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A positive and elementary stable nonstandard explicit scheme for a mathematical model of the influenza disease

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Abstract

In this paper, a nonstandard explicit discretization strategy is considered to construct a new nonstandard finite difference scheme for solving a mathematical model of the influenza disease. The new proposed scheme has some interesting properties such as high accuracy and ease of implementation, as well as some preserving properties of the exact theoretical solution of the SIRC system, like positivity and elementary stability. These characteristics make it suitable for solving efficiently the propose model. We provide some numerical comparisons to illustrate our results.

MSC: 65L05; 03H05; 03H10

Keywords: Nonstandard finite difference; Positivity; Elementary stability; Conservation law

1. Introduction

Ordinary differential equations (ODEs) are used extensively in the modeling of many biological and physical applications. They constitute a central component in applied mathematics and their numerical simulations are of fundamental importance in gaining the correct qualitative and quantitative information on the systems. Numerical methods based on the finite difference approximations [3,4,9,28], Taylor series expansion [29], and interpolation, such as Euler, Runge–Kutta and multistep methods [20–22,31], and some other methods [1,8,12,15,30–41], are widely used. Traditionally, important requirements in this context are, the investigation of the consistency of the discrete scheme with the original differential equation and linear stability analysis for problems with smooth solutions. These requirements are formulated to guarantee the convergence of the discrete solution to the exact one, but sometimes the essential qualitative properties of the solution are not transferred to the numerical solution. One way to tackle with this issue is to employ finite difference schemes that are nonstandard in the sense of Mickens' definition [18,25,27]. Nonstandard finite difference methods (NSFDs) in addition to the usual properties of the solutions such as consistency, stability and hence convergence, may also preserve essential properties of

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the solutions, like positivity, boundedness, monotonicity and total variation diminishing [2,6,7,16–19,23–25,27]. In this paper, we propose a new NSFD scheme for approximating the solution of the influenza disease system. The proposed scheme enables us to solve the examined problem accurately. An important feature of the new scheme is the positivity preservation of the produced solutions, which is an essential property in this context. We also prove that the new scheme is elementary stable.

The rest of the paper is organized as follows. In Section 2, we provide some preliminaries and definitions, including that of non-standard finite difference methods for ODEs, and a review of the general influenza disease model. [14]. In Section 3, we propose the new scheme and investigate its positivity and elementary stability. In Section 4, we compare the results obtained from the new scheme with the ones obtained from the classical fourth order Runge–Kutta method (we call it RK4), ode45, ode15s, the NSFD scheme in [13] and the NSFD scheme in [26]. Finally, we end the paper with some conclusions in Section 5.

2. Preliminaries and definitions

In this section, we give a brief summary of the NSFD methods for the numerical solution of initial value problems for systems of ODEs that can be written in the autonomous form

$$\frac{d}{dt}y(t) = F(y(t)), \quad (t \geq 0), \qquad y(t_0) = y_0, \tag{2.1}$$

where $y(t)$ may be a single function or a vector function of length k mapping $[t_0, T] \to \mathbb{R}^k$ and F is a single function or a vector function of length k mapping $\mathbb{R}^k \to \mathbb{R}^k$. By defining $t_n = t_0 + n\Delta t$, where Δt is a positive step size, the continuous differential equation (2.1) can be discretized as

$$\mathcal{D}_{\Delta t} y_n = \mathcal{F}_n(F, y_n), \tag{2.2}$$

where $y_n \approx y(t_n)$, $\mathcal{D}_{\Delta t} y_n$ represents the discretized version of $\frac{d}{dt}y(t)$ and $\mathcal{F}_n(F, y_n)$ approximates $F(y(t))$ at time t_n. In the sequel, we will consider the definition of the nonstandard finite-difference methods given in [2].

Definition 2.1 (*[2]*)**.** The method given in (2.2) is called a nonstandard finite-difference method if at least one of the following conditions is met:

- In the discrete derivatives $\mathcal{D}_{\Delta t} y_n$, the traditional denominator Δt is replaced by a nonnegative function $\varphi(\Delta t)$ such that

  $$\varphi(\Delta t) = \Delta t + O(\Delta t^2) \;\; as \;\; 0 < \Delta t \to 0, \tag{2.3}$$

  for example:

  $$\varphi(\Delta t) = 1 - \exp(-\Delta t), \qquad \varphi(\Delta t) = \tanh(\Delta t).$$

- $F(y(t))$ is approximated in a nonlocal way, i.e., by a suitable function of several points of the mesh. For instance, the terms y, y^2 and y^3 can be modeled as follows:

  $$y \approx a y_k + (1-a) y_{k+1}; \quad y \approx a(y_{k+1} + y_{k-1}) + (1-2a) y_k, \quad a \in \mathbb{R};$$
  $$y^2 \approx a y_k^2 + b y_k y_{k+1}, \quad a + b = 1, \quad a, b \in \mathbb{R}; \quad y^2 \approx y_k(\frac{y_{k+1} + y_{k-1}}{2});$$
  $$y^3 \approx a y_k^3 + (1-a) y_k^2 y_{k+1}, \quad a \in \mathbb{R},$$

  where y_{k+j} denotes an approximation of the true solution $y\left(t_{k+j}\right)$.

Definition 2.2. Any constant value $\tilde{y}_0$ satisfying $F(\tilde{y}_0) = 0$ is called an equilibrium point (a fixed-point or a critical point) of the differential equation given in (2.1). The constant solutions of the discretized system are also called equilibrium points.

Definition 2.3. The finite difference method given in (2.2) is called elementary stable, if for any value of the step size Δt, the only equilibrium points are those of the differential system (2.1), and the linear stability property of each one is the same for both, the differential system and its discretized version.

Table 1
Descriptions and values of the parameters used in the system (3.1).

Description	Parameter	Value
Cross-immune period	γ^{-1}	2
Infectious period	α^{-1}	$\frac{5}{365}$
Total immune period	δ^{-1}	1
Per capita birth rate	μ	$\frac{1}{50}$
Fraction of the exposed cross-immune individuals	σ	0.05

Definition 2.4. An equilibrium point $\tilde{y}_0$ of (2.1) is linearly

(i) stable iff $|Re\lambda_j| < 1$ for all j,
(ii) unstable iff $|Re\lambda_j| > 1$ for at least one j,

where the λ_j's are the eigenvalues of the Jacobian matrix of the system (2.1) evaluated at $\tilde{y}_0$.

3. A mathematical model of the influenza disease

In this section, we consider the mathematical model of the influenza disease, completely analyzed in [13,14,26], given in the form

$$\begin{aligned}
&\frac{dS(t)}{dt} = \mu - \mu S(t) - \beta S(t)I(t) + \gamma C(t),\\
&\frac{dI(t)}{dt} = \beta S(t)I(t) + \sigma\beta C(t)I(t) - (\mu+\alpha)I(t),\\
&\frac{dR(t)}{dt} = (1-\sigma)\beta C(t)I(t) + \alpha I(t) - (\mu+\delta)R(t),\\
&\frac{dC(t)}{dt} = \delta R(t) - \beta C(t)I(t) - (\mu+\gamma)C(t),\\
&S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0, \quad C(0) = C_0,
\end{aligned} \tag{3.1}$$

where S, I, R and C represent the proportion of susceptible, infective, recovered and cross-immune individuals at time t, respectively and β is the contact rate. The definitions of the other parameters present in the system (3.1) and the values used for them in this article can be found in Table 1. One of the main assumptions of this model is that the per capita birth rate is a constant $\mu > 0$ and the birth rate is the same as death rate. It implies that $S'(t) + I'(t) + R'(t) + C'(t) = 0$ (conservation law).

Theorem 3.1. *The solution* $(S(t), E(t), I(t), R(t))$ *of system* (3.1) *with positive initial condition is positive on* $[0, \infty)$.

Proof. Assume the solution $(S(t), I(t), R(t), C(t))$ with a positive initial condition exists and is unique on $[0, b)$, where $0 < b \leq \infty$ (see [11]). Since

$$I'(t) = [\beta S(t) + \sigma\beta C(t) - (\mu+\alpha)]\, I(t),$$

then

$$I(t) = I(0)\exp\left[\int_0^t [\beta S(\theta) + \sigma\beta C(\theta) - (\mu+\alpha)]\, d\theta\right] > 0.$$

So, for all $t \in [0, b)$ we have $I(t) > 0$. Now, for all $t \in [0, b)$, one must have $C(t) > 0$. Otherwise, there will exist a $t_1 \in (0, b)$ such that $C(t_1) = 0$ and $C(t) > 0$ in $(0, t_1)$. Thus, for any $t \in [0, t_1)$,

$$\begin{aligned}
S'(t) &= \mu - \mu S(t) - \beta S(t)I(t) + \gamma C(t)\\
&\geq \mu - \mu S(t) - \beta S(t)I(t)\\
&\geq -(\mu + \beta I(t))S(t).
\end{aligned}$$

Hence, for all $t \in (0, t_1)$,

$$S(t) \geq S(0) \exp \left[\int_0^t -(\mu + \beta I(\theta)) d\theta \right] > 0.$$

Now since $1 - \sigma \geq 0$, for all $t \in [0, t_1]$ we have

$$\begin{aligned} R'(t) &= (1-\sigma)\beta C(t) I(t) + \alpha I(t) - (\mu+\delta) R(t) \\ &\geq \alpha I(t) - (\mu+\delta) R(t) \\ &\geq -(\mu+\delta) R(t). \end{aligned}$$

Then, for all $t \in (0, t_1)$,

$$R(t) \geq R(0) \exp \left[\int_0^t -(\mu + \delta) d\theta \right] > 0.$$

Therefore, for $t \in [0, t_1]$ we can write

$$C'(t) = \delta R(t) - \beta C(t) I(t) - (\mu+\gamma) C(t) \geq -(\beta I(t) + (\mu+\gamma)) C(t).$$

Hence, by using a comparison argument we obtain that

$$C(t) \geq C(0) \exp \left[- \int_0^t (\beta I(\theta) + (\mu+\gamma)) d\theta \right] > 0,$$

and in particular, for $t = t_1$ we get

$$C(t_1) \geq C(0) \exp \left[- \int_0^{t_1} (\beta I(\theta) + (\mu+\gamma)) d\theta \right] > 0$$

which is a contradiction to $C(t_1) = 0$. So, for all $t \in [0, b)$, $C(t) > 0$. Using similar procedures, one can show that $R(t) > 0$ and $S(t) > 0$ for all $t \in [0, b)$. On the other hand, we have

$$\frac{dN}{dt} = \mu - \mu N(t), \quad N(t) = S(t) + I(t) + R(t) + C(t), \tag{3.2}$$

whose exact solution is

$$N(t) = 1 + (N(0) - 1)\, e^{-\mu t} = 1 - e^{-\mu t} + N(0) e^{-\mu t}, \tag{3.3}$$

where $N(0) = S(0) + I(0) + R(0) + C(0) > 0$. We have that for $t \in [0, b)$ it is

$$N(t) < 1 + N(0) e^{-\mu t} < 1 + N(0).$$

Thus, $S(t)$, $I(t)$, $R(t)$, $C(t)$ are bounded on $[0, b)$ and we have that $b = \infty$. This completes the proof. □

Following Definition 2.2, the system (3.1) has the equilibrium points [5]:

- the disease-free equilibrium (DFE), $E_0 = (1, 0, 0, 0)$,
- the positive endemic equilibrium (EE), $E^* = (S^*, I^*, R^*, C^*)$.

The stability of the these points is often described in terms of the *reproductive number* of the system. The reproductive number represents the number of secondary infections a primary infection generates on average over the course of its infectious period. The reproductive number for system (3.1) is,

$$R_0 = \frac{\beta}{\alpha + \mu},$$

and the stability of the equilibrium points is as follows:

- the disease free equilibrium, E_0, is asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$,
- the endemic equilibrium, E^*, is asymptotically stable if $R_0 > 1$.

4. Description and properties of the numerical scheme

By using the strategy of the nonstandard discretizations, we propose a new scheme for (3.1) given by:

$$\begin{aligned}
\frac{S_{i+1}-S_i}{\varphi(\Delta t)} &= \mu-\mu(2S_{i+1}-S_i)-\beta S_{i+1}I_i+\gamma C_i,\\
\frac{I_{i+1}-I_i}{\varphi(\Delta t)} &= \beta S_{i+1}I_i+\sigma\beta C_i I_i-\mu(2I_{i+1}-I_i)-\alpha I_{i+1},\\
\frac{R_{i+1}-R_i}{\varphi(\Delta t)} &= (1-\sigma)\beta C_i I_i+\alpha I_{i+1}-\mu(2R_{i+1}-R_i)-\delta R_{i+1},\\
\frac{C_{i+1}-C_i}{\varphi(\Delta t)} &= \delta R_{i+1}-\beta C_i I_i-\mu(2C_{i+1}-C_i)-\gamma C_i.
\end{aligned} \tag{4.1}$$

The explicit form of (4.1) can be written as

$$S_{i+1}=\frac{(1+\varphi(\Delta t)\mu)S_i+\varphi(\Delta t)\mu+\varphi(\Delta t)\gamma C_i}{1+2\varphi(\Delta t)\mu+\varphi(\Delta t)\beta I_i}, \tag{4.2}$$

$$I_{i+1}=\frac{(1+\varphi(\Delta t)\beta S_{i+1}+\varphi(\Delta t)\sigma\beta C_i+\varphi(\Delta t)\mu)I_i}{1+2\varphi(\Delta t)\mu+\varphi(\Delta t)\alpha}, \tag{4.3}$$

$$R_{i+1}=\frac{(1+\varphi(\Delta t)\mu)R_i+\varphi(\Delta t)(1-\sigma)\beta C_i I_i+\varphi(\Delta t)\alpha I_{i+1}}{1+2\varphi(\Delta t)\mu+\varphi(\Delta t)\delta}, \tag{4.4}$$

$$C_{i+1}=\frac{(1+\varphi(\Delta t)\mu-\varphi(\Delta t)\beta I_i-\varphi(\Delta t)\gamma)C_i+\varphi(\Delta t)\delta R_{i+1}}{1+2\varphi(\Delta t)\mu}. \tag{4.5}$$

Proposition 4.1. *The new scheme (4.1) preserves the conservation law.*

Proof. It can be obtained by using induction. You have that $S+I+R+C=1$, and thus for the initial values it is $S_0+I_0+R_0+C_0=1$. Using the above after summing the left hand sides, and the right hand sides in (4.1) for $i=0$ you get $S_1+I_1+R_1+C_1-1=2\varphi\mu(1-S_1+I_1+R_1+C_1)$, and thus $S_1+I_1+R_1+C_1=1$. The inductive procedure results in $S_{i+1}+I_{i+1}+R_{i+1}+C_{i+1}=1$, therefore the new scheme (4.1) preserves the conservation law. □

In the following, when $\varphi(\Delta t)=\Delta t$, the new method will be referred as NSFD-Δt, and if $\varphi(\Delta t)$ is different from Δt, the method will be referred as NSFD-$\varphi(\Delta t)$.

Theorem 4.2. *The new proposed scheme (4.2)–(4.5) is elementary stable and for a chosen $\varphi(\Delta t)$, the sufficient condition for positivity is*

$$\varphi(\Delta t)\geq\frac{1}{\beta+\gamma-\mu}.$$

Proof. Elementary stability: The equilibrium points for the new proposed scheme are exactly the points E_0 and E^* of the system (3.1). The Jacobian J of the scheme (4.1) has the form $J(S_i,I_i,R_i,C_i)=[j_{mn}(S_i,I_i,R_i,C_i)]_{4\times4}$, where

$$\begin{aligned}
j_{11}(S_i,I_i,R_i,C_i) &= \frac{1+\varphi(\Delta t)\mu}{1+2\varphi(\Delta t)\mu+\varphi(\Delta t)\beta I_i},\\
j_{12}(S_i,I_i,R_i,C_i) &= \frac{-\varphi(\Delta t)\beta[(1+\varphi(\Delta t)\mu)S_i+\varphi(\Delta t)\mu+\varphi(\Delta t)\gamma C_i]}{(1+2\varphi(\Delta t)\mu+\varphi(\Delta t)\beta I_i)^2},\\
j_{13}(S_i,I_i,R_i,C_i) &= 0,\\
j_{14}(S_i,I_i,R_i,C_i) &= \frac{\varphi(\Delta t)\gamma}{1+2\varphi(\Delta t)\mu+\varphi(\Delta t)\beta I_i},\\
j_{21}(S_i,I_i,R_i,C_i) &= \frac{\varphi(\Delta t)\beta I_i j_{11}}{1+2\varphi(\Delta t)\mu+\varphi(\Delta t)\alpha},
\end{aligned}$$

$$j_{22}(S_i, I_i, R_i, C_i) = \frac{1 + \varphi(\Delta t)\beta S_{i+1} + \varphi(\Delta t)\beta I_i j_{12} + \varphi(\Delta t)\sigma\beta C_i + \varphi(\Delta t)\mu}{1 + 2\varphi(\Delta t)\mu + \varphi(\Delta t)\alpha},$$
$$j_{23}(S_i, I_i, R_i, C_i) = 0,$$
$$j_{24}(S_i, I_i, R_i, C_i) = \frac{\varphi(\Delta t)\beta\sigma I_i + \varphi(\Delta t)\beta I_i j_{11}}{1 + 2\varphi(\Delta t)\mu + \varphi(\Delta t)\alpha},$$
$$j_{31}(S_i, I_i, R_i, C_i) = \frac{\varphi(\Delta t)\alpha j_{21}}{1 + \varphi(\Delta t)\delta + 2\varphi(\Delta t)\mu},$$
$$j_{32}(S_i, I_i, R_i, C_i) = \frac{\varphi(\Delta t)(1-\sigma)\beta C_i + \varphi(\Delta t)\alpha j_{22}}{1 + \varphi(\Delta t)\delta + 2\varphi(\Delta t)\mu},$$
$$j_{33}(S_i, I_i, R_i, C_i) = \frac{\varphi(\Delta t)(1-\sigma)\beta I_i + \varphi(\Delta t)\alpha j_{23}}{1 + \varphi(\Delta t)\delta + 2\varphi(\Delta t)\mu},$$
$$j_{34}(S_i, I_i, R_i, C_i) = \frac{1 + \varphi(\Delta t)\mu + \varphi(\Delta t)\alpha j_{24}}{1 + \varphi(\Delta t)\delta + 2\varphi(\Delta t)\mu},$$
$$j_{41}(S_i, I_i, R_i, C_i) = \frac{\varphi(\Delta t)\delta j_{31}}{1 + 2\varphi(\Delta t)\mu},$$
$$j_{42}(S_i, I_i, R_i, C_i) = \frac{-\varphi(\Delta t)\beta C_i + \varphi(\Delta t)\delta j_{32}}{1 + 2\varphi(\Delta t)\mu},$$
$$j_{43}(S_i, I_i, R_i, C_i) = \frac{1 + \varphi(\Delta t)\mu - \varphi(\Delta t)\gamma - \varphi(\Delta t)\beta I_i + \varphi(\Delta t)\delta j_{33}}{1 + 2\varphi(\Delta t)\mu},$$
$$j_{44}(S_i, I_i, R_i, C_i) = \frac{\varphi(\Delta t)\delta j_{34}}{1 + 2\varphi(\Delta t)\mu}.$$

By substituting $(S_0, I_0, R_0, C_0) = (1, 0, 0, 0) = E_0$, we have

$$J(E_0) = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{pmatrix},$$

where

$$a_1 = \frac{1 + \varphi(\Delta t)\mu}{1 + 2\varphi(\Delta t)\mu}, \qquad a_2 = \frac{-\varphi(\Delta t)\beta}{1 + 2\varphi(\Delta t)\mu}, \qquad a_3 = 0,$$
$$a_4 = \frac{\varphi(\Delta t)\gamma}{1 + 2\varphi(\Delta t)\mu}, \qquad a_5 = 0, \qquad a_6 = \frac{1 + \varphi(\Delta t)(\mu + \beta)}{1 + \varphi(\Delta t)(2\mu + \alpha)},$$
$$a_7 = 0, \qquad a_8 = 0, \qquad a_9 = 0,$$
$$a_{10} = \frac{\varphi(\Delta t)\alpha(1 + \varphi(\Delta t)(\mu + \beta))}{(1 + \varphi(\Delta t)(2\mu + \delta))(1 + \varphi(\Delta t)(2\mu + \alpha))},$$
$$a_{11} = \frac{1 + \varphi(\Delta t)\mu}{1 + \varphi(\Delta t)(2\mu + \delta)}, \qquad a_{12} = 0, \qquad a_{13} = 0,$$
$$a_{14} = \frac{\varphi(\Delta t)^2\delta\alpha(1 + \varphi(\Delta t)(\mu + \beta))}{(1 + \varphi(\Delta t)(2\mu + \alpha))(1 + \varphi(\Delta t)(2\mu + \delta))(1 + 2\varphi(\Delta t)\mu)},$$
$$a_{15} = \frac{\varphi(\Delta t)\delta(1 + \varphi(\Delta t)\mu)}{(1 + \varphi(\Delta t)(2\mu + \delta))(1 + 2\varphi(\Delta t)\mu)},$$
$$a_{16} = \frac{1 + \varphi(\Delta t)(\mu - \gamma)}{1 + 2\varphi(\Delta t)\mu}.$$

The eigenvalues of $J(E_0)$ are

$$\lambda_1 = \frac{1 + \varphi(\Delta t)\mu}{1 + 2\varphi(\Delta t)\mu}, \qquad \lambda_2 = \frac{1 + \varphi(\Delta t)\mu + \varphi(\Delta t)\beta}{1 + 2\varphi(\Delta t)\mu + \varphi(\Delta t)\alpha},$$
$$\lambda_3 = \frac{1 + \varphi(\Delta t)\mu}{1 + 2\varphi(\Delta t)\mu + \varphi(\Delta t)\delta}, \qquad \lambda_4 = \frac{1 + \varphi(\Delta t)\mu - \varphi(\Delta t)\gamma}{1 + 2\varphi(\Delta t)\mu}.$$

Table 2
Qualitative behavior with respect to E_0 of the schemes considered on the problem (3.1) with $\beta = 50$ and different step sizes Δt, $T = 60$.

Δt	$ode45$	$RK4$	$Euler$	$NSFD - \Delta t$	$NSFD - \varphi(\Delta t)$
0.01	Convergence	Convergence	Convergence	Convergence	Convergence
0.1	Convergence	Convergence	Divergence	Convergence	Convergence
1	Divergence	Divergence	Divergence	Convergence	Convergence
2	Divergence	Divergence	Divergence	Convergence	Convergence
3	Divergence	Divergence	Divergence	Convergence	Convergence
4	Divergence	Divergence	Divergence	Divergence	Convergence
5	Divergence	Divergence	Divergence	Divergence	Convergence
10	Divergence	Divergence	Divergence	Divergence	Convergence
50	Divergence	Divergence	Divergence	Divergence	Convergence
100	Divergence	Divergence	Divergence	Divergence	Convergence

It is clear that $|\lambda_1| < 1$, $|\lambda_3| < 1$, $|\lambda_4| < 1$ and if $R_0 < 1$ then $|\lambda_2| < 1$ too, and therefore $E_0 = (1, 0, 0, 0)$ is stable.

It is fair to say that for E^* we have no formal proof. But, the numerical results obtained by using the Math Toolbox software of MATLAB show that for any step-size $\Delta t > 0$ the equilibrium point (S^*, I^*, R^*, C^*) is stable (see Figs. 1–3 and Tables 3–4). These results guarantee the dynamical consistency between system (3.1) and the numerical scheme (4.1) around all the equilibrium points. Therefore, the new proposed scheme (4.1) is elementary stable.

Positivity: With positivity, we mean that the component-wise non-negativity of the initial vector is preserved in time for the approximated solution. Assuming $(S_0, I_0, R_0, C_0) \geq 0$, since all of the parameters are positive then $S_{i+1} > 0$ and $I_{i+1} > 0$. Also if $\sigma < 1$, then from (4.4) we have $R_{i+1} > 0$. Now for the positivity of C_{i+1}, it is sufficient to have

$$1 + \varphi(\Delta t)\mu - \varphi(\Delta t)\beta I_i - \varphi(\Delta t)\gamma \geq 0.$$

Since $I_i \leq 1$ it is sufficient to have

$$1 + \varphi(\Delta t)\mu - \varphi(\Delta t)\beta - \varphi(\Delta t)\gamma \geq 0,$$

which is equivalent to

$$1 + \varphi(\Delta t)(\mu - \beta - \gamma) \geq 0,$$

and

$$\varphi(\Delta t) \geq \frac{1}{\beta + \gamma - \mu}, \qquad \mu \leq \beta + \gamma.$$

Therefore the new proposed scheme is positive and elementary stable and this completes the proof. □

5. Numerical results

In this section, we present some numerical results to verify the properties of the proposed scheme and compare its performance with other methods available in the literature, namely, the RK4 method, ode45, ode15s, the NSFD method presented in [13] and the NSFD method presented in [26]. All the parameter values used in these simulations have been taken from [26] and we have considered $\varphi(\Delta t) = \tanh(\Delta t)$. For each experiment, the final value of the integration interval $[t_0, T]$ is specified on the graphs or the corresponding table.

It can be seen in Fig. 1 that the proposed scheme and the RK4 method with $\Delta t = 0.01$ preserve the stability of the equilibrium E_0. Furthermore, our new scheme converges to E_0 for large step-sizes, as can be seen in Figs. 2–5. By increasing the step-size, we can observe that ode45, the RK4 and Euler methods diverge, whereas NSFD-Δt converges for larger moderate values of Δt (until $\Delta t = 4$) and the NSFD-$\varphi(\Delta t)$ scheme converges for all step sizes. Table 2 shows the qualitative behavior of the considered schemes for different values of the step size.

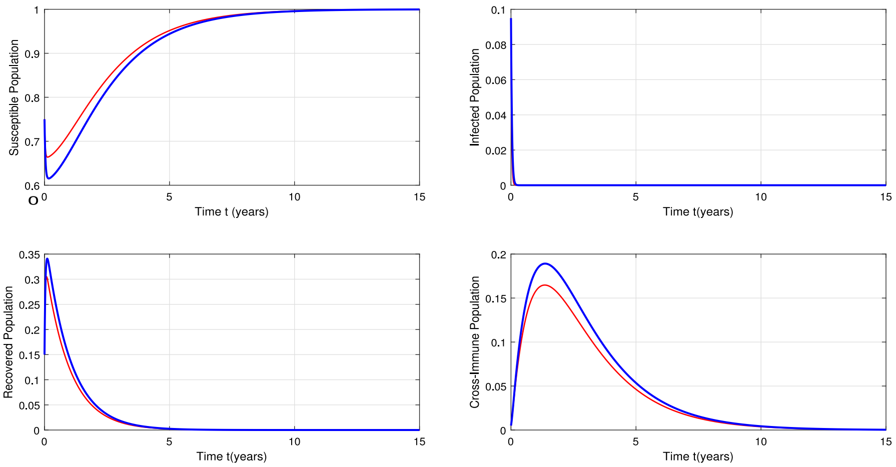

Fig. 1. Numerical results of the problem (3.1) by ode15s and the new scheme taking $\Delta t = 0.01$ and initial values $(S(0), I(0), C(0), R(0)) = (0.75, 0.095, 0.005, 0.15)$ with $\beta = 50$ $(R_0 < 1)$.

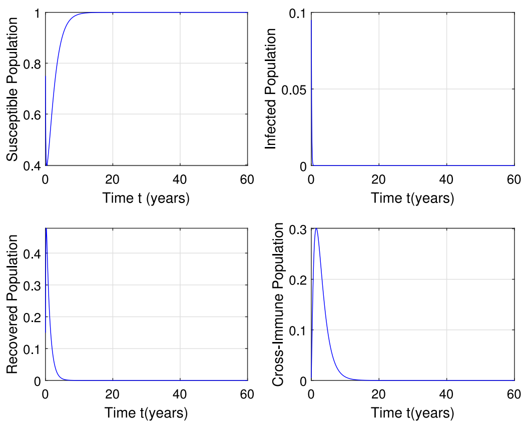

Fig. 2. Numerical results of the problem (3.1) by the new scheme taking $\Delta t = 0.1$ and initial values $(S(0), I(0), C(0), R(0)) = (0.75, 0.095, 0.005, 0.15)$ with $\beta = 50$ $(R_0 < 1)$.

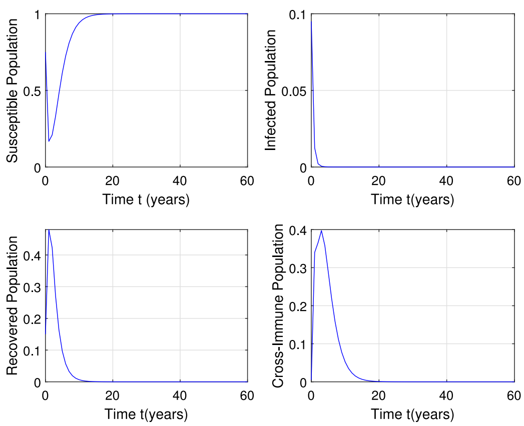

Fig. 3. Numerical results of the problem (3.1) by the new scheme taking $\Delta t = 1$ and initial values $(S(0), I(0), C(0), R(0)) = (0.75, 0.095, 0.005, 0.15)$ with $\beta = 50$ $(R_0 < 1)$.

Table 3
Qualitative behavior with respect to E^* of the schemes considered on the problem (3.1) for different step sizes with $\beta = 100$, $T = 45$.

Δt	$ode45$	$RK4$	$Euler$	$NSFD - \Delta t$	$NSFD - \varphi(\Delta t)$
0.01	Divergence	Divergence	Divergence	Convergence	Convergence
0.1	Divergence	Divergence	Divergence	Convergence	Convergence
1	Divergence	Divergence	Divergence	Convergence	Convergence
2	Divergence	Divergence	Divergence	Convergence	Convergence
3	Divergence	Divergence	Divergence	Convergence	Convergence
4	Divergence	Divergence	Divergence	Divergence	Convergence
5	Divergence	Divergence	Divergence	Divergence	Convergence
10	Divergence	Divergence	Divergence	Divergence	Convergence
50	Divergence	Divergence	Divergence	Divergence	Convergence
100	Divergence	Divergence	Divergence	Divergence	Convergence

Table 4
Spectral radius of the Jacobian matrix with respect to E^* with $\beta = 100$ and the parameter values in Table 1.

Δt	$\rho - NSFD - \Delta t$	$\rho - NSFD - \varphi(\Delta t)$
0.001	0.9999-Convergence	0.9999-Convergence
0.01	0.9997-Convergence	0.9997-Convergence
0.05	0.9989-Convergence	0.9989-Convergence
0.1	0.9979-Convergence	0.9980-Convergence
0.5	0.9901-Convergence	0.9921-Convergence
1	0.9806-Convergence	0.9875-Convergence
2	0.9628-Convergence	0.9831-Convergence
4	0.9308-Convergence	0.9831-Convergence
10	0.8567-Divergence	0.9806-Convergence
20	0.7771-Divergence	0.9806-Convergence
100	0.6760-Divergence	0.9806-Convergence

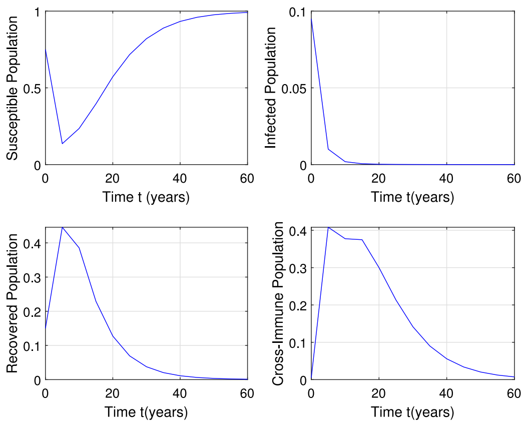

Fig. 4. Numerical results of the problem (3.1) by the new scheme taking $\Delta t = 5$ and initial values $(S(0), I(0), C(0), R(0)) = (0.75, 0.095, 0.005, 0.15)$ with $\beta = 50$ ($R_0 < 1$).

In the previous examples we have only emphasized the qualitative behavior of the solutions. It is obvious that the smaller the step size, the smaller the errors involved. In what follows we will show this aspect with respect to the equilibrium point E^*.

Fig. 6 shows that the new method preserves the stability of E^* for small step sizes. Similar behavior occurs for the NSFD method presented in [26] with $\Delta t = 0.01$, but when increasing the step size the scheme (4.1) converges to

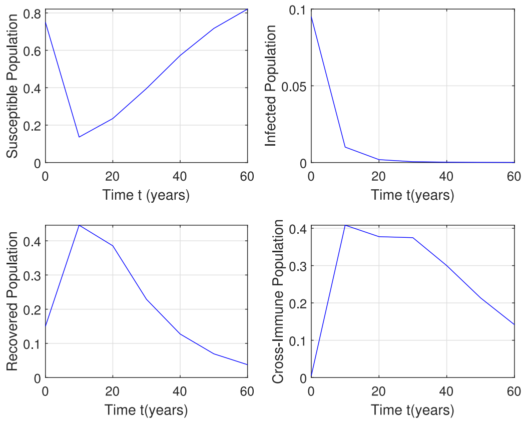

Fig. 5. Numerical results of the problem (3.1) by the new scheme taking $\Delta t = 10$ and initial values $(S(0), I(0), C(0), R(0)) = (0.75, 0.095, 0.005, 0.15)$ with $\beta = 50$ $(R_0 < 1)$.

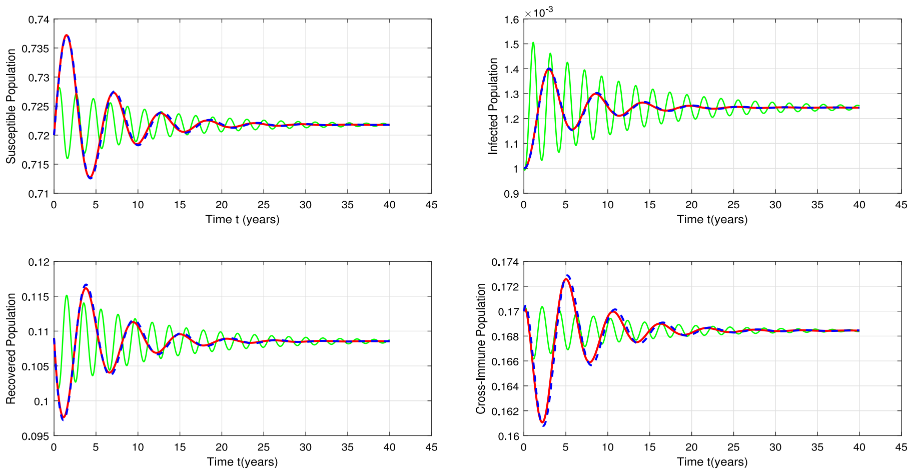

Fig. 6. Oscillatory behavior of the solution of problem (3.1) by ode15s (green line) taking $\Delta t = 0.01$, which is used as a reference solution. The proposed method in [13] (blue dashed line) and the new scheme (red line) present also this oscillatory behavior taking $\Delta t = 0.1$. The initial values are $(S(0), I(0), C(0), R(0)) = (0.72, 0.001, 0.17, 0.109)$ with $\beta = 100$ $(R_0 > 1)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the equilibrium E^* more accurately than the other methods. This can be seen in Fig. 7. Furthermore, the qualitative behavior of the considered schemes for different step sizes with respect to E^* is presented in Table 3. It can be seen that ode45, the RK4 and Euler methods diverge for all step-sizes but NSFD-$\varphi(\Delta t)$ is convergent. In Table 4 we observe that the spectral radius of the Jacobian matrix associated to the new scheme with respect to E^* are less than one showing that the scheme (4.1) is stable. Since we do not have an analytic solution for the nonlinear problem in (3.1), we use as a reference solution the one calculated with ODE15s method to represent our *true solution*. Figs. 8–11 include the absolute errors for different schemes and different values of Δt, showing that the proposed scheme is more accurate than the other methods.

Numerical simulations were developed for different representative values of $R0$ which can cover most of the possible realistic values. In Tables 3–4 we present some qualitative results and it can be observed that Scheme NSFD-$\varphi(\Delta t)$ converges to the equilibrium point for all the numerical simulations.

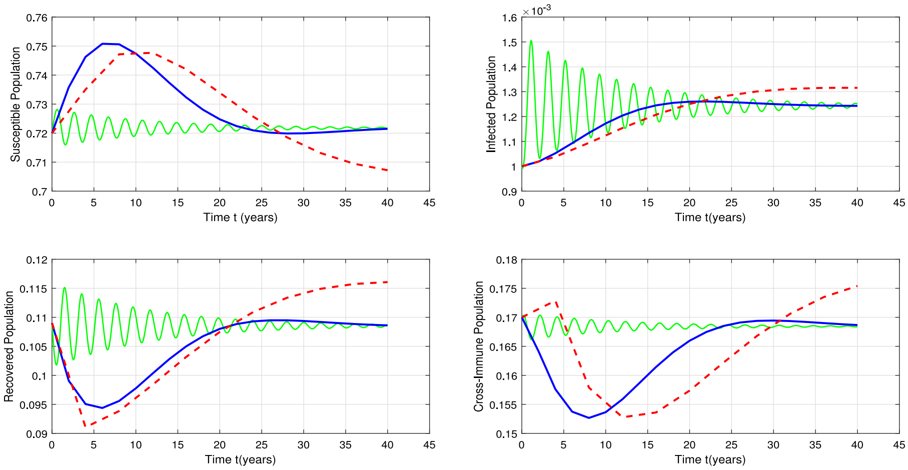

Fig. 7. Oscillatory behavior of the problem (3.1) by ode15s (green line) with $\Delta t = 0.01$, which is used as a reference solution. The proposed method in [13] (red dashed line) and the new scheme (blue line) present also this oscillatory behavior taking $\Delta t = 2$ and initial values $(S(0), I(0), C(0), R(0)) = (0.72, 0.001, 0.17, 0.109)$ with $\beta = 100$ $(R_0 > 1)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

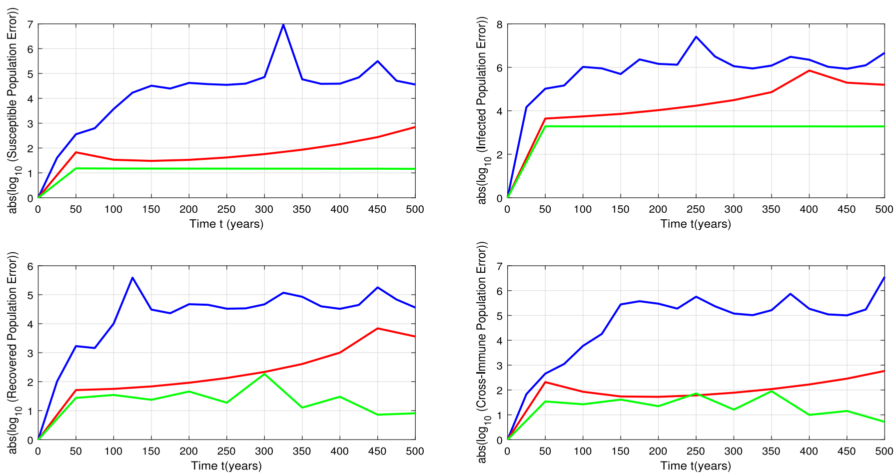

Fig. 8. Absolute errors for the problem (3.1) with $\Delta t = 0.01$ by the new scheme (blue line), the proposed method in [13] (red line) and the proposed method in [26] (green line) taking $\Delta t = 2$ and initial values $(S(0), I(0), C(0), R(0)) = (0.72, 0.001, 0.17, 0.109)$ with $\beta = 100$ $(R_0 > 1)$, using ode15s as a reference solution. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

As in the work of Gumel et al. [10], convergence of the proposed scheme has not been proved but based on all the developed numerical simulations, it seems to be unconditionally convergent to the equilibrium E^* of the SIRC model.

Conclusion

In this article, a nonstandard discretization approach is applied to solve numerically the influenza disease model analyzed in [26]. The new proposed scheme preserves the stability of all equilibrium points and the positivity of

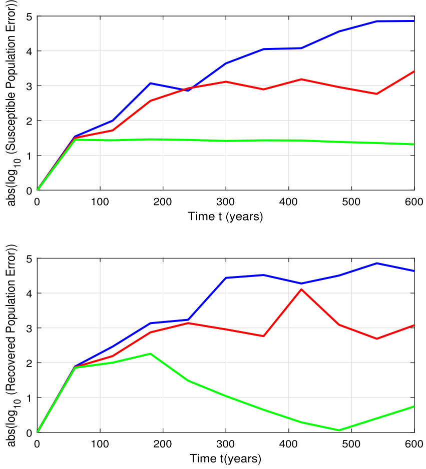

Fig. 9. Absolute errors of the problem (3.1) by the new scheme and the NSFD presented in [13] with $\Delta t = 10$, using ode15s as a reference solution with the initial values $(S(0), I(0), C(0), R(0)) = (0.72, 0.001, 0.17, 0.109)$ and $\beta = 100$.

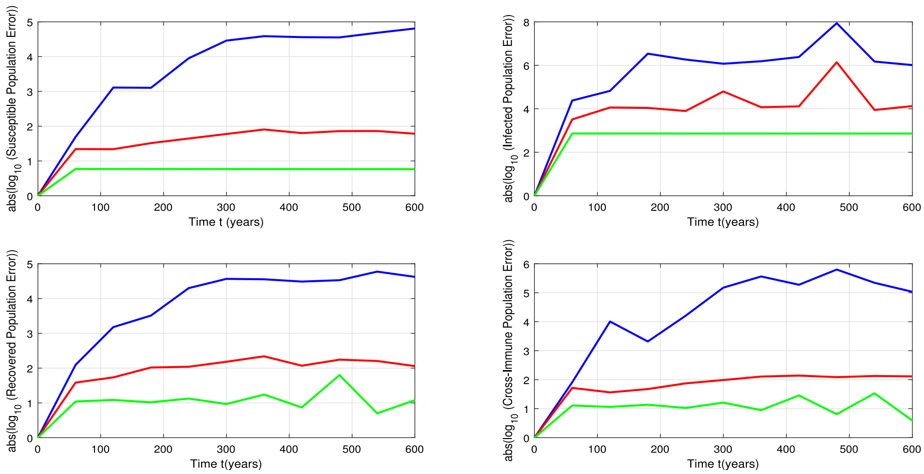

Fig. 10. Absolute errors of the problem (3.1) by the new scheme and the NSFD presented in [13] with $\Delta t = 15$, using ode15s as a reference solution with the initial values $(S(0), I(0), C(0), R(0)) = (0.72, 0.001, 0.17, 0.109)$ and $\beta = 100$.

solutions. Compared with the RK4, ode15s, the NSFD method presented in [26] and the NSFD method presented in [13], we show that the proposed scheme improves the accuracy and presents a better qualitative behavior for large step-sizes.

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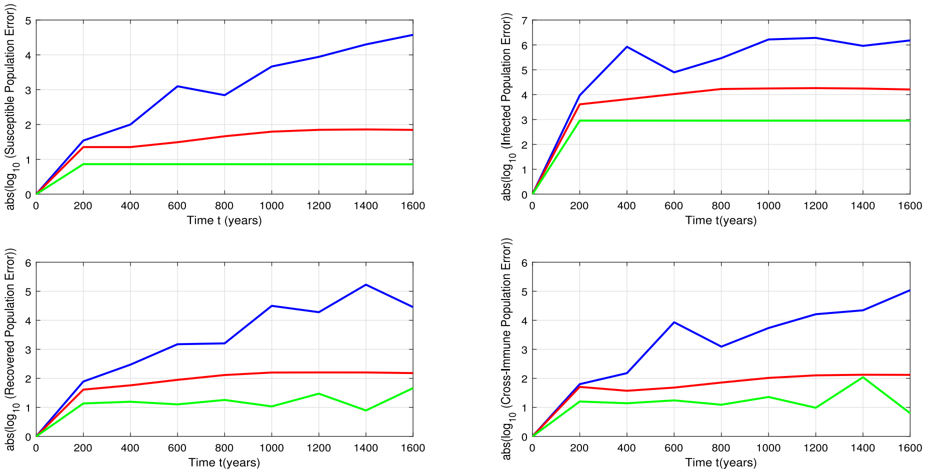

Fig. 11. Absolute errors of the problem (3.1) by the new scheme and the NSFD presented in [13] with $\Delta t = 100$, using ode15s as a reference solution with the initial values $(S(0), I(0), C(0), R(0)) = (0.72, 0.001, 0.17, 0.109)$ and $\beta = 100$.

References

[1] S. Abbas, M. Benchohra, N. Hamidi, J.J. Nieto, Hilfer and Hadamard fractional differential equations in Fréchet spaces, TWMS J. Pure Appl. Math. 10 (1) (2019) 102–116.

[2] R. Anguelov, J.M.S. Lubuma, Contributions to the mathematics of the nonstandard finite difference method and applications, Numer. Methods Partial Differential Equations 17 (2001) 518–543.

[3] A. Ashyralyev, D. Agirseven, R.P. Agarwal, Stability estimates for delay parabolic differential and difference equations, Appl. Comput. Math. 19 (2) (2020) 175–204.

[4] A. Ashyralyev, A.S. Erdogan, S.N. Tekalan, An investigation on finite difference method for the first order partial differential equation with the nonlocal boundary condition, Appl. Comput. Math. 18 (3) (2019) 247–260.

[5] R. Casagrandi, L. Bolzoni, S.A. Levin, V. Andreasen, The SIRC model and influenza A, Math. Biosci. 200 (2006) 152–169.

[6] W. Chen, C. Wang, X. Wang, S.M. Wise, Positivity-preserving, energy stable numerical schemes for the Cahn–Hilliard equation with logarithmic potential, J. Comput. Phys. X3 (2019) 100031.

[7] L. Dong, C. Wang, H.A. Zhang, Z. Zhang, A positivity-preserving, energy stable and convergent numerical scheme for the Cahn–Hilliard equation with a flory-huggins-degennes energy, Commun. Math. Sci. 17 (4) (2019) 921–939.

[8] T. Gadjiev, S. Aliev, Sh. Galandarova, A priori estimates for solutions to Dirichlet boundary value problems for polyharmonic equations in generalized Morrey spaces, TWMS J. Pure Appl. Math. 9 (2) (2018) 231–242.

[9] A. Golbabai, O. Nikan, M. Molavi-Arabshahi, Numerical approximation of time fractional advection–dispersion model arising from solute transport in rivers, TWMS J. Pure Appl. Math. 10 (1) (2019) 117–131.

[10] A.B. Gumel, T.D. Loewena, P.N. Shivakumara, B.M. Sahaib, P. Yuc, M.L. Garbad, Numerical modelling of the perturbation of HIV-1 during combination anti-retroviral therapy, Comput. Biol. Med. 31 (2001) 287–301.

[11] J.K. Hale, Ordinary Differential Equations, Wiley-Interscience, New York, 1969.

[12] S. Harikrishnan, K. Kanagarajan, E.M. Elsayed, Existence and stability results for differential equations with complex order involving Hilfer fractional derivative, TWMS J. Pure Appl. Math. 10 (1) (2019) 94–101.

[13] L. Jódar, R.J. Villanueva, A.J. Arenas, G. González, Nonstandard numerical methods for a mathematical model for influenza disease, Math. Comput. Simulation 79 (2008) 622–633.

[14] M.M. Khader, Sweilam, A.M.S. Mahdy, N.K. Abdel Moniem, Numerical simulation for the fractional SIRC model and influenza A, Appl. Math. Inf. Sci. 8 (3) (2014) 1029–1036.

[15] R.J. LeVeque, Numerical Methods for Conservation Laws, Birkhauser-Verlag, Basel, Boston, Berlin, 1992.

[16] M. Mehdizadeh Khalsaraei, An improvement on the positivity results for 2-stage explicit Runge–Kutta methods, J. Comput. Appl. Math. 235 (1) (2010) 137–143.

[17] M. Mehdizadeh Khalsaraei, Positivity of an explicit Runge–Kutta method, Ain Shams Eng. J. 6 (4) (2015) 1217–1223.

[18] M. Mehdizadeh Khalsaraei, F. Khodadoosti, A new total variation diminishing implicit nonstandard finite difference scheme for conservation laws, Comput. Methods Diff. Equ. 2 (2014) 85–92.

[19] M. Mehdizadeh Khalsaraei, F. Khodadoosti, Qualitatively stability of nonstandard 2-stage explicit Runge–Kutta methods of order two, Compu. Math. Phys. 56 (2) (2016) 235–242.

[20] M. Mehdizadeh Khalsaraei, A. Shokri, A new explicit singularly P-stable four-step method for the numerical solution of second order IVPs, Iran. J. Math. Chem. 11 (1) (2020) 17–31.

[21] M. Mehdizadeh Khalsaraei, A. Shokri, The new classes of high order implicit six-step P-stable multiderivative methods for the numerical solution of Schrödinger equation, Appl. Comput. Math. 19 (1) (2020) 59–86.

[22] M. Mehdizadeh Khalsaraei, A. Shokri, M. Molayi, The new high approximation of stiff systems of first ordinary IVPs arising from chemical reactions by k-step L-stable hybrid methods, Iran. J. Math. Chem. 10 (2) (2019) 181–193.

[23] M. Mehdizadeh Khalsaraei, R. Shokri Jahandizi, Positivity-preserving nonstandard finite difference schemes for simulation of advection-diffusion reaction equations, Comput. Methods Diff. Equ. 2 (4) (2014) 256–267.

[24] M. Mehdizadeh Khalsaraei, R. Shokri Jahandizi, Efficient explicit nonstandard finite difference scheme with positivity-preserving property, GU J. Sci. 30 (1) (2017) 259–268.

[25] R.E. Mickens, Nonstandard Finite Difference Models of Differential Equations, World Scientific, Singapore, 1994.

[26] R.E. Mickens, Numerical integration of population models satisfying conservation laws: NSFD methods, Biol. Dyn. 1 (4) (2007) 1751–1766.

[27] R.E. Mickens, P.M. Jordan, A positivity-preserving nonstandard finite difference scheme for the damped wave equation, Numer. Methods Partial Differential Equations 20 (2004) 639–649.

[28] M.I. Modebei, R.B. Adeniyi, S.N. Jator, H. Ramos, A block hybrid integrator for numerically solving fourth-order initial value problems, Appl. Math. Comput. 346 (2019) 680–694.

[29] Z. Odibat, Fractional power series solutions of fractional differential equations by using generalized Taylor series, Appl. Comput. Math. 19 (1) (2020) 47–58.

[30] S. Qureshi, H. Ramos, L-stable explicit nonlinear method with constant and variable step-size formulation for solving initial value problems, Int. J. Nonlinear Sci. Numer. Simul. 19 (7–8) (2018) 741–751.

[31] H. Ramos, Development of a new Runge–Kutta method and its economical implementation, Comput. Math. Methods 1 (2) (2019) e1016.

[32] H. Ramos, P. Popescu, How many k-step linear block methods exist and which of them is the most efficient and simplest one? Appl. Math. Comput. 316 (2018) 296–309.

[33] H. Ramos, M.A. Rufai, Third derivative modification of k-step block Falkner methods for the numerical solution of second order initial-value problems, Appl. Math. Comput. 333 (2018) 231–245.

[34] H. Ramos, M.A. Rufai, A third-derivative two-step block Falkner-type method for solving general second-order boundary-value systems, Math. Comput. Simulation 165 (2019) 139–155.

[35] H. Ramos, M.A. Rufai, Numerical solution of boundary value problems by using an optimized two-step block method, Numer. Algorithms 84 (1) (2020) 229–251.

[36] H. Ramos, G. Singh, A tenth order A-stable two-step hybrid block method for solving initial value problems of ODEs, Appl. Math. Comput. 310 (2017) 75–88.

[37] A. Shokri, M. Mehdizadeh Khalsaraei, A. Atashyar, A new two-step hybrid singularly P-stable method for the numerical solution of second-order IVPs with oscillating solutions, Iran. J. Math. Chem. 11 (2) (2020) 113–132.

[38] A. Shokri, M. Tahmourasi, A new two-step Obrechkoff method with vanished phase-lag and some of its derivatives for the numerical solution of radial Schrödinger equation and related IVPs with oscillating solutions, Iran. J. Math. Chem. 8 (2) (2017) 137–159.

[39] M.F. Simões Patrício, M. Patrício, H. Ramos, Extrapolating for attaining high precision solutions for fractional partial differential equations, Fract. Calc. Appl. Anal. 21 (6) (2018) 1506–1523.

[40] M.F. Simões Patrício, H. Ramos, M. Patrício, Solving initial and boundary value problems of fractional ordinary differential equations by using collocation and fractional powers, J. Comput. Appl. Math. 354 (2019) 348–359.

[41] G. Singh, H. Ramos, An optimized two-step hybrid block method formulated in variable step-size mode for integrating $y = f(x, y, y')$ numerically, Numer. Math. Theory Methods Appl. 12 (2) (2019) 640–660.