Ranked hesitant fuzzy sets for multi-criteria multi-agent decisions

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Abstract.

This paper introduces and investigates ranked hesitant fuzzy sets, a novel extension of hesitant fuzzy sets that is less demanding than both probabilistic and proportional hesitant fuzzy sets. This new extension incorporates hierarchical knowledge about the various evaluations submitted for each alternative. These evaluations are ranked (for example by their plausibility, acceptability, or credibility), but their position does not necessarily derive from supplementary numerical information (as in probabilistic and proportional hesitant fuzzy sets). In particular, strictly ranked hesitant fuzzy sets arise when no ties exist, i.e., when for any fixed alternative, each submitted evaluation is either strictly more plausible or strictly less plausible than any other submitted evaluation. A detailed comparison with similar models from the literature is performed. Then in order to produce a natural strategy for multi-criteria multi-agent decisions with ranked hesitant fuzzy sets, canonical representations, scores and aggregation operators are designed in the framework of ranked hesitant fuzzy sets. In order to help implementation of this model, *Mathematica* code is provided for the computation of both scores and aggregators. The decision-making technique that is prescribed is tested with a comparative analysis with four methodologies based on probabilistic hesitant fuzzy information. A conclusion of this numerical exercise is that this methodology is reliable, applicable and robust. All these evidences show that ranked hesitant fuzzy sets are an intuitive extension of the hesitant fuzzy set model designed by V. Torra, that can be implemented in practice with the aid of computationally assisted algorithms.

Keywords. Hesitant fuzzy set; aggregation operator; score; ranking; decision making.

1. Introduction

Semantics for the evaluation of information have evolved rapidly during the last few years. Linguistic assessments, hesitation and multi-evaluations, or temporal models have extended the purview of Zadeh's fuzzy sets tremendously (Zadeh, 1965). This paper contributes to this field of research with an intuitive extension of hesitant fuzzy sets (Torra, 2010) that despite its conceptual simplicity, is totally new in the literature. The new model will be called ranked hesitant fuzzy sets.

To motivate the need for this framework, first the characteristics of several important approaches to the modelization of uncertain knowledge will be recalled in section 1.1. Then section 1.2 will give facts about some models that are directly comparable to our proposal, in a critical review of literature concerning extensions of the hesitant fuzzy set model. That being established, the main traits of ranked hesitant fuzzy sets and research goals will be established in section 1.3. Section 1.4 outlines the rest of this paper.

1.1. A concise review on the modelization of uncertain knowledge

As is well known, Zadeh's fuzzy sets (together with related tools like fuzzy logic) enable us to produce formal arguments with partial memberships. The success of this model invited many authors to make substantial headway in modeling uncertain knowledge and practical problems. Direct extensions include type-2 (and type-n) fuzzy sets (Zadeh, 1975), intuitionistic fuzzy sets (Atanassov, 1986), or hesitant fuzzy sets (Torra, 2010). These models were differently inspired. Linguistic variables were the motivation behind Zadeh (1975): type-2 fuzzy sets enable us to incorporate linguistic uncertainties efficiently. Intuitionistic fuzzy sets split membership and non-membership, which is no longer determined by membership in its entirety. Hesitant fuzzy sets allow for hesitation: multiple membership degrees are allowed for each individual object. This model includes the case of interval-valued fuzzy sets (Zadeh, 1975) for which the membership degrees assigned to each object consist of closed intervals in $[0, 1]$. But alternative viewpoints produce altogether different approaches to the modelization of vaguely perceived or uncertainly defined objects. In this regard, rough sets (Pawlak, 1982) and soft sets (Molodtsov, 1999) stand out for their tractability and hybridization ability. Indeed many authors were quick to define and investigate fuzzy rough sets and rough fuzzy sets (Dubois & Prade, 1990), as well as fuzzy soft sets (Maji et al., 2001), as a short sample of hybrid models. Fundamental insights from some models have often been transferred into another model, and so for example, the idea of lower and upper approximations in rough set theory makes a convincing case that necessary and possible hesitant fuzzy sets are a helpful extension of hesitant fuzzy sets (Alcantud & Giarlotta, 2019). The crucial fact is that the lower and upper approximations of a 'roughly defined' set respectively capture the elements that necessarily and possibly belong to the set, and by the same token, this quality can be exported to hesitant evaluations.

Applications of various extensions of fuzzy sets (especially to multi-criteria decision making) abound, whether for intuitionistic fuzzy sets (Cheng et al., 2020), hesitant fuzzy sets (Farhadinia, 2013), or other related models. This article is especially motivated by the applicability of hesitant fuzzy sets and their evolved forms. Concerning managerial implications, a multi-criteria decision making model that combines hesitant and interval type-2 fuzzy sets has been proposed by Deveci et al. (2018), who apply it to improve the quality of service in three major Turkish airlines. And also Deveci et al. (2022) make further use of this framework to develop an entropy-based WASPAS approach multi-criteria decision-making method that yields an aircraft type selection. Recently, the ELECTRE-I approach has been extended to data with hesitant Pythagorean fuzzy information, and applied to risk evaluation, by Akram et al. (2022). Focusing on the model as originally envisaged by Torra, decisions with hesitant fuzzy information were the subject of Wang et al. (2021b).

Industrial applications of models that implement hesitant fuzziness have thrived in recent years too. Dinçer et al. (2019a) have used a hybrid hesitant fuzzy decision-making approach for the analysis of European energy investment policies. Narayanamoorthy et al. (2019) approached industrial robots selection by means of an interval-valued intuitionistic hesitant fuzzy entropy based VIKOR method. In the drone industry, Biswas et al. (2019) use intuitionistic hesitant fuzzy sets to achieve image enhancement for low illuminated dronograms. Dinçer et al. (2019b) have taken advantage of hesitant fuzzy information from experts to design a balanced scorecard based SERVQUAL model to rank competitors in the banking sector. Wang et al. (2021a) have set forth a hesitant fuzzy wind speed forecasting system. In the analysis of the reliability of engineering systems, Mahapatra et al. (2022) model the redundancy allocation under a hesitant fuzzy framework as a multi-objective problem. The hesitant fuzzy approach improves the degree of accuracy of the decision analysis model for outsourcing risk measurement proposed by Yazdani et al. (2021).

The possibilities of other models are endless. A shortlist of applications includes automated diagnosis of breast cancer with the assistance of fuzzy rough sets (Onan, 2015), selection of offshore wind farms sites with the assistance of interval-valued fuzzy rough sets (Deveci et al., 2020), bibliometric data analysis (Onan, 2019b), imbalanced learning (Onan, 2019a), and text classification and categorization (Onan et al., 2016; Onan, 2018).

As a sign of the rising popularity of hesitancy, many extensions were produced in addition to the aforementioned necessary and possible hesitant fuzzy sets and hesitant Pythagorean fuzzy sets. The next section focuses on a critical overview of certain enhanced versions of hesitant fuzzy sets. It is in this framework that this paper will contribute with an original model.

1.2. Related work on extensions of hesitant fuzzy sets

One of the reasons for hesitancy is that in a fuzzy context, the practitioners may have decided to avail themselves of various sources in order to obtain the appropriate membership degrees. Some authors expressed concerns that simply collecting all the evaluations to form a hesitant fuzzy set disregards the identities of the sources, which may have very different qualities. When the names of the sources should be preserved, then expanded or extended hesitant fuzzy sets can be employed (Alcantud & Santos-García, 2017; Zhu & Xu, 2016). If however, in this context people only want to retain how many sources support each membership degree (e.g., because they are all equally reliable), then a proportional hesitant fuzzy set can be used (Xiong et al., 2018). This model is simpler, yet includes supplemental information that is potentially valuable for a joint judgment of the assessments. A formally similar model stems from probabilistic considerations. In this case, it is assumed that in the hesitant assessment of every fixed alternative, there is a known probability for each membership degree to be the 'right' assessment. Then one has a probabilistic hesitant fuzzy set (Zhu & Xu, 2018). This model also applies if one decides to use a random device per alternative, in order to select which of its membership degrees should be accepted. When a residual probability may exist (for example, to capture the probability that neither of the membership degrees is right) then one has a weak probabilistic hesitant fuzzy set (Zhang et al., 2017). A controversial issue is that justifications pertaining to the proportional hesitant fuzzy spirit are sometimes expressed in probabilistic terms: see e.g., Zhang et al. (2017, Examples 2 and 3), Xu & Zhou (2017, Example 1) and Li et al. (2019, Section 4.2). Controversy aside, this issue proves the formal sameness of proportional and probabilistic hesitant fuzzy sets. Applications of the later model to decision making include Lin et al. (2020) and Jiang & Ma (2018) in a multi-criteria setting. Its spirit has exerted influence on the analysis of preferences in a hesitant fuzzy environment (Zhou & Xu, 2018). A critical examination of its uncertainty measures has been performed by Farhadinia et al. (2020). Correlation coefficients were defined and applied to medical diagnosis in Liu et al. (2021).

Other non-hesitant models radiate from situations where in a fuzzy context, the expers avail themselves of various sources. It may be the case that one can associate 'weights' with each source (for example, because of their reliability or expertise). Then models like n-dimensional fuzzy sets (Shang et al., 2010) or multi-fuzzy sets (Sebastian & Ramakrishnan, 2011) could be used. This is a different avenue that will not be pursued here.

Lastly, dual thinking produced dual hesitant fuzzy sets (Zhu et al., 2012a). They utilize both a membership and a nonmembership hesitancy function. This paper will not consider this avenue either, althouth it is worth explaining that probabilistic dual hesitant fuzzy sets have been defined and applied e.g., to risk evaluation (Hao et al., 2017).

Section 2 below will give technical details and a concise summary of the characteristics of all the relevant models. Let us say, for the moment, that the new model that this paper introduces lies in between hesitant fuzzy sets and probabilistic/proportional fuzzy sets. The next section motivates this model and states the main research targets of this article.

1.3. Ranked hesitant fuzzy sets: characteristics and research objectives

Let us revisit the case where in a fuzzy context, information from various sources is collected (cf., Section 1.2). The next situation is quite reasonable but no study has ever considered it: some sources are admittedly more plausible than others, but no *numerical* assessment (like weights or probabilities) supports this qualitative judgement. One can easily observe that neither of the models described in the literature captures all the features of this exact situation. Either hesitant fuzzy sets are used and then the ordinal comparison among sources is missing, or fictitious probabilities are brought into play in order to use probabilistic hesitant fuzzy sets, or proportions are computed in order to use proportional hesitant fuzzy sets. Should one use a proportional hesitant fuzzy set, he will be losing the information about the different credibilities of the evaluations because one can only register how many sources support each membership degree. Certainly some authors have resorted to fictitious probabilities to model the situation described above: see for example the aforementioned Zhang et al. (2017, Examples 2 and 3), Xu & Zhou (2017, Example 1) and Li et al. (2019, Section 4.2). But this is not tenable, as the choice of the probabilities that implement the reliability of the membership degrees determines all subsequent computations (and both concepts are semantically unrelated).

More faithful results will be derived if the researchers restrict themselves to the exact characteristics of the information that has been acquired. Ranked hesitant fuzzy sets will be the formal representation allowing everyone to take full advantage of the information described above, without distorting it with additional fictitious elements. Like hesitant fuzzy sets and its probabilistic and proportional improvements, ranked hesitant fuzzy sets admit hesitancy in the acquisition of fuzzy information. And they enhance Torra's hesitant fuzzy set model, quite like probabilistic and proportional hesitant fuzzy sets. However, unlike the later models, ranked hesitant fuzzy sets do not need to use additional numbers to express a prioritization of the fuzzy data.

Notice that the utilization of extended or expanded hesitant fuzzy sets raises the same problems as proportional hesitant fuzzy sets: those models simply register the sources of information without taking advantage of the qualitative information about them. They are insufficient to capture the information embodied by a ranked hesitant fuzzy set. Duality considerations are out of the question (and dual hesitant fuzzy sets disregard the ordinal information about the various sources too). Nonetheless all these models also enhance Torra's hesitant fuzzy sets, a feature that they share with ranked hesitant fuzzy sets.

Many real situations testify to the existence of ranked information. Hiring committees are typically formed by persons with varying ranks, and positions (like president or secretary) are often assigned by seniority. When a decision must be made in a company (like the launch of a marketing campaign or a product, or the selection of a site for a Volkswagen's battery factory for electric cars to be built in Spain[1]), a team leader is appointed that gathers input from

[1] See news at The Corner, 24th March 2022.

the team and then decides. In the United Nations, there are various ranks, with 5 permanent members and 10 elective members[2]. Also, Member States that are not members of the Security Council can take part in the Security Council through the figure of the Representative to the Council. When a committee of referees (for simplicity) evaluates a set of contributions for their inclusion in a special session, some opinions are more valuable than others (e.g., by seniority or closeness to the field).

The discussion above motivates the need for a new model of fuzzy evaluations under hesitancy. Ranked hesitant fuzzy sets will be defined to accommodate a novel concept of relative value, importance, or likelihood of the membership degrees associated with any alternative. By doing so, they provide a swift transition from hesitant fuzzy sets to numerically enhanced models like probabilistic or proportional hesitant fuzzy sets. Indeed this article will produce methods for these models to induce ranked hesitant fuzzy sets, and for them to induce hesitant fuzzy sets. Then meaningful constructions of scores for ranked hesitant fuzzy elements (the constituents of a ranked hesitant fuzzy set), and of aggregation operators for ranked hesitant fuzzy sets, will be presented. Scores will adhere to well-established arguments in the investigation of hesitant fuzzy elements. Aggregation of relatively ordered membership degrees will be achieved by the Borda rule. These elements will support the crucial steps of a novel decision-making mechanism that uses ranked hesitant fuzzy information in a multi-agent, multi-criteria framework. Thus this mechanism integrates and justifies the introduction of its accessory elements.

It is timely to explain that the special case with a strict prioritization of the evaluations will be the subject of a separate study in order to facilitate the comprehension of the general case (where various evaluations may be equally plausible) and of the tools that will be developed.

1.4. Outline of this paper

This study consists of the following parts. Section 2 gives preliminary facts concerning hesitant fuzzy sets and related notions like scores and probabilistic hesitant fuzzy sets. Section 3 gives the new definitions of both the general model that motivates this paper and a particular, simpler instance called strictly ranked hesitant fuzzy sets. Their relationships with other models are examined and graphically summarized. Then canonical representations are defined for both ranked hesitant fuzzy elements and sets. These representations are useful to define a new family of scores for ranked hesitant fuzzy elements. Here some special cases are studied too. *Mathematica* code is provided for the computation of the "standard" instance or S-score. With its help a comparison is performed with scores of probabilistic hesitant fuzzy elements and hesitant fuzzy elements. Section 4 establishes the procedure for the aggregation of ranked hesitant fuzzy elements, that takes advantage of canonical representations. Then it compares

[2]See UN Webpage.

the performance of this procedure with the aggregation of probabilistic hesitant fuzzy elements. *Mathematica* code for its calculation is provided too. To facilitate reproducibility of the results, both *Mathematica* notebooks can be downloaded from https://github.com/jcralcantud/RHFS. Section 5 shows how the tools described above permit to propose a novel decision-making mechanism for ranked hesitant fuzzy data. Its performance is contrasted with four solutions achieved by two other adaptable methodologies. This comparative analysis confirms the reliability of the novel method, and it is done with a revision of the case study in Li et al. (2019). Then it is used to analyze sensitivity and conclude that the new mechanism is robust. Section 6 offers some concluding remarks, as well as lines for future research.

2. Preliminaries

Let $I = [0, 1]$ denote the set of all possible membership degrees. Set $I^n_\neg = \{(r_1, \ldots r_n) \in I^n | r_i \neq r_j \text{ if } i \neq j\}$. Thus $I^n_\neg$ captures the set of all ordered vectors formed by n pairwise different membership degrees. Set also $E^n_\neg = \{(r_1, \ldots r_n) \in I^n | r_i < r_j \text{ if } i < j\}$. Thus $E^n_\neg \subset I^n_\neg$ captures the set of all ordered vectors formed by a strictly increasing sequence of n membership degrees. When $n = 1$, one has $I = I^n = I^n_\neg = E^n_\neg$.

The remaining of this article refers to a fixed non-empty set of alternatives X. Then $\mathcal{P}^*(X)$ denotes the set of its non-empty subsets, and $\mathcal{P}(X) = \mathcal{P}^*(X) \cup \{\varnothing\}$ is the set of all its subsets. In addition, $\mathcal{F}^*(X)$ denotes the set of non-empty finite subsets of X.

Now some relevant concepts from fuzzy sets and hesitant fuzzy sets will be recalled.

A fuzzy subset (FS) A of X is characterized by a function $\mu_A : X \to [0, 1]$. When $x \in X$, the number $\mu_A(x) \in [0, 1]$ is called the degree of membership of x in the subset. It represents the degree of truth of the statement "x belongs to A". Zadeh's fuzzy subsets of X are denoted by $\mathbf{FS}(X)$.

This paper originates with two related concepts, namely, hesitant fuzzy element and hesitant fuzzy set:

Definition 1. *(Xia & Xu, 2011)* A hesitant fuzzy element (HFE) is a non-empty, finite subset of $[0, 1]$.

The set of HFEs coincides with $\mathcal{F}^*([0, 1])$ by definition. It is standard practice to express an HFE as $h = \{h^1, ..., h^{l_h}\}$, with the convention $h^1 < \ldots < h^{l_h}$. This notation ensures that l_h is the cardinality of h. Particular cases include $h = \{1\}$, the full HFE, and $h = \{0\}$, the empty HFE.

Definition 2. *(Torra, 2010)* A *hesitant fuzzy set* (HFS) on X is a function $h_M : X \longrightarrow \mathcal{P}([0, 1])$.

The concept of typical hesitant fuzzy set is often used in applications. For each element of X, a HFSs on X gives a set of membership values, and they are HFEs when the HFS is typical:

Definition 3. *(Bedregal et al., 2014)* A *typical hesitant fuzzy set* (THFS) on X is $h_M : X \longrightarrow \mathcal{F}^*([0,1])$.

Henceforth $\mathbf{HFS}(X)$ denotes the set of all HFSs on X, and $\mathrm{HFS}^t(X)$ denotes the set of all THFSs on X. Notice that $\mathrm{HFS}^t(X) \subseteq \mathbf{HFS}(X)$.

Definitions 2 and 3 can be formally defined as follows:

i) a HFS is $M \subseteq X \times \mathcal{P}([0,1])$ such that for any $x \in X$, a unique $h_M(x) \in \mathcal{P}([0,1])$ exists (but it might be $\varnothing$) for which $(x, h_M(x)) \in M$.

ii) a THFS is $M \subseteq X \times \mathcal{F}^*([0,1])$ such that for any $x \in X$, a unique (non-empty) $h_M(x) \in \mathcal{F}^*([0,1])$ exists for which $(x, h_M(x)) \in M$.

A hesitant fuzzy set h_M is usually represented by $M = \{\langle x, h_M(x)\rangle \mid x \in X\}$. Notable examples include the *ideal* or *full* HFS, $M^* = \{\langle x, \{1\}\rangle \mid x \in X\}$, and the *anti-ideal* or *empty* HFS on X, $M^- = \{\langle x, \{0\}\rangle \mid x \in X\}$ (Torra, 2010).

Remark 1. *Any FS on X with membership function $\mu_M : X \longrightarrow [0,1]$ such that $\mu_M(x) = M_x$ can be identified with the THFS h_M described as $M = \{(x, h_M(x)) \mid x \in X, h_M(x) = \{M_x\}\}$. In this fashion one can naturally embed **FS**(X) into HFS$^t(X)$ and therefore into **HFS**(X).*

In other words, FSs are special THFSs with the natural identification explained above.

For each typical hesitant fuzzy set h_M on X, let

$$h_M(x) = \{h_M^1(x), ..., h_M^{l_M(x)}(x)\} \tag{1}$$

where $h_M^1(x) < \ldots < h_M^{l_M(x)}(x)$, and $l_M(x) = |h_M(x)|$, the *length* of $h_M(x)$, is the cardinality of the HFE $h_M(x)$. Since $h_M(x)$ is a set, repetitions are excluded by definition.

Typical hesitant fuzzy elements can be compared by their scores, i.e., the value of some functions that satisfy a minimal set of technical requirements (Alcantud & Giarlotta, 2019, Definition 8). They are conceived so that hesitant fuzzy elements consisting of low, resp. high, membership values have small, resp. high, scores. Next some examples are presented, although many other score functions can be thought of (Alcantud & Giarlotta, 2019, Example 1). More detailed analyses are found in Farhadinia (2013, 2014); Wang et al. (2019); Xia & Xu (2011).

Definition 4. For each $h = \{h^1, \ldots, h^n\}$, with $h^1 < \ldots < h^n$,

(i) its *Xia–Xu score* is $s_{\mathrm{X}}(h) = \frac{\sum_{i=1}^n h^i}{n}$,

(ii) its *Farhadinia score* associated with $\boldsymbol{\delta}$, where $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ is a fixed non-decreasing sequence of positive numbers, is $s_{\mathrm{F}}(h) = \frac{\sum_{i=1}^n (\delta_i h^i)}{\sum_{i=1}^n \delta_i}$, and

(iii) its *geometric mean score* is $s_{\mathrm{g}}(h) = \left(\prod_{i=1}^n h^i\right)^{1/n}$.

The most standard application of the Farhadinia score uses the sequence $\boldsymbol{\delta} = \{1, 2, 3, \ldots\}$. Notice that the Farhadinia score with $\delta_n = 1$ for all n produces the Xia–Xu score.

Some relationships among hesitant fuzzy sets and other soft computing models exist (Alcantud, 2016). Alcantud & Torra (2018) prove the first decomposition theorems and formulate extension principles for hesitant fuzzy sets.

Definition 5. *(Zhu & Xu, 2018; Zhang et al., 2017)* A *probabilistic hesitant fuzzy element* (PHFE) is denoted as $h_P = \{h_P^1(p^1), \ldots, h_P^l(p^l)\}$, where $h_P^i, p^i \in [0,1]$ for each $i = 1, \ldots, l$, and $\sum_{i=1}^l p^i \leqslant 1$. The set of all PHFEs will be denoted by $\mathbf{P}$.

A *probabilistic hesitant fuzzy set* (PHFS) on X is a mapping that associates a PHFE with each element of X. Briefly expressed,

$$P = \{(x, h_P(x)) \mid x \in X, h_P(x) \in \mathbf{P} \text{ for each } x \in X\}.$$

This definition adopts the terminology of Zhang et al. (2017) who relax the requirement $\sum_{i=1}^l p^i = 1$ in Zhu & Xu (2018). To maintain precision, the PHFS will be called full when the condition $\sum_{i=1}^l p^i(x) = 1$ holds for each $x \in X$. Zhang et al. (2017) define a normalization of probabilistic hesitant fuzzy sets that performs two operations. First for each $x \in X$, a normalization of the probabilities achieves a full PHFS. Then a conservative expansion adds evaluations with zero probability so that all probabilistic hesitant fuzzy elements have equal length. Notice that this process implicitly assumes some further structure, like the finiteness of X. Zhang et al. (2017, Example 5) illustrates this two-stage process.

As in the case of hesitant fuzzy elements, probabilistic hesitant fuzzy elements can be compared by their respective scores defined as follows:

Definition 6. *(Zhang et al., 2017, Definition 5)* If $h_P = \{h_P^1(p^1), \ldots, h_P^l(p^l)\}$ is a probabilistic hesitant fuzzy element, then its score is $\frac{\sum_{i=1}^l h_P^i p^i}{\sum_{i=1}^l p^i}$.

Applications of the probabilistic hesitant fuzzy model include Xu & Zhou (2017) and Jiang & Ma (2018), or Li et al. (2019) in a context of expression of preferences. A variation of this model uses proportions instead of probabilities, so a fundamentally semantical difference exists between both approaches:

Definition 7. *(Xiong et al., 2018, Definition 6)* If in Definition 5, the probabilities are replaced with proportions in such way that $\sum_{i=1}^l p^i = 1$, then we have *proportional hesitant fuzzy elements* (PrHFE) and *proportional hesitant fuzzy sets* (PrHFS).

Both PHFEs and PrHFEs produce associated HFEs by removing the information about probabilities or proportions. Thus the HFE associated with $h_P = \{h_P^1(p^1), \ldots, h_P^l(p^l)\}$, a PHFE, is $\{h_P^1, \ldots, h_P^l\}$.

The next section introduces a new model called ranked hesitant fuzzy sets. Table 1 summarizes the main features of the models presented above (and some related models that we mentioned in the Introduction), and it helps to fully grasp the novelty of ranked hesitant fuzzy sets and their main traits.

Table 1: Features of some models related to hesitant or multiple fuzzy information

Article	Model	Fixed n. of values	Ordering in values	Repeated values	Additional information or restrictions
Torra (2010)	HFS	No	No (except for notational convenience)	Not allowed	None
Zhu & Xu (2018); Zhang et al. (2017)	(Weak) Probabilistic HFS	No	No (implicit order)	Not allowed	Probabilities may sum up to less than 1 in the weak model.
Xiong et al. (2018)	Proportional HFS	No	No (implicit order)	Not allowed	Proportions are rational numbers. They sum up to 1. Multi-agent context.
Zhu & Xu (2016); Alcantud & Santos-García (2017)	Extended-Expanded HFS	Yes	No	Allowed	Who submitted. Multi-agent context.
Sebastian & Ramakrishnan (2011)	Multi-fuzzy sets	Yes. Possibly infinite	No	Allowed	None
Shang et al. (2010)	n-dimensional fuzzy sets	Yes	Smaller to higher (no semantics attached)	Allowed	None
This work	Ranked HFS	No	Yes	Not allowed	Order bespokes credibility. Equally plausible membership degrees allowed

3. Ranked hesitant fuzzy sets

Ranked hesitant fuzzy sets and elements are easy to understand but difficult to define and manipulate. To facilitate the understanding of their general structure a particular case will be explored first, namely, strictly ranked hesitant fuzzy sets and elements. This preliminary exploration is done because both the formal structure and manipulations are simpler in strictly ranked hesitant fuzzy sets than in the general model that we shall describe afterwards in section 3.2. However their fundamental analyses are very similar. Ranked hesitant fuzzy sets will be compared with other models from the literature in section 3.3. In section 3.4 their canonical representations will be defined. Then section 3.5 will introduce a family of scores, and afterwards their relationships with scores for hesitant fuzzy elements and probabilistic hesitant fuzzy elements will be explored in section 3.6.

3.1. Introducing strictly ranked hesitant fuzzy sets

In order to define this first original concept, the standard convention for the manipulation of hesitant fuzzy sets will be modified as follows:

Definition 8. A *strictly ranked hesitant fuzzy set* (SRHFS) on X consists of a set of pairs indexed by X, $\vec{S} = \{\langle x, h_{\vec{S}}(x)\rangle \mid x \in X\}$, such that

$$\begin{aligned} h_{\vec{S}}\colon\ X &\longrightarrow \bigcup_{a\in\mathbb{N}} I_{\neg}^{a} \\ x &\longmapsto h_{\vec{S}}(x) = (R_1(x), \ldots, R_{a(x)}(x)) \subseteq I_{\neg}^{a(x)}. \end{aligned} \tag{2}$$

The index $a(x)$ in Equation (2) is called the *amplitude* of x in $\vec{S}$.

Any vector from $\bigcup_{a\in\mathbb{N}} I_{\neg}^{a}$ is called a *strictly ranked hesitant fuzzy element* (SRHFE) of amplitude a.

The semantics of these concepts is very natural: for each $x \in X$, a strictly ranked hesitant fuzzy element is associated. This evaluation $h_{\vec{S}}(x)$ consists of an ordered list $(R_1(x), \ldots, R_{a(x)}(x))$ of $a(x)$ possible membership degrees which are (i) pairwise different, and (ii) ordered by their increasing plausibility. Strictly ranked hesitant fuzzy sets do not allow for equally plausible evaluations; this case will be modelled by the general concept of ranked hesitant fuzzy set. As explained above, the presentation of this model is postponed until section 3.2 due to its technical complexity.

The amplitude of x in $\vec{S}$ captures the height of the hierarchy that describes the comparative plausibilities of the membership degrees associated with x. It is not necessarily common to all $x \in X$. Put briefly, each $x \in X$ has its own amplitude which corresponds to the number of evaluations submitted for x.

It is very important to keep in mind that the ordering in the vector $h_{\vec{S}}(x)$ is not a convention, like in the case of the arrangement of elements in a HFE. Equation (1) describes the constituent $h_M(x) = \{h_M^1(x), ..., h_M^{l_M(x)}(x)\}$ of a HFS with the assumption $h_M^1(x) < \ldots < h_M^{l_M(x)}(x)$. Thus $h_M(x) = \{0.1, 0.9\}$ and

$h_M(y) = \{0.9, 0.1\}$ associate the same THFEs and the verdicts on x and y are the same. However when $\vec{S}$ is a SRHFS that declares $h_{\vec{S}}(x) = (0.1, 0.9)$ and $h_{\vec{S}}(y) = (0.9, 0.1)$, $\vec{S}$ is expressing that

(i) for x, two membership degrees are feasible, namely, 0.1 and 0.9; but 0.9 is strictly more credible than 0.1; and

(ii) for y, the only feasible membership degrees are 0.1 and 0.9; but 0.1 is strictly more credible than 0.9.

An inspection of this issue from a formal standpoint will be performed in section 3.4.

The practical differences between SRHFSs and HFSs are brought to light in the next example. It also clarifies some concepts and notation described above:

Example 1. *Let* $X = \{x, y\}$ *and let*

$$h_{\vec{S}} = \{\langle x, (0.7, 0.1, 0.3)\rangle, \langle y, (0.6, 0.3, 0.35, 0.4)\rangle\}$$

be a SRHFS on X*. Two SRHFEs are associated with* $h_{\vec{S}}$*, namely,* $(0.7, 0.1, 0.3) \in I^3_{\neg}$ *and* $(0.6, 0.3, 0.35, 0.4) \in I^4_{\neg}$*. Thus* $a(x) = 3$ *and* $a(y) = 4$*. The vector* $h_{\vec{S}}(x)$ *consists of 3 components, whereas* $h_{\vec{S}}(y)$ *has 4 components.*

The possible degrees of membership of x *are:* 0.7*;* 0.1 *(which is strictly more plausible than* 0.7*); and* 0.3 *(which is strictly more plausible than both* 0.7 *and* 0.1*).*

The possible degrees of membership of y *are:* 0.6*;* 0.3 *(which is strictly more plausible than* 0.6*);* 0.35 *(which is strictly more plausible than both* 0.6 *and* 0.3*); and* 0.4 *(which is strictly more plausible than* 0.6*,* 0.3*, and* 0.35*).*

The fundamental hallmarks of the model that will be investigated have been captured by SRHFSs, except one. Indeed, as explained above, the existence of membership degrees with exactly the same plausibility is not permitted in the evaluation of any fixed alternative by an SRHFS. The model in section 3.2 takes shape by allowing the possibility that two or more membership degrees are equally plausible in Definition 8.

3.2. The general concept of ranked hesitant fuzzy set

Section 3.1 has explained the semantics and formal definition of strictly ranked hesitant fuzzy sets. In the next definition the condition that the membership degrees in the assessment of each alternative must be strictly ranked will be relaxed, in such way that ties be allowed:

Definition 9. A *ranked hesitant fuzzy set* (RHFS) on X consists of a set of pairs indexed by X, $\vec{R} = \{\langle x, h_{\vec{R}}(x)\rangle \mid x \in X\}$, such that

$$\begin{aligned} h_{\vec{R}}\colon\ X &\longrightarrow \bigcup_{a\in\mathbb{N}} \mathcal{F}^*(X)\times \overset{a}{\ldots\ldots} \times\mathcal{F}^*(X) \\ x &\longmapsto h_{\vec{R}}(x) = (R_1(x), \ldots, R_{a(x)}(x)) \in \mathcal{F}^*(X)\times \overset{a(x)}{\ldots\ldots} \times\mathcal{F}^*(X) \end{aligned} \qquad (3)$$

with the condition that for each $x \in X$, $R_i(x) \cap R_j(x) = \varnothing$ for each $i, j \in \{1, \ldots, a(x)\}$ with $i \neq j$.

The index $a(x)$ in Equation (3) is called the *amplitude* of x in $\vec{R}$. For each $i = 1, \ldots, a(x)$, $l_i(x) = |R_i(x)|$ is called the *girth* of the ith hierarchical rank of x in $\vec{R}$.

Any vector from $\bigcup_{a \in \mathbb{N}} \mathcal{F}^*(X) \times \overset{a}{\ldots\ldots} \times \mathcal{F}^*(X)$ is called a *ranked hesitant fuzzy element* (RHFE) of amplitude a.

The semantics of the concepts given in Definition 9 are related to those of strictly ranked hesitant fuzzy sets and elements, with the proviso that now it is admitted that several evaluations be equally plausible. Therefore to summarize, for each $x \in X$, the vector $h_{\vec{R}}(x)$ –which itself is a RHFE– captures an ordered list of $a(x)$ subsets of possible membership degrees; these degrees are (i) pairwise different, and (ii) equally plausible when they belong to the same subset; in addition, (iii) degrees from different components are ordered by their increasing plausibility.

The amplitude of x in $\vec{R}$ captures the height of the hierarchical ranking that describes the comparative plausibilities of the membership degrees associated with x. And the number of equally plausible evaluations at each height gives its girth for x. In contrast with the case of SRHFSs, now the number of evaluations submitted for x is given by the sum of the $a(x)$ girths defined for x.

The next example illustrates the new concepts presented in this section.

Example 2. *Let $X = \{x, y\}$ and let*

$$h_{\vec{R}} = \Big\{ \langle x, (\{0.2, 0.5\}, \{0.25, 0.3, 0.35\}, \{0.8, 0.9, 0.95\}) \rangle, \\ \langle y, (\{0.7, 0.75, 0.8\}, \{0.15\}, \{0.25, 0.3\}, \{0.05, 0.1\}) \rangle \Big\}$$

be a RHFS on X. Two RHFEs are constituents of $h_{\vec{R}}$, namely,

$$(\{0.2, 0.5\}, \{0.25, 0.3, 0.35\}, \{0.8, 0.9, 0.95\}) \in \mathcal{F}^*(X) \times \mathcal{F}^*(X) \times \mathcal{F}^*(X)$$

and

$$(\{0.7, 0.75, 0.8\}, \{0.15\}, \{0.25, 0.3\}, \{0.05, 0.1\}) \in \mathcal{F}^*(X) \times \mathcal{F}^*(X) \times \mathcal{F}^*(X) \times \mathcal{F}^*(X).$$

They are respectively associated with x and y thus $a(x) = 3$ and $a(y) = 4$: vector $h_{\vec{R}}(x)$ consists of 3 components, whereas $h_{\vec{R}}(y)$ has 4 components.

The girths of the hierarchical ranks of x are $l_1(x) = 2$, $l_2(x) = 3$, and $l_3(x) = 3$. The information about the possible degrees of membership of x is that they are either

0.2 *and* 0.5 *(which are equally plausible), or*

0.25, 0.3, *and* 0.35 *(which are equally plausible, but each of them is strictly more plausible than* 0.2 *and* 0.5*), or*

0.8, 0.9, and 0.95 (which are equally plausible, but each of them is strictly more plausible than 0.2, 0.5, 0.25, 0.3, and 0.35).

The girths of the hierarchical ranks of y are $l_1(y) = 3$, $l_2(y) = 1$, and $l_3(y) = l_4(y) = 2$. The information about the possible degrees of membership of y is that they are either

0.7, 0.75 and 0.8 (which are equally plausible), or

0.15 (which is strictly more plausible than 0.7, 0.75 and 0.8), or

0.25 and 0.3 (which are equally plausible, but each of them is strictly more plausible than 0.7, 0.75, 0.8, and 0.15), or

0.05 and 0.1 (which are equally plausible, but each of them is strictly more plausible than 0.7, 0.75, 0.8, 0.15, 0.25 and 0.3).

The next concept will prove useful in the study of decision-making with ranked hesitant fuzzy information:

Definition 10. The *complement* of the ranked hesitant fuzzy set $\vec{R} = \{\langle x, h_{\vec{R}}(x)\rangle \mid x \in X\}$ described in Definition 9 is $\vec{R}^c = \{\langle x, h^c_{\vec{R}}(x)\rangle \mid x \in X\}$, such that

$$\begin{aligned} h^c_{\vec{R}} \colon \ X \ &\longrightarrow \ \bigcup_{a\in\mathbb{N}} \mathcal{F}^*(X) \times \overset{a}{\ldots\ldots} \times \mathcal{F}^*(X) \\ x \ &\longmapsto \ h^c_{\vec{R}}(x) = ((1-R_1)(x), \ldots, (1-R_{a(x)})(x)) \end{aligned} \tag{4}$$

where $(1-R_j)(x) = \{1-y \mid y \in R_j(x)\}$, for each $x \in X$ and $j = 1, \ldots, a(x)$.

The intuitive description of complements is that at each point, the evaluations submitted by the ranked hesitant fuzzy set are subtracted from one, and the results are ranked in the same order.

3.3. Some relationships

From the comparison between Definitions 8 and 9, one can draw the conclusion that strictly ranked hesitant fuzzy sets can be identified with ranked hesitant fuzzy sets for which $l_i(x) = 1$ for all $x \in X$ and $i \in \{1, \ldots, a(x)\}$. The process simply identifies each membership degree m with the singleton $\{m\}$. The next example insists on this trivial fact:

Example 3. *Let $X = \{x, y\}$ and*

$$h_{\vec{R}} = \{\langle x, (\{0.7\}, \{0.1\}, \{0.3\})\rangle, \langle y, (\{0.6\}, \{0.3\}, \{0.35\}, \{0.4\})\rangle\}$$

be a RHFS on X. All girths are 1. This RHFS can be identified with the SRHFS defined in Example 1.

Of course, ranked hesitant fuzzy elements whose girths are 1 can be identified with strictly ranked hesitant fuzzy elements by the same procedure.

Similarly, hesitant fuzzy sets can be identified with ranked hesitant fuzzy sets for which $a(x) = 1$ for all $x \in X$. The idea is that when all amplitudes are 1, there is no evidence that some membership degrees are more credible than others. The next example insists on this situation:

Example 4. *Let $X = \{x, y\}$ and*

$$h_{\vec{R}} = \{\langle x, (\{0.1, 0.3, 0.7\})\rangle, \langle y, (\{0.3, 0.35, 0.4\})\rangle\}$$

be a RHFS on X. All amplitudes are 1. This RHFS can be identified with the hesitant fuzzy set $h_M = \{\langle x, \{0.1, 0.3, 0.7\}\rangle, \langle y, \{0.3, 0.35, 0.4\}\rangle\}$. Intuitively, the reason is that there is no clue as to which of the three feasible membership degrees for x, resp. y, are more tenable.

As shown in Example 4, ranked hesitant fuzzy elements of amplitude 1 are hesitant fuzzy elements.

Now suppose that a probabilistic HFS on X (cf., Definition 5) is given. If one only retains its ordinal information about how likely each membership degree is, and the cardinal measurement embodied in a probabilistic expression is discarded, then a RHFS on X obtains. The same is true for probabilistic HFEs, which become RHFEs when only the comparative information about the degrees is preserved. And similar facts hold for the proportional hesitant fuzzy model (Xiong et al., 2018). An example from the interesting case study in Zhang et al. (2017, Section 4) illustrates this relationship.

Example 5. *Three experts provide their opinions on five car brands, denoted by $X = \{A_1, \ldots, A_5\}$. They are asked to submit assessments for five elements of their safety systems $\{C_1, \ldots, C_5\}$. Afterwards an aggregate probabilistic HFS is constructed for each attribute (Zhang et al., 2017, Section 4). The exercise concentrates on the aggregate PHFS given for the first one, namely, C_1 (brake system). The values collectively assigned to each car brand are the following* *probabilistic HFEs:*

$$h_P(A_1, C_1) = \{0.56(0.0375), 0.59(0.0625), 0.61(0.075), 0.62(0.025), 0.63(0.125),$$
$$0.64(0.0375), 0.66(0.1125), 0.67(0.0375), 0.68(0.075), 0.69(0.0875),$$
$$0.7(0.125), 0.71(0.025), 0.72(0.05), 0.73(0.0375), 0.74(0.0625), 0.76(0.025)\}$$
$$h_P(A_2, C_1) = \{0.52(0.125), 0.55(0.075), 0.56(0.3125), 0.59(0.1875), 0.6(0.1875),$$
$$0.62(0.1125)\}$$
$$h_P(A_3, C_1) = \{0.59(0.0375), 0.61(0.0875), 0.63(0.05625), 0.64(0.13125), 0.65(0.0225),$$
$$0.67(0.14625), 0.68(0.2525), 0.7(0.07875), 0.72(0.05625), 0.73(0.13125)\}$$
$$h_P(A_4, C_1) = \{0.72(0.075), 0.73(0.1), 0.74(0.075), 0.75(0.045), 0.76(0.135),$$
$$0.77(0.145), 0.78(0.075), 0.79(0.075), 0.8(0.1), 0.81(0.075), 0.82(0.03),$$
$$0.83(0.04), 0.84(0.03)\}$$
$$h_P(A_5, C_1) = \{0.64(0.125), 0.67(0.125), 0.68(0.1), 0.69(0.125), 0.71(0.225),$$
$$0.72(0.1), 0.73(0.025), 0.74(0.1), 0.75(0.025), 0.76(0.025), 0.78(0.025)\}$$

This information can be used to generate a ranked hesitant fuzzy set on X as follows:

$$h^1_{\vec{R}} = \{\langle A_1, (\{0.62, 0.71, 0.76\}, \{0.56, 0.64, 0.67, 0.73\}, \{0.72\}, \{0.59, 0.74\},$$
$$\{0.61, 0.68\}, \{0.69\}, \{0.66\}, \{0.63, 0.7\})\rangle,$$
$$\langle A_2, (\{0.55\}, \{0.62\}, \{0.52\}, \{0.59, 0.6\}, \{0.56\})\rangle$$
$$\langle A_3, (\{0.65\}, \{0.59\}, \{0.63, 0.72\}, \{0.7\}, \{0.61\}, \{0.64, 0.73\}, \{0.67\}, \{0.68\})\rangle$$
$$\langle A_4, (\{0.82, 0.84\}, \{0.83\}, \{0.75\}, \{0.72, 0.74, 0.78, 0.79, 0.81\}, \{0.73, 0.8\}, \{0.76\}, \{0.77\})\rangle$$
$$\langle A_5, (\{0.73, 0.75, 0.76, 0.78\}, \{0.68, 0.72, 0.74\}, \{0.64, 0.67, 0.69\}, \{0.71\})\rangle\}$$

Intuitively, for each car brand all the evaluations for which the probabilities coincide are gathered; and then they have been ranked from lowest to highest probability.

Remark 2. *The complement (Xu & Zhou, 2017, Definition 4) of a probabilistic hesitant fuzzy element $h_P = \{h^1_P(p^1), \ldots, h^l_P(p^l)\}$, where $h^i_P, p^i \in [0,1]$ for each $i = 1, \ldots, l$, and $\sum_{i=1}^{l} p^i \leqslant 1$, is $h^c_P = \{(1 - h^1_P)(p^1), \ldots, (1 - h^l_P)(p^l)\}$. Complements of probabilistic hesitant fuzzy sets are the natural expansion of this concept. One can readily observe that the complement of the ranked hesitant fuzzy set derived from a probabilistic hesitant fuzzy set coincides with the ranked hesitant fuzzy set derived from the complement of the probabilistic hesitant fuzzy set. The same statement is true at the level of (probabilistic/ranked) hesitant fuzzy elements.*

Similarly, if an RHFS on X is given and one dispenses with its ordinal information about how likely each membership degree is, an HFS on X is produced that is called its original HFS (or OHFS). The same is true for ranked hesitant fuzzy elements: they induce hesitant fuzzy elements by discarding the ordering imposed on the degrees, and they will be called their original hesitant fuzzy elements (or OHFEs). The terminology is borrowed from Zhang et al. (2017). In fact, the original hesitant fuzzy set of a probabilistic hesitant fuzzy (Zhang et al., 2017) is the original hesitant fuzzy set of the ranked hesitant fuzzy set induced by it. And the same can be said about their constituent elements.

Example 6. *Consider $h_{\vec{R}}$, the RHFS defined in Example 2. Then $h_{\vec{R}}$ induces the following OHFS on $X = \{x, y\}$:*

$$h_M = \Big\{ \langle x, \{0.2, 0.25, 0.3, 0.35, 0.5, 0.8, 0.9, 0.95\} \rangle, \\ \langle y, \{0.05, 0.1, 0.15, 0.25, 0.3, 0.7, 0.75, 0.8\} \rangle \Big\}$$

The two RHFEs associated with $h_{\vec{R}}$ (one for x, one for y) respectively induce the OHFEs $\{0.2, 0.25, 0.3, 0.35, 0.5, 0.8, 0.9, 0.95\}$ *and* $\{0.05, 0.1, 0.15, 0.25, 0.3, 0.7, 0.75, 0.8\}$.

Figure 1 summarizes the relationships among models that have been stated above.

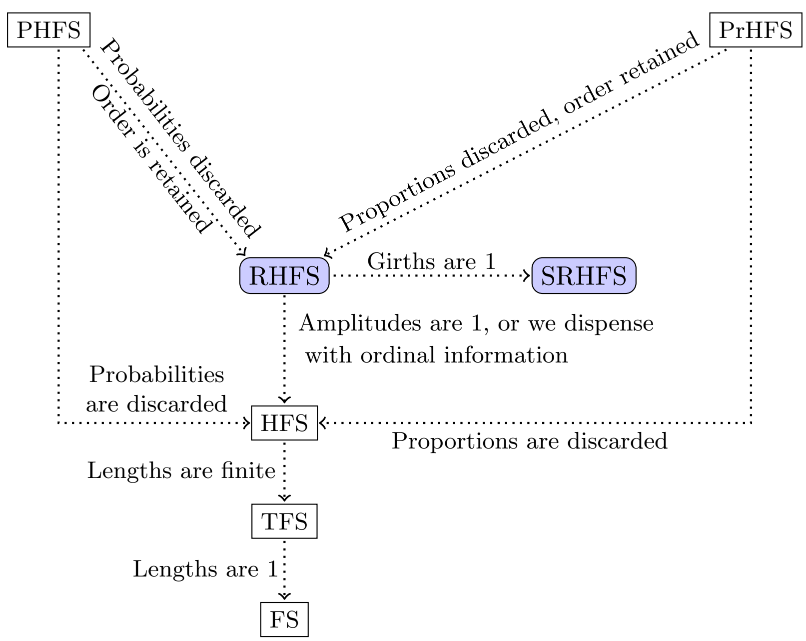

Figure 1: A summary of relationships among various models, including the new models defined in Section 3 (in coloured boxes with rounded corners). A dotted arrow means 'becomes under the condition(s)'.

3.4. Canonical representations

To better understand the formal differences among the concepts defined above, their respective 'canonical' representations will become useful. Intuitively speaking, 'canonical' means that membership degrees that are equally ranked will be arranged by ascending order, i.e., from smallest to largest. For illustration, the inspirational case of hesitant fuzzy elements (where all membership degrees are equally ranked) and hesitant fuzzy sets will be considered firstly.

Any hesitant fuzzy element can be identified with a unique element from $\bigcup_{n\in\mathbb{N}} E_\neg^n$. For example, both $\{0.1, 0.9\}$ and $\{0.9, 0.1\}$ are canonically represented by $(0.1, 0.9) \in E_\neg^2$. This process gives a canonical representation for HFSs too. In this way a hesitant fuzzy set on X is canonically represented by

$$h_M\colon\ X \longrightarrow \bigcup_{n\in\mathbb{N}} E_\neg^n \tag{5}$$

As $E_\neg^n \subset I_\neg^n$ for all n, Equations (2) and (5) warrant a canonical embedding of HFSs into SRHFSs. This is a simply algebraic fact, and of course there is no claim that HFS 'are' SRHFSs. Section 3.3 explained that HFS 'are' RHFS whose amplitudes are always 1.

In a similar fashion, a canonical representation for ranked hesitant fuzzy elements and sets will be suggested. It is theoretically important that a uniquely defined way to view all RHFEs and RHFSs becomes available in order to avoid uncertainty in subsequent definitions. In addition, note that section 3.5 will make explicit use of this canonical form.

Definition 11. A ranked hesitant fuzzy set on X can be uniquely identified with $\vec{R} = \{\langle x, h_{\vec{R}}(x)\rangle \mid x \in X\}$, another ranked hesitant fuzzy set whose ORHFS is the same and possesses the property that for each $x \in X$,

$$h_{\vec{R}}(x) = (\vec{R}_1(x), \ldots, \vec{R}_{a(x)}(x)) \subseteq E_\neg^{l_1(x)} \times \ldots \times E_\neg^{l_{a(x)}(x)} \tag{6}$$

with the condition that if $\vec{R}_i(x) = (R_i^1(x), \ldots, R_i^{l_i(x)}(x)) \in E_\neg^{l_i(x)}$ for each $i = 1, \ldots, a(x)$, then $R_i^j(x) \neq R_k^l(x)$ when either $i \neq k$ or $j \neq l$. This is the *canonical representation of* $\vec{R}$. Implicit in this construction is the notion of canonical representation of ranked hesitant fuzzy elements.

In order to illustrate these natural concepts, the next situation may be useful (see also Example 9).

Example 7. *Let* $X = \{x, y\}$ *and*

$$h'_{\vec{R}} = \{\langle x, (\{0.7, 0.3, 0.1\})\rangle, \langle y, (\{0.35, 0.4, 0.3\})\rangle\}$$

be a RHFS on X*. Then the canonical form of* $h'_{\vec{R}}$ *is the RHFS defined in Example 4, i.e.,* $h_{\vec{R}} = \{\langle x, (0.1, 0.3, 0.7)\rangle, \langle y, (0.3, 0.35, 0.4)\rangle\}$.

Analogously, both the RHFS defined in Example 2 and the next one

$$h_{\vec{R}'} = \Big\{\langle x, (\{0.5, 0.2\}, \{0.35, 0.3, 0.25\}, \{0.8, 0.95, 0.9\})\rangle,$$
$$\langle y, (\{0.8, 0.75, 0.7\}, \{0.15\}, \{0.25, 0.3\}, \{0.1, 0.05\})\rangle\Big\}$$

have the same canonical representation, namely,

$$\Big\{\Big\langle x, \Big((0.2, 0.5), (0.25, 0.3, 0.35), (0.8, 0.9, 0.95)\Big)\Big\rangle, \\ \Big\langle y, \Big((0.7, 0.75, 0.8), (0.15), (0.25, 0.3), (0.05, 0.1)\Big)\Big\rangle\Big\}$$

Hence some consequences can be drawn in terms of canonical representations of ranked hesitant fuzzy elements. Indeed it is implicitly stated that $\Big((0.2, 0.5), (0.25, 0.3, 0.35), (0.8, 0.9, 0.95)\Big)$ *is the canonical representation of both the RHFEs* $(\{0.2, 0.5\}, \{0.25, 0.3, 0.35\}, \{0.8, 0.9, 0.95\})$ *and* $(\{0.5, 0.2\}, \{0.35, 0.3, 0.25\}, \{0.8, 0.95, 0.9\})$*. And similarly, it is implicitly claimed that* $\Big((0.7, 0.75, 0.8), (0.15), (0.25, 0.3), (0.05, 0.1)\Big)$ *canonically represents both the RHFEs* $(\{0.7, 0.75, 0.8\}, \{0.15\}, \{0.25, 0.3\}, \{0.05, 0.1\})$ *and* $(\{0.8, 0.75, 0.7\}, \{0.15\}, \{0.25, 0.3\}, \{0.1, 0.05\})$*.*

3.5. A family of scores for ranked hesitant fuzzy elements

This section defines a family of scores that shall provide us with a uniform yardstick for RHFE comparisons. The inspiration is the family of Farhadinia scores (cf., Definition 4). As done in the presentation of the new model, the expression that defines our family on strictly ranked hesitant fuzzy elements only is discussed first, in order to maintain accessibility. Once its design and necessary computations have become known, the expression will be extended so that it can be applied with ranked hesitant fuzzy elements as well.

Definition 12. Fix $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$, a non-decreasing sequence of positive numbers. Let $e_{\vec{S}} = (R_1, \ldots, R_a) \in I_{\neg}^{a}$ be a SRHFE. Then

$$s_{\boldsymbol{\delta}}(e_{\vec{S}}) = \frac{\sum_{i=1}^{a} \delta_i R_i}{\sum_{i=1}^{a} \delta_i} \tag{7}$$

defines the $\boldsymbol{\delta}$-score of $e_{\vec{S}}$.

In particular, the *S-score* (for standard score) on SRHFEs is defined by the sequence $\boldsymbol{\delta} = \{1, 2, 3, \ldots\}$. And the *A-score* (for average score) on SRHFEs is defined by the sequence $\boldsymbol{\delta} = \{1, 1, 1, \ldots\}$.

Some numerical computations show that the S-score can be calculated by the formula

$$s_{\boldsymbol{S}}(e_{\vec{S}}) = \frac{2}{a(a+1)} \sum_{i=1}^{a} (i \cdot R_i) \tag{8}$$

It is apparent that the $\boldsymbol{\delta}$-score adapts the spirit of the Farhadinia score associated with $\boldsymbol{\delta}$ to SRHFEs. Farhadinia's scores were designed to operate without regard of any ordinal information about which degrees were more likely to appear. Hence they assign higher weights to higher degrees. The substance of this idea is retained and adapted to the circumstances of SRHFEs: a non-decreasing sequence of positive numbers captures the relative importances of the evaluations, but now more plausible degrees receive higher leverage. In this way an adjustable family of procedures is obtained for the comparison of strictly ranked hesitant fuzzy elements.

Recall that a hesitant fuzzy element can be derived from any (strictly) ranked hesitant fuzzy element, and that Example 6 illustrates this basic conversion. Remarkably, the A-score of a SRHFE coincides with the Xia–Xu score of the hesitant fuzzy element associated with it. And if by any chance, the rank in a SRHFE is given by the standard order of its evaluations, then its $\boldsymbol{\delta}$-score coincides with the Farhadinia score associated with $\boldsymbol{\delta}$ of its derived hesitant fuzzy element.

The next example computes S-scores. But it also argues that the application of the formula in (7) is more appropriate than the utilization of scores of derived HFEs as a first resort.

Example 8. *Two SRHFEs are defined in Example 1, namely,* $(0.7, 0.1, 0.3) \in I^3_\neg$ *and* $(0.6, 0.3, 0.35, 0.4) \in I^4_\neg$*. Their respective S-scores are computed as follows:*

$$\frac{1 \cdot 0.7 + 2 \cdot 0.1 + 3 \cdot 0.3}{1 + 2 + 3} = 0.3,$$

$$\frac{1 \cdot 0.6 + 2 \cdot 0.3 + 3 \cdot 0.35 + 4 \cdot 0.4}{1 + 2 + 3 + 4} = 0.385.$$

Notice that if these SRHFEs are converted into ordinary HFEs (cf., Example 4) and then Farhadinia's score with the same sequence of numbers is applied to the result, then one obtains the following figures:

$$\frac{1 \cdot 0.1 + 2 \cdot 0.3 + 3 \cdot 0.7}{1 + 2 + 3} = 0.4667,$$

$$\frac{1 \cdot 0.3 + 2 \cdot 0.35 + 3 \cdot 0.4 + 4 \cdot 0.6}{1 + 2 + 3 + 4} = 0.46.$$

Observe that discarding the information about the priorities in the allocation of membership degrees has reversed the comparison: the second SRHFE is deemed higher than the first SRHFE, whereas the HFE associated with the first SRHFE is deemed higher that the HFE associated with the second.

The conclusion stems that dispensing with the ordinal information about rankings in a SRHFE is not innocuous.

Definition 12 provides the groundwork for the construction of a score for ranked hesitant fuzzy elements. This extension abides by two principles. First, if two evaluations are equally plausible, then they are weighted by the same number. Second, this number is the average of the δ_i's corresponding to their positions in its canonical representation. The next example explains the details of this reasonable extension. Then a compact expression for the S-score will be explored, which of course is defined by the sequence $\boldsymbol{\delta} = \{1, 2, 3, \ldots\}$ as in the particular case of SRHFEs.

Example 9. *Let $\boldsymbol{\delta} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots\}$ be the Fibonacci sequence.*

Consider the RHFEs defined in Example 2 (see also Example 6), whose canonical representations are

$$\Big((0.2, 0.5), (0.25, 0.3, 0.35), (0.8, 0.9, 0.95)\Big) \text{ and}$$
$$\Big((0.7, 0.75, 0.8), (0.15), (0.25, 0.3), (0.05, 0.1)\Big).$$

Placing the eight membership degrees submitted by the first RHFE in order of non-decreasing plausibility, one has a list $(0.2, 0.5, 0.25, 0.3, 0.35, 0.8, 0.9, 0.95)$ *with the first two tied, the next three tied, and the last three tied. Take the first eight components of $\boldsymbol{\delta}$, which are* $(1, 1, 2, 3, 5, 8, 13, 21)$. *Then the averages of the first two numbers, then the next three, and the last three shall be applied to each of the elements of the three groups of evaluations.*

After these considerations, the $\boldsymbol{\delta}$-score of the first RHFE is calculated as

$$\frac{\frac{1+1}{2}(0.2 + 0.5) + \frac{2+3+5}{3}(0.25 + 0.3 + 0.35) + \frac{8+13+21}{3}(0.8 + 0.9 + 0.95)}{1 + 1 + 2 + 3 + 5 + 8 + 13 + 21} = 0.7556,$$

Similar computations show that the second RHFE has the following $\boldsymbol{\delta}$-score:

$$\frac{\frac{1+1+2}{3}(0.7 + 0.75 + 0.8) + 3 \cdot 0.15 + \frac{5+8}{2}(0.25 + 0.3) + \frac{13+21}{2}(0.05 + 0.1)}{1 + 1 + 2 + 3 + 5 + 8 + 13 + 21} = 0.177.$$

Two particular cases are worth discussing.

As in the case of SRHFEs, the A-score of a RHFE arises from the sequence $\boldsymbol{\delta} = \{1, 1, 1, \ldots\}$ by definition. Clearly, such an A-score coincides with the Xia–Xu score of the hesitant fuzzy element associated with the RHFE.

The other particular case is the S-score that arises from the sequence $\boldsymbol{\delta} = \{1, 2, 3, \ldots\}$. The next Proposition produces an explicit formula for the S-score of a generic ranked hesitant fuzzy element (cf., Definitions 9 and 11):

Proposition 1. *The S-score of $e_{\vec{R}}$, a ranked hesitant fuzzy element whose canonical form is $e_{\vec{R}} = (\vec{R}_1, \ldots, \vec{R}_a)$ with $\vec{R}_i = (R_i^1, \ldots, R_i^{l_i})$ for each $i = 1, \ldots, a$, can be computed as*

$$s_{\mathrm{S}}(e_{\vec{R}}) = \frac{(1 + l_1)\sum_{k=1}^{l_1} R_1^k + \sum_{j=2}^{a}\Big((2(l_1 + l_2 + \ldots + l_{j-1}) + l_j + 1)\sum_{k=1}^{l_j} R_j^k\Big)}{A(A+1)} \tag{9}$$

where $A = l_1 + l_2 + \ldots + l_a$.

Proof. The formula derives from some arithmetic computations. By construction,

$R_1^1, \ldots, R_1^{l_1}$ are averaged by $\dfrac{1 + \ldots + l_1}{l_1}$ which is $\dfrac{1 + l_1}{2}$.

$R_2^1, \ldots, R_2^{l_2}$ are averaged by $\dfrac{l_1 + 1 + l_2 + 1 + \ldots + l_1 + l_2}{l_2}$ or $\dfrac{2l_1 + l_2 + 1}{2}$.

$R_3^1, \ldots, R_3^{l_3}$ are averaged by $\dfrac{l_1 + l_2 + 1 + \ldots + l_1 + l_2 + l_3}{l_3}$ or $\dfrac{2l_1 + 2l_2 + l_3 + 1}{2}$.

By a sequential argument, we conclude that when $i = 2, 3, \ldots, a$, all evaluations $R_i^1, \ldots, R_i^{l_i}$ are averaged by $\dfrac{2(l_1 + \ldots + l_{i-1}) + l_i + 1}{2}$.

Hence because there are a total of $A = l_1 + l_2 + \ldots + l_a$ degrees, $s_{\text{S}}(e_{\vec{R}})$ is calculated as

$$\frac{(R_1^1 + \ldots + R_1^{l_1})\frac{1+l_1}{2} + (R_2^1 + \ldots + R_2^{l_2})\frac{2l_1+l_2+1}{2} + \ldots + (R_a^1 + \ldots + R_a^{l_a})\frac{2(l_1+\ldots+l_{a-1})+l_a+1}{2}}{1 + 2 + 3 + \ldots + (l_1 + l_2 + \ldots + l_a)}$$

which can be simplified by the expression in (9). Observe that because the denominator is a sum of consecutive elements $1, 2, 3, \ldots, l_1 + l_2 + \ldots + l_a = A$ of an arithmetic progression, it can be expressed as $\frac{1+A}{2}A$. □

To maintain accessibility and to ensure replicability, Listing 1 provides the *Mathematica* code that produces the S-score of any ranked hesitant fuzzy element. As it stands, it computes the S-score of the first RHFE in Example 9. However it is ready to operate on any other RHFE and to do that the reader can simply replace the input in Line 1 with the desired RHFE.

```
1   ClearAll;
2   HFE = {{0.2, 0.5}, {0.25, 0.30, 0.35}, {0.8, 0.9, 0.95}}; (* Input the ranked
      hesitant fuzzy element *)
3   RHFE = Map[Sort, HFE]; (* Produce its canonical representation *)
4   a = Length[RHFE];
5   ls = Map[Length, RHFE];
6   A = Total[ls];
7   sums = Total[RHFE, {-1}];
8   scoreHFE = ((1 + ls[[1]])*sums[[1]] + Sum[(2*(Sum[ls[[i]], {i, 1, j - 1}]) +
     ls[[j]] + 1)*sums[[j]], {j, 2, a}])/(A*(1 + A))
```

Listing 1: Mathematica code for the computation of the S-score of any ranked hesitant fuzzy element. For illustration, it produces the S-score of the first RHFE in Example 9.

Remark 3. *Although the formula of the A-score is very simple, the reader may be interested in a* Mathematica *code that calculates it. To this purpose, we only need to replace the last input line in Listing 1 by*

`Total[sums]/Total[ls]`

3.6. A score-based comparative analysis

Recall that probabilistic hesitant fuzzy elements produce ranked hesitant fuzzy elements where only the ordinal information is kept. Besides, the A-score of a ranked hesitant fuzzy element coincides with the Xia–Xu score of its associated (original) hesitant fuzzy element.

The next Example will produce a comparison of the score of probabilistic hesitant fuzzy elements, the S-score of the ranked hesitant fuzzy elements that are respectively associated with them, and the Xia–Xu score of the hesitant fuzzy elements that are respectively associated with them.

Example 10. *This example revisits the case study in Zhang et al. (2017, Section 4). It was concerned with three opinions on $X = \{A_1, \ldots, A_5\}$ (five car brands) and assessments for $\{C_1, \ldots, C_5\}$ (elements of their safety systems). Example 5 has given a glimpse of its formulation: it has recalled the aggregate PHFS that arises from the combination of opinions of three experts on C_1. Now one can compute the S-scores of the five ranked hesitant fuzzy elements generated from this PHFS in Example 5, with the assistance of the* Mathematica *program presented above. One readily obtains*

$s_{\text{S}}(h^1_{\vec{R}}(A_1)) = 0.665699,$
$s_{\text{S}}(h^1_{\vec{R}}(A_2)) = 0.574524,$
$s_{\text{S}}(h^1_{\vec{R}}(A_3)) = 0.669455,$
$s_{\text{S}}(h^1_{\vec{R}}(A_4)) = 0.769835,$ *and*
$s_{\text{S}}(h^1_{\vec{R}}(A_5)) = 0.70.$

It is surely unnecessary to insist on the details that produce $h^2_{\vec{R}}$, $h^3_{\vec{R}}$, $h^4_{\vec{R}}$, and $h^5_{\vec{R}}$, the RHFSs corresponding to the PHFSs that respectively capture the membership degrees for C_2, C_3, C_4, and C_5 in Zhang et al. (2017, Section 4). Suffice to say that the procedure is an exact replication of what has been done in Example 5 for C_1.

Zhang et al. (2017, Table 4) show the scores of all the PHFEs involved in this exercise (cf., Definition 6). Besides, Zhang et al. (2017, Table 5) show the Xia–Xu scores of all the HFEs derived from the aforementioned PHFEs (cf., Definition 4).

Table 2 summarizes all these figures. In addition to the S-scores of the five ranked hesitant fuzzy elements computed above, this table also shows the computation of the S-scores of the other twenty RHFEs considered in the case study in Zhang et al. (2017, Section 4).

One can now visually compare the performance of these scores on three different modelizations with decreasing level of complexity. Remarkably, a comparison of the S-scores of all the RHFEs with the scores of the PHFEs that produced them shows that only in two of the 25 comparisons, the difference appears to be as high as 0.01*. More accurate comparisons cannot be made since Zhang et al. (2017) only kept two decimal places in their calculations. It is remarkable that the results obtained here with a conceptually simpler model are so extraordinarily similar. By contrast, a comparison of the Xia–Xu scores of all the HFEs with the scores of the PHFEs that produced them shows that the difference is* 0.02 *in four cases,* 0.03 *in one case, and even* 0.04 *in another case (see boldfaced figures in Table 2).*

In conclusion, this analysis shows evidence that the ranked hesitant approach produces a conveniently simple hesitant model that does not require extra numerical assessments, but is almost indistinguishable (in terms of evaluation by scores) of its probabilistic enhancement. Put shortly: simplicity does not come at the cost of less accurate evaluations by scores.

Table 2: The comparison of scores for three models (PHFE, RHFE, HFE) in Example 10.

	A_1			A_2			A_3			A_4			A_5		
	P[1]	R[2]	H[3]	P[1]	R[2]	H[3]	P[1]	R[2]	H[3]	P[1]	R[2]	H[3]	P[1]	R[2]	H[3]
C_1	0.67	0.666	0.67	0.57	0.575	0.57	0.67	0.669	0.66	0.78	0.770	0.78	0.70	0.700	0.72
C_2	**0.66**	0.668	**0.70**	0.58	0.581	0.59	0.60	0.599	0.61	0.70	0.695	0.70	0.61	0.615	0.60
C_3	0.62	0.620	0.63	0.65	0.653	0.64	0.55	0.544	0.54	0.75	0.750	0.76	0.64	0.647	0.63
C_4	0.55	0.548	0.55	0.64	0.650	0.66	0.65	0.652	0.65	0.58	0.582	0.57	0.70	0.701	0.71
C_5	0.73	0.724	0.71	0.66	0.662	0.69	0.68	0.681	0.67	0.75	0.751	0.75	0.77	0.769	0.75

[1] Score of PHFE: v., Definition 6

[2] S-Score of RHFE: v., Definition 12

[3] Xia–Xu score of HFE: v., Definition 4 (i)

In terms of sensitivity, this numerical experiment confirms the suitability of ranked hesitant fuzzy sets for robust score-based assessments. Scores of probabilistic hesitant fuzzy sets were robust to small changes in the probabilities associated with the membership degrees. Notice that sufficiently small changes do not affect the underlying ranked hesitant fuzzy set, as their relative importances do not change. Hence the scores of ranked hesitant fuzzy sets derived from a probabilistic hesitant fuzzy set remain unaffected by sufficiently small perturbations of the probabilities.

To conclude this section, particular forms of the S-score that appeal to two notable cases will be investigated. The second will yield a direct comparison with a popular score for HFEs.

Notice first that when a ranked hesitant fuzzy element is in fact stricly ranked, then Equation (9) reduces to Equation (8). In this case $A = a$ since $l_i = 1$ for $i = 1, \ldots a$, and (9) becomes

$$\frac{(1+1)\sum_{k=1}^{1} R_1^k + \sum_{j=2}^{a}\left((2(1+1+\overset{j-1}{\ldots}+1)+1+1)\sum_{k=1}^{1} R_j^k\right)}{a(a+1)} =$$

$$2\frac{R_1^1 + \sum_{j=2}^{a}\left(jR_j^1\right)}{a(a+1)} = \frac{2}{a(a+1)}\sum_{j=1}^{a}\left(jR_j^1\right)$$

which is (8).

Besides, it has been explained that hesitant fuzzy elements are ranked hesitant fuzzy elements of amplitude 1 (v., section 3.3). Notice that in fact, when one plugs $a = 1$ in Equation (9) and denoting $l_1 = l$ for simplicity, the expression reduces to

$$s_{\mathrm{S}}(e_{\vec{R}}) = \frac{(1+l)\sum_{k=1}^{l} R_1^k}{l(l+1)} = \frac{\sum_{k=1}^{l} R_1^k}{l}$$

This is the standard form of the Xia–Xu score of $e_{\vec{R}}$ (with $a = 1$). Hence the next property has been proven:

Lemma 1. *The S-score of a hesitant fuzzy element coincides with its Xia–Xu score.*

In conclusion, the S-score constitutes a rightful extension of the Xia–Xu score when a prioritization of membership degrees is added.

4. Aggregation of ranked hesitant fuzzy sets

In this section social choice meets ranked hesitant fuzzy sets. Techniques imported from social choice will be employed for the aggregation of ranked hesitant fuzzy sets. Then a *Mathematica* code will be provided to implement the procedure that stems from this interaction.

Aggregation of hesitant fuzzy elements typically resorted to operators originating with algebraic manipulations. For example, (weighted) arithmetic and geometric means, and quasi-arithmetic means, produce the corresponding aggregation operators for HFEs (Xia & Xu, 2011). The Einstein sum and product are the germ of hesitant fuzzy Einstein aggregation operators, and the Frank sum and product give raise to hesitant fuzzy Frank (arithmetic, geometric) aggregation operators of various types. Other sources of inspiration include the Maclaurin symmetric mean, or the Bonferroni, Hamacher, and Choquet aggregators. This generic technique can be exported to numerical enhancements like probabilistic hesitant fuzzy elements (Zhang et al., 2017, Definitions 10 and 11).

However this paper operates in a different framework which resembles more of a ranking aggregation method (Ding et al., 2018). In this case a list of elements (like the membership degrees associated with an alternative) are ranked and then an aggregate rank on these elements is derived. In particular, the Borda method is particularly attractive because it has already been used with intuitionistic fuzzy sets (Cheng et al., 2020) and probabilistic linguistic term sets (Liao et al., 2019; Wu et al., 2018). As a proxy for a consensus measure, (weighted) Borda counts have been applied in a context of hesitant fuzzy linguistic information (Liao et al., 2020).

For the purposes of this research the key concept that must be known beforehand is the Borda indices of a completely ordered list of alternatives. A complete preorder is a complete and transitive binary relation. Informally, it is an extension of a linear order for which ties are allowed. Then the Borda index of a fixed alternative is the difference between the number of options that are worse than it, and the number of options that are better than it (Gärdenfors, 1973). Example 12 below illustrates this construction in the context of ranked hesitant fuzzy elements: notice that according to its semantics, a ranked hesitant fuzzy element produces a complete preorder on the membership degrees belonging to its original hesitant fuzzy element.

In order to give a formal definition of our aggregation procedure, the following auxiliary concept will be needed. It is a normalization process whose verbal explanation comes right afterwards and is much simpler to understand:

Definition 13. Let $E = \{e^1_{\vec{R}}, \ldots, e^p_{\vec{R}}\}$ be a list of ranked hesitant fuzzy elements whose respective OHFEs are $h^1, \ldots, h^p$. Denote by h the union of these OHFEs, $h = h^1 \cup \ldots \cup h^p$, which is another HFE. The normalized list of E is $N(E) = \{n^1_{\vec{R}}, \ldots, n^p_{\vec{R}}\}$, a list of ranked hesitant fuzzy elements such that:

1. When $h^j = h$, $n^j_{\vec{R}} = e^j_{\vec{R}}$.
2. When $h^j \subsetneq h$, if the canonical form of $e^j_{\vec{R}}$ is $(\vec{R}^j_1, \ldots, \vec{R}^j_a(j))$ with $\vec{R}^j_i = (R^{j,1}_i, \ldots, R^{j,l_i}_i)$ for each $i = 1, \ldots, a(j)$, then the canonical form of $n^j_{\vec{R}}$ is $(\vec{R}^j_0, \vec{R}^j_1, \ldots, \vec{R}^j_a)$ with $\vec{R}^j_0$ containing the elements in $h - h^j$.

Intuitively, the normalization of a list of p RHFEs produces another list of p RHFEs with the following characteristics. First, the OHFEs associated with the RHFEs in the normalized list are always h. This means that after normalization, all the ranked hesitant evaluations consist of exactly the same degrees, which are all the membership degrees pertaining to some of the p evaluations given in

E. Secondly, when an RHFE in E lacks some degrees that pertain to some of the evaluations given in E, all these missing elements are added at the bottom rank of its corresponding RHFE in the normalized list. An example clarifies this simple procedure:

Example 11. *Consider $E = \{e^1_{\vec{R}}, e^2_{\vec{R}}, e^3_{\vec{R}}\}$, the ranked hesitant fuzzy elements whose respective canonical forms are*

$$\Big((0.1, 0.2), (0.7), (0.5, 0.6)\Big)$$

$$\Big((0.7), (0.2, 0.3), (0.4, 0.5), (0.6)\Big)$$

$$\Big((0.1, 0.2, 0.7), (0.3), (0.4, 0.5, 0.6)\Big).$$

Then $h^1 = \{0.1, 0.2, 0.5, 0.6, 0.7\}$, $h^2 = \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}$, and $h^3 = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}$. Therefore $h = h^3$.

The normalized list of E is $N(E) = \{n^1_{\vec{R}}, n^2_{\vec{R}}, n^3_{\vec{R}}\}$ whose respective canonical forms are

$$\Big((0.3, 0.4), (0.1, 0.2), (0.7), (0.5, 0.6)\Big)$$

$$\Big((0.1), (0.7), (0.2, 0.3), (0.4, 0.5), (0.6)\Big)$$

$$\Big((0.1, 0.2, 0.7), (0.3), (0.4, 0.5, 0.6)\Big).$$

The aggregation of ranked hesitant fuzzy elements proceeds in the following four steps:

Step 1. Normalize the list of ranked hesitant fuzzy elements (cf., Definition 13).

Step 2. Compute the Borda index of each membership degree in each normalized ranked hesitant fuzzy element.

Step 3. For each membership degree in each normalized ranked hesitant fuzzy element, Compute the sum of its Borda indices.

Step 4. Rank the membership degrees by their increasing aggregate Borda indices.

Now the application of these steps will be illustrated with a synthetic example:

Example 12. *In order to compute the aggregation of the three ranked hesitant fuzzy elements in Example 11, the following steps should be taken.*

Step 1 was achieved in Example 11.

Step 2 computes three Borda indices (one associated with each $n^i_{\vec{R}}$) for each alternative in $h = \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}$.

The Borda indices associated with $n^1_{\vec{R}}$ are:
$0 - 5 = -5$ *for the membership degrees* 0.3 *and* 0.4

$2-3=-1$ *for the membership degrees* 0.1 *and* 0.2
$4-2=2$ *for the membership degree* 0.7
$5-0=5$ *for the membership degrees* 0.5 *and* 0.6

The Borda indices associated with $n^2_{\vec{R}}$ *are:*
$0-6=-6$ *for the membership degree* 0.1
$1-5=-4$ *for the membership degree* 0.7
$2-3=-1$ *for the membership degrees* 0.2 *and* 0.3
$4-1=3$ *for the membership degrees* 0.4 *and* 0.5
$6-0=6$ *for the membership degree* 0.6

The Borda indices associated with $n^3_{\vec{R}}$ *are:*
$0-4=-4$ *for the membership degrees* 0.1*,* 0.2 *and* 0.7
$3-3=0$ *for the membership degree* 0.3
$4-0=4$ *for the membership degrees* 0.4*,* 0.5 *and* 0.6

Step 3 simply sums the three Borda indices computed, for each membership degree. The result is:
$-1-6-4=-11$ *for the membership degree* 0.1
$-1-1-4=-6$ *for the membership degree* 0.2
$-5-1+0=-6$ *for the membership degree* 0.3
$-5+3+4=2$ *for the membership degree* 0.4
$5+3+4=12$ *for the membership degree* 0.5
$5+6+4=15$ *for the membership degree* 0.6
$2-4-4=-6$ *for the membership degree* 0.7

Step 4. Use these assessments to produce a ranked hesitant fuzzy element. In conclusion, $\Big((0.1),(0.2,0.3,0.7),(0.4),(0.5),(0.6)\Big)$ *is the canonical form of the aggregate RHFE of the three ranked hesitant fuzzy elements in Example 11.*

In order to facilitate calculations, Listing 2 provides the *Mathematica* code that produces the aggregation of any number of ranked hesitant fuzzy elements. It computes the aggregate RHFE given in Example 12. This program is designed to print all intermediate computations, like the successive Borda indices of each membership degree, their sums accross membership degrees, and the aggregate ranked hesitant fuzzy element. For subsequent applications, the reader can simply replace the input in Line 1 with the desired list of RHFEs.

```
1  ClearAll;
2  MRHFE = {{{0.1, 0.2}, {0.7}, {0.5, 0.6}},
3            {{0.7}, {0.2, 0.3}, {0.4, 0.5}, {0.6}},
4            {{0.1, 0.2, 0.7}, { 0.3}, {0.4, 0.5, 0.6}}
5            }; (* Input the ranked hesitant fuzzy elements *)
6  p = Length[MRHFE];
7  MOHFE = Map[Flatten, MRHFE]; (* Produces their OHFEs *)
8  HFEValues = Sort[DeleteDuplicates[Catenate[MOHFE]]];
9  A = Length[HFEValues];
10 OHFEMissing = Range[p];
11 Do[OHFEMissing[[i]] = Complement[HFEValues, MOHFE[[i]]], {i, 1, p}]
12 RHFETotal = Range[p];
13 Do[RHFETotal[[i]] = Join[{OHFEMissing[[i]]}, MRHFE[[i]]], {i, 1, p}];
```

```
Mls = Table[Length[RHFETotal[[x, y]]], {x, 1, p}, {y, 1, Length[RHFETotal[[x
   ]]]}];
Clear[i];
For[i = 1, i <= A, i++,
  Print["Computations for value: ", HFEValues[[i]]]
    For[j = 1, j <= p, j++,
      Print["The degree ", HFEValues[[i]], " appears at rank position ",
   Position[RHFETotal, HFEValues[[i]]][[j, 2]], " of assessment ", j, ", hence
   its Borda index is ", Apply[Plus,
    Mls[[j, 1 ;; Position[ RHFETotal, HFEValues[[i]] ][[j, 2]] ]]]
   -
   Apply[Plus, Mls[[j, Position[ RHFETotal, HFEValues[[i]] ][[j, 2]] ;;
   Length[RHFETotal[[j]]]]] ] ]
      ]
  ]
MlsN = Table[
Apply[Plus, Mls[[j, 1 ;; Position[ RHFETotal, HFEValues[[i]] ][[j, 2]] ]]] -
Apply[Plus, Mls[[j, Position[ RHFETotal, HFEValues[[i]] ][[j, 2]] ;;
    Length[RHFETotal[[j]]]]]  ], {i, 1, A}, {j, 1, p}
    ];
Borda = Total[MlsN, {-1}];
Clear[i];
For[i = 1, i <= A, i++,
  Print["The degree ", HFEValues[[i]], " receives a total Borda index of ",
   Borda[[i]]]
  ]
BordaScores = Sort[DeleteDuplicates[Borda]];
NBordaScores = Length[BordaScores];
Print["The aggregate ranking of the degrees (from worst to best) is: "]
For[i = 1, i <= NBordaScores, i++,
    Print["Position ", NBordaScores - i + 1, " is occupied by ",
    HFEValues[[#]] &[Flatten[Position[Borda, BordaScores[[i]]]]] ]
  ]
Print["Here are the top ranked membership degrees."]
```

Listing 2: *Mathematica* code for the computation of the the aggregation of any number of ranked hesitant fuzzy elements. For illustration, it produces the solution given in Example 12.

Lastly in this section, a qualitative difference makes the aggregation of ranked hesitant fuzzy elements more approachable than its counterparts for probabilistic strengthenings, like Zhang et al. (2017, Definitions 10 and 11). As stated above, in the new aggregation mechanism all the membership degrees associated with an alternative are prioritized by means of an aggregate rank. Therefore it does not introduce new membership degrees that are supported by no evidence. This is a marked contrast to algebraic approches, since they produce considerable expansions of the membership degrees involved in the calculations. Both practical and methodological advantages are apparent. For illustration, the next example is given:

Example 13. *Example 5, revisited in Example 10, has reinterpreted some elements from the case study in Zhang et al. (2017, Section 4). As in those cases, attention is restricted to the evaluation of the safety element C_1 (or brake system). The exercise in this example consists of computing the aggregate ranked hesitant fuzzy elements that stem from the aggregation of the ranked opinions of* *the three experts. This exercise is done for the five car brands. Table 3 displays the elements that are needed for these calculations. The opinions by the three experts given in Zhang et al. (2017, Section 4) have been first converted into their ranked versions (this simplification was utilized in Example 10 too), so that ranked hesitant fuzzy elements make their opinions plainer. Their canonical expressions are shown in Table 3. Now the aggregation procedure described in this section produces the respective five aggregate outputs.*

Table 3: Aggregation of the data in Example 13.

	Opinions about C_1			
	From expert 1	From expert 2	From expert 3	Aggregate RHFE
A_1	$\big((0.6, 0.8), (0.7)\big)$	$\big((0.5, 0.7)\big)$	$\big((0.8), (0.6), (0.7)\big)$	$\big((0.5), (0.8), (0.6), (0.7)\big)$
A_2	$\big((0.5, 0.6)\big)$	$\big((0.5), (0.6)\big)$	$\big((0.7), (0.6)\big)$	$\big((0.7), (0.5), (0.6)\big)$
A_3	$\big((0.8), (0.7)\big)$	$\big((0.5), (0.6), (0.7)\big)$	$\big((0.5), (0.6)\big)$	$\big((0.8), (0.5), (0.7), (0.6)\big)$
A_4	$\big((0.9), (0.85), (0.8)\big)$	$\big((0.7, 0.8)\big)$	$\big((0.5, 0.7), (0.6)\big)$	$\big((0.9), (0.5), (0.85), (0.6), (0.7), (0.8)\big)$
A_5	$\big((0.65, 0.75)\big)$	$\big((0.8), (0.7), (0.6)\big)$	$\big((0.7, 0.8)\big)$	$\big((0.65, 0.75), (0.6), (0.8), (0.7)\big)$

Now one can compare the last column of Table 3 with the five probabilistic hesitant fuzzy elements $h_P(A_1, C_1)$, ..., $h_P(A_5, C_1)$ that produced the aggregation of the respective three probabilistic hesitant fuzzy elements in Zhang et al. (2017, Section 4). Example 5 has recalled $h_P(A_1, C_1)$, ..., $h_P(A_5, C_1)$. Observe that they are much longer than the corresponding RHFEs displayed in Table 3, hence they require more computational load. And they are formed by membership degrees that have not been supported by the opinion of any of the experts, which is not the case of the aggregate ranked hesitant fuzzy elements that we have obtained.

Besides, the aggregate output of our calculations can be condensed into one ranked hesitant fuzzy set, namely,

$$h_{\vec{R}} = \{\langle A_1, (\{0.5\}, \{0.8\}, \{0.6\}, \{0.7\})\rangle, \langle A_2, (\{0.7\}, \{0.5\}, \{0.6\})\rangle,$$
$$\langle A_3, (\{0.8\}, \{0.5\}, \{0.7\}, \{0.6\})\rangle, \langle A_4, (\{0.9\}, \{0.5\}, \{0.85\}, \{0.6\}, \{0.7\}, \{0.8\})\rangle,$$
$$\langle A_5, (\{0.65, 0.75\}, \{0.6\}, \{0.8\}, \{0.7\})\rangle\}.$$

This expression can also be compared with $h^1_{\vec{R}}$, the aggregate output obtained from the simplification of the aggregate output given by Zhang et al. (2017) as a probabilistic hesitant fuzzy element. The conclusions of this comparison are similar to the comparison of their respective elements.

5. Making decisions with ranked hesitant fuzzy information: the multi-criteria multi-agent case

The fundamental theory of ranked hesitant fuzzy sets has been established with the design of complements, scores and aggregation operators. It is now time to apply these ingredients to forge a multi-criteria group decision making procedure. Section 5.1 will give a glimpse to the multi-criteria multi-agent case study in Li et al. (2019) for both motivation and comparison. In fact these authors already compared the conclusions of their analysis with those of the methodology presented by Xu & Zhou (2017). Then the algorithm for multi-criteria multi-agent decisions with ranked hesitant fuzzy information will be stated in section 5.2, and a comparison with the aforementioned methodologies is performed in section 5.3.

5.1. *Statement of the problem*

The structure of the new model will be apparent with the following practical illustration.

Tables 4, 5 and 6 capture the ranked hesitant fuzzy information representing the opinion of three overseas experts that estimate the potential for investment in four alternatives, namely, A_1 (automobile sector), A_2 (food industry), A_3 (clothing industry), and A_4 (computer industry). The experts were concerned about three criteria, namely, c_1 (profits), c_2 (growth), and c_3 (environmentally friendly). These tables are directly derived from Li et al. (2019, Tables 3,4,5). The later tables represent respective probabilistic hesitant fuzzy sets in Li et al.'s case study. To deduce Tables 4, 5 and 6 from them, the simplification explained in Section 3.3 has been used (see Example 5 for illustration).

Table 4: Ranked information derived from data by Overseas Expert 1 (canonical form).

	A_1	A_2	A_3	A_4
c_1	$\big((0.6, 0.8), (0.7)\big)$	$\big((0.6), (0.5)\big)$	$\big((0.6, 0.7), (0.8)\big)$	$\big((0.3), (0.4)\big)$
c_2	$\big((0.5, 0.6)\big)$	$\big((0.4, 0.5), (0.6)\big)$	$\big((0.7, 0.8)\big)$	$\big((0.7), (0.6)\big)$
c_3	$\big((0.8), (0.7)\big)$	$\big((0.3), (0.4, 0.5)\big)$	$\big((0.6), (0.7)\big)$	$\big((0.5), (0.6)\big)$

Table 5: Ranked information derived from data by Overseas Expert 2 (canonical form).

	A_1	A_2	A_3	A_4
c_1	$\big((0.5, 0.7)\big)$	$\big((0.9), (0.8), (0.7)\big)$	$\big((0.7), (0.8)\big)$	$\big((0.5, 0.7), (0.6)\big)$
c_2	$\big((0.5, 0.6)\big)$	$\big((0.7), (0.6)\big)$	$\big((0.8), (0.6), (0.5)\big)$	$\big((0.6, 0.7)\big)$
c_3	$\big((0.5, 0.7), (0.6)\big)$	$\big((0.8), (0.7)\big)$	$\big((0.5, 0.7)\big)$	$\big((0.7), (0.8), (0.6)\big)$

As in Li et al. (2019), and previously Xu & Zhou (2017), the current target is to prioritize the alternatives. This will be done with the assistance of a new multi-criteria multi-agent methodology. Its main novelty is that it is capable of integrating a multiplicity of sources, each providing a hierarchy of membership degrees, and then it performs a score-based comparison of the aggregate results. The algorithm is flexible at this penultimate step, and the solution will illustrate this property with the application of two different scores. The final step

Table 6: Ranked information derived from data by Overseas Expert 3 (canonical form).

	A_1	A_2	A_3	A_4
c_1	$\big((0.8),(0.6),(0.7)\big)$	$\big((0.6,0.7)\big)$	$\big((0.6),(0.7)\big)$	$\big((0.6),(0.7)\big)$
c_2	$\big((0.7),(0.6)\big)$	$\big((0.5,0.7),(0.6)\big)$	$\big((0.6,0.7),(0.5)\big)$	$\big((0.8),(0.6)\big)$
c_3	$\big((0.5),(0.6)\big)$	$\big((0.5,0.7)\big)$	$\big((0.7),(0.8)\big)$	$\big((0.6),(0.7)\big)$

combines the scores by a weighted average, which allows to consider attributes with different importances.

5.2. Solution to the problem with ranked hesitant fuzzy information

It is time to establish the procedure for selecting a best decision investment from the data provided. In fact a complete preorder on the four alternatives $X = \{A_1, A_2, A_3, A_4\}$ will be given. For that purpose Algorithm 1 is put forward. Of course, this Algorithm is not restricted to this specific situation. It is capable of acting on any multiplicity of ranked hesitant fuzzy sets.

In relation with Algorithm 1, two comments are in order.

First, one can observe that the Algorithm is flexible as a score may be selected that best suits the needs of the decision maker. This selection is used at *Step 5*. Such feature will be useful to test the sensitivity of the algorithm to the choice of a score. The application to solve the problem stated in the previous section will use S-scores and A-scores at *Step 5*.

Notice also that Li et al. (2019) computed the weights of the criteria from their data. However a similar methodology cannot be used to derive weights endogenously, because the current formulation of the problem dispenses with the numerical probabilities from which they stem. This accounts for the fact that weights are inputted in Step 1. Relatedly, Li et al. (2019) produced two solutions to their problem, depending on whether the dominance degree matrices were additive or multiplicative consistent. Although these assumptions produced different weights, the two ranking solutions were ultimately identical: $A_1 \succ A_2 \succ A_3 \succ A_4$.

The details of the application of this decision-making procedure to the case at hand are as follows.

Step 1. In order to compare the results from our study with those of Li et al. (2019), select the weights that they had calculated in their two cases: first proceed with $w_1 = 0.3627$, $w_2 = 0.2696$, $w_3 = 0.3662$, and then with $w_1 = 0.3845$, $w_2 = 0.2434$, $w_3 = 0.3721$. Also, a solution with arithmetic averages will be computed, i.e., with the simple case $w_1 = w_2 = w_3 = \frac{1}{3}$. This multiple choice will produce a more complete sensitivity analysis.

Step 2 consists of Tables 4, 5 and 6. There are no cost attributes, thus *Step 3* requires no further action.

Step 4 requires the aggregation of 12 separate lists of 3 ranked hesitant fuzzy elements. The *Mathematica* code in Listing 2 can be used in order to produce the aggregate data displayed in Table 7.

Algorithm 1. MCGDM procedure based on ranked hesitant fuzzy information

Elements of the problem: alternatives $X = \{x_1, x_2, \cdots, x_p\}$, attributes $C = \{c_1, c_2, \cdots, c_q\}$, group of k experts.
Elective element of the solution: score for the evaluation of ranked hesitant fuzzy elements.

1: Input weights of the attributes $w_1, w_2, \cdots, w_q$.

2: Input for each agent, a table of ranked hesitant fuzzy elements. Its cell (i, j) contains the evaluation of alternative j with respect to attribute c_i.

3: For cost attributes, replace the evaluations of all alternatives by their complements (cf., Definition 10) throughout all tables.

4: For each alternative and attribute: aggregate the list of k ranked hesitant fuzzy elements submitted by the experts by the procedure explained in Section 4.
The result is a unique aggregate table whose cells are ranked hesitant fuzzy elements.

5: Compute the scores $s(e_{ij})$ of these ranked hesitant fuzzy elements, for all $c_i \in C$, $x_j \in X$.

6: Compute the weighted score s_j of each alternative $x_j \in X$, which is $s_j = w_1 s(e_{1j}) + w_2 s(e_{2j}) + \cdots + w_q s(e_{qj})$.

Finally, rank the alternatives in X by their respective weighted scores.
Any of the alternatives for which $s_i = \max_{j=1,2,\cdots,p} s_j$ can be chosen.

Table 7: Ranked information derived from the aggregation of the data in Tables 4, 5 and 6.

	A_1	A_2	A_3	A_4
c_1	$\big((0.5),(0.8),(0.6),(0.7)\big)$	$\big((0.9),(0.5,0.8),(0.6),(0.7)\big)$	$\big((0.6),(0.7),(0.8)\big)$	$\big((0.3,0.5),(0.4),(0.7),(0.6)\big)$
c_2	$\big((0.7),(0.5),(0.6)\big)$	$\big((0.4),(0.5,0.7),(0.6)\big)$	$\big((0.8),(0.6,0.7),(0.5)\big)$	$\big((0.8),(0.7),(0.6)\big)$
c_3	$\big((0.8),(0.5),(0.7),(0.6)\big)$	$\big((0.3),(0.8),(0.4),(0.5,0.7)\big)$	$\big((0.6),(0.5),(0.8),(0.7)\big)$	$\big((0.5),(0.8),(0.7),(0.6)\big)$

At *Step 5*, suppose first that the S-score is used to evaluate each ranked hesitant fuzzy element in Table 7. In order to produce these evaluations, the *Mathematica* code in Listing 1 can be utilized. The results are shown in Table 8. Its last three rows average these scores by the respective vectors of weights selected at Step 1. Therefore this is the consequence of *Step 6*.

Table 8: The S-scores and weighted scores of the ranked hesitant fuzzy elements in Table 7.

	A_1	A_2	A_3	A_4
c_1	0.67	0.67	0.7333	0.5467
c_2	0.5833	0.58	0.605	0.6667
c_3	0.63	0.5667	0.68	0.66
Weighted score 1	0.630623	0.606555	0.677729	0.619322
Weighted score 2	0.634013	0.609656	0.682239	0.618067
Arithmetic score	0.627767	0.605567	0.672767	0.624467

Then the exercise is repeated with the A-score at *Step 5*. Remark 3 explains that one can also use a *Mathematica* code that slightly modifies the code in Listing 1. The results are shown in Table 9 and again, they are derived from the weighted scores calculated at *Step 6*.

Table 9: The A-scores and weighted scores of the ranked hesitant fuzzy elements in Table 7.

	A_1	A_2	A_3	A_4
c_1	0.65	0.7	0.7	0.5
c_2	0.6	0.55	0.65	0.7
c_3	0.65	0.54	0.65	0.65
Weighted score 1	0.635185	0.599588	0.66677	0.60768
Weighted score 2	0.63783	0.603954	0.669225	0.604495
Arithmetic score	0.633333	0.596667	0.666667	0.616667

In conclusion, the three selections of weights coincide to recommend the ranking $A_3 \succ A_1 \succ A_4 \succ A_2$, both when we use S-scores and A-scores. Hence A_3 is the unique alternative that is recommended.

5.3. Comparative analysis and sensitivity

Table 10 compares the result achieved in section 5.2 with the solutions provided by other four methodologies that acted on the original probabilistic hesitant fuzzy information (v., section 5.1). In this regard, note that Li et al. (2019) already compared their conclusions for that problem with those of Xu & Zhou (2017). The alternatives recommended by each study were different (A_1 versus A_3), and the rankings produced by their methodologies are rather opposed to each other. Here their discussion is complemented with the observation that Algorithm 1 produces always the same ranking, whose two top alternatives are precisely A_3 and A_1. This holds true even when equal weights are used, which further simplifies the computations, and the score is varied.

Thus Algorithm 1 has consistently detected the two preferred alternatives that the other four methodologies had computed. Remarkably, an inspection of

Table 10: A comparison of results.

Methodology	Source	Variants	Ranking	Selection
Xu & Zhou (2017)	Probabilistic	HPFWA/HPFWG operator	$A_3 \succ A_4 \succ A_1 \succ A_2$	A_3
Li et al. (2019)	Probabilistic	Additive/Multiplicative consistency	$A_1 \succ A_2 \succ A_3 \succ A_4$	A_1
Algorithm 1	Ranked	Weights from Li et al. (2019) / equal weights. S-score	$A_3 \succ A_1 \succ A_4 \succ A_2$	A_3
Algorithm 1	Ranked	Weights from Li et al. (2019) / equal weights. A-score	$A_3 \succ A_1 \succ A_4 \succ A_2$	A_3

Table 10 shows that neither of the other procedures settled for these two options at the top of their recommendation.

In relation with sensitivity, these conclusions show strong evidence of the robustness of the methodology presented in this paper.

In addition, it has been highlighted that the model defined here is simpler and less demanding than the probabilistic hesitant fuzzy structure. This is the model that both Li et al. (2019) and Xu & Zhou (2017) had used in their analysis. The simplified requirement in the model by ranked hesitant fuzzy sets has allowed to streamline the computations, which have been mostly referred to *Mathematica* programs. Altogether this article has produced an user-friendly methodology with accurate recommendations from a new non-technical model.

6. Concluding remarks

The ultimate goal of hesitant fuzzy sets is the representation of hesitation about the fuzzy degrees of membership of some alternatives. This should not come at the cost of dispensing with supplementary features embodied in those evaluations. Researchers with different takes on this research question have studied probabilistic, proportional, extended, or expanded versions of the model famously shown by Torra (2010). But so far the literature has not dealt with the case of a hierarchy of evaluations based on their plausibility. This paper has provided a formal framework for encoding this situation. With this model as a starting point, new scores and aggregation procedures advance the mathematical background of ranked hesitant fuzzy sets. To facilitate reproducibility of the results in this paper, *Mathematica* codes implementing these two auxiliary tools that act on ranked hesitant fuzzy elements have been offered.[3] They are the fundamental constituents of a flexible Algorithm that gives a ranking of alternatives defined by multiple attributes (of possibly unequal importance) in a multi-agent context. A case study has shown that its computational simplicity is not at odds with accuracy.

[3] The corresponding *Mathematica* notebooks can be downloaded from https://github.com/jcralcantud/RHFS.

As a consequence of this contribution, hesitant fuzzy sets have become a simplified ranked hesitant fuzzy structure. In the opposite direction, the ranked version of hesitant fuzzy sets has been contrasted with other extensions of Torra's original model. In particular, ranked hesitant fuzzy sets become a simplified version of probabilistic and proportional hesitant fuzzy sets that retain their most important trait, namely, the prioritization of some evaluations over others. In relation with the practical implications of the present study, it is remarkable that ranked hesitant fuzzy sets have proved exceptionally adept in terms of evaluative ability: scores of probabilistic hesitant fuzzy elements remain largely unaffected by their reduction to a ranked version. Thus in practice, less information is needed to produce very similar score-based comparisons. Advantages in terms of aggregation have been found too, as this process has proven to be more accessible in RHFEs than in PHFEs. And a sensitivity analysis has shown the robustness of the decision-making methodology proposed in this paper.

Concerning limitations, our study has not yet considered other technical elements that might help understand and apply ranked hesitant fuzzy sets in the future. Entropy, deviation degrees, and correlation coefficients can be defined, discussed and applied. Inspirational studies exist in related settings (Su et al., 2019; Wang et al., 2019).

Lastly, it is likely that novel procedures for the aggregation of ranked hesitant fuzzy elements may be defined by resort to the Sugeno integral, precisely because this is an ordinal aggregator (Beliakov et al., 2019).

Also the managerial and industrial applications of ranked hesitant fuzzy sets still remain to be explored. Expectations are bolstered by successful approaches like Deveci et al. (2018) or Deveci et al. (2022) (management) and Dinçer et al. (2019a) or Narayanamoorthy et al. (2019) (industry).

In a different vein, ranked hesitant fuzzy sets can be made compatible with more structured information in the future. For example, it has been mentioned that probabilistic dual hesitant fuzzy sets exist since Hao et al. (2017), see also Garg & Kaur (2020) for a recent update. It is therefore feasible that ranked dual hesitant fuzzy sets can be similarly motivated and that new applications can be found. Another possible line for future inspection is the case of missing information, for which antecedents in the probabilistic hesitant fuzzy case exist (Zhang et al., 2017, Example 8).

Declaration of competing interest

The author declares no conflict of interest.

Ethical approval

This article does not contain any studies with human participants or animals performed by the author.

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