Hyperintensional Ω-Logic*

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Abstract

This essay examines the philosophical significance of Ω-logic in Zermelo-Fraenkel set theory with choice (ZFC). The categorical duality between coalgebra and algebra permits Boolean-valued algebraic models of ZFC to be interpreted as coalgebras. The hyperintensional profile of Ω-logical validity can then be countenanced within a coalgebraic logic. I argue that the philosophical significance of the foregoing is two-fold. First, because the epistemic and modal and hyperintensional profiles of Ω-logical validity correspond to those of second-order logical consequence, Ω-logical validity is genuinely logical. Second, the foregoing provides a hyperintensional account of the interpretation of mathematical vocabulary.

1 Introduction

This essay examines the philosophical significance of the consequence relation defined in the Ω-logic for set-theoretic languages. I argue that, as with second-order logic, the hyperintensional profile of validity in Ω-Logic enables the property to be epistemically tractable. Because of the duality between coalgebras and algebras, Boolean-valued models of set theory can be interpreted as coalgebras. In Section **2**, I demonstrate how the hyperintensional profile of Ω-logical validity can be countenanced within a coalgebraic logic. Finally, in Section **3**, the philosophical significance of the characterization of the hyperintensional profile of Ω-logical validity for the philosophy of mathematics is examined. I argue (i) that Ω-logical validity is genuinely logical, and (ii) that it provides a hyperintensional account of formal grasp of the concept of 'set'. Section **4** provides concluding remarks.

2 Definitions

In this section, I define the axioms of Zermelo-Fraenkel set theory with choice. I define the mathematical properties of the large cardinal axioms which can

*This is a substantially revised version of a paper, "Modal Ω-Logic", which was published in Matteo Vincenzo D'Alfonso and Don Berkich (eds.), *On the Cognitive, Ethical, and Scientific Dimensions of Artificial Intelligence*, Springer (2019), pp. 65-82. Last revised: December 8, 2023.

be adjoined to ZFC, and I provide a detailed characterization of the properties of Ω-logic for ZFC. Coalgebras are dual to Boolean-valued algebraic models of Ω-logic. Modal and hyperintensional coalgebras are then argued to provide a precise characterization of the modal and hyperintensional profiles of Ω-logical validity.

2.1 Axioms[1]

- Extensionality

  $\forall x,y.(\forall z.z{\in}x \iff z{\in}y) \rightarrow x = y$

- Empty Set

  $\exists x.\forall y.y{\notin}x$

- Pairing

  $\forall x,y.\exists z.\forall w.w{\in}z \iff w = x \vee w = y$

- Union

  $\forall x.\exists y.\forall z.z{\in}y \iff \exists w.w{\in}x \wedge z{\in}w$

- Powerset

  $\forall x.\exists y.\forall z.z{\in}y \iff z \subseteq x$

- Separation (with $\overrightarrow{x}$ a parameter)

  $\forall \overrightarrow{x},y.\exists z.\forall w.w{\in}z \iff w{\in}y \wedge A(w,\overrightarrow{x})$

- Infinity

  $\exists x.\emptyset{\in}x \wedge \forall y.y{\in}x \rightarrow y \cup \{y\}{\in}x$

- Foundation

  $\forall x.(\exists y.y{\in}x) \rightarrow \exists y{\in}x.\forall z{\in}x.z{\notin}y$

- Replacement

  $\forall x,\overrightarrow{y}.[\forall z{\in}x.\exists!w.A(z,w,\overrightarrow{y})] \rightarrow \exists u.\forall w.w{\in}u \iff \exists z{\in}x.A(z,w,\overrightarrow{y})$

- Choice

  $\forall x.\emptyset{\notin}x \rightarrow \exists f{\in}(x \rightarrow \cup x).\forall y{\in}x.f(y){\in}y$

[1] For a standard presentation, see Jech (2003). The presentation here follows Avigad (2021). For detailed, historical discussion, see Maddy (1988,a).

2.2 Large Cardinals

Borel sets of reals are subsets of ω^ω or $\mathbb{R}$, closed under countable intersections and unions.[2] For all ordinals, a, such that $0 < a < \omega_1$, and $b < a$, Σ^0_a denotes the open subsets of ω^ω formed under countable unions of sets in Π^0_b, and Π^0_a denotes the closed subsets of ω^ω formed under countable intersections of Σ^0_b.

Projective sets of reals are subsets of ω^ω, formed by complementations (ω^ω – u, for u$\subseteq\omega^\omega$) and projections [p(u) = $\{\langle x_1, \ldots, x_n\rangle \in \omega^\omega \mid \exists y \langle x_1, \ldots, x_n, y\rangle \in u\}$]. For all ordinals a, such that $0 < a < \omega$, Π^1_0 denotes closed subsets of ω^ω; Π^1_a is formed by taking complements of the open subsets of ω^ω, Σ^1_a; and Σ^1_{a+1} is formed by taking projections of sets in Π^1_a.

The full power set operation defines the cumulative hierarchy of sets, V, such that $V_0 = \varnothing$; $V_{a+1} = \wp(V_0)$; and $V_\lambda = \bigcup_{a<\lambda} V_a$.

In the inner model program (cf. Woodin, 2001, 2010, 2011; Kanamori, 2012,a,b), the definable power set operation defines the constructible universe, L($\mathbb{R}$), in the universe of sets V, where the sets are transitive such that a$\in$C $\iff$ a$\subseteq$C; $L(\mathbb{R}) = V_{\omega+1}$; $L_{a+1}(\mathbb{R}) = \mathrm{Def}(L_a(\mathbb{R}))$; and $L_\lambda(\mathbb{R}) = \bigcup_{a<\lambda}(L_a(\mathbb{R}))$.

Via inner models, Gödel (1940) proves the consistency of the generalized continuum hypothesis, ${\aleph_a}^{\aleph_a} = \aleph_{a+1}$, as well as the axiom of choice, relative to the axioms of ZFC. However, for a countable transitive set of ordinals, M, in a model of ZF without choice, one can define a generic set, G, such that, for all formulas, ϕ, either ϕ or $\neg\phi$ is forced by a condition, f, in G. Let $M[G] = \bigcup_{a<\kappa} M_a[G]$, such that $M_0[G] = \{G\}$; with $\lambda < \kappa$, $M_\lambda[G] = \bigcup_{a<\lambda} M_a[G]$; and $M_{a+1}[G] = V_a \cap M_a[G]$.[3] G is a Cohen real over M, and comprises a set-forcing extension of M. The relation of set-forcing, $\Vdash$, can then be defined in the ground model, M, such that the forcing condition, f, is a function from a finite subset of ω into {0,1}, and $f \Vdash$ u$\in$G if f(u) = 1 and $f \Vdash$ u$\notin$G if f(u) = 0. The cardinalities of an open dense ground model, M, and a generic extension, G, are identical, only if the countable chain condition (c.c.c.) is satisfied, such that, given a chain – i.e., a linearly ordered subset of a partially ordered (reflexive, antisymmetric, transitive) set – there is a countable, maximal antichain consisting of pairwise incompatible forcing conditions. Via set-forcing extensions, Cohen (1963, 1964) constructs a model of ZF which negates the generalized continuum hypothesis, and thus proves the independence thereof relative to the axioms of ZF.[4]

Gödel (1946/1990: 1-2) proposes that the value of Orey sentences such as the GCH might yet be decidable, if one avails of stronger theories to which new axioms of infinity – i.e., large cardinal axioms – are adjoined.[5] He writes that: 'In set theory, e.g., the successive extensions can be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinatorial

[2]See Koellner (2013), for the presentation, and for further discussion, of the definitions in this and the subsequent paragraph.

[3]See Kanamori (2012,a: 2.1; 2012,b: 4.1), for further discussion.

[4]See Kanamori (2008), for further discussion.

[5]See Kanamori (2007), for further discussion. Kanamori (op. cit.: 154) notes that Gödel (1931/1986: fn48a) makes a similar appeal to higher-order languages, in his proofs of the incompleteness theorems. The incompleteness theorems are examined in further detail, in Section **3.2**, below.

and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e. the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory ... is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from present axioms plus some true assertion about the largeness of the universe of sets'.

For cardinals, x,a,C, $C \subseteq a$ is closed unbounded in a, if it is closed [if $x < C$ and $\bigcup(C \cap a) = a$, then $a \in C$] and unbounded ($\bigcup C = a$) (Kanamori, op. cit.: 360). A cardinal, S, is stationary in a, if, for any closed unbounded $C \subseteq a$, $C \cap S \neq \emptyset$ (op. cit.). An ideal is a subset of a set closed under countable unions, whereas filters are subsets closed under countable intersections (361). A cardinal κ is regular if the cofinality of κ is identical to κ. Uncountable regular limit cardinals are weakly inaccessible (op. cit.). A strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^{\lambda} < \kappa$ (op. cit.).

Large cardinal axioms are defined by elementary embeddings.[6] Elementary embeddings can be defined thus. For models A,B, and conditions ϕ, j: $A \to B$, $\phi\langle a_1, \ldots, a_n\rangle$ in A if and only if $\phi\langle j(a_1), \ldots, j(a_n)\rangle$ in B (363). A measurable cardinal is defined as the ordinal denoted by the critical point of j, crit(j) (Koellner and Woodin, 2010: 7). Measurable cardinals are inaccessible (Kanamori, op. cit.).

Let κ be a cardinal, and $\eta > \kappa$ an ordinal. κ is then η-strong, if there is a transitive class M and an elementary embedding, j: $V \to M$, such that crit(j) = κ, $j(\kappa) > \eta$, and $V_{\eta} \subseteq M$ (Koellner and Woodin, op. cit.).

κ is strong if and only if, for all η, it is η-strong (op. cit.).

If A is a class, κ is η-A-strong, if there is a j: $V \to M$, such that κ is η-strong and $j(A \cap V_{\kappa}) \cap V_{\eta} = A \cap V_{\eta}$ (op. cit.).

κ is a Woodin cardinal, if κ is strongly inaccessible, and for all $A \subseteq V_{\kappa}$, there is a cardinal $\kappa_A < \kappa$, such that κ_A is η-A-strong, for all η such that κ_{η}, $\eta < \kappa$ (Koellner and Woodin, op. cit.: 8).

κ is superstrong, if j: $V \to M$, such that crit(j) = κ and $V_{j(\kappa)} \subseteq M$, which entails that there are arbitrarily large Woodin cardinals below κ (op. cit.).

Large cardinal axioms can then be defined as follows.

$\exists x \Phi$ is a large cardinal axiom, because:

(i) Φx is a Σ_2-formula, where 'a sentence ϕ is a Σ_2-sentence if it is of the form: There exists an ordinal α such that $V_{\alpha} \Vdash \psi$, for some sentence ψ' (Woodin, 2019);

(ii) if κ is a cardinal, such that $V \models \Phi(\kappa)$, then κ is strongly inaccessible; and

(iii) for all generic partial orders $\mathbb{P} \in V_{\kappa}$, $V^{\mathbb{P}} \models \Phi(\kappa)$; I_{NS} is a non-stationary

[6] The definitions in the remainder of this subsection follow the presentations in Koellner and Woodin (2010) and Woodin (2010, 2011).

ideal; A^G is the canonical representation of reals in L($\mathbb{R}$), i.e. the interpretation of A in M[G]; H(κ) is comprised of all of the sets whose transitive closure is < κ (cf. Woodin, 2001: 569); and $L(\mathbb{R})^{\mathbb{P}max} \models \langle H(\omega_2), \in, I_{NS}, A^G\rangle \models$ 'ϕ'. $\mathbb{P}$ is a homogeneous partial order in L($\mathbb{R}$), such that the generic extension of $L(\mathbb{R})^{\mathbb{P}}$ inherits the generic invariance, i.e., the absoluteness, of L($\mathbb{R}$). Thus, $L(\mathbb{R})^{\mathbb{P}max}$ is (i) effectively complete, i.e. invariant under set-forcing extensions; and (ii) maximal, i.e. satisfies all Π_2-sentences and is thus consistent by set-forcing over ground models (Woodin, ms: 28).

Assume ZFC and that there is a proper class of Woodin cardinals; $A \in \mathbb{P}(\mathbb{R}) \cap L(\mathbb{R})$; ϕ is a Π_2-sentence; and V(G), s.t. $\langle H(\omega_2), \in, I_{NS}, A^G\rangle \models$ 'ϕ': Then, it can be proven that $L(\mathbb{R})^{\mathbb{P}max} \models \langle H(\omega_2), \in, I_{NS}, A^G\rangle \models$ 'ϕ', where 'ϕ' := $\exists A \in \Gamma^{\infty} \langle H(\omega_1), \in, A\rangle \models \psi$.

The axiom of determinacy (AD) states that every set of reals, $a \subseteq \omega^{\omega}$ is determined.

Woodin's (1999) Axiom (*) can be thus countenanced:

$AD^{L(\mathbb{R})}$ and $L[(\mathbb{P}\omega_1)]$ is a $\mathbb{P}$max-generic extension of L($\mathbb{R}$),

from which it can be derived that $2^{\aleph_0} = \aleph_2$. Thus, ¬CH; and so CH is absolutely decidable.

In more recent work, Woodin (2019) provides evidence that CH might, by contrast, be true. The truth of CH would follow from the truth of Woodin's Ultimate-L conjecture. The following definitions are from Woodin (op. cit.): 'A transitive class is an inner model if[, for the class of ordinals Ord, - HK] Ord $\subset$ M, and M $\Vdash$ ZFC'. L, the constructible reals, and HOD, the hereditarily ordinal definable sets, are inner models. 'Suppose N is an inner model and that $[a]$ is an uncountable (regular) cardinal of V. N has the $[a]$-cover property if for all $\sigma \subset$ N, if $|\sigma| < [a]$ then there exists $\tau \in$N such that: $\sigma \subset \tau$ and $|\tau| < [a]$. N has the $[a]$-approximation property if for all sets X $\subset$ N, the following are equivalent: (i) X$\in$N and (ii) For all $\sigma \in$N, if $|\sigma| < [a]$, then $\sigma \cap X \in$N. Suppose N is an inner model and that $\sigma \subset$ N. Then N[σ] denotes the smallest inner model M such that N $\subseteq$ M and $\sigma \in$M. Suppose that N is an inner model and $[a]$ is strongly inaccessible. Then N has the $[a]$-genericity property if for all $\sigma \subseteq [a]$, if $|\sigma| < [a]$then $N[\sigma] \cap V_a$ is a Cohen extension of $N \cap V_a$. The axiom for V = Ultimate-L states then that '(i) There is a proper class of Woodin cardinals, and (ii) For each Σ_2-sentence ϕ, if ϕ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that $HOD^{L(A,\mathbb{R})} \Vdash \phi$, where a set is universally Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}^n$, the preimage of A by π has the property of Baire in the space Ω'. The property of Baire holds if, for a subset of a topological space A $\subseteq$ X, there is an open set U $\subset$ X such that A Ξ U is a meagre subset, where Ξ is the symmetric difference, i.e. the union of relative complements, and a subset of a topological space is meagre if it is a countable union of nowhere dense sets, where nowhere dense subsets of the topology hold if their union with an open set is not dense.[7] The Ultimate-L Conjecture is then as follows: 'Suppose that $[a]$ is an extendible cardinal. $[a]$ is

[7]https://en.wikipedia.org/wiki/PropertyofBaire, https://en.wikipedia.org/wiki/Symmetricdifference, https://en.wikipedia.org/wiki/Meagreset.

an extendible cardinal if for each $\lambda > [a]$ there exists an elementary embedding j : $V_{\lambda+1} \to V_{j(\lambda)+1}$ such that CRT(j) = $[a]$ and j($[a]$) $> \lambda$. Then provably there is an inner model N such that: 1. N has the $[a]$-cover and $[a]$-approximation properties. 2. N has the $[a]$-genericity property. 3. N $\Vdash$ 'V = Ultimate-L" (Woodin, op. cit.).

2.3 Ω-Logic

For partial orders, $\mathbb{P}$, let $V^{\mathbb{P}} = V^{\mathbb{B}}$, where $\mathbb{B}$ is the regular open completion of ($\mathbb{P}$).[8] $M_a = (V_a)^M$ and $M_a^{\mathbb{B}} = (V_a^{\mathbb{B}})^M = (V_a^{M^{\mathbb{B}}})$. *Sent* denotes a set of sentences in a first-order language of set theory. $T\cup\{\phi\}$ is a set of sentences extending ZFC. *c.t.m* abbreviates the notion of a countable transitive $\in$-model. *c.B.a.* abbreviates the notion of a complete Boolean algebra.

Define a *c.B.a.* in V, such that $V^{\mathbb{B}}$. Let $V_0^{\mathbb{B}} = \emptyset$; $V_\lambda^{\mathbb{B}} = \bigcup_{b<\lambda} V_b^{\mathbb{B}}$, with λ a limit ordinal; $V_{a+1}^{\mathbb{B}} = \{f\colon X \to \mathbb{B} \mid X \subseteq V_a^{\mathbb{B}}\}$; and $V^{\mathbb{B}} = \bigcup_{a\in On} V_a^{\mathbb{B}}$.

ϕ is true in $V^{\mathbb{B}}$, if its Boolean-value is $1^{\mathbb{B}}$, if and only if

$V^{\mathbb{B}} \models \phi$ iff $[\![\phi]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Thus, for all ordinals, a, and every *c.B.a.* $\mathbb{B}$, $V_a^{\mathbb{B}} \equiv (V_a)^{V^{\mathbb{B}}}$ iff for all $x\in V^{\mathbb{B}}$, $\exists y\in V^{\mathbb{B}} [\![x = y]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$ iff $[\![x\in V^{\mathbb{B}}]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Then, ${V^{\mathbb{B}}}_a \models \phi$ iff $V^{\mathbb{B}} \models$ '$V_a \models \phi$'.

Ω-logical validity can then be defined as follows:

For $T\cup\{\phi\}\subseteq Sent$,

$T \models_\Omega \phi$, if for all ordinals, a, and *c.B.a.* $\mathbb{B}$, if $V_a^{\mathbb{B}} \models T$, then $V_a^{\mathbb{B}} \models \phi$.

Supposing that there exists a proper class of Woodin cardinals and if $T\cup\{\phi\}\subseteq Sent$, then for all set-forcing conditions, $\mathbb{P}$:

$T \models_\Omega \phi$ iff $V^T \models$ '$T \models_\Omega \phi$',

where $T \models_\Omega \phi \equiv \emptyset \models$ '$T \models_\Omega \phi$'.

The Ω-Conjecture states that $V \models_\Omega \phi$ iff $V^{\mathbb{B}} \models_\Omega \phi$ (Woodin, ms). Thus, Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic universe.

The soundness of Ω-Logic is defined by universally Baire sets of reals. For a cardinal, e, let a set A be e-universally Baire, if for all partial orders $\mathbb{P}$ of cardinality e, there exist trees, S and T on ω X λ, such that A = p[T] and if $G\subseteq\mathbb{P}$ is generic, then $p[T]^G = \mathbb{R}^G - p[S]^G$ (Koellner, 2013). A is universally Baire, if it is e-universally Baire for all e (op. cit.).

Ω-Logic is sound, such that $V \vdash_\Omega \phi \to V \models_\Omega \phi$. However, the completeness of Ω-Logic has yet to be resolved.

A **E**-coalgebra is a pair $\mathbb{A} = (A, \mu)$, with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\to$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390).

$\mathbb{A} = \langle A, \mu\colon A \to \mathbf{E}(A)\rangle$ is dual to the category of algebras over the functor μ (417-418). If μ is a functor on categories of sets, then coalgebraic models are dual to Boolean-algebraic models of Ω-logical validity.

[8]The definitions in this section follow the presentation in Bagaria et al. (2006).

Leach-Krouse (ms) defines the modal logic of Ω-consequence as satisfying the following axioms:

For a theory $\mathbf{T}$ and with $\Box\phi := \mathbf{T}^{\mathbb{B}}_{\alpha} \Vdash \text{ZFC} \Rightarrow \mathbf{T}^{\mathbb{B}}_{\alpha} \Vdash \phi$,

ZFC $\vdash \phi \Rightarrow$ ZFC $\vdash \Box\phi$

ZFC $\vdash \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

ZFC $\vdash \Box\phi \to \phi \Rightarrow$ ZFC $\vdash \phi$

ZFC $\vdash \Box\phi \to \Box\Box\phi$

ZFC $\vdash \Box(\Box\phi \to \phi) \to \Box\phi$

$\Box(\Box\phi \to \psi) \vee \Box(\Box\psi \wedge \psi \to \phi)$, where this clause added to GL is the logic of 'true in all V_{κ} for all κ strongly inaccessible' in ZFC.

2.4 Two-dimensional Hyperintensionality and Ω-logic

Finally, the axioms of the modal logic of Ω-consequence can be rendered hyperintensional as follows:

For a theory $\mathbf{T}$ and with A($\Box\phi$) := for all t∈P there is a t'∈P such that t' ⊔ t ∈P and t' ⊢ '$\mathbf{T}^{\mathbb{B}}_{\alpha} \Vdash \text{ZFC} \Rightarrow \mathbf{T}^{\mathbb{B}}_{\alpha} \Vdash \phi$', where $\Box$ is interpreted as $\mathbf{T}^{\mathbb{B}}_{\alpha} \Vdash$ ZFC $\Rightarrow \mathbf{T}^{\mathbb{B}}_{\alpha} \Vdash \phi$,

ZFC $\vdash \phi \Rightarrow$ ZFC $\vdash$ A($\Box\phi$)

ZFC $\vdash$ A[$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$]

ZFC $\vdash$ A($\Box\phi$) $\to \phi \Rightarrow$ ZFC $\vdash \phi$

ZFC $\vdash$ A($\Box\phi$) $\to$ A($\Box\Box\phi$)

ZFC $\vdash$ A[$\Box(\Box\phi \to \phi)$] $\to$ A($\Box\phi$)

A[$\Box(\Box\phi \to \psi) \vee \Box(\Box\psi \wedge \psi \to \phi)$]. As with the two-dimensional hyperintensional profile of the Epistemic Church-Turing Thesis [see Bowen (2023)], the two-dimensional hyperintensional profile of Ω-logical consequence can be countenanced by adding a topic-sensitive truthmaker from a metaphysical state space and making its value dependent on the value of the epistemically necessary truthmaker A(ϕ) [see Fine (2017a-c), for a presentation of truthmaker semantics, and Bowen (op. cit.) for further details, development, and applications of the semantics].

3 Discussion

This section examines the philosophical significance of coalgebras and the Boolean-valued models of set-theoretic languages to which they are dual. I argue that, similarly to second-order logical consequence, (i) the 'mathematical entanglement' of Ω-logical validity does not undermine its status as a relation of pure logic; and (ii) both the modal profile and model-theoretic characterization of Ω-logical consequence provide a guide to its epistemic tractability.[9] I argue, then, that there are several considerations adducing in favor of the claim that the interpretation of the concept of set constitutively involves hyperintensional notions. The role of coalgebras in (i) characterizing the modal profile of Ω-logical

[9]The phrase, 'mathematical entanglement', is owing to Koellner (2010: 2) who attributes the phrase to Parsons.

consequence, and (ii) being constitutive of the hyperintensional understanding-conditions for the concept of set, provides, then, support for a realist conception of the cumulative hierarchy.

3.1 Ω-Logical Validity is Genuinely Logical

Frege's (1884/1980; 1893/2013) proposal – that cardinal numbers can be explained by specifying a biconditional between the identity of, and an equivalence relation on, concepts, expressible in the signature of second-order logic – is the first attempt to provide a foundation for mathematics on the basis of logical axioms rather than rational or empirical intuition. In Frege (1884/1980. cit.: 68) and Wright (1983: 104-105), the number of the concept, **A**, is argued to be identical to the number of the concept, **B**, if and only if there is a one-to-one correspondence between **A** and **B**, i.e., there is a bijective mapping, R, from **A** to **B**. With Nx: a numerical term-forming operator,

- $\forall\mathbf{A}\forall\mathbf{B}[\text{Nx: }\mathbf{A} = \text{Nx: }\mathbf{B} \equiv \exists\text{R}[\forall\text{x}[\mathbf{A}\text{x} \to \exists\text{y}(\mathbf{B}\text{y} \wedge \text{Rxy} \wedge \forall\text{z}(\mathbf{B}\text{z} \wedge \text{Rxz} \to \text{y} = \text{z}))] \wedge \forall\text{y}[\mathbf{B}\text{y} \to \exists\text{x}(\mathbf{A}\text{x} \wedge \text{Rxy} \wedge \forall\text{z}(\mathbf{A}\text{z} \wedge \text{Rzy} \to \text{x} = \text{z}))]]]$.

Frege's Theorem states that the Dedekind-Peano axioms for the language of arithmetic can be derived from the foregoing abstraction principle, as augmented to the signature of second-order logic and identity.[10] Thus, if second-order logic may be counted as pure logic, despite that domains of second-order models are definable via power set operations, then one aspect of the philosophical significance of the abstractionist program consists in its provision of a foundation for classical mathematics on the basis of pure logic as augmented with non-logical implicit definitions expressed by abstraction principles.

There are at least three reasons for which a logic defined in ZFC might not undermine the status of its consequence relation as being logical. The first reason for which the mathematical entanglement of Ω-logical validity might be innocuous is that, as Shapiro (1991: 5.1.4) notes, many mathematical properties cannot be defined within first-order logic, and instead require the expressive resources of second-order logic. For example, the notion of well-foundedness cannot be expressed in a first-order framework, as evinced by considerations of compactness. Let E be a binary relation. Let m be a well-founded model, if there is no infinite sequence, $a_0, \ldots, a_i$, such that $Ea_0, \ldots, Ea_{i+1}$ are all true. If m is well-founded, then there are no infinite-descending E-chains. Suppose that T is a first-order theory containing m, and that, for all natural numbers, n, there is a T with $n + 1$ elements, $a_0, \ldots, a_n$, such that $\langle a_0, a_1\rangle, \ldots, \langle a_n, a_{n-1}\rangle$ are in the extension of E. By compactness, there is an infinite sequence such that that $a_0 \ldots a_i$, s.t. $Ea_0, \ldots, Ea_{i+1}$ are all true. So, m is not well-founded.

By contrast, however, well-foundedness can be expressed in a second-order framework:

[10]Cf. Dedekind (1888/1963) and Peano (1889/1967). See Wright (1983: 154-169) for a proof sketch of Frege's theorem; Boolos (1987) for the formal proof thereof; and Parsons (1964) for an incipient conjecture of the theorem's validity.

$\forall X[\exists x Xx \rightarrow \exists x[Xx \wedge \forall y(Xy \rightarrow \neg Eyx)]]$, such that m is well-founded iff every non-empty subset X has an element x, s.t. nothing in X bears E to x.

One aspect of the philosophical significance of well-foundedness is that it provides a distinctively second-order constraint on when the membership relation in a given model is intended. This contrasts with Putnam's (1980) claim, that first-order models *mod* can be intended, if every set s of reals in *mod* is such that an ω-model in *mod* contains s and is constructible, such that – given the Downward Lowenheim-Skolem theorem[11] – if *mod* is non-constructible but has a submodel satisfying 's is constructible', then the model is non-well-founded and yet must be intended. The claim depends on the assumption that general understanding-conditions and conditions on intendedness must be co-extensive, to which I will return in Section **3.2**

A second reason for which Ω-logic's mathematical entanglement might not be pernicious, such that the consequence relation specified in the Ω-logic might be genuinely logical, may again be appreciated by its comparison with second-order logic. Shapiro (1998) defines the model-theoretic characterization of logical consequence as follows:

'(10) Φ is a logical consequence of [a model] Γ if Φ holds in all possibilities under every interpretation of the nonlogical terminology which holds in Γ' (148).

A condition on the foregoing is referred to as the 'isomorphism property', according to which 'if two models M, M' are isomorphic vis-a-vis the nonlogical items in a formula Φ, then M satisfies Φ if and only if M' satisfies Φ' (151).

Shapiro argues, then, that the consequence relation specified using second-order resources is logical, because of its modal and epistemic profiles. The epistemic tractability of second-order validity consists in 'typical soundness theorems, where one shows that a given deductive system is truth-preserving' (154). He writes that: '[I]f we know that a model is a good mathematical model of logical consequence (10), then we know that we won't go wrong using a sound deductive system. Also, we can know that an argument is a logical consequence ... via a set-theoretic proof in the metatheory' (154-155).

The modal profile of second-order validity provides a second means of accounting for the property's epistemic tractability. Shapiro argues, e.g., that: 'If the isomorphism property holds, then in evaluating sentences and arguments, the only 'possibility' we need to 'vary' is the size of the universe. If enough sizes are represented in the universe of models, then the modal nature of logical consequence will be registered ... [T]he only 'modality' we keep is 'possible size', which is relegated to the set-theoretic metatheory' (152).

Shapiro's remarks about the considerations adducing in favor of the logicality of non-effective, second-order validity generalize to Ω-logical validity. In the previous section, the modal profile of Ω-logical validity was codified by the duality between the category, $\mathbb{A}$, of coalgebraic modal logics and complete Boolean-valued algebraic models of Ω-logic. As with Shapiro's definition of logical consequence, where Φ holds in all possibilities in the universe of models

[11] The Downward Lowenheim-Skolem theorem claims that for any first-order model M, M has a submodel M' whose domain is at most denumerably infinite, s.t. for all assignments s on, and formulas $\phi(x)$ in, M', M,$s \Vdash \phi(x) \iff M'$,$s \Vdash \phi(x)$.

and the possibilities concern the 'possible size' in the set-theoretic metatheory, the Ω-Conjecture states that $\mathrm{V} \models_{\Omega} \phi$ iff $\mathrm{V}^{\mathbb{B}} \models_{\Omega} \phi$, such that Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic universe.

Finally, the epistemic tractability of Ω-logical validity is secured, both – as on Shapiro's account of second-order logical consequence – by its soundness, but also by its being the dual of coalgebras.

3.2 Hyperintensionality and the Concept of Set

In this section, I argue, finally, that the hyperintensional profile of Ω-logic can be availed of in order to account for the understanding-conditions of the concept of set.

Putnam (op. cit.: 473-474) argues that defining models of first-order theories is sufficient for both understanding and specifying an intended interpretation of the latter. Wright (1985: 124-125) argues, by contrast, that understanding-conditions for mathematical concepts cannot be exhausted by the axioms for the theories thereof, even on the intended interpretations of the theories. He suggests, e.g., that:

'[I]f there really were uncountable sets, their existence would surely have to flow from the concept of set, as intuitively satisfactorily explained. Here, there is, as it seems to me, no assumption that the content of the ZF-axioms cannot exceed what is invariant under all their classical models. [Benacerraf] writes, e.g., that: 'It is granted that they are to have their 'intended interpretation': '∈' is to mean set-membership. Even so, and conceived as encoding the intuitive concept of set, they fail to entail the existence of uncountable sets. So how can it be true that there are such sets? Benacerraf's reply is that the ZF-axioms are indeed faithful to the relevant informal notions only if, in addition to ensuring that '∈' means set-membership, we interpret them so as to observe the constraint that 'the universal quantifier has to mean all or at least all sets' (p. 103). It follows, of course, that if the concept of set does determine a background against which Cantor's theorem, under its intended interpretation, is sound, there is more to the concept of set that can be explained by communication of the intended sense of '∈' and the stipulation that the ZF-axioms are to hold. And the residue is contained, presumably, in the informal explanations to which, Benacerraf reminds us, Zermelo intended his formalization to answer. At least, this must be so if the 'intuitive concept of set' is capable of being explained at all. Yet it is notable that Benacerraf nowhere ventures to supply the missing informal explanation – the story which will pack enough into the extension of 'all sets' to yield Cantor's theorem, under its intended interpretation, as a highly non-trivial corollary' (op. cit).

In order to provide the foregoing explanation in virtue of which the concept of set can be shown to be associated with a realistic notion of the cumulative hierarchy, I will argue that there are several points in the model theory and epistemology of set-theoretic languages at which the interpretation of the concept of set constitutively involves hyperintensional notions. The hyperintensionality at issue is consistent with realist positions with regard to both truth values and the ontology of abstracta.

One point is in the coding of the signature of the theory, T, in which Gödel's incompleteness theorems are proved (cf. Halbach and Visser, 2014). The choice of coding bridges the numerals in the language with the properties of the target numbers. The choice of coding is therefore intensional, and has been marshaled in order to argue that the very notion of syntactic computability – via the equivalence class of partial recursive functions, λ-definable terms, and the transition functions of discrete-state automata such as Turing machines – is constitutively semantic (cf. Rescorla, 2015). Further points at which hyperintensionality can be witnessed in the phenomenon of self-reference in arithmetic are introduced by Reinhardt (1986). Reinhardt (op. cit.: 470-472) argues that the provability predicate can be defined relative to the minds of particular agents – similarly to Quine's (1968) and Lewis' (1979) suggestion that possible worlds can be centered by defining them relative to parameters ranging over tuples of spacetime coordinates or agents and locations – and that a theoretical identity statement can be established for the concept of the foregoing minds and the concept of a computable system. A hyperintensional semantics for provability logic is suggested in Bowen (2023).

A second point at which understanding-conditions may be shown to be constitutively hyperintensional can be witnessed by the conditions on the epistemic entitlement to assume that the theory in which Gödel's second incompleteness theorem is proved is consistent (cf. Dummett, 1963/1978; Wright, 1985). Wright (op. cit.: 91, fn.9) suggests that '[T]o treat [a] proof as establishing consistency is implicitly to exclude any doubt ... about the consistency of first-order number theory'. Wright's elaboration of the notion of epistemic entitlement, appeals to a notion of rational 'trust', which he argues is recorded by the calculation of 'expected epistemic utility' in the setting of decision theory (2004; 2014: 226, 241). Wright notes that the rational trust subserving epistemic entitlement will be pragmatic, and makes the intriguing point that 'pragmatic reasons are not a special genre of reason, to be contrasted with e.g. epistemic, prudential, and moral reasons' (2012: 484). Crucially, however, the very idea of expected epistemic utility in the setting of decision theory makes implicit appeal to epistemically possibly worlds or hyperintensional epistemic states.

A third consideration adducing in favor of the thought that grasp of the concept of set might constitutively possess a hyperintensional profile is that the concept can have a hyperintension – i.e., a function from states to extensions. The modal similarity types in the coalgebraic modal logic may then be interpreted as dynamic-interpretational modalities, where the dynamic-interpretational modal operator has been argued to entrain the possible reinterpretations both of the domains of the theory's quantifiers (cf. Fine, 2005, 2006), as well as of the intensions of non-logical concepts, such as the membership relation (cf. Uzquiano, 2015). A hyperintensional semantics for dynamic-interpretational modalities is countenanced in Bowen (2023).

The fourth consideration avails directly of the hyperintensional profile of Ω-logical consequence. While the above dynamic-interpretational states will suffice for possible reinterpretations of mathematical terms, the absoluteness of the consequence relation is such that, if the Ω-conjecture is true, then Ω-logical validity is invariant in all possible set-forcing extensions of ground models in the set-theoretic universe. The truth of the Ω-conjecture would thereby place an indefeasible necessary condition on a formal understanding of the hyperintension for the concept of set.

4 Concluding Remarks

In this essay I have examined the philosophical significance of the duality between coalgebras and Boolean-valued algebraic models of Ω-logic. I argued that – as with the property of validity in second-order logic – Ω-logical validity is genuinely logical. I argued, then, that modal and hyperintensional coalgebras, which characterize the hyperintensional profile of Ω-logical consequence, are constitutive of the interpretation of mathematical concepts such as the membership relation.

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