On Dieks against the Received View

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Abstract

Dennis Dieks addressed some criticisms of the so-called Received View (RV) of non-individual quantum objects in a series of papers. His main concern is that the RV doesn't fit the *practice of physics* since in some situations the physicist assumes that quantum objects can be treated individually, imitating standard objects (individuals) in classical physics. In this paper, we revise his argumentation, showing that it involves some misunderstandings regarding the objectives of the RV.

Dieks also proposes an Alternative View (AV) which he thinks is more in accordance with the way physicists proceed. We argue that the AV is not conflating the RV, but is complementary to it, namely, substitutes it when quantum objects are sufficiently apart and can be treated as obeying classical logic. Thus, from the point of view of the practice of physics, in most cases, we can opt for the Alternative View, but the RV is more adequate when we are looking for logical and foundational analyses.

Keywords: identity, individuality, non-individuality, quantum objects, Received View, Alternative View, Dennis Dieks

> If you wish to converse with me, define your terms.
>
> Voltaire

1 Introduction

Dennis Dieks is one of the top philosophers of physics of the moment. His positions and opinions are always centred on fundamental topics and are led with care and competency. In particular, he advanced a series of papers, alone

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or with collaborators, against the so-called Received View (RV) of quantum non-individuality which sees quantum objects as *non-individuals*, this being understood as entities to which the *standard theory of identity* (STI) of classical logic should not apply [FK06]. Dieks is not clear about the meaning of the term 'identity' he uses in his works, even when he speaks about the RV. Surely this causes some misunderstandings about the precise claim of the RV that it is a specific notion of identity ascribed by STI which is supposed not to hold in the quantum domain. The reasons will be recalled soon.[1]

In this paper, we revise his main points and try to make clear what is going on with the RV, showing that his view (termed the 'Alternative View', AV) is not *against* the RV but can be considered as *complementary* to it, being applied in certain situations more in accordance, as he says, with the practice of physics while, in contrast, the RV would hold in general from the logical and foundational point of view. In going to a view adapted to the 'practice of physics', we argue that supposing that it is possible to attribute identities to the considered particles requires a move that falls in the same logical mistake someone does when 'dispenses with' infinitesimals in some application of the earlier calculus ('the move' being to neglect the interference terms).

To start, let us say what *we* understand by identity. We say that an object *has identity*, or that it is an *individual* if it obeys STI (see below and our informal definition of an individual in the next section). This implies at least the following things that are important for the discussion: (i) any two or more objects obeying STI are *different* and this entails that (ii) being different, the objects can be distinguished by some *monadic* property, for instance their 'identities' summed up by the following definition; the identity of the object a is given by the predicate $I_a(x) := x \in \{a\}$. When objects can be distinguished by a monadic property, we call them *absolutely discernible* according to the usual literature; when they cannot be discerned in any way, we say that they are *completely indiscernible*. Notice that we are in a set theory such as the ZFC system which encompasses STI. Soon we shall discuss why some philosophers don't accept this predicate as providing the identity of a; (iii) fundamentally, if in a context we exchange an object by another, even quite 'similar' to it (in some sense of this word), the context changes.

Of course, when speaking of the identity (or of the lack of) of some objects, someone would explain what she understands by such a concept. The informal dictum that 'identity is *something* an object has which distinguishes it from something else' is vague and redundant to be used in logical analyses. We need a *theory of identity*, and here we assume STI.

STI is formalized in first-order languages with a primitive binary relation '=' by two postulates, namely, reflexivity ($\forall x(x = x)$) and substitutivity, that is, for *any* formula α, $\forall x \forall y(x = y \to (\alpha(x) \leftrightarrow \alpha(y)))$, with the usual restrictions

[1]Anyway, is there some 'other identity'? The informal characterisation that identity would be *that thing* an individual has that makes it what it is in distinction to any *other* thing, or that it shares just with itself is quite vague. Furthermore, we repute attempts such as Geach's *relative identity* [Gec67] and Quine's 'definition'(borrowed from Hilbert and Bernays [HB34, §5] as nothing more than indiscernibility relative to the predicates of the language [Qui86, Chap.6].

(see [Men97, p.95]). If our theory is some 'standard' set theory such as the ZFC system, we add an axiom of extensionality $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$, and a further axiom $\forall x \forall y(\forall z(x \in z \leftrightarrow y \in z) \rightarrow x = y)$ if there are atoms involved (the ZFA set theory — see [Sup72]).

In higher-order languages, identity can be defined in the Whitehead-Russell's style, here done for second-order languages. This is Leibniz Law: $x = y := \forall F(F(x) \leftrightarrow F(y))$, where x and y are individual variables and F is a variable for properties of individuals. Notice that none form of *haecceity* is assumed, that is, all we have are the individual's properties and relations.

This has consequences, as suggested above. Every entity described by such a mathematical framework encompassing STI is an individual and can be discerned from anything else by a monadic property, that is, *absolutely*. Thus, if we wish to admit the existence of completely indiscernible things, that is, things that cannot be discerned in any way, we cannot use a mathematical framework encompassing STI (but see the subsection 7.2 for alternatives).

Let us fix some terminology and suppositions. Dieks agrees that the RV is grounded on a non-classical mathematics termed *quasi-set theory*, '$\mathfrak{Q}$' for short. Using this theory, we can express in a more precise way what the RV claims. This theory encompasses two kinds of ur-elements, the M-atoms, which behave as the ur-elements of ZFA (the Zermelo-Fraenkel set theory with Atoms [Sup72]), and the m-atoms, to which STI does not apply. This means that if either x or y are m-atoms, then the expressions of the form $x = y$ or $x \neq y$ have no meaning; in particular, the theory does not provide any meaning for $x = x$.[2] There may exist m-atoms of 'different kinds', which may be *distinguishable* among them, $x \not\equiv y$ in symbols. The basic primitive notions are membership ('$\in$') and indistinguishability ('$\equiv$') and this one has the properties of an equivalence relation. But it is not a congruence, since $x \in y$ and $x \equiv z$ does not entail that $z \in y$ (for details, see [FK06, dBHK23]). The universe is populated by such atoms and the quasi-sets ('qsets' for short) and the postulates extend those of the theory ZFA. A concept of identity, termed 'extensional identity', '$=_E$', is defined for both M-atoms, when they belong to the same qsets and to qsets when they have the same elements. Some qsets may have a cardinal, termed its *quasi-cardinal* in a way that the existence of a quasi-cardinal does not entail that the elements of the qset are discernible (this will be considered later — see section 7.4). The Axiom of Weak Extensionality (WEA) says that if qsets A and B have 'the same quantity' (expressed using the quasi-cardinals) elements 'of the same sort' (that is, indiscernible among them), then they are indiscernible ($A \equiv B$). Extensionally identical things (or just 'identical things') are indiscernible, but not the other way around. From now on we shall use these notions. For details about this theory, see [FK06, dBHK23].

[2] This does not entail that we cannot define an identity for the m-atoms, as we show in subsection 7.3, but $\mathfrak{Q}$ does not assume that.

2 The Received View

The RV is not occupied with 'the practice of physics' in the same sense that the definitions of limit and continuity in Calculus were not occupied with the practice of the engineer or the applied mathematician when informally using the notion of infinitesimal (say when she speaks of an 'infinitesimal element of volume' for instance). Below we revise this analogy.

The RV treats some entities as *individuals*, and this is informally characterised as follows: an individual is something that (i) is a unity of a kind, say a table, a person, a pen; (ii) it has *genidentity*, that is, it can be re-identified *as such* individual in different contexts. This second requisite is fundamental and usually forgotten by philosophers (see for instance [Cau14]). Julius Caesar was a man and *the same man* when in Rome and when in Egypt, at least we usually accept that.[3]. If something obeys STI, it is an individual in this sense and can be said *to have an identity*; the natural number two (once defined for instance in the sense of von Neumann) is the same natural number two when we list it as an element of the set of the even natural numbers and when we refer to the set-theoretical successor of one, that is, the set $1 \cup \{1\}$, being $1 = \{\emptyset, \{\emptyset\}\}$.

Quantum objects can be *isolated* for instance in quantum traps, but this does not make them individuals. As David Hume has reminded us, "One single object conveys the idea of unity, not that of identity." [Hum88, p.200]. Really, quantum objects lack (ii); once one has left the trap, it will never more be identified as *that* entity of before. Usually, the notions of *identity*, *individuality* and *individuation* (or 'isolation') are conflated and taken as being synonymous while they should be discerned on each other (see below, subsection **??**). For more details about these notions and their distinctions, see [KAB22].

The RV has been accused of entering *factorism*, a notion to be discussed below. Roughly speaking, this is characterised by assuming that the labels we use in the quantum formalism name particles and not only the Hilbert spaces which comprise their state vectors. As we shall see, in our opinion this is quite natural, for quantum theory deals with particles, despite referring to their states; as Sunny Auyang has claimed, "physical theories are about things" [Auy95, p.], obviously, *physical things*. So, despite the initial entities may be fields or strings, the target entities, those that enter in certain states and are accelerated in the particle accelerators, are *particles* — more on this in what follows, but it could be recalled the *metamorphoses* that the concept of particle passed from classical physics to the Standard Model of particle physics, as reported by B. Falkenburg [Fal07, Chap.6].

For the sake of clarity, some general remarks would be recalled:

1. **The objectives of the RV —** The RV is occupied with the logical and metaphysical foundations of quantum theories. The main book where such a

[3] David Hume, for instance, claims that we *attribute* identity to an object when we observe that there is a continual succession of perceptions in our mind [Hum88, p.65]. His Principle of Individuation "is nothing but the *invariableness* and *uninterruptedness* of any object"' — *idem*, p.201.

view is outlined has a subtitle 'A historical, philosophical, and formal analysis' [FK06]. This should be enough to show its main finalities: a formal (logical) analysis of the view that quantum objects can be viewed as *non-individuals*. Notice in addition that the RV does not pray for such a view, accepting that there are alternatives and talks about the non-individuals view as one of the possibilities. In the same mentioned book, it is argued that quantum objects can also be viewed as *individuals*, provided that only some states are available. This is more or less in the direction Dieks suggests the physicist goes in her usual 'practice'. But, as we shall enlighten below, to be careless with the underlying logic is a fault regarding the logical foundations of any physical discipline, despite it can be dispensed with in the first moment during applications.

2. **Why to question STI —** The criticism of the STI assumed by the RV has a reason. According to this theory, whenever we have more than one object, they are *different* ('$a \neq b$') and this means that a and b can be discerned by some property. That is if $a \neq b$, then there exists P such that $P(a)$ but $\neg P(b)$ or the other way around. There is no escape to this conclusion, which is imposed by logic, once one assumes a logic encompassing STI. But take bosons in a bosonic condensate, a BEC (Bose-Einstein Condensate). They may be milliards, all in the same quantum state. It is assumed by the quantum theory that *there are no differences among them*, and as far as quantum mechanics works, no differences among them can be found. The obedience to quantum statistics (in this case, Bose-Einstein statistics) provides also an argument favouring the RV: without assuming substratum or something else beyond the properties, how can something obey such statistics under the validity of STI? In our opinion, this makes no sense: if it is assumed that bosons (or other quantum entities) cannot be discerned *in any way*, how can STI hold to them? Notice that we agree with the claim that the physicist can, *momentaneously*, treat them as individuals endowed with identity, but such an identity has no sense in the wide aspect and should be understood as just a *mock identity*, as advanced (and acknowledged by Dieks) by Toraldo di Francia [TdF76, TdF98] (see below).

3. **Weak discernibility and discerning properties —** Some philosophers have proposed that in certain situations quantum entities (both bosons and fermions) cannot be discerned 'absolutely', that is, by a monadic predicate, but just by an irreflexive and symmetric relation. They call it *weak discernibility* (see [DL22] for references and discussion). We have discussed elsewhere such a proposal and will not revise it here, but just make some general remarks; for details, see [Kra10]. The fact is that if the underlying logic encompasses STI, as classical logic these philosophers are assuming does, there is no escape to the conclusion that for any object y (say in the universe of sets and atoms) we can define the property 'identity of y' by positing something like $I_y(x) := x \in \{y\}$. Since the unitary set $\{y\}$ can be assumed to exist for every y, then y is the only object having such a property. There is no surprise concerning this, for it is exactly what STI says. However, the mentioned philosophers are not satisfied with such a logical imposition. They refuse that a 'logical property' can individuate a thing, saying that the discerning property must be 'physical', something

'measurable'. This supposition introduces a lot of other difficult questions, such as (a) what is a 'physical property'? (b) what is 'measurement'? (c) why such a bias to avoid logical implications in a theory? — see below, section 3. There are no clear answers to these questions. We think that by 'measurable property' we can understand whatever property to which we can ascribe the epithet 'true' or 'false' when compared with some value or set of values (for instance, whether a certain observable has a value in some specific Borelian set). For instance, 'the volume of that portion of water is half a litre' can be true or false depending on the portion of water. But also 'the spin of the electron in the z direction is UP' and 'x belongs to the unitary set of y' are 'measurable' in this sense. So, we don't see any reason to restrict the properties (hence the formulas) involved in the axioms of STI. As Shoenfield insists, "the symbol '$\forall$' [used in the definition of identity in STI] means *for all*" [Sho67, p.13], and we could add 'and not *for some*'. Thus, when we say (even if in the metalanguage) that in being identical x and y satisfy all the same formulas, we are not restricting the phrase to 'some formulas' (or predicates).

On the contrary, the theory of quasi-sets may be the right place to formalise the weak discernibility claim. We can suppose a qset with two indiscernible m-atoms and with quasi-cardinal two so that there is an irreflexive but symmetric relation holding between them. No monadic property is supposed to exist. For instance, the classical example of the two electrons of a neutral Helium atom and the property 'to have spin opposite to'. The idea that electrons are so discerned but are not discerned by a monadic property can be formalised in $\mathfrak{Q}$, but not in ZFC.

4. **Isolation —** We have remarked already that with regard to quantum objects being distinct in certain situations conflates the notions of identity and *isolation*, or *idividuation*. Identity is a logical notion, given by some 'theory of identity' such as STI; individuation is an epistemological notion of taking something in a situation that can give us *the impression* that we are facing an isolated individual with an identity. For instance, a case also explored by Dieks speaks of isolation and not of identity. Quantum objects located in distant places (say the South Pole and the North Pole) are *isolated* or *individuated*, being able to be momentaneously treated as individuals, but they are not individuals since they do not satisfy re-identification, something required by something 'having identity'. Later we shall further analyse this example.

5. **Factorism —** Dieks criticizes the RV also because this view presupposes that quantum objects are the basic tenet of the theory and labels such as '#1', '#2', etc are given to these entities, causing what he calls the *factorizing* account (a term borrowed from A. Caulton [Cau14]). According to this view, labels such as '#1' and '#2' refer to particles and this would cause them to be able to be *absolutely discerned* by a monadic property. As Dieks and Lubberdink say, "the factor space labels [in the N-fold tensor product of Hilbert spaces] should *not* be thought of as referring to single particles" and the tensor product, in their interpretation, do not have the form of a concatenation of one-particle states

[DL22]. They attribute such a 'complication' to the symmetrization postulates:

> "As the symmetrization postulates apply universally and globally to all particles of a given kind (. . .), 'factorists' must hold that each single electron [say] is equally present at all positions in the universe."

Not at all! The 'identical' particles sharing the Bell state (their equation (1))

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\Big(|L\rangle_1 \, |R\rangle_2 \pm |R\rangle_1 \, |L\rangle_2 \Big) \tag{Bell}$$

are not both at L (left) together with all other particles of the universe (of the same kind). They share that state *while a measurement is not made*, but as a vector of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, this does not imply that we are identifying *that* particle (say, 'Peter') as the particle whose states are represented by vectors at $\mathcal{H}_1$ and which is at L. No, the tensor product just says that we have *one* particle in each position (or around it) without saying which one is it. The identification makes no sense, as is well known. In fact, they add that "[i]t is therefore impossible to individuate the 'factorist particles' via different physical characteristics."

Well, while sharing the Bell state there is no way to individuate them. By the way, we think that this is one of the main results of quantum physics! the identification, say that one is at L, comes only with the measurement. Maybe this emphasizes the role of the underlying logic (or mathematics); the two particles are *two* and there are no *physical* characteristics that discern them, but there are *logical* ones once we remain with STI. But if we wish, as it seems clear, to maintain that *before measurement* the particles are 'really' indiscernible, then STI must be placed aside.

In general, the authors forget that, even implicitly, they assume something like ZFC since they seem to reason with the classical logic canons. Thus, in a system of N quantum systems, the state space is the 'factored' tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \mathcal{H}_N$ of the Hilbert spaces of each system. The indices $1, \ldots, N$ are labelling both the H-spaces and the particles. But this is not so in the $\mathfrak{Q}$-spaces [DHK08, DHKK10] built in $\mathfrak{Q}$. There, there are no particle labels that significantly, tag the particles. Maybe this would satisfy Dieks et al.

The RV intends to consider entities of the kind just referred to; it assumes that, whatever their origin, they are better characterized as non-individuals, even if resulting from quantum fields or strings. We reinforce that the RV does not refuse the alternatives that the literature presents us, Dieks' inclusive, but proposes a different approach. If Dieks argues that the RV is *wrong*, then there is no other alternative than to say that it is so wrong as the approach he proposes.

6. **Isolation does not entail identity —** In the RV, 'identical particles' (in the physicist's jargon), that is, indistinguishable particles of the same species, can have different properties, say when localized in different traps, or when having different directions of spin. As seen before, Dieks suggests that the RV entails that all such particles have exactly the same properties. We never found where

French and Krause (some of the main proponents of the RV) are saying that or making such a hypothesis. The two electrons in a neutral Helium atom are fermions and have opposite values of spin in a given direction, so cannot have all the same properties and values. Their indistinguishability results because they share the same entangled state, a state that cannot be separated into two states, one in each factor of the tensor product, as the Bell state above. If there is some quantum system (a positron, say) in a quantum trap. it should be regarded as an *isolated* entity, not as an individual. The notion of 'mock' (or 'fake') identity is useful here. The non-individuality is linked more with the impossibility of re-identification, but *isolation* is possible, as discussed in the case of Hans Dehmelt's positron 'Priscilla' (see [Die23, Kra11]). Just to summarize, Priscilla was a positron trapped in Dehmelt's laboratory and supposedly, she had an identity because of that. But this is not so; recalling David Hume once more, a single object gives us the idea of unity, not of identity [Hum88, p.200]. If Priscilla were an individual, she would carry her identity whenever she goes, but this is not what happens; as we know from quantum theory, quantum entities don't have *genidentity*, something acknowledged by Dieks himself. Individuals can have proper names which serve to re-identify them in other contexts; these proper names act as *rigid designators* in Kripke's sense, something not available in quantum physics [DCTdF93].

6. **The bank account analogy —** Dieks goes further. He criticizes the theory of quasi-sets by stating that to use such a theory to discuss the bank account analogy would be "an extravagant decision". Not at all! We think that quasi-sets can explain nicely what happens with such an analogy, something that a standard set theory cannot do. Let us explain, even if briefly. The bank account analogy was introduced by Schrödinger [Sch98] and used for instance by Paul Teller [Tel83] to exemplify quantum non-individuality. Suppose I have €100 in my bank account. Is there a sense in asking for my particular euros, that is, to suppose that there are in the bank some particular euros that are exactly my hundred euros? Not at all, and here enters an important feature of the RV: what import are the *qualities* (of the things, that is, their *kinds*), say 'euros', and their *quantities*, say hundred. No particular euro exists as being mine. The same happens in chemistry and, in a more general situation, with quantum theory. In a water molecule H_2O, what import is that we have *two* Hydrogen atoms and *one* Oxygen atom distributed in a certain way, and not which particular atoms we have. *Kinds and quantities.* It is precisely this which the theory of quasi-sets enables us to consider. Symbolically, we could write in terms of quasi-sets the above molecule as $\langle H, O; 2, 1\rangle$, while my account is associated with $\langle euros; 100\rangle$. If we have two accounts in two distinct banks, one in New York with \$ 200 and another in Paris with €100, we can write $\langle dollars, euros; 200, 100\rangle$. No identification, no identity.

Quasi-set theory enlarges standard set theory (say the ZFC system) by enabling us to consider collections (quasi-sets) with a cardinal, but with no associated ordinal, as explained already. It is a theory that goes in the direction of the problem posed by Yuri Manin when he asked for a 'more general' theory of collections to cope with indistinguishable quantum objects (see [Man76, FK06] and section 7.4 below).

8. **Permutations —** In a comment on quasi-sets, Dieks claims that "if labels cannot be defined, permutations as ordinarily defined makes no sense". This is correct; a quasi-set with N indistinguishable elements of kind k is *indistinguishable* (and not 'identical') to any quasi-set with N indistinguishable elements of the same kind k. This makes sense the fact that there may exist several 'distinct' BECs with elements of the same kind and estimated to have the same quantity of elements (termed 'N').

The idea can be extended to the situation with more kinds of things so that we can say that two sulfur acid molecules are indistinguishable $H_2SO_4 \equiv H_2SO_4$ in the quasi-sets notation. A permutation can modify things in this case; although Dieks did not comment on this point, we suppose it is relevant and deserves mention. In the acid molecule, of course there is no sense in permuting Oxygen atoms among them or Hydrogen atoms among them. But we can modify the *format* of some molecules by re-arranging their components and getting different things. This is the case, for instance, when isomers are considered, that is, substances that have the same molecular formulas with the same number of atoms of each element, but the atoms are arranged differently in space. To consider the *form* of something constituted by different kinds of things and their respective quantities, our suggestion is to develop a *quantum mereology*, something not achieved yet (see [Kra12, Kra17, HJ23]). Thus, by specifying how a whole is formed by its parts, maybe we arrive at a way to approach the form of an entity.

3 The role of the underlying logic

Any theory has an underlying logic, even if it is not made explicit. Usually, scientists assume that what is called *classical logic*, or at least a part of it as their basic logic, perhaps because the ways we reason are more conformed to such a logic (of course we should say the opposite: classical logic was created to cope with the standard ways we reason). By classical logic we understand the standard 'classical' first-order predicate logic with or without equality, some subsystem of this calculus such as the classical propositional logic and even those system of *Magna Logica*, encompassing higher-order 'classical' systems and set theories. Categorical logic ([Hat82]) can also be included in this schema. What characterizes a logical system as 'classical' is the obedience to some basic principles, such as the following ones: the excluded middle law, the law of non-contradiction, the explosion rule, the principle of identity, the double negation rule, Peirce's law, the 'classical' reductio at absurdum (in distinction to the intuitionistic reduction at absurdum, which by the way holds also in classical logic), some form of Leibniz's Principle of the Identity of Indiscernibles, an axiom of extensionality in set theory, compositionality (the truth value of a complex sentence is a function of the truth values of its component formulas), etc. In particular, classical logic (in whatever form) encompasses a *theory of identity* which we term the Standard Theory of Identity (STI) discussed earlier. Hence it is a theorem of any theory based on classical logic that if $a \neq b$, then there exists F (either a predicate or a set) such that $F(a)$ ($a \in F$ in the case of set theories) and $\neg F(b)$. Thus, a and b can be discerned *absolutely*, if by this term we understand the existence of a *monadic* predicate obeying the indicated conditions. So, one should pay attention to this, to be emphasised below: even if two particles represented in a mathematical scheme comprising STI cannot be discerned by what philosophers wish to call 'physical properties', they are discerned (absolutely) by 'logical' properties. Since the theorems of the underlying logic are also theorems of the physical theory, we need to conclude that, with STI, even 'identical' particles are discerned absolutely.

Really, the role of the theory's underlying logic is to give an account of the theory's acceptable inferences (deductions in the case of deductive logics), and all theorems of the underlying logic are theorems of the theory itself. So, if the theory's underlying logic is classical logic, for any formulas A and B, the formula $A \to (B \to A)$ is a theorem, but it is not if the logic is some suitable quantum logic [DCGG04]. So, we may say that the theorems of a theory T can be divided up into two classes: (i) the *logical* theorems, which do not make use of any specific notion of the theory, and (ii) the *specific* theorems, which are typical of the considered theory. So, take T as classical particle mechanics as formulated by McKinsey, Sugar and Suppes in 1953 (see [Sup02]). Kepler's laws are theorems of T and belong to the second group of theorems, but the above $A \to (B \to A)$ is also a theorem of such mechanics since it is supposed to be settled on classical set theory (hence encompassing the classical propositional calculus). What respect to quantum mechanics, let us assume a formulation such as that presented in some standard book like [Ish95].

In this sense, a theory T can be viewed as an ordered pair $T = \langle F, \mathscr{M} \rangle$, where F is a 'formalism', which means the mathematical counterpart of T, and $\mathscr{M}$ is a class of mathematical structures, the *models* of T. We are not considering here 'models' other than set-theoretical structures, such as toy models or mockups. Furthermore, we assume that at least in principle every scientific theory can be axiomatized by a set-theoretical predicate [Sup02].

Thus we may assume that F is formulated in the language of a set theory such as the ZFC system, which is enough for most of the physical theories that are important here, enlarged by specific (proper of the theory) concepts, such as 'electron', 'wave-function', and so on. Being mathematical structures, the models of T must be built *in some place*. That is, we need a meta-theory to give rise to models; after all, $T = \langle F, \mathscr{M} \rangle$ is a *set* in some set theory. If we are modelling also ZFC, what would happen if we assume that the theory's logic is precisely ZFC, then the models need to be constructed in a strong theory, such as the KM (Kelley-Morse) system [Kel55, Rub67].

The choice of the theory's underlying logic, as the choice of the theory's primitive notions and axioms, is a task of the scientist. But once chosen, the axioms, both of the logic and of the theory, become *normative*: they determine what can be accepted as legitimate. If something goes wrong, the scientist needs to revise her assumptions. But before this step, the scientist is free to choose the logic and the postulates she wants, and despite there being no specific rules in this direction, in general, there is an underlying metaphysics being assumed, even if unconsciously. For instance, in classical physics one makes assumptions, such as (i) determinism, (ii) individuals, (iii) impenetrability, (iv) the existence of trajectories, (v) locality, (vi) separability. all of this in a certain way subsumed in a 'classical' setting.

4 The infinitesimals analogy

It is well documented that Newton's fluxions and fluents and Leibniz's infinitesimals led to contradictions [DB15, Vic13]. Let us summarise a little using updated language. The idea is to calculate what later was called the derivative of a function $y = f(x)$ (a fluent). In Leibniz's terms, this would require calculating the quotient of the increments given to the independent variable x and that one got by the dependent variable, that is, $\frac{\mathrm{d}y}{\mathrm{d}x}$. Take $y = x^2$ as a paradigmatic case. Given an infinitesimal increment dx to x, we get $y + dy = (x + dx)^2$, that is, $y + dy = x^2 + 2xdx + (dx)^2$, so $dy = 2xdx + (dx)^2$. Dividing both terms by dx (which is supposed to be not null since it is an increment), we get $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + dx$. And here is the tricking step: since dx is *arbitrarily small*, we can dispense it and arrive at the derivative, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$. Of course, there is a contradiction since dx was taken both as not zero and as zero. The right way to provide such a calculation came only later with Cauchy and Weierstrass with the notion of limit.

The practical results of the Calculus were correct despite this fact. Still today an engineer may use infinitesimal elements of areas or volumes in her reasoning in the old style, sometimes even by ignoring the right definitions in terms of limits or the existence of Non-Standard Analysis, introduced in 1960 by A. Robinson, who gave a re-birth to the notion of infinitesimals without the old problems (see the above references). Berkeley, who made a serious criticism of the use of infinitesimals, acknowledged that the results of the Calculus are right and did not criticise them, *but its logical deficiencies*. This is important to our case, as we shall see now. Anyhow, it is clear that to neglect the infinitesimals constitutes a logical mistake.

Dieks argues that sometimes physicists consider two quantum entities as described by independent wave functions ψ_1 and ψ_2 when they are sufficiently apart. In this case, as he recalls, all happens as if they were two distinct and isolated entities behaving as classical physics says. Let us summarize the argument given in [DCTdF93] with a simplified example. We shall not use the bra-ket notation here for simplicity. Suppose we have two elementary particles of the same kind located at different points A and B, say the North Pole and the South Pole of Earth. Being x_1 and x_2 their coordinates,[4] let $\psi_A(x_1)$ and $\psi_B(x_2)$

[4]Notice that the coordinates do not provide identity to the particles, but just say that one is in the North Pole while the other is in the South pole; the identity of the particles don't matter, mainly if they are of the same kind.

the wave-functions of the particles. Then the joint probability amplitude for finding the first particle at x_1 and the second at x_2 might be thought as being done by the tensorial product $\psi_A(x_1)\psi_B(x_2)$, but it is not! Since the particles are indiscernible, nothing different would get if they are exchanged, that is that the joint probability amplitude would be $\psi_A(x_2)\psi_B(x_1)$, which is *different* from the first product once the tensor product is not commutative. As Dalla Chiara and Toraldo di Francia said, "this would go against the indistinguishability principle".[5]

The acknowledged right vector for describing the joint probability amplitude is

$$\psi_{12} = \frac{1}{\sqrt{2}}\Big(\psi_A(x_1)\psi_B(x_2) \pm \psi_A(x_2)\psi_B(x_1)\Big),$$

where the plus sign holds for bosons and the minus sign holds for fermions. The joint probability density is then given by

$$\left\|\psi_{12}\right\|^2 = \frac{1}{2}\Big(\left\|\psi_A(x_1)\right\|^2\left\|\psi_B(x_2)\right\|^2 + \left\|\psi_A(x_2)\right\|^2\left\|\psi_B(x_1)\right\|^2 \pm 2Re\left\langle \psi_A(x_1)\psi_B(x_2)\middle|\psi_A(x_2)\psi_B(x_1)\right\rangle\Big),$$

where the last term $2Re(\ldots)$ is the *interference term*. This term, for the 'practice of physics', can be eliminated, since the overlap of the two wave functions becomes appreciable only when the distance between the particles is not much larger than the de Broglie's wavelength. As Dalla Chiara and Toraldo di Francia emphasize,

> "This is the reason why an engineer, when discussing a drawing, can *temporarily* make an exception to the anonymity principle[6] and say for instance: 'Electron *a* issued from point *S* will hit the screen at *P* while electron *b* issued from *T* hits it at *Q*.' But this mock individuality of the particles has very brief duration. When the electron hits the screen (...) it meets with other electrons with substantial overlapping, and the individuality is lost. In fact the de Broglie wavelength of an electron inside an atom is on the same order of magnitude as the atomic diameter."

This shows that the supposition that the interference term can be neglected is similar to the supposition that infinitesimals can be dispensed with. The results in quantum physics, so as those in the Calculus, are right (as far as we know), but from the logical foundational point of view the logical mistake is evident.

[5]This principle states that for all vectors (states) $|\psi\rangle$, all operators $\hat{A}$, and all particle label permutation operators P, we have $\langle\psi|\hat{A}|\psi\rangle = \langle P\psi|\hat{A}|P\psi\rangle$, that is, the expectation values are the same before and after a permutation.

[6][According to them, quantum physics is the land of anonymity, where proper names make no sense since they to not play the role of rigid designators, as it would be if the involved entities were individuals.]

We emphasize here, again, that the Received View is not occupied with the practice of physics, but with its logical foundations. This is why we think that the interference term cannot be dispensed with in such an analysis.

5 Dieks' proposal: the Alternative View

Dieks et al. propose an Alternative View (AV) to substitute the RV which, they say, is more in conformity with the practice of physics. In this section, we revise his main claims that conduce to such a view and add some 'comments'.

Dieks accuses the RV of using labels to name the particles (in a join system with N of them). The labels $1, 2, \ldots, N$ serve to name the corresponding Hilbert spaces in the tensor product but also name the particles. He considers anti-symmetrized states (we use Dieks' numeration for the equation below):

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\Big(|\psi\rangle_1 |\phi\rangle_2 - |\phi\rangle_1 |\psi\rangle_2 \Big) \tag{2.1}$$

He says that such equations (this one and similar) are so that "the symmetry of [the equations] implies that each of those particles is in exact the same state, a mixed state that can be obtained by the procedure of 'partial tracing'.[7] This sameness of physical states entails that switching the labels has no significance: all statistical predictions of quantum mechanics are invariant under permutations of particle labels." [Die23, p.15]

This is the Indistinguishability Postulate, a core notion in quantum theory. French and Krause conclude that the labels are 'otiose', and Dieks explores that, by insisting that "all quantum particles of the same sort are in exactly the same physical state and possess exactly the same physical properties and then cannot be distinguished and individuated by any physical process."

Comment — Fermions cannot share the same quantum states. For instance, the two electrons of a neutral He atom share the same entangled state, but given a certain direction, one of them has spin UP and the other has spin DOWN. They do not have 'exactly' the same properties; the case is that when they share an entangled state such as (2.1), we cannot specify which is which, that is, to know *which* electron has spin UP, something that will be revealed only after a measurement. We cannot speak of the electrons as isolated individuals; it seems that this is a quantum fact. So, neither quantum mechanics nor the RV assumes that *all* the particles (of the universe) of the same kind are in the same quantum state and have the same properties.

Dieks takes for granted that when we follow the path of a particle in a bubble chamber, or when a single particle is trapped in a potential well or even

[7] The partial tracing is an operation used when the join system is described by a density operator and enables us to consider the trace (which gives us the expectation values) of some component of the total system even if the whole state is entangled. But it should be remarked that if the systems are of the same kind, it is not relevant which is the particular system we are taking into apart.

when just one electron is fired and later detected on a screen, we are assigning them identities. We have discussed the case of trapped particles elsewhere (see [Kra11]).

Comment — Again, the claim depends on what one understands by identity. What Dieks means by that? Of course, we suppose anyone will agree, that the particles do not obey STI for in this case they would carry 'permanent identities', which in physical terms might be thought of as *genidentity*. The above assumption can be made *provisorily* by the physicists, but one should agree that it is an extrapolation of what quantum mechanics says.

In proposing the Alternative View, Dieks says that the particles are not defined by reference to their labels but by observable physical properties which give them their states. He recalls Schrödinger in that it is the state that confers a particle its *momentaneous* (our add) individuality. So, according to him, states such as (2.1) represent two individual particles possessing well-defined individual properties, one characterized by the state $\left|\phi\right\rangle$ and the other by the orthogonal state $\left|\psi\right\rangle$. He reports to a paper by Ghirardi et al. [GMW02] in which they have analysed such a situation the particles behave *in many respects* like product states representing individual particles (our emphasis). This is the core of the AV: to be closer to the practice of physics. A nice example helps in clarifying the issue. Dieks supposes two particles, one at the left L and another at the right R of a certain apparatus whose details to not import here. The entangled state is similar to (2.1) with $|L\rangle$ for $\left|\psi\right\rangle$ and $|R\rangle$ for $\left|\phi\right\rangle$. Then he states that "the states $|L\rangle$ and $|R\rangle$ do the job of identifying the particles in this alternative approach. (...) The particles are clearly distinguishable."

Comment — Being in $|L\rangle$ or being in $|R\rangle$ do not provide *identity* to the particles. If we close our eyes for a moment and an evil genius appears and says that he probably has permuted the particles, how could we know whether he is telling the truth? Any measurement will do the same result independently of *which* particle is in the left (right). This is not compatible with the identity ascribed by STI. So, we will never be able to know if the genius is telling the truth.

Dieks suggests that the attribution of identities to the particles (without specifying the identity he is considering) "is in accordance with how states of this kind *are interpreted in the physical practice*." (our emphasis). "By the contrary, he continues, according to the RV the state (2.1) represents two particles with exactly the same location, 'smered out' overly over L and R."

Comment — Not at all! The RV never said or assumed that! As said already, the RV assumes that there is *one* particle at L and *one* at R, but quantum physics cannot give them identities in the sense of STI. In our opinion, the practice of physics requires precisely this: one at left and one at right, but their identities do not matter (being of the same kind): *kinds and quantities*, not individualities. This is well exemplified by Dieks himself, but we also refer to the above mentioning of Dalla Chiara and Toraldo di Francia.

6 Conclusion

The RV and the AV are not conflicting interpretations. Instead, we regard them as complementary. Once one assumes that quantum entities can be viewed as non-individuals, failing to obey STI, then an underlying mathematics such as that provided by quasi-set theory is in order. But in order to account with the practice of physics, *one can reason as if* the particles can be localized and have (momentary) identities so that *all happens* as if they were individuals. But we insist: on neglect the interference term, as mentioned above, we are committing the same error than in the early calculus when they have neglect the infinitesimal increments.

Dieks et al. analyses are of course good, and their points are clever. But we insist that the RV is not proposed to cope with the practice of physics but with its logical foundations. Thus, as the quantum logicians have agreed, if one wishes to preserve the non-distributivity law needs to go outset a Boolean structure, we argue that if someone wishes to consider 'legitimate' (and not 'fake') indistinguishable things she needs to go out of a mathematical framework that encompasses STI. Quasi-set theory is of course an alternative.

7 Appendix

In this Appendix, we recall some other questions put by philosophers of physics which seem to be against the RV, some of them endorsed by Dieks himself. The comments are just to enlighten the main points, which are developed in other works mentioned below.

7.1 The relevance of the notion of identity

We have seen that contrary to what Otávio Bueno, Francisco Berto and others say, something endorsed by Dieks et al., the notion of identity is not 'essential' for the meaning of the concept of an entity.[8] By an entity, we understand everything that can be referred to by a suitable language either by a proper name or by some description. An electron is an entity, and so are all quantum particles.

We really can suppose the existence of *entitites* to which (at least) the standard notion of identity (given by STI) does not hold. There is no logical contradiction in supposing that, except if the theory's logic says the opposite. But since in the general discussions in the philosophy of physics the logic being used is

[8]A response to Bueno is presented in [KA19]. Again recurring to the example of a BEC, would the *entities* that form a BEC (atoms, say) not be 'entities' of some kind?

rarely made explicit, we are free to suppose that there are also non-individuals in our metaphysical pantheon.

Quantum entities seem to provide a paradigmatic example of non-individuals (entities that fail to obey STI). From a logical point of view, this was also assumed by F. P. Ramsey when he criticized Whitehead and Russell's definition of identity (Leibniz Law) saying that it is logically possible the existence of more than one thing sharing all their properties [Ram50, p.31]. There is no logical contradiction here once we leave STI.

Perhaps what is involved in the atavistic criterion that *we need* identity to construct our theories, so it cannot be dispensed with is a fallacy. The same would apply in the construction of paraconsistent or intuitionistic frameworks not regarding identity, but respectively the notions of non-contradiction and the excluded middle. In the metalanguage, we assume the validity of the Principle of Non-Contradiction even when we develop a system in which it is not universally valid; the same happens with the Principle of the Excluded Middle and some constructive frameworks (see also below). So, we usually *assume* an intuitive identity to start with, but later we can dispense it in favour of a theory encompassing non-individuals.

Summing up, we regard identity as a useful concept (perhaps even a necessary one) to conceptualize *individuals*, but of course not for *any* kind of entity.

7.2 Emulating non-individuals

STI is incompatible with (completely) indistinguishable but not identical things. This does not entail that we cannot *mimic* them within a theory encompassing STI, as the ZFC system. This is the case of taking deformable or non-rigid structures. A set-theoretical structure $\mathfrak{A}$ is *rigid* if its only automorphism is the identity function, the trivial automorphism, otherwise, it is deformable. For instance, the additive group of the integers, $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$ is deformable, since the application $h(x) = -x$ is an automorphism, as is easy to prove. So, *within* $\mathcal{Z}$, the integers 2 and -2 are indistinguishable. Thus, in a deformable structure, we can make things happen *as if* some individuals were indistinguishable by the canons of the structure, as it is supposed to happen when we opt by eliminating surplus structures in favour of just symmetric and anti-symmetric ones. With this move, we can work inside a 'standard' logical framework as usually done; by the way, all standard books of quantum physics are developed with standard mathematics by 'mimicking' indistinguishability by the use of symmetry postulates.[9]

But every structure in ZFC (and in similar theories) can be extended to a rigid structure. So, even if two elements are indiscernible inside a structure, in the extended one they can be realized to be individuals. Furthermore, the whole universe of sets is rigid [Jec03, p.66]. That is, *everything* represented in a theory like ZFC is an individual and we do not need any argument other than

[9] Yuri Manin says that quantum mechanics (and quantum physics in general) has no 'proper' language, making use of a fragment of the standard functional calculus [Man77, p.84].

logical ones to state that.

7.3 Identity for m-atoms

Can we define identity for m-atoms in the theory $\mathfrak{Q}$? Of course we can. So, why not to do that? The answer is similar to that one a paraconsistent logician might give when asked why she does not accept the universal validity of the Principle of Non-Contradiction or then the answer given by an intuitionistic logician when asked why she does not accept the universal validity of the Principle of the Excluded Middle: the reason is that we don't wish to do it! We are presenting a logical system that supports a metaphysics comprising non-individuals, understood as entities that fail to obey STI. So, no relation that turns out to be equivalent to the identity of STI is to be allowed, as those presented below (without the proofs), and the reasons were put before.

Let us suggest some ways to define an identity among m-atoms so that the theory $\mathfrak{Q}$ would turn equivalent to ZFA. Let x and y be m-atoms. Then we pose

Definition 1. $x =_a y := \forall z(x \in z \leftrightarrow y \in z)$

Definition 2. $x =_b y := x \in [\![y]\!] \wedge y \in [\![x]\!]$,

where $[\![w]\!]$ is a strong singleton of w, let us recall, a qset whose quasi-cardinal is one and whose only element is indistinguishable from w.[10] This definition can be put another way as follows

Definition 3. $x =_c y := [\![x]\!] =_E [\![y]\!]$,

being $=_E$ the extensional identity introduced earlier. All these alternatives conduce to the identity of STI as it seems to be immediate but, as said before, we do not intend to introduce either of these (or other) identities for m-atoms which turn out to be 'classical identity' (given by STI).

7.4 Quantity by not ordering

Francisco Berto for instance [Ber17], claims something accepted by most of the philosophers we are considering, namely, that we cannot have a collection of entities with a definite cardinality if these entities do not possess self-identity that makes them different from each other. Here we sketch a way to do it with a qset of N indiscernible things devoid of self-identity (N being a natural number; the infinite case will be not touched here).

What we shall do is to enlarge the theory $\mathfrak{Q}$ with additional axioms that give us natural numbers. Of course, we have a copy of the standard model of Peano Arithmetics (PA) in $\mathfrak{Q}$, but the natural numbers we shall consider came not from this model, but from the Peano Arithmetics we *add* to $\mathfrak{Q}$ as a step-theory. If we take the natural numbers from this model, they would be ordinals, something

[10]But recall once more that we cannot assert that such an element *is* w for to say that we need identity: the element of the qset is just 'identical' to w.

we wish to avoid. This is similar to adding the theory of fields to that of vector spaces as usually done. Let us call $\mathfrak{Q}'$ this new theory, whose postulates are those of $\mathfrak{Q}$ plus those of PA. Hence the natural numbers to be considered here *are not* ordinals, but just 0, $s0$, $ss0$ and so on, where s stands for the successor function and 0 is the natural number 'zero'. Now we can assume the existence of a binary functional symbol qc and write $qc(x, N)$ to mean that the qset x has quasi-cardinal N, being N one of such natural numbers.[11] The postulates of this new notion are the following, where x and y range over qsets and M, N over natural numbers.

1. $\forall x(qc(x, 0) \leftrightarrow x = \emptyset)$.
2. $\forall x \forall y(qc(x, N) \wedge qc(y, 1) \wedge x \cap y =_E \emptyset \to qc(x \cup y, N + 1))$
3. $\forall x(qc(x, N) \to qc(\mathcal{P}(x), 2^N))$
4. $\forall x(qc(x, N) \to \forall M(M < N \to \exists y(y \in \mathcal{P}(x) \wedge qc(y, M)))$

The way we attribute a natural number N to a qset x is not a logical problem, being left to the physical theory. For instance, chemistry has a way of attributing natural numbers to the orbitals. The important thing here is that we can assume that a certain qset has a finite number of elements.

So, $\mathfrak{Q}'$ shows that we really can consider collections of completely indiscernible things with a cardinal. For the sake of making things clear, consider the third axiom and let $qc(x, 4)$. Then $qc(\mathcal{P}(x), 16)$. So, *we can reason as if* there is one subqset with no element, four subqsets with one element each, six subqsets with two elements, also four with three elements and one with the four elements. The counting is usual. The interesting fact is that we cannot discern among the subqsets with the same quasi-cardinality but just to state that they are indiscernible (by WEA). Since their elements are (by hypothesis) indiscernible, all we have are their quantities given by the axioms.

But notice that in $\mathfrak{Q}'$, so as in $\mathfrak{Q}$, the fact that we cannot discern either the elements of some qsets or the qsets themselves, does not make them identical as STI would require.

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[11] See also the alternative approach proposed by E. Wajch in [Waj23].

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