On coordinate-based and coordinate-free approaches to Maxwellian spacetime

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Abstract

I discuss and clarify the relationship between the recent wave of 'intrinsic' coordinate-free approaches to Maxwell gravitation and the coordinate-based discussions of Saunders (2013) and Wallace (2020).

Contents

1 Introduction

In recent years, philosophers of physics have considered afresh the question of the appropriate spacetime setting for Newtonian gravitation theory. At the centre of this debate have been two apparently conflicting proposals for what one should take this geometry to be: on the one hand, Saunders's (2013) proposal that Corollary VI to the Laws of Motion in Newton's *Principia* reveals that Maxwellian spacetime is the correct setting for Newtonian physics, and on the other hand, Knox's (2014) proposal that Corollary VI motivates a transition to a geometrised formulation of Newtonian gravitation, known as Newton-Cartan theory. Their claims have sparked a series of discussions of theories of Newtonian gravitation set on Maxwellian spacetime, and their relation to Newton-Cartan theory.[1]

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1. See Weatherall (2016), Teh (2018), Wallace (2020), Jacobs (2023), March, Wolf, and Read (2024).

One focus of these discussions has been on how Maxwellian spacetime—which is supposed to be equipped with a standard of rotation, but *not* a standard of absolute acceleration—should best be characterised. Earman (1989) originally defined the standard of rotation in terms of an equivalence class of derivative operators, and Dewar (2018) also adopted this definition. But a number of authors have voiced concerns about this approach. For example, Weatherall (2018, 34) notes that it "makes reference to structure that one does not attribute to spacetime," Jacobs (2022) argues that it is not suitably "intrinsic" and so fails to offer a perspicuous formalism from which we can read off the theory's ontology,[2] and Wallace (2019, 2020) goes so far as to suggest that the awkwardness of standard differential-geometric presentations of Maxwellian spacetime obscures the similarity between Newton-Cartan theory and theories of Newtonian gravitation set on Maxwellian spacetime, and (more generally) shows that coordinate-free differential geometry is not an intuitive way of characterising certain spacetime structures. In response to these concerns, Weatherall (2018) developed an 'intrinsic' characterisation of a standard of rotation, and Chen (2023) and March (2023b) have recently (and independently) shown that this object can be used to write down dynamics for Newtonian gravitation on Maxwellian spacetime (Maxwell gravitation).

However, this new wave of (coordinate-free differential-geometric) presentations of Maxwell gravitation are somewhat removed from Saunders' original coordinate-based 'vector relationism'. It would be of interest to see how these fit together. It also remains unclear how exactly Wallace's own (2020) (also coordinate-based) discussion of vector relationism and Newton-Cartan theory relates to the approaches outlined above.

In this paper, I aim to fill in these remaining pieces of the puzzle, by (a) making precise the relationship between vector relationism and Maxwell gravitation, and (b) translating Wallace's argument into the language of coordinate-free differential geometry. I thereby (i) clarify how Wallace's argument relates to other arguments concerning the (in)equivalence of Maxwell gravitation and Newton-Cartan theory in the literature, and (ii) address Wallace's concern that coordinate-free presentations of Maxwell gravitation obscure its similarities to Newton-Cartan theory. Indeed, I will argue, the same similarities to which Wallace alludes can be seen very naturally from a coordinate-free differential-geometric standpoint. Finally, this (iii) gives us the resources to connect up to Teh's (2018) discussion of Wallace and vector relationism, in which he also claims put Wallace into the language of coordinate-free differential geometry.

In more detail, the structure of this paper will be as follows. In §2, I present some basic details of Maxwell gravitation and Newton-Cartan theory, and the relationship between them. I then turn to the task of connecting these coordinate-free approaches with the work of Saunders (2013) and Wallace (2020). In §3, I present Saunders' vector relationism, and make precise its relationship to Maxwell gravitation. §4 reconstructs Wallace's argument that vector relationism and Newton-Cartan theory are equivalent; §5 aims to dispel the remainder of Wallace's concerns about coordinate-free presentations of Maxwellian spacetime by showing that the same argument can be made in the language of coordinate-free differential geometry. To end, in §6, I compare my approach to that of Teh (2018). §7 concludes.

2. See also Dürr & Read (2019, 1094-1096), who raise similar concerns.

2 Background: coordinate-free approaches to Newtonian gravitation on Maxwellian spacetime

Let M be a smooth four-manifold (assumed connected, Hausdorff, and paracompact). A temporal metric t_a on M is a smooth, closed, non-vanishing 1-form;[3] a spatial metric h^{ab} on M is a smooth, symmetric, rank-$(2,0)$ tensor field which admits, at each point in M, a set of four non-vanishing covectors $\overset{i}{\sigma}_a$, $i = 0, 1, 2, 3$, which form a basis for the cotangent space and satisfy $h^{ab}\overset{i}{\sigma}_a\overset{j}{\sigma}_b = 1$ for $i = j = 1, 2, 3$ and 0 otherwise. A spatial and temporal metric are compatible iff $h^{an}t_n = 0$. A vector field σ^a is spacelike iff $t_n\sigma^n = 0$, and timelike otherwise. Given the structure defined here, t_a induces a foliation of M into spacelike hypersurfaces, and relative to any such hypersurface, h^{ab} induces a unique spatial derivative operator D such that $D_a h^{bc} = 0$.[4] h^{ab} is flat just in case for any such spacelike hypersurface, D commutes on spacelike vector fields, i.e. $D_{[a}D_{b]}\sigma^c = 0$ for all spacelike vector fields σ. Finally, let ∇ be a connection on M. ∇ is compatible with the metrics just in case $\nabla_a t_b = 0$ and $\nabla_a h^{bc} = 0$.

With these structures in place, we can introduce Earman's (1989) original definition of a standard of rotation. Let t_a, h^{ab} be compatible temporal and spatial metrics on M, and let ∇, ∇' be a pair of flat derivative operators on M, both compatible with the metrics. ∇ and ∇' are *rotationally equivalent* just in case for any unit timelike vector field η^a on M, $\nabla^{[a}\eta^{b]} = 0 \Leftrightarrow \nabla'^{[a}\eta^{b]} = 0$. Then a standard of rotation compatible with t_a and h^{ab} is an equivalence class $[\nabla]$ of rotationally equivalent compatible flat derivative operators.

Within this framework, Dewar (2018) shows that one can formulate Newtonian gravitation theory as follows. Let t_a, h^{ab} be compatible temporal and spatial metrics on M, $[\nabla]$ an equivalence class of rotationally equivalent compatible flat derivative operators, and T^{ab} the Newtonian mass-momentum tensor for whichever matter fields are present. Let $\rho := t_a t_b T^{ab}$ be the scalar mass density field. Then $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ is a model of *Maxwell-Dewar gravitation*[5] just in case for all points $p \in M$ where $\rho \neq 0$, the following equations hold at p:

$$t_a \nabla_n T^{na} = 0 \tag{1a}$$

$$\nabla_m(\rho^{-1}\nabla_n T^{nm}) = -4\pi\rho \tag{1b}$$

$$\nabla^c(\rho^{-1}\nabla_n T^{na}) - \nabla^a(\rho^{-1}\nabla_n T^{nc}) = 0, \tag{1c}$$

where ∇ is an arbitrary member of $[\nabla]$.

Recently, however, Weatherall (2018, 34) has queried this definition of a standard of rotation, noting that it "makes reference to structure that one does not attribute to spacetime." Weatherall points to two criticisms of this approach. First, if a standard of rotation is defined as an equivalence class of derivative operators, then we must select an arbitrary member of this class to perform calculations. But some of the terms in these calculations may depend on the choice of derivative operator, and it is not clear how these should be interpreted. Secondly, one might worry that the appeal to derivative operators somehow obscures the structure of Maxwellian spacetime.

3. Here and throughout, abstract indices are written in Latin script; component indices are written in Greek script, with the exception of i, j, k, which are reserved for the spatial components of tensor fields in some coordinate basis; and the Einstein summation convention is used. Round brackets denote symmetrisation, square brackets antisymmetrisation.

4. See Weatherall (2018, 37–38) and Malament (2012, §4.1) for further details.

5. Note that Dewar (2018) calls this theory Maxwell gravitation; here, I reserve that name for presentations of the theory which do not make reference to any structure which is not definable from that of Maxwellian spacetime, such as the theory presented in section 3.

In response, Weatherall offers an alternative definition: if t_a, h^{ab} are compatible temporal and spatial metrics on M, a standard of rotation $\circlearrowright$ compatible with t_a and h^{ab} is a map from smooth vector fields ξ^a on M to smooth, anti-symmetric rank-$(2,0)$ tensor fields $\circlearrowright^b \xi^a$ on M, such that

1. $\circlearrowright$ commutes with addition of smooth vector fields;
2. Given any smooth vector field ξ^a and smooth scalar field α, $\circlearrowright^a (\alpha\xi^b) = \alpha \circlearrowright^a \xi^b + \xi^{[b} d^{a]}\alpha$;
3. $\circlearrowright$ commutes with index substitution;
4. Given any smooth vector field ξ^a, if $d_a(\xi^n t_n) = 0$ then $\circlearrowright^a \xi^b$ is spacelike in both indices; and
5. Given any smooth spacelike vector field σ^a, $\circlearrowright^a \sigma^b = D^{[a}\sigma^{b]}$.

One can then define a Maxwellian spacetime as a structure $\langle M, t_a, h^{ab}, \circlearrowright\rangle$, where $\circlearrowright$ is compatible with t_a and h^{ab}.

Now fix a spacetime $\langle M, t_a, h^{ab}\rangle$, and let ∇ and $\circlearrowright$ be a connection and standard of rotation on M, both compatible with the metrics. In what follows, we will often want to consider connections and standards of rotation which 'agree' with one another in the following sense: for any vector field η^a on M, $\nabla^{[a}\eta^{b]} = \circlearrowright^a \eta^b$. In this case, I will say that the connection and standard of rotation are *compatible*.[6] Likewise, a connection ∇ is compatible with a spacetime $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ just in case it is compatible with the metrics and $\circlearrowright$. Finally, I will say that a spacetime $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ is *rotationally flat* just in case h^{ab} is flat and there exists a unit timelike vector field ξ^a on M such that $\circlearrowright^a \xi^b = 0$ and $\pounds_\xi h^{ab} = 0$,[7] or equivalently, just in case some flat derivative operator is compatible with $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ (Weatherall 2018, Proposition 1). Where there is no ambiguity over the temporal and spatial metrics in question, I will sometimes drop talk of the metrics and simply refer to $\circlearrowright$ instead.

Finally, we need to say something about the Newtonian mass-momentum tensor T^{ab}. We have already seen that we can extract the scalar mass density field ρ from T^{ab} using the temporal metric. But in Maxwell-Dewar gravitation, we also used derivative operators to extract vector fields from T^{ab}. In what follows, we will likewise want to extract vector fields from T^{ab}, but without the use of derivative operators. To do this, we first impose the 'Newtonian mass condition': whenever $T^{ab} \neq 0$, $T^{nm}t_n t_m > 0$. This captures the idea that the matter fields we are interested in are massive, in the sense that there can only be non-zero mass-momentum in spacetime regions where the mass density is strictly positive.[8] Since T^{ab} is symmetric, the Newtonian mass condition guarantees that whenever $T^{ab} \neq 0$, we can uniquely decompose T^{ab} as

$$T^{ab} = \rho\xi^a\xi^b + \sigma^{ab} \tag{2}$$

where $\xi^a = \rho^{-1}t_n T^{na}$ is a smooth unit timelike future-directed vector field (interpretable as the net four-velocity of the matter fields F), and σ^{ab} is a smooth symmetric rank-$(2,0)$ tensor field which is spacelike in both indices (interpretable as the stress tensor for F).

6. This idea is made precise by Weatherall (2018, Proposition 1); the basic fact is that any connection determines a unique compatible standard of rotation, but a standard of rotation does not similarly determine a unique compatible connection.

7. Here and throughout, $\pounds$ denotes the Lie derivative.

8. For example, Weatherall (2012, 211) suggests that "[one] might take [the Newtonian mass condition] to be a benign and unsurprising characterisation of what we mean by "massive particle" in Newtonian gravitation."

We are now in a position to formulate Newtonian gravitation theory in terms of Weatherall's standard of rotation. Let $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$ be a Maxwellian spacetime, and let T^{ab} be the Newtonian mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ is a model of *Maxwell gravitation* just in case

(i) $\langle M, t_a, h^{ab}, \circlearrowleft\rangle$ is rotationally flat; and

(ii) For all points $p \in M$ such that $\rho \neq 0$, the following equations hold at p:

$$\pounds_\xi \rho - \frac{1}{2}\rho \hat{h}_{mn} \pounds_\xi h^{mn} = 0 \tag{MG1}$$

$$\frac{1}{3}\sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \Delta_n(\xi^m \Delta_m \overset{i}{\lambda}{}^r) = -\frac{4}{3}\pi\rho - \frac{1}{3}D_m(\rho^{-1}D_n\sigma^{nm}) \tag{MG2}$$

$$\pounds_\xi(\circlearrowleft^c \xi^a) + 2(\circlearrowleft^n \xi^{[c})\hat{h}_{nm}\pounds_\xi h^{a]m} + \circlearrowleft^c(\rho^{-1}D_n\sigma^{na}) = 0, \tag{MG3}$$

where $\hat{h}_{ab}$ is the spatial metric relative to ξ^a,[9] the $\overset{i}{\lambda}{}^a$ are three orthonormal connecting fields for ξ^a, and Δ is the 'restricted derivative operator' defined in Weatherall (2018). This acts on arbitrary spacelike vector fields σ^a at a point p according to

$$\eta^n \Delta_n \sigma^a := \pounds_\eta \sigma^a + \sigma_n \circlearrowleft^n \eta^a - \frac{1}{2}\sigma_n \pounds_\eta h^{an} \tag{5}$$

where η^a is a unit timelike vector at p (the Lie derivative is taken with respect to any extension of η^a off of p). It also has the property that $\eta^n\Delta_n\sigma^a = \eta^n\nabla_n\sigma^a$ for any derivative operator ∇ compatible with $\circlearrowleft$ (Weatherall 2018). The relationship between Maxwell gravitation and Maxwell-Dewar gravitation is summarised by the following pair of propositions (March 2023b; Chen 2023):

Proposition 1. *Let $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ be a model of Maxwell gravitation. Then there exists a unique equivalence class of rotationally equivalent flat derivative operators $[\nabla]$ such that all the $\nabla \in [\nabla]$ are compatible with $\circlearrowleft$ and $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ is a model of Maxwell-Dewar gravitation.*

Proposition 2. *Let $\langle M, t_a, h^{ab}, [\nabla], T^{ab}\rangle$ be a model of Maxwell-Dewar gravitation. Then there exists a unique standard of rotation $\circlearrowleft$ such that all the $\nabla \in [\nabla]$ are compatible with $\circlearrowleft$ and $\langle M, t_a, h^{ab}, \circlearrowleft, T^{ab}\rangle$ is a model of Maxwell gravitation.*

Finally, we can extend our discussion to Newton-Cartan theory. Let $\langle M, t_a, h^{ab}\rangle$ be a non-relativistic spacetime, ∇ a metric-compatible derivative operator on M, and T^{ab} the mass-momentum tensor for whichever matter fields are present. Then $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of *Newton-Cartan theory* just in case

$$\nabla_n T^{na} = 0 \tag{NCT1}$$

$$R_{ab} = 4\pi\rho t_a t_b \tag{NCT2}$$

$$R^a{}_b{}^c{}_d = R^c{}_d{}^a{}_b \tag{NCT3}$$

$$R^{ab}{}_{cd} = 0. \tag{NCT4}$$

The relation between Maxwell gravitation and Newton-Cartan theory is summarised by the following pair of propositions:

9. That is, the unique symmetric tensor field on M such that $\hat{h}_{an}\xi^n = 0$ and $h^{an}\hat{h}_{nb} = \delta^a{}_b - t_b\xi^a$.

Proposition 3. *Let $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ be a model of Newton-Cartan theory. Then there exists a unique standard of rotation $\circlearrowright$ such that ∇ is compatible with $\circlearrowright$ and $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ is a model of Maxwell gravitation.*

Proposition 4. *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of Maxwell gravitation. Then there exists a derivative operator ∇ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory. Moreover, the derivative operator ∇ is not unique. If ∇ is such a derivative operator, then so is $(\nabla, t_b t_c \sigma^a)$, where σ^a is any spacelike, twist-free, and divergence-free vector field such that $\rho\sigma^a = 0$.*

Corollary 4.1 (Chen 2023)**.** *Let $\langle M, t_a, h^{ab}, \circlearrowright, T^{ab}\rangle$ be a model of Maxwell gravitation such that $\rho \neq 0$ throughout some open set O. Then there exists a unique derivative operator ∇ such that $\langle M, t_a, h^{ab}, \nabla, T^{ab}\rangle$ is a model of Newton-Cartan theory.*

3 Maxwell gravitation and vector relationism

In §2, I have reviewed the recent literature on coordinate-free approaches to MG, and their relationship to NCT. But as noted in §1, these presentations of MG are rather distant from Saunders' original (2013) discussion of Newtonian gravitation on Maxwellian spacetime. This distance has three sources, which will occupy us for the rest of this section:

- Saunders' preferred characterisation of the appropriate setting for his vector relationist dynamics is as an affine space which he calls Newton-Huygens spacetime, rather than a differentiable manifold with differential-geometric objects defined thereon;
- Saunders' dynamics are presented in the coordinate-based framework; and
- Saunders' theory concerns only the dynamics of point particles, rather than continua.

Now, the first of these is easily dealt with—as Saunders himself notes, the idea of Newton-Huygens spacetime is just that Maxwellian spacetime *á la* Earman can be redescribed as an affine space, albeit one in which affine structure is appropriately restricted to spacelike hypersurfaces. And we have already seen that Earman's Maxwellian spacetime is equivalent to Weatherall's characterisation using a primitive rotation standard, along with the extra condition of rotational flatness. However, it is worth probing a little more deeply into the motivation for this spacetime structure. Saunders is not completely explicit on this, but the issue is addressed by Wallace (2020). First, we need to recall the details of Saunders' theory. Saunders presents vector relationism as a theory of the displacement vectors between point particles, formulated with reference to some Maxwellian coordinate system. The dynamics are specified by the following pair of equations:

$$\mathbf{r}_{ij} = \mathbf{X}_i - \mathbf{X}_j \tag{VR1}$$

$$\frac{d^2\mathbf{r}_{ij}}{dt^2} = \frac{1}{m_i}\sum_{k\neq i}\mathbf{F}_{ik} - \frac{1}{m_j}\sum_{k\neq j}\mathbf{F}_{jk}, \tag{VR2}$$

where $\mathbf{X}_i(t)$ denotes the position of particle i at time t with respect to such a coordinate system, m_i its mass, and the $\mathbf{F}_{ij}$ denote interparticle forces. These are taken to be antisymmetric in i and j (this is the import of Newton's third law) and functions of $\mathbf{r}_{ij}$ only. The equations (VR) are invariant under the Maxwell group (Pooley 2013)—transformations of the form

$$t \to t + \tau \tag{13a}$$

$$x^i(t) \to R^i{}_j x^j(t) + a^i(t), \tag{13b}$$

where $R^i{}_j$ is an arbitrary 3D rotation matrix, $a^i(t)$ an arbitrary vector-valued function of time, and τ an arbitrary scalar.

To argue that Maxwellian spacetime is the appropriate setting for vector relationism, Wallace makes tacit appeal to Earman's (1989) "adequacy conditions" on the construction of spacetime theories.[10] These demand that there be a match between the spacetime and dynamical symmetries of a theory, in the following sense:

SP1: Any dynamical symmetry of T is a spacetime symmetry of T.

SP2: Any spacetime symmetry of T is a dynamical symmetry of T.

It is straightforward to show that if a Maxwellian spacetime is rotationally flat, its automorphism group is the Maxwell group (one can simply make use of the argument given in e.g. Earman (1989, ch. 2.3) or Jacobs (2023, Proposition 5), noting that diffeomorphisms preserve flatness of any compatible connection). This justifies the claim that Maxwellian spacetime is the appropriate setting for vector relationism.

With this in hand, I will now address the second two bullet points by examining the relationship between the equations (MG) and (VR). First, following Wallace (2020, 11), we can decompose the forces in (VR2) into 'universal' and 'non-universal' components—characterised, respectively by whether the ratio q_i/m_i is constant for that force, where m_i is the inertial mass of a particle and q_i its charge. For the case of only potential forces, (VR) may then be written as

$$\begin{aligned} \frac{d^2\mathbf{X}_i}{dt^2} - \frac{d^2\mathbf{X}_j}{dt^2} &= -\sum_{k\neq i} \nabla\phi(\mathbf{X}_i - \mathbf{X}_k) + \sum_{k\neq j} \nabla\phi(\mathbf{X}_j - \mathbf{X}_k) \\ &\quad - \frac{q_i}{m_i}\sum_{k\neq i} \nabla V(\mathbf{X}_i - \mathbf{X}_k) + \frac{q_j}{m_j}\sum_{k\neq j} \nabla V(\mathbf{X}_j - \mathbf{X}_k), \end{aligned} \tag{14}$$

where ϕ is the potential associated with the universal force, and V the potential for the non-universal force (there could be multiple such; I omit them for simplicity). Now consider the continuum limit, where point-particle trajectories are parametrised by some continuous spatial parameter $\mathbf{x}$. In this limit, (14) becomes

$$\begin{aligned} \partial_i\left(\frac{d^2\mathbf{X}(\mathbf{x},t)}{dt^2}\right)\delta x^i &= -\partial_i \int d^3\mathbf{x}' \nabla\phi(\mathbf{x}-\mathbf{x}',t)\delta x^i \\ &\quad - \partial_i \int d^3\mathbf{x}' \tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\nabla V(\mathbf{x}-\mathbf{x}',t)\delta x^i, \end{aligned}$$

where $\rho(\mathbf{x},t)$ is the mass density, and $\tilde{\rho}(\mathbf{x},t)$ the charge density associated with the non-universal interaction, so that

$$\partial_i\left(\frac{d^2X^j(\mathbf{x},t)}{dt^2}\right) = -\partial_i \int d^3\mathbf{x}' (\partial^j\phi(\mathbf{x}-\mathbf{x}',t) + \tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\partial^j V(\mathbf{x}-\mathbf{x}',t)). \tag{15}$$

10. For recent discussion of the status of these conditions, see Myrvold (2019). Note that Myrvold considers these conditions to be *analytically* true.

When ϕ is the familiar gravitational potential, we have

$$\phi(\mathbf{x}-\mathbf{x}',t)=\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|},$$

so that

$$\partial_i\left(\frac{d^2X^j(\mathbf{x},t)}{dt^2}\right)=-\partial_i\int d^3\mathbf{x}'\rho(\mathbf{x}',t)\partial^j(|\mathbf{x}-\mathbf{x}'|)^{-1} - \partial_i\int d^3\mathbf{x}'\tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\partial^j V(\mathbf{x}-\mathbf{x}',t). \qquad (16)$$

We have seen that the appropriate spacetime setting for vector relationism is a rotationally flat Maxwellian spacetime, $\langle M,t_a,h^{ab},\circlearrowleft\rangle$. Since we can always (if M is simply connected) find a globally defined scalar field t such that $d_a t=t_a$, we can then set up an arbitrary Maxwellian coordinate system x^μ on M as follows: we take $x^\mu=(t,x^i)$, where t is as above and the x^i are three smooth scalar fields such that the vector fields $(\partial/\partial x^i)^a$ are spacelike, orthonormal, and twist-free (with respect to $\circlearrowleft$).[11]

Let x^μ be such a coordinate system, and let ∇ be the coordinate derivative operator on M canonically associated with x^μ.[12] ∇ is flat (since it is a coordinate derivative operator); it is compatible with t_a by construction, and is compatible with h^{ab} since the $(\partial/\partial x^i)^a$ are spacelike and orthonormal. Moreover, since the $(\partial/\partial x^\mu)^a$ are all twist-free with respect to $\circlearrowleft$ and $\circlearrowleft$ is rotationally flat, ∇ is also compatible with $\circlearrowleft$.[13] Now consider a smooth unit timelike vector field ξ^a on M. The integral curves ξ of any such field can always be parametrised by their temporal length, which differs from t by at most an arbitrary additive constant. Then on any such curve ξ, we have

$$\xi^a=\frac{dx^\mu(\xi(t))}{dt}\left(\frac{\partial}{\partial x^\mu}\right)^a$$

so that, since ∇ is flat

$$\xi^n\nabla_n\xi^a=\frac{d^2x^\mu(\xi(t))}{dt^2}\left(\frac{\partial}{\partial x^\mu}\right)^a.$$

Clearly, the only non-vanishing d^2x^μ/dt^2 are the d^2x^i/dt^2. Moreover, if σ^{ab} is a (symmetric) tensor field which is spacelike in both indices, then we can write

$$D_n\sigma^{na}=\partial_\mu\sigma^{\mu\nu}\left(\frac{\partial}{\partial x^\nu}\right)^a$$

where the only non-vanishing $\partial_\mu\sigma^{\mu\nu}$ are the $\partial_\mu\sigma^{\mu i}$. If we now take ξ^a to represent the four velocity field of a fluid, and σ^{ab} the stress tensor for that fluid, then these suggest the following identifications:

$$\xi^n\nabla_n\xi^m(d_m x^i)=\frac{d^2X^i(\mathbf{x},t)}{dt^2} \qquad (17a)$$

$$\rho^{-1}D_n\sigma^{nm}(d_m x^i)=\int d^3\mathbf{x}'\tilde{\rho}(\mathbf{x},t)\rho^{-1}(\mathbf{x},t)\partial^i V(\mathbf{x}-\mathbf{x}',t). \qquad (17b)$$

11. If M is not simply connected then the same analysis goes through locally; I suppress it here for reasons of brevity.

12. That is, the unique derivative operator such that all the $\nabla_a(\partial/\partial x^\mu)^b=0$.

13. Note that $(\partial/\partial t)^a$ is twist-free by construction, since t_a is closed.

Why? Take (17a). We are looking for something with which to identify the (non-zero) components of the acceleration vector field of a fluid $\xi^n \nabla_n \xi^m (d_m x^i)$ with respect to the coordinate derivative operator canonically associated with some Maxwellian coordinate system x^μ. Not only is this precisely what the $d^2 X^i(\mathbf{x}, t)/dt^2$ represent, we have also seen that when ∇ is such a derivative operator, the $\xi^n \nabla_n \xi^m (d_m x^i) = d^2 x^i(\xi(t))/dt^2$ take this same form. Now consider (17b). The left hand side of this equation are the (non-zero) components of a spacelike vector field which is supposed to describe the acceleration due to non-gravitational interactions—think of (the geometrised version of) Newton's second law

$$\rho \xi^n \nabla_n \xi^a = -\nabla_n \sigma^{na}. \tag{NII}$$

And this is precisely the role of the term on the right hand side. We can then write (16) as

$$\nabla_r (\xi^n \nabla_n \xi^m)(d_m x^j) \left(\frac{\partial}{\partial x^i} \right)^r = -\partial_i \int d^3\mathbf{x}' \rho(\mathbf{x}', t) \partial^j (|\mathbf{x} - \mathbf{x}'|)^{-1} - D_r(\rho^{-1} D_n \sigma^{nm})(d_m x^j) \left(\frac{\partial}{\partial x^i} \right)^r. \tag{19}$$

Now consider the case where $i = j$. In this case, carrying out the differentiation in the right hand side of (19) gives

$$\nabla_m (\xi^n \nabla_n \xi^m) = -4\pi\rho - D_m(\rho^{-1} D_n \sigma^{nm})$$

where we have used the fact that $\xi^n \nabla_n \xi^a$ and $\rho^{-1} D_n \sigma^{na}$ are both spacelike. This immediately yields (MG2). Meanwhile, if we take $i \neq j$ in (19), then differentiating and raising indices we have

$$\nabla^r (\xi^n \nabla_n \xi^m)(d_m x^j)(d_r x^i) = \int d^3\mathbf{x}' \rho(\mathbf{x}', t) \left(3 \frac{(x^j - x'^j)(x^i - x'^i)}{|\mathbf{x} - \mathbf{x}'|^5} \right) - D^r(\rho^{-1} D_n \sigma^{nm})(d_m x^j)(d_r x^i),$$

so that, since $\circlearrowright^a$ $(\xi^n \nabla_n \xi^b)$ is spacelike in both indices,

$$\nabla^a (\xi^n \nabla_n \xi^b) - \nabla^b (\xi^n \nabla_n \xi^a) = -D^a(\rho^{-1} D_n \sigma^{nb}) + D^b(\rho^{-1} D_n \sigma^{na}),$$

which, given the continuity equation (MG1) and the fact that ∇ is flat by construction, entails (MG3) (see the proof of proposition 2). For (MG1) itself, note that in Newtonian point particle mechanics, mass is transported only by particles along their (continuous) worldlines, and is *a fortiori* locally conserved.

Conversely, it is also possible to recover (VR) from (MG). Given the identifications (17), we can use (MG) to derive expressions for $\partial_i (d^2 X^i/dt^2)$ and $\partial^{[i} (d^2 X^{j]}/dt^2)$ in any Maxwellian coordinate system x^μ on M. These are sufficient to specify (15) uniquely, providing that $\partial_i (d^2 X^i/dt^2)$ and $\partial^{[i} (d^2 X^{j]}/dt^2)$ fall off at least as $1/r^2$ at spatial infinity. If we then specialise to the case of a point-particle distribution (which justifies making the above assumptions about $d^2 X^i/dt^2$), this gives $\phi(\mathbf{x} - \mathbf{x}', t) \to \phi(\mathbf{x} - \mathbf{x}', t) \sum_i \delta^3(\mathbf{x}' - \mathbf{X}_i(t))$ and analogously for V. Hence,

$$\partial_i \left(\frac{d^2 X^j(\mathbf{x}, t)}{dt^2} \right) = -\partial_i \sum_k \partial^j \phi(\mathbf{x} - \mathbf{X}_k, t) - \partial_i \sum_k \tilde{\rho} \rho^{-1} \partial^j V(\mathbf{x} - \mathbf{X}_k, t). \tag{20}$$

Since $\tilde{\rho}\rho^{-1} = \sum_i q_i/m_i \delta^3(\mathbf{x} - \mathbf{X}_i(t))$, (14) then follows from integrating along any path between $\mathbf{X}_i(t)$ and $\mathbf{X}_j(t)$.

So for all their surface-level differences, there is a close relationship between Maxwell gravitation and vector relationism. Both are set on rotationally flat Maxwellian spacetime. Moreover, the equations of Maxwell gravitation emerge naturally in the continuum limit of vector relationism, whilst vector relationism is precisely what results from restricting Maxwell gravitation to the point particle sector. In turn, this appears to support Dewar's (2018) claim that "Maxwell gravitation [...] represents the natural extension of Saunders' remarks to the field-theoretic context." Dewar argues for this on the basis that Maxwell gravitation, like vector relationism, collapses the distinction between materially identical models of Newton-Cartan theory. However, the fact that Maxwell gravitation can be recovered in the continuum limit of vector relationism, and *vice versa*, provides a more direct route to this conclusion.

4 Wallace on vector relationism and Newton-Cartan theory

With the relationship between vector relationism and Maxwell gravitation on a firmer footing, I will now turn to Wallace's (2020) discussion of vector relationism and Newton-Cartan theory. Here, Wallace claims to show that "mathematically speaking, there is no real distinction between Newton-Cartan theory [...] and vector relationism" (24), and suggests that any differences between the two theories are partly an artefact of the awkwardness of standard differential-geometric presentations of Maxwellian spacetime (28). As a result, Wallace adheres to a coordinate-based presentation of both theories in setting out his argument.

Wallace's discussion of vector relationism and Newton-Cartan theory centres on the behaviour of dynamically isolated subsystems of particles embedded in a larger universe—showing that within vector relationism, such systems exhibit emergent inertial behaviour which can be idealised in terms of test particles. This forms the basis of his argument that vector relationism and Newton-Cartan theory are equivalent. When non-gravitational interactions vanish, the equations governing the relative acceleration vectors of infinitesimally separated test particles can be written to take the same form as the (coordinate-based) equation of geodesic deviation in Newton-Cartan theory, and thus, Wallace claims, may equally well be interpreted as such (§8).

Wallace is not explicit about the standard of theoretical equivalence he is working with here. But it is fairly straightforward to reconstruct from his remarks what he may have in mind. Having recovered the Newton-Cartan equation of geodesic deviation within vector relationism, Wallace claims of the two theories that

> both are built using Maxwellian spacetime as a background; both have dynamics that can be expressed as a set of inertial trajectories defined by the matter distribution and in turn constraining the matter distribution via a matter dynamics according to which material particles follow those trajectories except when acted on by non-gravitational forces. (Wallace 2020, 24)

Similarly, in his concluding remarks, Wallace argues that

> there is essentially no difference between Newton-Cartan theory [...] and Saunders's relational version of Newtonian dynamics: at the formal level, the latter can be reformulated as the former; at the substantive level, the inertial structure of Saunders's theory is well defined

> and coincides with that defined by the Newton–Cartan connection. (Wallace 2020, 28)

From these remarks, one can isolate three points which Wallace takes to bear on whether Maxwell gravitation and Newton-Cartan theory are equivalent:

1. They have the same background spacetime structure.
2. Their central dynamical equations can be (re)written so as to appear mathematically identical.
3. They have the same inertial structure.

For our purposes, we can elevate this to a criterion of theoretical equivalence, though it should be borne in mind both that Wallace does not explicitly endorse this, and that such a criterion may be more or less well-suited to theories other than vector relationism and Newton-Cartan theory. I will now make several comments on this criterion, all of which will indicate refinements of the points 1-3 above.

First, on point 1, what does the 'background spacetime structure' of a theory consist in? In the literature, there are various competing schools of thought about how this is to be identified. For example, one might take 'spacetime structure' to be objects of a certain object-type that appear between the angle brackets of a theory's models *á la* e.g. Earman (1989) or Friedman (1983), or one might invoke a criterion such as Knox's (2013) spacetime functionalism, according to which 'spacetime structure' is just whatever it is that encodes the local structure of inertial frames.[14] Again, Wallace's remarks give some hint as to what he may have in mind here:

> [In] Newton–Cartan theory, the connection does double duty, imposing both the rotation standard (a piece of absolute structure) and the inertial structure (something dynamical and contingent). One purpose of my somewhat idiosyncratic presentation of Newton–Cartan theory is to emphasize the fact that the Newton–Cartan connection is naturally understood as an additional piece of structure added to Maxwellian spacetime; indeed, as the Maxwellian version of the affine connection. (Wallace 2020, 29)

This suggests, for the purposes of point 1, that we should take the 'background spacetime structure' of a theory to be its absolute objects. The absolute objects of a theory are those which are the same in all its DPMs, where 'sameness' is sameness up to isomorphism (see e.g. Earman (1989, 45)). If this is the right precisification of 1, then Maxwell gravitation and Newton-Cartan theory do indeed have the same background spacetime structure as Wallace claims—see March (2023a).

Second, on 2, one might worry about the restriction to the 'central' dynamical equations of a theory. Whilst I won't attempt to address the question of what it means for some equation or other to be 'central' to a theory here, note that this restriction is needed because Wallace does not explicitly consider all the equations of Newton-Cartan theory in his analysis (and as we will see in section 5, not all the equations of Maxwell gravitation and Newton-Cartan theory (or vector relationism and Newton-Cartan theory, for that matter) can be written so as to appear mathematically identical).

14. Though note that Knox's spacetime functionalism cannot be the right criterion if we are looking to identify Maxwellian spacetime as the background spacetime structure of Maxwell gravitation and Newton-Cartan theory, since Maxwellian spacetime by itself lacks a full inertial frame structure.

Continuing with point 2, one might also ask what it means for the central equations of two theories to 'be rewritten so as to appear mathematically identical'. For our purposes, we can take this mean: we can re-express the dynamics of the theories so that they have some non-empty set of equations (the 'central' ones) in common, whilst preserving solutionhood.

Finally on point 3, how, according to Wallace, are we to identify the inertial structure of a theory? Here, Wallace closely follows Knox (2013): the inertial structure of a theory is whatever it is that encodes the local structure of inertial frames, i.e. those with respect to which gravitating but otherwise force-free bodies move with constant velocities, in which the equations governing non-gravitational interactions take their simplest form, and which are universal (in the sense that all bodies and interactions pick out this same class of frames). Crucially, in the case of non-relativistic theories, this means that if there exists a connection such that (NII) is satisfied, then this (amongst other things) qualifies it as encoding the inertial structure of that theory.

5 Understanding Wallace from a coordinate-free perspective

Having presented Wallace's argument, I will now show that with Maxwell gravitation in hand, the same argument can be made in the language of coordinate-free differential geometry. For point 1, we have already noted that Maxwell gravitation and Newton-Cartan theory have the same absolute object structure (March 2023a). For point 2, following Wallace, let us compare the equations of Maxwell gravitation with those of Newton-Cartan theory. Let $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$ be a Maxwellian spacetime. Then for any derivative operator ∇ compatible with $\langle M, t_a, h^{ab}, \circlearrowleft \rangle$, the following implications hold (illustrated in Figure 1). Statements and proofs of these equivalences are contained in appendix A.

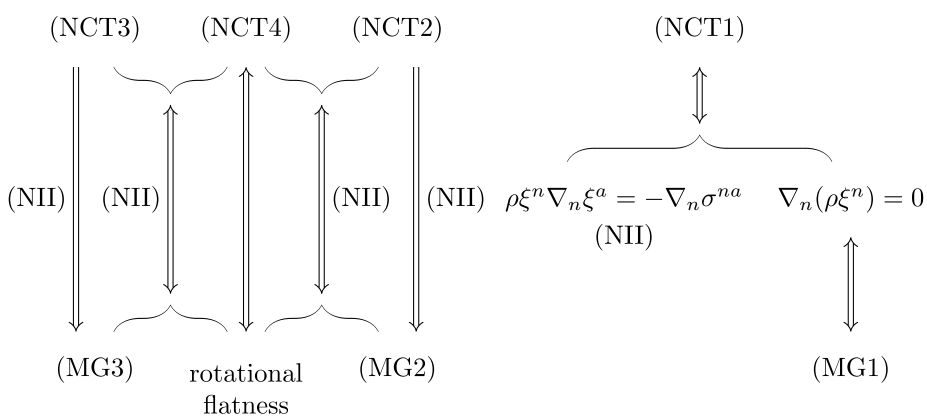

Figure 1: Relationships between the equations of Maxwell gravitation and Newton-Cartan theory. Labelled arrows are to be understood as in the scope of a conditional—so e.g. the first arrow from the left says that if (NII) holds, then (NCT3) implies (MG3)

There are several features of Figure 1 worth noting. First, whilst (NCT4) is equivalent to the rotational flatness condition, there is no similarly sharp correspondence between (NCT3) and (MG3). (NCT3) and (NII) jointly imply (MG3), but (MG3) and (NII) do not imply (NCT3). This points to the fact that rotational flatness plays double duty in relating the two theories. From (MG3) and (NII) we can infer that $\xi^n\xi^m(R^c{}_n{}^a{}_m - R^a{}_m{}^c{}_n) = 0$; the rotational flatness condition allows us to further infer that $\xi^n h^{bm}(R^c{}_n{}^a{}_m - R^a{}_m{}^c{}_n) = 0$, which yields (NCT3).[15]

Secondly, although (NCT2) and (MG2) are not in general equivalent, they are equivalent on assumption of (NII) and rotational flatness. Likewise, given (NII), rotational flatness and (MG3) are equivalent to (NCT4) and (NCT3). As such, once (NII) has been fixed, we can then move freely between the remaining pairs of equations.

Now recall point 2 of Wallace's argument: for an idealised congruence of test particle trajectories, the equations (VR) can be rewritten so as to take the same form as the equation of geodesic deviation in Newton-Cartan theory. But we have just seen that this has an obvious analogy for Maxwell gravitation and Newton-Cartan theory: by replacing (NCT2) with the expression for the average radial acceleration (MG2), we can reformulate the two theories so that their central dynamical equations appear mathematically identical. Within Newton-Cartan theory, (MG2) encodes the relative acceleration of neighbouring fluid elements due to both spacetime curvature and non-gravitational interactions, so represents the natural generalisation of Wallace's geodesic deviation equation to non-test matter. And just as in Wallace's example, the only difference, as far as this pair of equations is concerned, is the interpretation of (MG2)—in Newton-Cartan theory, the $-4/3\pi\rho$ term is naturally understood as a manifestation of geodesic deviation in curved spacetime, whereas in Maxwell gravitation it is not.

Moreover, once we move from vector relationism to Maxwell gravitation, the case for regarding this disagreement as merely verbal appears even stronger. After all, in vector relationism, the gravitational field is explicitly represented elsewhere in the formalism. But in Maxwell gravitation, we do not even have that. Of course, we are always free to ascribe the $-4/3\pi\rho$ term in (MG2) to 'the gravitational field'—but without some further indication of what this is supposed to be, the gravitational field is simply that whereby neighbouring test particles have non-zero relative acceleration. And since this is precisely the role of the Newton-Cartan spacetime curvature, the difference between the two begins to look insubstantive. As such, we seem to have in the relationship between (MG2) and (NCT2) a coordinate-free realisation of point 2 of Wallace's argument.

However, we can also say a little more about this reasoning. Given the relationships illustrated in Figure 1, not only are we free to replace (NCT2) with (MG2) in Newton-Cartan theory, we can also replace (NCT3) with (MG3), (NCT4) with the rotational flatness condition, and rewrite (NCT1) as the conjunction of (NII) and (MG1). From this perspective, the only difference between these sets of equations is the presence of (NII) in Newton-Cartan theory, whose role is essentially to provide a (partial) gauge fixing of the connection. This provides a further sense in which point 2 of Wallace's argument is strengthened when we move from vector relationism to Maxwell gravitation—*all* the equations of Newton-Cartan theory, with the exception of (NII), can be written so as to appear mathematically identical to the equations of Maxwell gravitation.

Note that this also highlights why it is that Newton-Cartan theory cannot be the continuum limit of vector relationism. If one assumes that the dynamics for test particles in Newton-Cartan theory are given by the geodesic equation, then it is possible to show that in both Newton-Cartan theory and the continuum limit of vector relationism, test particles satisfy the equation of geodesic deviation. But precisely what one cannot recover in the continuum limit of

15. Recall that $h^{dn}h^{bm}(R^c{}_n{}^a{}_m - R^a{}_m{}^c{}_n) = 0$ in any classical spacetime.

vector relationism is the geodesic equation itself—or rather its generalisation to non-test matter (NII).

Finally, this brings us to point 3, viz. the inertial structure of Maxwell gravitation, such as it is. For this, it is helpful to recall proposition 4. Proposition 4 tells us that, providing there is sufficient matter in one's spacetime, there exists a unique Newton-Cartan connection which satisfies (NII), i.e. such that massive test bodies follow geodesics. Moreover, providing that the test bodies of interest are sufficiently far from other massive matter (which we can idealise as meaning at spatial infinity), then this connection will, at least locally, be well-approximated by a flat connection. This allows us to recover (and expand upon) Wallace's claims about the emergence of inertial structure in Maxwell gravitation, in three ways.

First, suppose that we say, with Wallace, that what it is to encode the inertial structure of a theory just is to be the unique connection such that massive test bodies follow geodesics. Then it follows that, whenever the conditions of corollary 4.1 are satisfied, a model of Maxwell gravitation does indeed come equipped with an inertial structure, which coincides with the Newton-Cartan connection. So whilst Maxwellian spacetime lacks full inertial structure by itself, there is an emergent such structure to be had for those models in which there is sufficient matter available.

Second, continuing with the above theme, if we have antecedent reasons for adopting (NII) as an implicit definition of the inertial structure of a theory, then we might as well go ahead and add this as an extra condition to those models of Maxwell gravitation in which there are open sets throughout which the mass density field is non-vanishing. In that case, one can also recover the dynamics of Newton-Cartan theory from those of Maxwell gravitation. So whilst one cannot rewrite the dynamics of Maxwell gravitation to appear mathematically identical to those of Newton-Cartan theory *by themselves*, there is a natural sense in which the dynamics of Maxwell gravitation plus definitions *are* sufficient to recover the dynamics of Newton-Cartan theory, again providing there is sufficient matter in one's spacetime. This provides a way of making sense of Wallace's claim that "Saunders's vector relational version of Newtonian dynamics [...] can be reformulated as [Newton-Cartan theory]" (Wallace 2020, 28).

Third, corollary 4.1 clarifies just what is needed for the emergence of this inertial structure. In particular, sufficient for this is that there exist open sets throughout which the mass density field is non-vanishing. So providing that we are doing non-vacuum continuum mechanics (or even for certain point particle distributions—see March (2023a)) then the above arguments can be made; one does not need to consider a full congruence of particle trajectories.

All this serves to blunt the force of Wallace's (2019; 2020) recent claims that Maxwellian spacetime is not naturally characterised in coordinate-free differential geometric terms, and that this is partly what obscures the similarities between Maxwell gravitation and Newton-Cartan theory. Rather, we have seen that once cast in terms of Weatherall's standard of rotation, the formal similarities which Wallace discusses re-emerge from a coordinate-free perspective. As a result, one might suspect that the problem lies not with coordinate-free differential geometry *per se*, but with formulating a theory in terms of geometric objects which cannot be defined from the structure it ascribes to the world.[16]

But it does suggest an alternative moral. Both Maxwell and Maxwell-Dewar gravitation are formulated in the language of coordinate-free differential geom-

16. c.f. Pitts (2012, 2022, 2006). For an extended discussion of other possible issues relating to this in the context of the interpretation vs. motivation and reduction vs. sophistication debates, see Jacobs (2022).

etry. But the fact that a theory has been formulated in a coordinate-free way does not automatically mean that this is a perspicuous way of presenting that theory. When working with coordinate-free differential geometry, as ever, it is important to be attentive to this possibility.

6 The link with Teh

Finally, I will consider the relationship between my discussion of Wallace and that of Teh (2018), who adopts a rather different strategy for diffusing Wallace's concerns about the coordinate-free framework. Teh's approach begins by noting that compatible connections on a classical spacetime can be represented by means of a special connection (for some unit timelike vector field ξ^a) and a two-form Ω_{ab} (see Malament (2012, Propositions 4.3.4, 4.1.3). Providing the connection of interest satisfies (NCT3), this two-form is closed, and so can (at least locally) be specified by a one-form A_a, defined up to exact one-form shifts. Since ξ^a is geodesic with respect to its special connection, one can therefore view ξ^a as encoding a 'background inertial structure', and Ω_{ab} as encoding the forces experienced by bodies relative to this inertial structure. Alternatively, one can view Ω_{ab} as encoding the force differences between different idealised congruences of particle trajectories, and so as realising Saunders' vector relationist dynamics (Teh 2018, 207).

How does this allow one to make Wallace's argument, and in what ways does this address Wallace's concerns about the coordinate-free framework? On this, one can identify three points:

- As Teh himself (2018) notes, suppose we are given an equivalence class $[\nabla]$ of rotationally equivalent (not necessarily flat) connections which satisfy (NCT3). Any such connection will be the special connection for some unit timelike vector field ξ^a. Now suppose that we are given another special connection ∇. Then all the ξ^a have the same rotation tensor with respect to ∇. This, Teh claims, furnishes the notion of rotational equivalence with a physical interpretation in terms of representations which share the same vorticity.

- Now suppose that the connections in this equivalence class are, in addition, flat. Then the choice of such a connection is equivalent to a choice of inertial frame (since the ξ^a in question must now be rigid). So the equivocation involved in defining Maxwellian spacetime *á la Earman* (one might think) is no worse than that involved in equivocating between Maxwellian coordinate systems when writing down e.g. Saunders' vector relationist dynamics.

- Teh's framework emphasises the way in which the two-form Ω_{ab} used to pick out the Newton-Cartan connection of interest can always be reinterpreted as encoding either forces experienced by test bodies relative to the inertial structure defined by ξ^a, or as encoding a connection relative to which those same test bodies exhibit geodesic motion. Or in other words, that there is no mathematical difference between the universal forces of vector relationism and the geodesic motion in curved spacetime of Newton-Cartan theory, as Wallace argues.[17]

Nevertheless, we have seen that we can do better. For example, the fact that rotationally equivalent connections can be given a physical interpretation

17. For a slightly different take on this issue, see Weatherall and Manchak (2014).

as Teh suggests does not eliminate the need to take equivalence classes—it just amounts to insisting that it is physically meaningful to do so! So if Wallace's concern about the 'awkwardness' of Maxwellian spacetime was about the need to invoke equivalence classes *at all*, Teh's discussion does not help with this. Likewise, insofar as Teh's framework highlights the fact that equivocating between rotationally equivalent flat connections is the same as equivocating between inertial frames, this might just seem like grist to Wallace's mill: wasn't one of the advantages of the coordinate-free approach supposed to be that it avoids all this need for equivocation, since we can just talk about the objects of interest directly?

So whilst Teh's discussion has much to recommend it, it is not clear that it is sufficient to address Wallace's concerns. On the other hand, we have seen that Maxwell gravitation *does* address these issues. Moreover, the discussion of §5 highlights which of Teh's constructions carry over to Maxwellian spacetime characterised 'intrinsically' and which do not. In particular, Teh's 'proto-Maxwell spacetime'—which he defines using an equivalence class of rotationally equivalent connections all of which satisfy (NCT3)—cannot be defined using just Weatherall's standard of rotation.

7 Conclusions

In this paper, my aim has been to connect up the recent wave of 'intrinsic' coordinate-free approaches to Maxwellian spacetime with the coordinate-based discussions of Saunders (2013) and Wallace (2020). By doing so, I have clarified the relationship between vector relationism and Maxwell gravitation (the latter is just the continuum limit of the former, as one would have hoped); I have also explained why Newton-Cartan theory is not the continuum limit of vector relationism, *contra* the appearance of Wallace's discussion. Finally, I have shown how the similarities between vector relationism and Newton-Cartan theory which Wallace discusses can also straightforwardly be seen using the coordinate-free approach, and used this both to assess Wallace's argument, and the extent to which Teh's discussion of Wallace makes good on its aims.

In many ways, the upshot of all this is irenic. The coordinate-free framework undoubtedly has its advantages: one is that it allows one to talk about the objects of interest *directly* (rather than just their components in some coordinate basis), it is undoubtedly necessary for a fully-rigorous treatment of certain topics, and it minimises certain opportunities for confusion (e.g. between active and passive coordinate transformations, or coordinate transformations and diffeomorphisms, or which properties of an object are viciously coordinate-dependent and which are not, or...). On the other hand, Wallace (2019) has made the case that the coordinate-based framework allows for better cohesion with physics practice, and is better suited to discussions of Brown's dynamical approach to spacetime theories (among others). One might also—and somewhat more plausibly!—take Wallace's claims about the supposed 'awkwardness' of Maxwellian spacetime as really offering an argument in favour of the coordinate-based approach in cases where no 'intrinsic' coordinate-free characterisation of some spacetime structure of interest is as yet available (or as an incentive to develop one).[18] In any case, I hope to have laid to rest the idea that the example of Maxwellian spacetime provides a reason to prefer one approach over the other.

18. c.f. Pitts (2012) on spinors.

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A Statements and proofs of equivalences

Let $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ be a Maxwellian spacetime, ∇ any compatible connection, and T^{ab} the mass-momentum tensor, which we assume to satisfy the Newtonian mass condition. That (NCT4) holds iff $\langle M, t_a, h^{ab}, \circlearrowright\rangle$ is rotationally flat is shown by Malament (2012, Proposition 4.2.4); that (NCT1) holds iff (NII) and (MG1) hold is shown by Malament (2012, 266), noting that $\xi^n \nabla_n \rho = \pounds_\xi \rho$ and $\nabla_n \xi^n = -1/2 \hat{h}_{nm} \pounds_\xi h^{nm}$.

For the remaining four implications, assume that (NII) holds. A straightforward computation shows that we can use ∇ to rewrite (MG3) as

$$\xi^n \nabla_n (\omega^{ca}) = 2\omega^{n[c} \theta_n{}^{a]} - \nabla^{[c}(\rho^{-1} \nabla_n \sigma^{|n|a]})$$

where ω^{ab}, θ^{ab} are the rotation and expansion tensors for ξ^a, respectively. It follows that, given (NII)

$$\xi^n \nabla_n (\omega^{ca}) = 2\omega^{n[c} \theta_n{}^{a]} + \nabla^{[c}(\xi^{|n|} \nabla_n \xi^{a]}). \tag{21}$$

Likewise (MG2) can be rewritten as

$$\frac{1}{3} \sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \nabla_n (\xi^m \nabla_m \overset{i}{\lambda}{}^r) = -\frac{4}{3}\pi\rho + \frac{1}{3} \nabla_m (\xi^n \nabla_n \xi^m). \tag{22}$$

Now we just need to do some calculations, which follow the proofs of propositions 4.3.6, 1.8.5, and 4.3.2 of Malament (2012) closely. First

$$\begin{aligned}
\xi^n \nabla_n (\omega^{ca}) &= \nabla^{[c}(\xi^{|n|} \nabla_n \xi^{a]}) - (\nabla^{[c} \xi^{|n|})(\nabla_n \xi^{a]}) + (R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)\xi^n \xi^m \\
&= 2\omega^{n[c} \theta_n{}^{a]} + \nabla^{[c}(\xi^{|n|} \nabla_n \xi^{a]}) + (R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)\xi^n \xi^m
\end{aligned}$$

where we have made use of the fact that ω^{ab} is spacelike in both indices. So if (NCT3) holds, (MG3) immediately follows. Conversely, if (MG3) holds then comparison with (21) yields that $(R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n)\xi^n \xi^m = 0$. Then to establish (NCT3), we just need to show that $(R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n) h^{nb} \xi^m = 0$ (since $(R^a{}_n{}^c{}_m - R^c{}_m{}^a{}_n) h^{nb} h^{md} = 0$ in any classical spacetime). This, in turn, follows from rotational flatness (using the symmetries of the Riemann tensor). Note that rotational flatness is needed here because ξ^a need not be twist-free. Next,

$$\begin{aligned}
\frac{1}{3} \sum_{i=1}^{3} \overset{i}{\lambda}_r \xi^n \nabla_n (\xi^m \nabla_m \overset{i}{\lambda}{}^r) &= \frac{1}{3} \sum_{i=1}^{3} \overset{i}{\lambda}_r (\overset{i}{\lambda}{}^n \nabla_n (\xi^m \nabla_m \xi^r) + R^r{}_{nms} \xi^n \overset{i}{\lambda}{}^m \xi^s) \\
&= \frac{1}{3} \nabla_n (\xi^m \nabla_m \xi^n) - \frac{1}{3} R_{nm} \xi^n \xi^m
\end{aligned}$$

so that if (NCT2) holds, so does (MG2). Conversely, if (MG2) holds, then $R_{nm} \xi^n \xi^m = 4\pi\rho$. If we then assume rotational flatness we also have that $R^a{}_n \xi^n = R^{ab} = 0$, which gives us (NCT2).

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