presented from a geometrical construction. More precisely, we consider a discretized space-time where the evolution equations are translated into *recurrent vector equations*; then, given the acting force and any set of initial conditions, solutions at any future time are computed by iteration and graphically represented. In this way, the dynamical equations of both ordinary and unconventional physical systems are presented as *recurrence prescriptions*. The continuous space-time limit is then qualitatively obtained by looking at the solutions on a scale where discretization becomes unnoticeable, avoiding in the first place all mathematical details. This kind of reasoning is naturally implemented on a spreadsheet and gradually towards more elaborate software in order to simplify the computational aspect and adapt it to computer skills possessed by high-school students. Agreeing with a rich bibliography [6-9] we propose the spreadsheet as the main didactic tool because it is largely used in school contexts and it is also an ideal environment for an intuitive approach to the general problem of solving dynamical equations. Moreover, without having previous knowledge of programming languages, students can easily implement solution algorithms and focus their attention on the actual behaviour of solutions.

But the focus of our proposal is more on the *reasoning* involved than on the *tool* actually employed. We wish to stress that our primary aim is to stimulate an intuitive understanding of the *meaning* and *conceptual role* of evolution laws in the *general* study of physical systems, relying on and putting to profit the key idea of *iterative process*. Our plan is not merely to replace a *continuous* mathematical language with a *discrete* one: the necessity of a *theory of limits* to give a rigorous meaning to the evolution equations in a spatio-temporal *continuum* should be discussed and clarified. In particular, it should be made explicit that this further step requires new mathematical tools. However, whenever approximate solutions are explicitly computable, the physical meaning and the explanatory power of a theory may be equally perceived in a discrete space-time setting([1]).

In the following sections, we detail the discrete version of the evolution equations we propose to consider. In particular, we will use the simplest form of discretization, without going into any issue concerning the stability of numerical calculation. As a final remark, we wish to share the view supported in [12] and in [13], where, in opposition to conceptual fragmentation, the authors emphasize the importance of a phenomenological approach that aims to connect experimental and theoretical perspectives, without excluding the relevant epistemological problems inevitably arising in the "game of modeling".

2. – Classical dynamics

Geometric introduction. – The basic idea of Newton's dynamical theory of motion consists in attributing a *cause* to the motions of bodies in terms of the *forces* acting on them. His three "laws of motion" detail quantitatively the *effects* of these forces on the motions of bodies. The first law states that under the action of the sole *vis insita* or *inertia* (*i.e.*, in the case of an approximately isolated body) the motion is rectilinear with constant speed. The second law states that the effect of the interaction between bodies, *i.e.*, of *forces*, is a *change* in the velocity of the body (*e.g.*, in the direction and magnitude of its motion). In particular, Newton showed how a large class of natural motions could be explained through the single *hypothesis* of the existence of a *reciprocal attraction* acting along the line connecting physical bodies. In this way, he was able to account in a unified way for a vast range of phenomena, including the Keplerian motions of planets, satellites and comets, tidal phenomena, falling bodies and oscillatory motions (namely, the motions of pendula).

In Newton's system, *space* and *time* form a homogeneous and isotropical frame that pre-exists matter and is completely unaffected by it. Physical bodies move inside this *absolute* (*i.e.*, *independent from anything else*) space-time frame: what Newton calls *space* is the (continuous) set of *places* occupied by material bodies, while the term *time* indicates, in essence, a (continuous) *uniformly flowing* parameter which can be used to track the motion of all the bodies in the universe([2]). In order to find the motion of bodies under some given condition, one is faced with the mathematical problem of *constructing* the successive positions and velocities of a material body resulting from a *continuous infinity of velocity variations* (*i.e.*, the continuous impulses of the external

([1]) In this regard, we want to mention the book [10], where the authors apply a similar approach to analyze qualitative and quantitative features of the evolution of classic dynamical systems, with particular emphasis to the onset of chaotic behaviors. Another clear text in discrete mathematical physics intended for an audience of mathematics students at College level is [11].
([2]) In principle, this absolute *time* would be exhibited to observation by the motion of a body isolated from any other, but this is only a limiting condition that is never attained in the physical world. An accessible overview of Newton's mathematics and philosophy is [14].

forces), starting from some assumed values of initial position and velocity([3]).

As well known, in the *Principia* Newton solved this problem using a purely geometrical approach. His method consisted essentially in the construction of diagrams made according to *rules* that embodied his laws of motion, *i.e.*, the inertia principle and the proportionality between force and acceleration. Following Feynman [15], we will call this kind of dynamical diagrams *Newtonian diagrams*.

The procedure for the construction of a Newtonian diagram may be summarized as follows:

1) The motion is analyzed into a finite succession of rectilinear and uniform motions, covering equal intervals of time and separated by *impulsive* forces that cause instantaneous changes in the magnitude and direction of the velocity of the body.
2) The net change of velocity Δv of a material body velocity during a "short" interval of time Δt is proportional to the *impulse* of the force F acting on the body during this interval ($\Delta v \propto F\Delta t$). Δt is "short" in the sense that, by assumption, during this time the force F does not change "significantly".
3) From the polygonal trajectory thus obtained, in which the motion is linear and uniform in each interval, the continuous path in space is obtained in the limit of "vanishingly short" intervals of time([4]).

As a paradigmatic example of this kind of process, we briefly sketch Newton's proof of the assertion that in presence of a central force a moving body sweeps with respect to the center of attraction equal areas in equal times (*Principia*, Prop. 1). This result, which is the cornerstone of Newton's dynamics, evidently includes Kepler's *area law* about the motion of planets around the Sun, provided this latter is regarded as a center of force.

Refer to fig. 1. Consider a succession of points (A, B, C), representing the successive positions of a planet moving around the Sun, this latter being placed at the fixed position S; we assume that these positions are separated by equal intervals of time Δt. The body starts in position A, and in the time Δt goes to B with a constant speed. If no force acted, by the first law the body would proceed inertially on a straight line with the same speed, ending up after another Δt in the position c along AB such that $AB = Bc$. However, because of the pull from the Sun, after $2\Delta t$ from the initial time the body will be on some point C placed on the line parallel to BS and passing through c. The line cC represents the change in velocity occurred in the time $2\Delta t$ due to the pull from the Sun, and by the second law of motion is proportional to the force acting in this time. The areas of the triangles SAB and SBC are the areas described by the radius vector of the planet in two successive equal intervals of time, and it is easily demonstrated (by *Elements*, I.35) that they are equal by construction. Therefore, *in two successive equal intervals of time the radius vector covers equal areas.*

([3]) For historical reasons, problems of this kind are sometimes called *inverse problems*, as contrasted to *direct* ones, where the motion is *given* and it is required to find the acting forces. For example, the *direct Kepler problem* may be stated as follows: *given a body moving in an ellipse around a fixed body occupying one of its focii and with a constant areal velocity with respect to it, to find the force acting between the two bodies.* On the other hand, the *inverse Kepler problem* is the following: *given two bodies attracting each other with an inverse-square law of force, to find the motion of a one body with respect to the other, assumed stationary in space.* In the present paper we deal exclusively with *inverse* problems.

([4]) This is only one of the possible approximate methods for the construction of a dynamical diagram. For a detailed analysis on the various methods used by Newton in the *Principia* see [16].

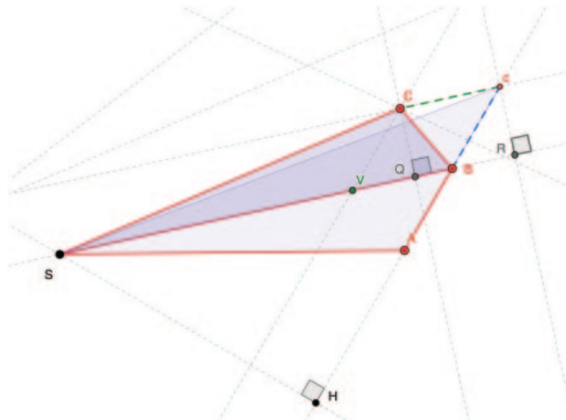

Fig. 1. – Proof that the triangles SAB, SBc and SBC have equal areas. SAB and SBC have the same base because $AB = Bc$ and a common altitude SH, so they have the same area; also SBC has the same area as SBc, because they have the same base SB and are comprised between the same parallel line; therefore, $SAB = SBc = SBC$.

If one *iterates* the same construction for the successive triplets of equal-time positions (B, C, D), (C, D, E), etc., the same result will hold, provided the force (which in general is allowed to be different for each triplet) is always directed to the same point S (fig. 2). So, in such a *polygonal approximation* the planet moves in each interval at a constant velocity along the lines AB, BC, CD, DE which represent the (mean) velocity of the body in the corresponding intervals of the orbit; at positions B, C, D, E the body undergoes sudden changes of velocity, and the lines $BV, CW, DX, \dots,$ represent these changes, which are the net effect of the gravitational pull of the Sun in each interval. In the limit of smaller and smaller Δt, the approximate polygonal path approaches without limit a curved trajectory, described by the radius vector in such a way that equal *areas* correspond to equal *times*. QED.

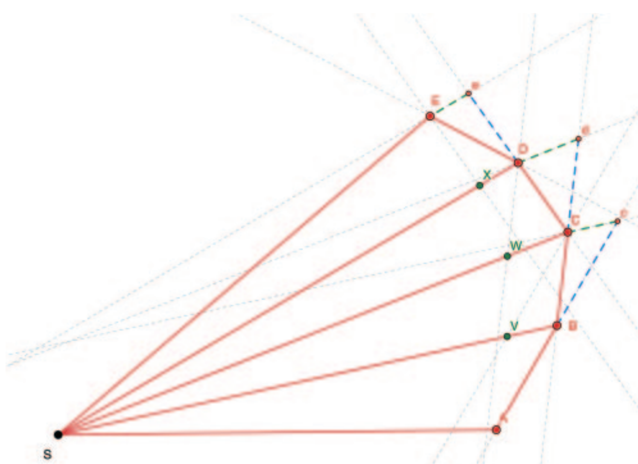

Fig. 2. – Reproduction of Newton's diagram for *Proposition 1* in the *Principia*. The positions A, B, C, D, E are separated by equal times. The lines AB, BC, CD, DE are the (mean) velocities in those intervals; the lines $BV, CW, DX, \dots,$ are the changes of velocity occurring at the positions B, C, D; the areas $SAB, SBC, SCD, SDE, \dots,$ are all equal.

If one makes some further assumption about the character of the force acting on the moving body (*e.g.*, that it decreases with the square of the length of the radius vector) one gets a definite *rule* for the construction of the successive vectors $BV, CW, DX, \ldots$, and, therefore, of the successive velocities and positions of the moving body starting from some initial values (*e.g.*, a Keplerian ellipse, with the Sun occupying one of the focii). In this way, one can *draw* the successive positions and velocities of the moving body and thus *construct* the solution of the dynamical problem at hand.

For example, under the assumption of an inverse-square law it easy to see that the velocity vectors trace a *circle* around the center of force, which in general is eccentric to the circle itself. This peculiar dynamical symmetry of the inverse-square law was first recognized in 1846 by William Rowan Hamilton [17] and independently rediscovered many times, most notably by Feynman [18]. Hamilton called this geometrical picture of the inverse-square law *law of the circular hodograph*, the *hodograph* being defined in general as the curve traced by the velocity vectors of a moving point, when they are all drawn from a common origin. The general method of the hodograph, which allows for very simple and elegant solutions to many relevant dynamical problems, may be very useful in high-school teaching and we used it extensively in our activities([5]).

Symbolical translation. – The geometrical construction of any Newtonian diagram may be easily translated into symbols using the language of vectors. Refer, for example, to fig. 3, where the case of a central force acting on a unit mass is illustrated. Allowing also for more general conditions, if we make the identifications

$$\begin{aligned} SA &= \mathbf{x}(t) \\ SB &= \mathbf{x}(t+\Delta t) \\ AB &= Bc = \mathbf{v}(t)\Delta t \\ BC &= \mathbf{v}(t+\Delta t) \\ cC &= \mathbf{v}(t+\Delta t) - \mathbf{v}(t) = \frac{1}{m}\mathbf{F}[\mathbf{x}(t)]\Delta t, \end{aligned}$$

where the function $\mathbf{F}(\mathbf{x}(t), t)$ is the force acting on the point-like body when its position is $\mathbf{x}(t)$, we see that the construction of the Newtonian diagram solving an (inverse) dynamical problem amounts to solving for "short" Δt the following pair of vector equations:

$$\begin{cases} \mathbf{x}(t+\Delta t) - \mathbf{x}(t) = \mathbf{v}(t)\,\Delta t \\ \mathbf{v}(t+\Delta t) - \mathbf{v}(t) = \dfrac{1}{m}\mathbf{F}[\mathbf{x}(t), t]\,\Delta t \end{cases} \tag{1}$$

the function $\mathbf{F}(\mathbf{x}(t), t)$ being given([6]).

Notice that, also in the most general case, the only (kinematical) prerequisites of this analysis are the vector identities expressing the total displacement and the total velocity

([5]) On the LES website `http://www.les.unina.it/?page_id=4784` the reader may find additional material in this regard.
([6]) Notice that a possible dependence of the force on the velocity (for example a friction force) is not going to complicate the computational procedure used to find solutions of 1.

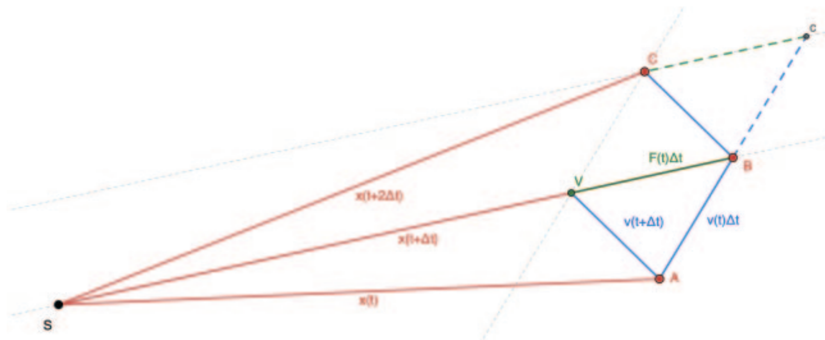

Fig. 3. – Construction of the Newtonian diagram under the assumption of central force directed toward S. A, B, C are successive positions of the moving body separated by equal times Δt; SA, SB, SC are the vectors of relative position; $AB, BC = AV$ are successive vectors of velocity, *before* and *after* B; $BV = cC$ is the change of velocity occurring in B due to the attraction from S. The triangles SAB, SBc, SBC have equal areas by construction.

variation of the point particle as a sum of position and velocity changes in each time interval $[t_{i-1}, t_i]$, no matter how coarse or fine the subdivision of the total time interval is

$$\begin{aligned} \mathbf{x}(t_N) - \mathbf{x}(t_0) &= \sum_{i=1}^{i=N} [\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})] = \sum_{i=1}^{i=N} \frac{\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})}{t_i - t_{i-1}} (t_i - t_{i-1}) \\ &= \sum_{i=1}^{i=N} \mathbf{v}_{t_{i-1}, t_i} (t_i - t_{i-1}), \end{aligned} \tag{2}$$

the "average velocity" between times t and $t' > t$ being defined as $\mathbf{v}_{t,t'} \equiv \frac{\mathbf{x}(t') - \mathbf{x}(t)}{t' - t}$. In the same way, the total velocity variation is expressed as

$$\begin{aligned} \mathbf{v}(t_N) - \mathbf{v}(t_0) &= \sum_{i=1}^{i=N} [\mathbf{v}(t_i) - \mathbf{v}(t_{i-1})] = \sum_{i=1}^{i=N} \frac{\mathbf{v}(t_i) - \mathbf{v}(t_{i-1})}{t_i - t_{i-1}} (t_i - t_{i-1}) \\ &= \sum_{i=1}^{i=N} \mathbf{a}_{t_{i-1}, t_i} (t_i - t_{i-1}), \end{aligned} \tag{3}$$

where the "average acceleration" between times t and $t' > t$ is defined as $\mathbf{a}_{t,t'} = \frac{\mathbf{v}(t') - \mathbf{v}(t)}{t' - t}$.

It is noteworthy that within such a discretized time scheme one can easily define *work*, *kinetic* and *potential energy*, as well as the crucial *conservativeness* of harmonic and gravitational forces. Then, the dynamical law (1) and the kinematical identities (2), (3) allow proving the conservation of total energy apart from terms that become negligible when the time step of the chosen discretization is "small", *i.e.*, when the approximation defined by (1) is made finer and finer.

A trivial example of this procedure is the following "proof" of the work-energy theorem. From the identity

$$\vec{v}_{i+1} \cdot \vec{v}_i = \frac{1}{2}|\vec{v}_{i+1}|^2 + \frac{1}{2}|\vec{v}_i|^2 - \frac{1}{2}(\vec{v}_{i+1} - \vec{v}_i) \cdot (\vec{v}_{i+1} - \vec{v}_i),$$

one gets

$$\begin{aligned}\vec{F}_i \cdot \vec{v}_i &= \frac{m}{\tau}(\vec{v}_{i+1} - \vec{v}_i) \cdot \vec{v}_i = \frac{m}{2\tau}|\vec{v}_{i+1}|^2 - \frac{m}{2\tau}|\vec{v}_i|^2 - \frac{m}{2\tau}|\vec{v}_{i+1} - \vec{v}_i|^2 \\ &= \frac{1}{\tau}(E_{cin_{i+1}} - E_{cin_i}) - \frac{m\tau}{2}|\vec{a}_i|^2.\end{aligned} \tag{4}$$

For bounded values of the force (meaning in turn bounded values of the acceleration) and for small time intervals τ, (4) proves that the equality between the kinetic energy variation velocity and the power of the force ($\equiv \vec{F}_i \cdot \vec{v}_i$) holds. Multiplying by τ and summing over all i's one finds that the total work differs from the variation of the kinetic energy by a term which is proportional to τ^2, and becomes negligible if the time step is taken sufficiently small.

We want to emphasize the concreteness, easiness and effectiveness of the discrete approach coupled with an affordable way to compute solutions to equations (1)([7]). Implementing the recurrence (1) on a spreadsheet it is possible to compute the point mass motion as a function of time. More precisely, when position and velocity (and, in turn, force) are known at the initial time, the recurrence equations return position and velocity at any time t. Each computational step consists in fact in the copy and paste of the previous line, where the recursion formulas are written via relative references (apart from fixed parameters appearing as absolute references).

In the following, we show how the procedure outlined above applies to classical systems which are usually considered too complicated to be presented in an elementary physics course. Our aim is to show that a qualitative and quantitative understanding of relevant features of the evolution of complex system is surely within the reach of high-school students.

Gravitation: the three-body problem. – As is widely known, Newton used the law of universal gravitation together with the second law of motion to analyze the two-body problem. He characterized all possible orbits and, in particular, deduced from his assumptions Kepler's laws of planetary motion. As astonishing as this result was, it was little thing if compared, for example, to the theoretical discovery of Neptune (first observed by Le Verrier in 1846), when the existence of a new planet was inferred by its perturbative effect on the motion of Uranus. This was probably the most astonishing confirmation of Newton's theory of gravitation. More generally, we believe it is in the treatment of *non-Keplerian* motions that the power of classical dynamics is most effectively perceived, since it allows making useful and precise predictions also when no closed and general solution to the equations of motion may be found. Also, it was the success of Newtonian dynamics in many-body problems that established *celestial mechanics* as the basis of the general *mechanical world-view* that prevailed during the Age of Enlightenment and reached its peak with Laplace's determinism.

Indeed, most post-Newtonian theoretical developments came from the effort to solve problems involving more than two bodies. Already in the *Principia*, Newton made a first attempt to deal with the three-body problem in order to calculate the effects of the

([7]) However, it is clear that if no absolute minimal time interval is given (as it is the case when a continuous model of space-time is assumed) the value of Δt remains unspecified and a rigorous theory is missing. As we already remarked, this point is important and should not be overlooked.

Sun on the Moon's motion around the Earth, a problem related, among other things, to the prediction of tides. Not long after, Euler attacked the problem using a simplified model, later denoted by Poincaré as the "circular restricted three-body problem". In this model, three point-like bodies interact via gravitational forces. One of them has a mass which is negligible with respect to the masses of the other two, and these two "heavy" bodies follow circular orbits around their common center of mass. The circular restricted three-body problem marked the birth of *perturbation theory* in celestial mechanics, which allowed to understand (and compute) the secular variations of planetary motions, and started the investigations about the stability of the solar system.

Along the lines above it is possible to very easily investigate the evolution of a simplified circular restricted three-body system. For the sake of simplicity, we assume:

- That the heaviest body (the Sun) has a mass M_S so large to be subjected to negligible acceleration. Its fixed position will be then taken as the origin of the Cartesian coordinate system inside which the motion of the two planets is described.

- A planet of large mass M_e is assumed to follow a circular orbit around the Sun. Its position at time t is denoted by $\mathbf{x}_e(t) = (x_e(t), y_e(t))$.

- A second planet, of mass m negligible with respect to the first, has an initial velocity in the plane containing the Sun and the two planets at the initial time, in such a way that its motion will always develop on this plane. The initial distance of the light planet will be taken smaller than the radius of the orbit of the heavy one, and for this reason the two planets will be referred respectively as the *inner* and *outer* planet. The coordinates of the inner planet will be denoted by $\mathbf{x}(t) = (x(t), y(t))$.

Under these assumptions, it is possible to use (1) to examine the motion of the light planet subject to the gravitational action of the Sun and of the outer planet, for various initial conditions and mass ratios.
In the following example, the components of the outer planet positions are taken to be

$$x_e(t) = R_e \cos \omega_e t; \quad y_e(t) = R_e \sin \omega_e t.$$

In this case, eqs. (1) read

$$\begin{aligned} \mathbf{x}(t+\Delta t) - \mathbf{x}(t) &= \mathbf{v}(t)\,\Delta t, \\ \mathbf{v}(t+\Delta t) - \mathbf{v}(t) &= \frac{1}{m}\left[\mathbf{F}_{M_S m}(\mathbf{x}(t)) + \mathbf{F}_{M_e m}(\mathbf{x}(t), t)\right]\Delta t, \end{aligned} \tag{5}$$

where the forces on the inner planet due, respectively, to the Sun and to the outer planet are

$$\mathbf{F}_{M_S m}(\mathbf{x}(t)) = -\frac{G M_S m}{||\mathbf{x}(t)||^3}\mathbf{x}(t),$$

$$\mathbf{F}_{M_e m}(\mathbf{x}(t), t) = -\frac{G M_S m}{||\mathbf{x}_e(t) - \mathbf{x}(t)||^3}[\mathbf{x}_e(t) - \mathbf{x}(t)]. \tag{6}$$

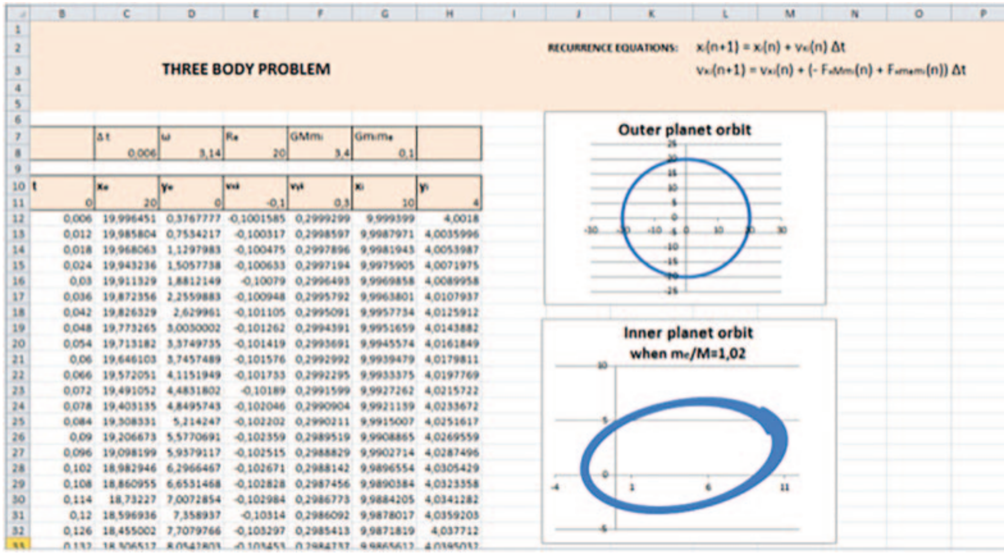

Fig. 4. – An example of a worksheet showing computation and plots of the planet orbits in a simplified circular restricted three-body planetary system.

The x components of these forces are

$$F_{M_S m}(\mathbf{x}(t)) = \frac{GM_S m}{[(x^2(t) + y^2(t))]^{\frac{3}{2}}}\, x(t),$$

$$F_{M_e m}(\mathbf{x}(t), t) = \frac{GM_e m}{[(x_e(t) - x(t))^2 + (y_e(t) - y(t))^2]^{\frac{3}{2}}}\, [x_e(t) - x(t)] \,. \tag{7}$$

The y components can be easily obtained by replacing x with y.

By implementing these equations on a spreadsheet (fig. 4), it is possible to examine the motion of the inner planet for various values of the dynamical parameters (namely, the ratios M_e/M_S and m/M_S) and for different initial conditions. In particular, students should be able to analyze the regime of "small perturbations" when both planetary orbits are Keplerian, characterize the range of mass ratios when the orbits of the inner planet are slowly perturbed and the onset of a chaotic behavior, investigating in this last regime the strong dependence on the initial conditions. It is important to remark that a space discretization is inevitably implied by the choice of the maximum number of decimal places allowed in the spreadsheet.

As soon as students are aware of the possibility to compute approximate solutions to the evolution equations via intuitive recurrence procedures, they may be confronted with newer and more efficient ways to estimate the approximations made and to investigate better and more stable procedures (*e.g.*, two-point approximations). In particular, at some point one can introduce object-oriented software to examine complex behaviors of many-body gravitational systems. On the basis of previous experience, students should be led to understand that the animations produced by the software are not the outputs of a magic and inscrutable black-box (as many applets one may find on the web tend to

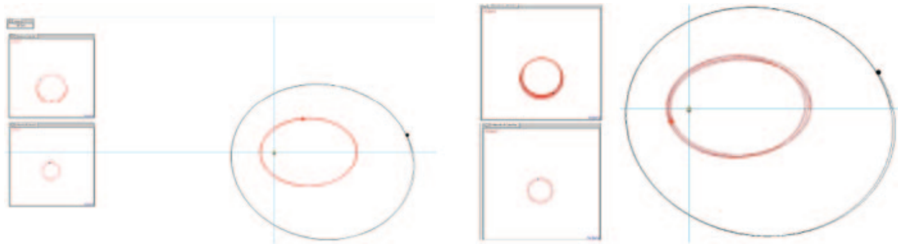

Fig. 5. – The mass of each planet is very small with respect to the mass of the Sun. On the left, their masses are equal; on the right, the mass of the outer planet is ten times that of the inner planet. The orbits of the planets are almost Keplerian.

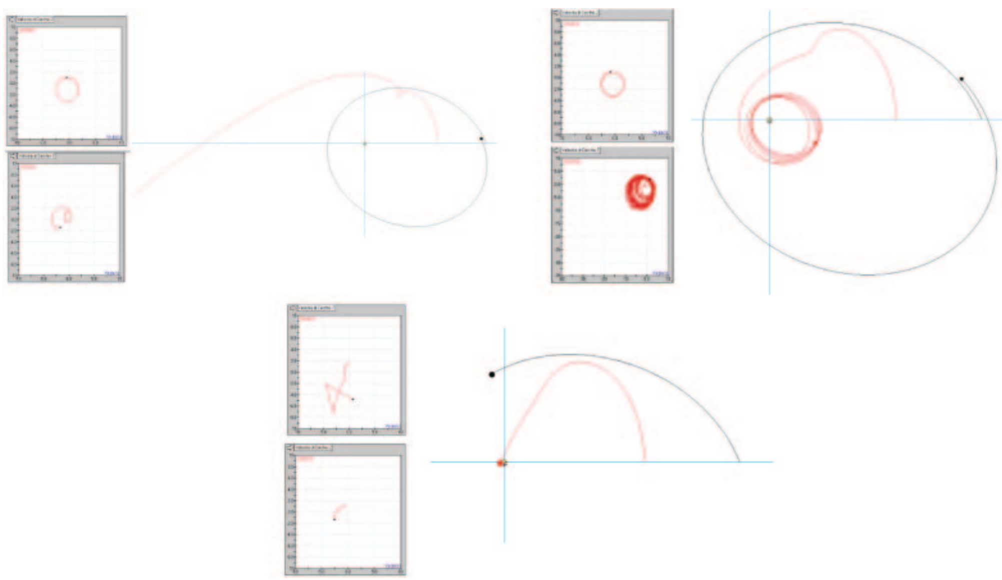

Fig. 6. – The mass of the outer planet is comparable to the mass of the Sun. For small differences in initial conditions there are very different long-term outcomes for the inner planet. On the upper-left, it is captured in a gravitational slingshot and is pushed out of the system; on the upper-right, it enters an orbit much closer to the Sun. In the last figure, it finally collides with the Sun.

be perceived), but the result of computational procedures not so far in their conceptions from those they themselves had implemented.

Solutions of the evolution laws relative to different gravitational systems and their animations, obtained with both a spreadsheet and object-oriented software, may be found at `http://www.les.unina.it/?page_id=4784`. A freely modifiable spreadsheet modeling the three-body problem may be found at `https://rb.gy/9jo4`. Here below, we inserted a selection of figures exhibiting characteristic features of the three-body dynamics; see figs. 5 and 6.

Oscillations: pendulum motion and Van der Pol model. – Besides planetary motion, another gravitation-induced dynamics is the motion of the pendulum, an example of oscillation problem that is, by itself, extremely relevant for the history of measurement, conceptualization and mathematical modeling of *time*. As an aside, it is worth mentioning that textbooks often report confusing tautologies about the "operational" definition of time intervals and of its measurement units. First, it is stressed the necessity to *find* (rather than *define*) a "periodic" phenomenon; then, the "small" oscillations of pendulum are experimentally found isochronous using a chronometer, whose working principle is based on the existence of another isochronous phenomenon that no one knows how could be assessed. In this way, the very important observation of the *universal proportionality* between periods of completely distinct and independent systems oscillating around a stable equilibrium position is overlooked in favor of a tautological procedure that leaves the theoretical problem of time measurement untouched.

Pendulum motion can be thoroughly examined via the discretization procedure described in the previous sections. In particular, the process can be implemented in a spreadsheet and allows analyzing the oscillatory motion of a pendulum for whatever initial conditions, also in the presence of viscous friction and forcing terms (*e.g.*, the one meant to model the escapement of the pendulum clock). Indeed, the recurrence relations (1) are usable also in the case of oscillations driven by forces depending non-linearly on the displacement from the equilibrium position.

The recurrence equations for a pendulum in the time interval $[0, T]$, in the presence of viscous friction and a forcing term, are

$$\begin{aligned} \alpha_i &= -\frac{\beta}{m}\omega_i - \frac{g}{l}\sin\theta_i, \\ \omega_i &= \omega_{i-1} + \alpha_{i-1}\tau, \\ \theta_i &= \theta_{i-1} + \omega_{i-1}\tau, \end{aligned}$$

where α_i (respectively, ω_i, θ_i) is the angular acceleration (respectively, angular velocity, angular position) at time $i\tau = T/N$, β is the damping coefficient. One may also add a forcing term to reproduce the effect of the escapement mechanism, for example of the kind

$$F_{esc} = \mu e^{-10\theta_i^2}(1 + \operatorname{sgn}\omega_i),$$

where μ is a strength coefficient. This is a brief impulse affecting the pendulum motion which is non-negligible only when the pendulum is very close to the vertical position; the term $(1+\operatorname{sgn}\omega_i)$ makes the escapement force active only when the pendulum crosses the vertical position from left to right.

Students can examine several features of the motion. In particular, it is possible to analyze the dependence of the period T on the initial conditions and on the dynamic parameters and find out that:

- the period shows negligible changes for small oscillation amplitudes ($\theta < 5°$);
- isochronism is lost for large oscillation amplitudes;
- the period depends on a negligible way on damping: even if the oscillation amplitude decreases, the small oscillation period remains essentially constant;
- the escapement restores the energy lost by viscous friction without changing the oscillation period.

Simulations and graphical presentations of results can be found at `http://www.les.unina.it/?page_id=4812`, where also modifiable spreadsheets are available.

Another interesting example of a non-linear oscillating system which can be treated with the same approach is the *Van der Pol oscillator*, sometimes called *the model of models*. It was introduced in 1927 by electrical engineer Balthazar Van der Pol to describe oscillations in a triode, and is one of the first examples of self-sustained oscillations, with applications in very different fields of biology and technology [19].

One can regard this model as representing a spring-mass system whose energy is pumped in or drained out depending on the mass position. The discretized equations have the same structure as the pendulum equation for small oscillations ($\sin(x) \sim x$) and without the escapement term; the coefficient that multiplies the velocity, however, can take positive or negative values depending on the position of the mass. Therefore, the model predicts transitions between regimes in which energy grows and decays. More precisely,

$$
\begin{aligned}
a_i &= -\beta(1 - x_i^2)v_i - x_i, \\
v_i &= v_{i-1} + a_{i-1}\tau, \\
x_i &= x_{i-1} + v_{i-1}\tau,
\end{aligned}
$$

where x, v, and a are, respectively, (linear) position, velocity, and acceleration of the oscillating mass, $\beta > 0$ and $\beta(1 - x^2)$ is positive or negative depending on whether the mass is close or distant from the origin. When $\beta = 0$, the equation describes a simple harmonic oscillator characterized by the period $T = 2\pi$ ($\omega = 1$). Solutions to the Van der Pol equation obtained with a spreadsheet may be found at `http://www.les.unina.it/?page_id=4812`, where also modifiable spreadsheets are available.

Fields: the elastic string. – With the same procedure outlined above, it is also possible to investigate the dynamics of classical *fields*, the only addition being the *explicit* discretization of the spatial coordinates. Thus, field amplitudes become functions on a discrete space-time lattice, and the evolution equations are given as recursive transition matrices connecting space lattices at different times.

The simplest example of this kind of systems is an elastic string vibrating longitudinally, which is also a good introduction to the dynamics of deformable bodies. A discretized model of this system is a chain of N point masses which interact with their neighbors through massless springs of elastic constant k and rest length equal to the spatial lattice spacing Δx. The oscillator chain that leads to the propagation of a wave is the final step in an educational path on coupled oscillators with which we have experimented. The path starts from the experiences and modeling of two oscillators coupled with the "discovery" that the generic motion (however bound by the conservation of mechanical energy) is a linear combination of normal modes with a suitable choice of initial conditions. The oscillators are pendula or carts coupled with springs and the motion is detected with a motion detector in real time. In the transition to many oscillators, one also works with dynamic systems for the realization of simulations and animations. Denoting with $s(j\Delta x, n\Delta t)$ the displacement with respect to the equilibrium position of the j^{th} mass at time $n\Delta t$, the force acting on that mass m is

$$\begin{aligned} F_{j\Delta x,n\Delta t} &= k[s((j+1)\Delta x, n\Delta t) - s(j\Delta x, n\Delta t)] - k[s(j\Delta x, n\Delta t) - s((j-1)\Delta x, n\Delta t)] \\ &= k[s((j+1)\Delta x, n\Delta t) - 2s(j\Delta x, n\Delta t) + s((j-1)\Delta x, n\Delta t)]. \end{aligned} \tag{8}$$

The equations of motion become

$$\begin{cases} s(j\Delta x, (n+1)\Delta t) - s(j\Delta x, n\Delta t) = v(n\Delta t)\,\Delta t, \\ v((n+1)\Delta t) - v(n\Delta t) = \dfrac{1}{m}\, F_{j\Delta x,n\Delta t}\,\Delta t, \end{cases} \tag{9}$$

with $j \in -N/2, \dots, N/2$ and $n \in \mathbb{Z}$. Given the displacements $s(j\Delta x)$ and the velocities $v(n\Delta x)$ at time $t = 0$, the solution of the recurrence equations return displacements and velocities of any point mass at any time $n\Delta t$.

Students are required to examine the way evolution depends on the initial conditions and on the dynamical parameters, and to investigate traveling and standing solutions of the wave equations. Examples may be found at `http://www.les.unina.it/?page_id=4784`.

3. – Comments and conclusions

The aim of this article is to propose a unifying instructional strategy for the study of dynamic systems for high-school students. The treatment of the theoretical and computational aspects of our proposal is oriented to unify the teaching methodology. The technical difficulties involved in the rigorous treatment of differential equations may be overcome by the use of recursive relations that are easily implemented on a spreadsheet. In our opinion, the advantages of such a presentation are:

- Dynamical laws are not simply *given*, but *analyzed* on the basis of their effectiveness in modelling the step-by-step evolution of physical systems. This fact is first *seen* by a (historically motivated) geometrical approach, and then *translated* into the language of vectors.

- Students are enabled to *autonomously* study the dynamical features of complex systems whose analysis is generally considered too advanced for a high-school or even undergraduate audience (*e.g.*, many-body gravitational systems, large amplitude pendulum oscillations, etc.). The numerical computation of real-time experiments makes it possible to address topics that are usually studied in university courses. They are provided with some preliminary skills for the study of how solutions depend on the initial conditions and other dynamical parameters, a key element for the study of non-linear and chaotic systems.

- The method is extendible to stochastic and even quantum dynamical laws. As an example, the dynamics of a quantum particle can be described by means of a discretized Schrödinger equation on a lattice, presented as a wave equation for a two-dimensional vector field, avoiding the use of complex variables that are not yet mastered by high-school students.

- In Italian secondary schools, the mathematics and physics programs are not aligned. In particular, infinitesimal analysis is addressed in the last year and in physics textbooks, the concepts of limit, derivative and integral are often hastily introduced. The approach we propose, based on discretization, develops starting from middle school and refers to the experience of Emma Castelnuovo([8]), and this allows us to treat the laws of evolution of physical systems in a coherent way already in the first years of secondary school.

- The proposal highlights how in the manual and automatic data taking (with in-line sensors) and in the animations and simulations created with "professional" systems, the models are all discrete and in particular the use of the spreadsheet can help in acquiring articulated skills on numerical aspects of modelling. A significant element of the proposal is the possibility of dealing with non-linear and relatively complex phenomena with the same approach, giving the possibility of grasping the meaning of the schematizations presented in the standard examples of textbooks. In fact, when working with analytical solutions, physics books are forced to deal with standard cases which are not always suitable for modelling the phenomena encountered in the laboratory. Moreover, they do not allow working with fundamental concepts and phenomena for the construction of theories (*e.g.*, non-linear oscillators, three-body problem, etc.).

- The introduction to dynamical problems by the use of geometric diagrams coupled with spreadsheets may be an effective prevention of the "black-box mindset" when one moves to more sophisticated object-oriented simulation software. Here students have full control over specific geometric and dynamical properties, evolution rules, interactions among objects and generic initial conditions. Programs then compute and display the successive evolution even for systems of great complexity. These are important resources that allow students to explore on their own the predictions of theoretical models, but too often the sophisticated computational protocols behind the software remain concealed and are never seen by students not experienced in programming languages. Our proposal also aims to bridge this gap, using simplified forms of recursive calculation in order to make as clear as possible how computational methods work.

The proposal outlined in the paper was presented in the university course "Didactics of Physics" addressed to second-year students in physics and mathematics, in various teacher training courses. In 2021-2022 school year, it was tested in middle-school and high-school classes with great interest from the students and teachers of the classes involved([9]). The activities carried out with high-school students were analyzed with the teachers of mathematics and physics in a training course organized by our group. In the discussions, for the part concerning gravitational systems, the teachers underlined the absence of teaching materials and textbooks which deal in a unitary way with the discrete modelling of laws of evolution for high-school students.

([8]) See `https://en.wikipedia.org/wiki/Emma_$Castelnuovo$` for an introduction to the life and work of Emma Castelnuovo. See also `https://www.mathunion.org/icmi/awards/emma-castelnuovo-award`.

([9]) See `http://www.les.unina.it/?p=3563` for additional material.

REFERENCES

[1] http://www.indire.it/lucabas/lkmw_file/licei2010/indicazioni_nuovo_impaginato/_decreto_indicazioni_nazionali.pdf.

[2] Sjøberg S., *Science and technology education current challenges and possible solutions*, in *Innovations in Science and Technology Education*, Vol **8**, edited by Jenkins E. (UNESCO, Paris) 2002.

[3] Vistnes A. I. and Hjorth-Jensen M., *Numerical Methods as an Integrated Part of Physics Education*, in *9th Workshop on Multimedia in Physics Teaching and Learning (Graz, Austria 9-11 September 2004)*, arXiv:physics/0505116 [physics.ed-ph] (2005).

[4] Weber J. and Wilhelm T., *Eur. J. Phys.*, **41** (2020) 034003.

[5] Benacka J., *Eurasia J. Math., Sci. Technol. Educ.*, **12** (2016) 947.

[6] Landau R. H., *Comput. Phys. Commun.*, **177** (2007) 191.

[7] Fauzi A., *J. Phys.: Conf. Ser.*, **909** (2017) 012056.

[8] Misner C. W., Cooney P. J. and Kinderman J. V., *Am. J. Phys.*, **60** (1992) 94.

[9] Quale A., *Phys. Educ.*, **47** (2012) 355.

[10] Della Corte A. and Russo L., *La bottega dello scienziato. Introduzione al metodo scientifico* (Il Mulino, Bologna) 2016.

[11] Graffi S. and Degli Esposti M., *Fisica matematica discreta* (Springer Science & Business Media) 2003.

[12] Levrini O. and Fantini P., *Sci. Educ.*, **22** (2013) 1895.

[13] Ogborn J., Hanc J. and Taylor E., in *Proceedings of the GIREP Conference 2006: Modelling in Physics and Physics Education* (University of Amsterdam) 2006.

[14] Guicciardini N., *Isaac Newton. Filosofo della Natura, interprete della Scrittura, cronologo degli Antichi Regni* (Carocci, Roma) 2021.

[15] Feynman R., *The Feynman Lectures of Physics* (California Institute of Technology) 1989.

[16] Brackenridge J. B., *The Key to Newton's Dynamics. The Kepler Problem and the Principia* (University of California Press) 1995.

[17] Hamilton W. R., *Proc. R. Irish Acad.*, **3** (1846) 344.

[18] Goodstein D. and Goodstein J., *Feynman's Lost Lecture: The Motion of Planets Around the Sun* (W. W. Nortonm, New York) 1996 (Trad. italiana, *Il moto dei pianeti intorno al Sole. Una lezione inedita di Richard Feynman* (Zanichelli, Bologna) 1997).

[19] Israel G., *La visione matematica della realtà. Introduzione ai temi e alla storia della modellistica matematica* (Laterza, Roma-Bari) 1996.