The semantics of N-soft sets, their applications, and a coda about three-way decision

José Carlos R. Alcantud[a,*]

[a]BORDA Research Group and Multidisciplinary Institute of Enterprise (IME), University of Salamanca, 37007 Salamanca, Spain

ARTICLE INFO

Keywords:
N-soft set
Aggregation
Three-way decision
Decision making
Multi-valued logic

ABSTRACT

This paper presents the first detailed analysis of the semantics of N-soft sets. The two benchmark semantics associated with soft sets are perfect fits for N-soft sets. We argue that N-soft sets allow for an utterly new interpretation in logical terms, whereby N-soft sets can be interpreted as a generalized form of incomplete soft sets. Applications include aggregation strategies for these settings. Finally, three-way decision models are designed with both a qualitative and a quantitative character. The first is based on the concepts of V-kernel, V-core and V-support. The second uses an extended form of cardinality that is reminiscent of the idea of scalar sigma-count as a proxy of the cardinality of a fuzzy set.

1. Introduction

Uncertain knowledge may be voiced in many declinations. There are situations where we can adequately describe some objects or alternatives in terms of whether they fulfil some perceivable attributes. If there is a unique attribute to check for, the mathematical concept that stems is a subset (which is formed by the objects that have the required attribute). When the number of attributes is unrestricted, a soft set is defined [1]. But things are not always black or white. Thus to allow for several properties that can happen in different categories, the mathematical concept that stems is called N-soft set [2].

Uncertain decisions are a natural consequence of imperfect knowledge. Bulding on probabilistic rough set theory, three-way decision takes a cautious approach. This theory was formulated by Yao [3, 4, 5, 6] to model situations where enforcing a two-way end result (acceptance vs. rejection) has a negative bearing on the verdict on some alternatives. A third alternative allows us to defer the final decision about a part of the available options. The introduction of a non-commital outcome bears comparison with decision rules derived from rough set theory through the negative, boundary, and positive regions of a target set [7].

Divisions of a set into three regions had been attempted from the perspective of shadowed sets [8] and interval sets [9] after Zadeh's insight that a division of the membership degrees of an object (the interval [0, 1]) could be used to indicate either belongingness or non-belongingness to a fuzzy set, or another indeterminate state [10]. However this strategy has been strongly reinvigorated by three-way decision in the last decade.

A study recently performed by Yang and Yao [11] has linked soft sets and three-way decision. In passing, these authors have produced the first sound explanation of the semantics of soft sets, and of the sort of unresolvedness that they can model. Our article has been motivated by their conclusions [11, Section 6], which claim that 'one may investigate three-way approximations in other theories of uncertainty for the purpose of three-way decision'. We have taken up the baton to conduct a more general analysis in the flourishing framework of N-soft sets. Consequently, we shall explore the semantical interpretation of N-soft sets. We shall argue that in addition to the natural extension of the two semantics of soft sets suggested in [11], a totally new interpretation by degrees of truth connects N-soft sets with many-valued logics. Applications to the aggregation of incomplete soft sets, and of N-soft sets, ensue from this remark. Finally, we will exploit these facts to produce three-way decision approaches with information modeled by N-soft sets. Both the qualitative and qualitative three-way decision strategies presented by Yang and Yao [11] will ve conveniently extended to the case of N-soft sets.

The rest of the paper is organized as follows. Section 2 gives background in various fields: N-soft sets, semantics of soft sets, three-way decision, and many-valued logics. Section 3 introduces our three semantics for N-soft sets. Section

* *Email address:* jcr@usal.es
ORCID(s): 0000-0002-4533-9281 (J.C.R. Alcantud)

Table 1
The tabular representation of an N-soft set.

(F,T,N)	t_1	$\dots\dots$	t_q
o_1	r_{11}	$\dots\dots$	r_{1q}
$\vdots$	$\vdots$		$\vdots$
o_p	r_{p1}	$\dots\dots$	r_{pq}

4 discusses the applicability of the suggested semantics to aggregation. Section 5 produces increasingly general models for three-way decision with N-soft sets. Conclusions are drawn in Section 6.

2. Antecedents and related work

In this section, we briefly recall the essentials of N-soft sets and the semantics of soft sets, three-way decisions, and many-valued logic. We first describe the model that characterizes a set of alternatives by multinary assessments of their attributes. We also summarize the semantical interpretations given to the particular case of binary evaluations, i.e., the model called soft sets. Then we present the fundamental facts about three-way decisions and its validity. And finally, we set forth some non-technical elements from many-valued logic.

Henceforth $\mathcal{P}(X)$ represents the set of all subsets of a set X.

2.1. The setting: N-soft sets and particular cases

We define our framework by the following items. Some alternatives or objects O are investigated in relation with their characteristics T. The figures $0, 1, 2, \dots, N-1$ are a convenient default for the representation of the 'level of satisfaction' of each characteristic, and we assume $N \in \{2, 3, \dots\}$. For illustration, in the 5-star rating system for hotels, a 5-soft set captures the classification of a list of hotels, by way of the identification $0 = 1$ star, $1 = 2$ stars, ... $4 = 5$ stars. Restaurants are graded on their quality by rating systems like the 'stars' in the Michelin guide, or the 'Suns' in the Spanish Repsol guide. Both rating systems use 3 items for grading, and they are updated yearly.

A detailed inspection of the semantical interpretation of the N-soft set model will be given later on in this paper. We should bear in mind that the case $N = 2$ produces the soft set case [1], and that Yang and Yao [11] have first explored the semantics of soft sets. Let us now recall the technical description and value of N-soft sets. Afterwards we shall summarize Yang and Yao's interpretations for the soft set case in section 2.2.

Definition 1. [2] *An N-soft set over O is a triple (F,T,N), where F is a mapping from T to $2^{O\times G}$ and $G = \{0, 1, \dots, N-1\}$. It is requested that F satisfies the condition that for each $t \in T$ and $o \in O$ there must be exactly one pair $(o, g_t) \in O \times G$ such that $(o, g_t) \in F(t)$, $g_t \in G$.*

When $O = \{o_1, \dots, o_p\}$ and $T = \{t_1, \dots, t_q\}$ are finite, a user-friendlier presentation of an N-soft set proceeds as follows. For any $t_j \in T$, Definition 1 submits exactly one grade from G for each $o_i \in O$. This it is the unique r_{ij} that satisfies $(o_i, r_{ij}) \in F(t_j)$. In other words, $F(t_j)(o_i) = r_{ij} \in G$ is a shorthand for $(o_i, r_{ij}) \in F(t_j)$. Therefore the information embodied in an N-soft set can be visually captured in a table whose cells are figures from G. Table 1 shows the appearance of a generic N-soft set with this convention. Besides, we can transform input data produced in another format (like the Michelin stars or the Repsol Suns systems of classification of selected restaurants or otherwise), to the standardized expression displayed by Table 1. Many other real examples have been mentioned elsewhere [2, 12, 13].

N-soft sets were proposed by Fatimah *et al.* [2] to enhance the informational ability of soft sets, which are the particular case with $N = 2$. Hence they make multinary assessments possible, beyond the case of simply binary evaluations. A soft set is usually defined as a mapping $F' : T \longrightarrow \mathcal{P}(O)$ and this is shortened by (F', T). It is often represented as the collection of pairs $\{(t, F'(t)) \mid t \in T\}$ and the subset $F'(t) \subseteq O$ is called the t-approximate set of O. The interpretation in terms of their tabular representations is that in a soft set, Table 1 only contains 0's and 1's. To transform Definition 1 into a standard soft set when $N = 2$, we define

$$F'(t) = \{o \in O : F(t)(o) = 1\}.$$

Thus when $N = 2$, $F(t)(o) = 1$ is equivalent to claiming $o \in F'(t)$.

Since their appearance, N-soft sets have been generalized and hybridized with characteristics like bipolarity, fuzziness and multi-fuzziness, hesitancy, roughness, and other depictions of uncertain information. Recently, aggregation of N-soft sets has been first posed and solved with the help of OWA operators for the purpose of decision-making [13]. Their links with rough set theory have been highlighted too [12]. N-soft sets have produced expansions of the research on soft topologies, since they are the germ of N-soft topologies and M-parameterized N-soft topologies.

The value of all these models has been justified by applications inclusive of medical problems [14]. Another practical issue that has been investigated is parameter reduction [15].

Incomplete data produce incomplete N-soft sets [2] which have been little studied hitherto. However there is abundant literature on the specific case of incomplete soft sets (v., [16] and the references therein). Their conceptual interpretation in tabular terms is that Table 1 can contain three values, usually represented as 0, 1, and * (for 'unknown' or 'indeterminate'). Therefore after a trivial notational adaptation, an incomplete soft set is a 3-soft set for which 0 designates false, 2 true, and 1 holds for the 'indeterminate' case. Likewise, incomplete N-soft sets can be identified with $(N + 1)$-soft sets in various manners (depending on the figure representing the 'unknown' or 'indeterminate' state).

Example 1. *Figures 1 and 2 show data from two real examples. They are part of the reports on papers respectively submitted by anonymous authors to journals of the MDPI (Multidisciplinary Digital Publishing Institute) and IEEE (Institute of Electrical and Electronics Engineers) publishing groups. In the case of MDPI, the various characteristics of the papers are evaluated with three values, but 'No answer' or 'Not applicable' can be selected. Thus the ouputs for any list of submissions can be modelled by incomplete* 3*-soft sets. IEEE allows for five evaluations for each feature of the paper. Thus in this case, the ouputs for any list of submissions can be modelled by* 5*-soft sets. We shall now give specific descriptions and details with the help of other fictitious reports.*

Reports in the MDPI example use the set of attributes (some of them are slightly simplified here) $T_M = \{t_1 = \text{Originality/Novelty}, t_2 = \text{Significance of content}, t_3 = \text{Quality of presentation}, t_4 = \text{Scientific soundness}, t_5 = \text{Interest to the readers}, t_6 = \text{Overall merit}, t_7 = \text{Introduction}, t_8 = \text{Research design}, t_9 = \text{Methods}, t_{10} = \text{Clarity}, t_{11} = \text{Conclusions supported by results}\}$.

The set of grades is associated with $N = 4$*. As explained above, the grades* $G = \{0, 1, 2, 3\}$ *are convenient defaults. Being a case of an incomplete* 3*-soft set,* 0 *shall represent the 'No answer' (attributes* t_1 *to* t_6*) or 'Not applicable' (attributes* t_7 *to* t_{11}*) assessments. Besides, for the purpose of illustration,* 3 *represents 'High' when referred to the attributes* t_1 *to* t_6*, whereas it represents 'Yes' when referred to* t_7 *to* t_{11}*. Likewise,* 1 *represents 'Low' in the first case, and 'Must be improved' in the second.*

We produce an incomplete 3*-soft set* $(F_M, T_M, 4)$ *when we gather the reports for all the relevant submissions in a structured manner. Figure 1 captures the raw information for the first paper, denoted* p_1*. It becomes row* 1 *in the tabular representation given by Table 2. Its rows* 2 *and* 3 *represent (fictitious) assessments for respective drafts submitted to the same journal. We observe that there is no definite information about the second paper* (p_2) *concerning* t_2*, nor about the third paper* (p_3) *concerning* t_{11}*.*

Let us now explain a complete example in the case of the IEEE publisher.

The set of attributes is $T_I = \{t_1 = \text{Importance}, t_2 = \text{Content}, t_3 = \text{Depth}, t_4 = \text{Style}, t_5 = \text{Organization}, t_6 = \text{Presentation}, t_7 = \text{References}, t_8 = \text{Overall evaluation}\}$.

The set of grades is associated with $N = 5$*, hence they are* $G = \{0, 1, 2, 3, 4\}$ *by default. For illustration,* 4 *represents 'valuable' when referred to the attribute 'Importance', whereas it represents 'Original' when referred to 'Content'. Likewise,* 0 *represents 'Useless' in the first case, and 'Derivative' in the second.*

We produce a 5*-soft set* $(F_I, T_I, 5)$ *when the reports for a set of submissions are gathered together in a structured manner. Figure 2 captures the raw information for one paper, denoted* p_1*. It becomes row* 1 *in the tabular representation given by Table 3. The other rows of this table represent (fictitious) assessments for two more drafts submitted to the same journal.*

2.2. Tips on the semantics of soft sets

From a technical viewpoint, soft sets are simply multi-valued mappings from the set of attributes to the set of alternatives. It is their semantics that makes them so appealing. But the first complete exploration of this issue has been made only recently by Yang and Yao [11]. These authors have set forth two plausible semantics for soft sets, namely, a 'multi-context' and a 'possible worlds' sematics. Suffice here to say that section 3 explains them in the course of our semantical investigation of N-soft sets.

Report 1

	High	Average	Low	No Answer
Originality / Novelty	(x)	( )	( )	( )
Significance of Content	(x)	( )	( )	( )
Quality of Presentation	( )	(x)	( )	( )
Scientific Soundness	(x)	( )	( )	( )
Interest to the readers	(x)	( )	( )	( )
Overall Merit	(x)	( )	( )	( )

Overall Recommendation

( ) Accept in present form
(x) Accept after minor revision (corrections to minor methodological errors and text editing)
( ) Reconsider after major revision (control missing in some experiments)
( ) Reject (article has serious flaws, additional experiments needed, research not conducted correctly)

English Language and Style

( ) Extensive editing of English language and style required
(x) Moderate English changes required
( ) English language and style are fine/minor spell check required
( ) I don't feel qualified to judge about the English language and style

	Yes	Can be improved	Must be improved	Not applicable
Does the introduction provide sufficient background and include all relevant references?	(x)	( )	( )	( )
Is the research design appropriate?	(x)	( )	( )	( )
Are the methods adequately described?	(x)	( )	( )	( )
Are the results clearly presented?	( )	(x)	( )	( )
Are the conclusions supported by the results?	(x)	( )	( )	( )

Figure 1: Anonymized section of a report on a paper submitted to a journal of the MDPI publishing group.

PAPER EVALUATION: COMMENTS RETURNED TO AUTHOR(S)

TECHNICAL MERIT:							
Importance	Valuable	○	○	◉	○	○	Useless
Content	Original	○	○	◉	○	○	Derivative
Depth	Deep	○	○	◉	○	○	Shallow
PRESENTATION:							
Style	Readable	○	○	◉	○	○	Incoherent
Organization	Precise	○	◉	○	○	○	Ambiguous
Presentation	Orderly	○	◉	○	○	○	Confusing
References	Complete	○	◉	○	○	○	Incomplete
OVERALL:							
Overall Evaluation	Excellent	○	○	◉	○	○	Dreadful

Figure 2: Anonymized section of a report on a paper submitted to a journal of the IEEE publishing group.

2.3. Three-way decision

Bulding on probabilistic rough set theory, Yao [3] set forth three-way decision theory. The ethos of this discipline concerns information processing and problem solving in terms of triples. The last decade has witnessed a remarkable growth of both its theoretical background, methodology, and applicability [4, 5, 6].

After the advent of three-way decisions with many particular models, applications to numerous fields have emerged. Aranda-Corral *et al.* [17] have introduced Knowledge Harnessing, together with a three-way decision model to approach this novel problem. In relation with context-aware recommender systems, Abbas *et al.* [18] improved the classification of items in order to avoid their allocation to irrelevant contextual groups caused by missing non-binary ratings. Other fields of application include conflict analysis [19], sentiment analysis [20], clustering [21], data classification [22], clinical decision support systems [23, 24], and information filtering [25]. In multi-attribute decision making, Zhang *et al.* [26] produce a TOPSIS model with the help of three-way decision models, Ye *et al.* [27] use decision information systems in a fuzzy probabilistic rough set model, Deng, Zhan and Wu [28] introduce

Table 2
The tabular representation of the incomplete 3-soft set $(F_M, T_M, 4)$ in Example 1. Row 1 corresponds to the raw information displayed in Figure 1.

$(F_M, T_M, 4)$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}
p_1	3	3	2	3	3	3	3	3	3	2	3
p_2	3	0	3	3	1	2	2	1	3	2	2
p_3	1	2	2	2	2	2	3	2	3	2	0

Table 3
The tabular representation of the 5-soft set $(F_I, T_I, 5)$ in Example 1. Row 1 corresponds to the raw information displayed in Figure 2.

$(F_I, T_I, 5)$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
p_1	2	2	2	2	3	3	3	2
p_2	3	4	3	3	4	4	4	4
p_3	1	2	2	2	2	2	4	2

three-way decision into a multi-scale decision information system, and Wang, Dai and Xu [29] take advantage of fuzzy complementary preference relations based on additive consistency. In hesitant fuzzy environments, Wang *et al.* [30, 31, 32] have studied multi-attribute three-way decision making too. Within the realm of fuzzy hesitancy, Qiao [33] have discussed the applicability of hesitant relations for the production of decision evaluation functions in three-way decision spaces, and Feng *et al.* [34] have studied three-way decision based on canonical soft sets of hesitant fuzzy sets. We find applications in an intuitionistic fuzzy framework too, e.g., Wang, Zhan and Mi [35].

A more complete list of topics for which three-way decision has provided inspiration is given in [6, Section 1].

For our purposes it is three-way decision models for information given by soft sets which is of interest. They have been designed with both a qualitative and quantitative character [11]. Here we only give a conceptual level description and postpone a technical presentation to section 5.1.

The qualitative approach to three-way decision is based on the concepts of core and support of a soft set [36]. These subsets of the set of alternatives are nested (the core of a soft set is always a subset of its support). The core of the soft set defines its positive region. The complement of the support gives its negative region. The quantitative approach uses two thresholds and a ratio between cardinalities that stems from two successive transformations. First the soft set is converted into a fuzzy set. Then this fuzzy set is converted into a shadowed set that immediately produces the desired three regions. The choice of extreme thresholds results into the qualitative three-way decision explained above.

2.4. Many-valued logics

Despite the fact that Aristotelian logic remained unchallenged until the 20th century, Aristotle himself did not fully accept the ‘law of excluded middle’, by which the negation of propositions that are not true must be true. Since 1920, many-valued (or multi-valued) logics were created, first with the introduction of a third truth value that denotes ‘possibility’ [37], afterwards with any finite number of truth values [38] and even an infinity of truth values [39].

Special instances of many-valued logics deserve special attention, particularly because of its significance in computer science.

2.4.1. Three-valued logics

Structured Query Language (SQL) has become the standard language for retrieving, updating, and removing information from relational databases. SQL implements three logical results. A state or marker, identified by the reserved word NULL, indicates that a data value is not found in the database. Here we assume that the database does not store a complete knowledge of the world. Then a specialized three-valued logic (3VL) caters for actual SQL implementations. In fact, the truth tables that SQL applies for the combination of logical states (AND or $\wedge$, OR or $\vee$, and NOT or $\neg$) correspond to the Kleene and Łukasiewicz three-valued logics. Suppose that the truth values in

Table 4
A representation of the conjunction and disjunction connectives in Kleene's 3-valued logic.

$\wedge$	T	N	F
T	T	N	F
N	N	N	F
F	F	F	F

$\vee$	T	N	F
T	T	T	T
N	T	N	N
F	T	N	F

Table 5
A representation of the conjunction and disjunction connectives in Belnap's 4-valued logic.

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	B	N	F
T	T	T	T	T
B	T	B	T	B
N	T	T	N	N
F	T	B	N	F

three-valued logic are dubbed $\{T, F, N\}$ (true, false, and none or NULL in SQL). Then negation acts as expected ($\neg T = F$, $\neg F = T$, $\neg N = N$), and Table 4 represents the truth values of conjunction and disjunction in Kleene's logic.

In relation with the representation of qualitative information, Yao [40] produced a model for the extension of a partially-known concept by interval sets. Soon afterwards Yao and Li [41] gave a possible-worlds analysis of Kleene's logic that uses the interval-set model.

2.4.2. Four-valued logics

IEEE established a four-valued logic with the standard IEEE 1364 (Verilog) in order to model signal values in digital circuits. Four states define Boolean false and true, open circuit/high impedance (Z), and no effect/unknown logic value (X). Belnap [42, 43] considered the issue of reasoning about incomplete and inconsistent information in the context of computer-assisted question answering. Truth and falseness are retrieved from various sources (like databases or multi-person inputs). Incomplete information happens when no answer is found. Simultaneous false and true answers produce contradictory information.

The truth values in Belnap's four-valued logic are frequently dubbed $\{T, F, N, B\}$ (true, false, none, both). Negation acts as expected: $\neg T = F$, $\neg F = T$, $\neg B = B$, $\neg N = N$. Table 5 represents the truth values of conjunction and disjunction in Belnap's logic.

2.4.3. Other many-valued logics

Knowledge bases in computer networks have benefitted from 16-valued systems like the logic developed by Shramko and Wansing [44]. It was designed with an aim at information processing, especially for the case of a net of hierarchically interconnected computers. Odintsov [45] has contributed to the axiomatization of the Shramko-Wansing 16-valued logic.

3. The semantics of N-soft sets

Any examination of this issue should start with a summary of the soft set case. In this regard, Yang and Yao [11] have made the first explicit exploration of the semantics of soft sets.

Their semantical analysis considered two approaches. First and foremost, Yang and Yao explore the original interpretation of a soft set that they call the multi-context semantics. This idea can be exported to the semantical analysis of N-soft sets verbatim (cf., section 3.1). Then they suggest an additional interpretation, namely, a 'possible-worlds' semantics. In section 3.2 we complement their arguments with additional support from the literature about decision under uncertainty. As in the case of the multi-context semantics, this second interpretation can be exported to the case of N-soft sets by the same token. Importantly, in section 3.3 we offer a completely new semantics for N-soft sets which links them to the field of many-valued logic. Section 3.4 discusses the relationships among these interpretative semantics.

3.1. The 'multi-context' semantics of N-soft sets

The potential of soft sets resides in the idea of parameterization [1]. In its original semantics, a soft set offers a taxonomy: it classifies, describes or categorizes the alternatives based on their characteristic features. Species in biology share common attributes, and an ample collection of qualities serve the experts to isolate any single specie. Similarly, complex objects can be perceived from various positions. Each perspective gives us an approximate description. This idea, which was superficially presented in the seminal paper on the topic, was subsequently taken further by authors like Feng *et al.* [46].

In contrast with soft sets, N-soft sets are distinguished by their ability to rate the level of satisfaction of each attribute; the multi-context interpretation can be replicated, *ceteris paribus*, for N-soft sets. T continues to encapsulate the possible contexts in which the objects can be interpreted. When t is an attribute and we declare $F(t)(o) = r$, then we construe that o belongs to the approximate set of alternatives that in the context t, have a performance of r. From the founding [2], it transpires that this is the original interpretation of the N-soft set model. And it is the description that underlies Example 1. All the elements of this example serve as typical cases of the constituents described above.

3.2. 'Possible worlds' vs. 'states of nature' semantics of N-soft sets

According to this explication, the set of attributes is formed by possible worlds for the interpretation of a partially-known concept [11]. And *exactly one* of those possible worlds is the actual world defining the set of instances of the concept. Objects may either be or not be an instance of it and we do not always know the status of all objects.

This interpretation is reminiscent of the idea of 'states of nature' introduced by Savage [47] in order to formalize the concept of resolution in decision under uncertainty. The 'states of nature' semantics subtly modifies the idea of 'possible worlds'. A state of nature refers to some future event which the decision-maker cannot control. Their probabilities are unknown. Marked examples include descriptions of natural phenomena (temperature during our future holidays, frequency and forces of the next storms to hit our home, et cetera) and a person's state of health (which is possibly perceived by the results of some tests and its symptoms). Other simple examples that illustrate the idea are sports (e.g., horse races) or contests (e.g., beauty contests or bids for the Olympic games).

Yang and Yao [11] argued that interval sets may sometimes illustrate this semantics of soft sets. Let us recall their definition:

Definition 2. *[9] Let X be a nonempty set. When $X_l \subseteq X_u \subseteq X$, an interval set is $[X_l, X_u] = \{A \subseteq X | X_l \subseteq A \subseteq X_u\}$.*

Notice however that this representation is not universal, but only a possible tool for presenting soft sets. Let us now revisit [11, Example 3] in order to introduce a slight modification under this perspective.

Example 2. *[11, Example 3] A host invites five persons for a party. The persons are denoted by the symbols $\{x_1, \dots, x_5\}$. Both x_1 and x_2 confirm that they will come to the party, however x_3 made her excuses and declined the invitation, whereas x_4 and x_5 did not reply.*

Yang and Yao [11] model this situation by in interval set [9]. We can consider $X = \{x_1, \dots, x_5\}$ and the interval set defined by $X_l = \{x_1, x_2\}$ and $X_u = \{x_1, x_2, x_4, x_5\}$. Afterwards this interval set is transformed into a soft set, the set of parameters being $2^{\{x_4, x_5\}} = \mathcal{P}(\{x_4, x_5\})$. Thus for example, the possible world represented by $\{x_5\}$ captures the case where x_5 attends the party but x_4 does not. The $\{x_5\}$-approximate set is therefore $\{x_1, x_2, x_5\}$.

However there is a common situation that this model overlooks. Consider the case where x_4 and x_5 are a couple and they are together in a foreign country. Both are willing to attend the party. But their assistance is conditional upon a series of events (like transit flights or testing negative for COVID in an airport). In this case, the possible worlds represented by $\{x_4\}$ and $\{x_5\}$ are meaningless thus we need a more general representation that allows us to omit it. The interval set representation obliges us to consider these 'possible worlds' that are in fact, impossible.

Of course, one can resort to the 'possible worlds' $P = \{\emptyset, \{x_4, x_5\}\}$ and the soft set (F, P) where $F(\emptyset) = \{x_1, x_2\}$, $F(\{x_4, x_5\}) = \{x_1, x_2, x_4, x_5\}$, in order to model this situation. The point of our argument is that we cannot derive this representation from interval sets.

Notice that interval sets cannot be used for the representation of N-soft sets, except when an N-soft set is in fact a soft set. The subset structure upon which interval sets depend is not fine enough to capture multinary evaluations of the attributes, except when it is in fact binary.

A final remark is that the 'possible worlds' and 'states of nature' semantics are directly applicable to N-soft sets. Neither of the clarifications and provisos in this section affect the validity of Yang and Yao's arguments when the attributes are assessed at items from a graded scale.

3.3. The 'values of truth' semantics of N-soft sets

The 'multi-context' semantics of soft sets has been discussed and extended to N-soft sets in section 3.1. The basic assumption is that a soft set models the situation where every object can be unequivocally associated to each and every characteristic that it possesses. Regardless, this premise is directly influenced by Aristotelian logic. Both philosophers and logicians were freed from this constraint after the introduction of many-valued logical systems. Already in 3-valued systems of propositional logic, propositions must not be either true or false. The rejection of the law of excluded middle in our discussion means that objects that neither satisfy nor do not satisfy a property are within the purview of N-soft set theory with $N \geqslant 3$.

What is more important is that this situation is very common. Many of the real examples discussed in the literature can be interpreted in these terms. We often watch films that we cannot describe as either 'funny' or 'not funny'. Hotels are sometimes neither 'comfortable' nor 'not comfortable'. We are often unsure whether a color is cool or warm, so we are undecided when asked if some objects have a warm color. Therefore we cannot use soft sets when these statements determine our portrayal of complex objects. However N-soft sets are perfectly suited to the purpose of describing objects for which the question whether they possess a feature may be neither true nor false.

This discussion is directly related to incomplete soft sets. They have been used to encapsulate the case where we cannot always decide if an alternative satisfies or does not satisfy a property. We have explained in section 2.2 that an incomplete soft set is a 3-soft set for which 0, 2 and 1 respectively designate false, true, and 'all other situations'. However, the utilization of a unique value (either 1 or *) to represent a wide range of situations is a possible weakness of incomplete soft sets. The 'values of truth' semantics of N-soft sets extends the representation ability of incomplete soft sets, much like multi-valued logics extend three-valued logic. Whereas 0 and $N-1$ represent the 'falsest' and 'truest' values, the rest of the values correspond to intermediate degrees of uncertainty (like 'undefined', 'unknown' or 'contradictory'). Therefore we can rightfully claim that N-soft sets can be regarded as a generalized form of incomplete soft sets. The next example gives a streamlined case.

Example 3. *Consider the situation of Example 2, or [11, Example 3]. It can be represented by an incomplete soft set, since the status of both x_4 and x_5 are not known.*

However it could be the case that x_4 has confirmed that she will do her best to attend, whereas x_5 believes that it is unlikely that he can cancel another appointment for the same day. The degrees of indeterminacy are clearly different, and a 4-soft set might be better suited to capture the uncertain knowledge about the problem.

A hands-on introduction to this novel semantics requires a discussion of the formal presentation of our inputs. At this point we should bear in mind that the labels in the graded scale $G = \{0, 1, \dots, N-1\}$ are a convenient default. This remark is important because Łukasiewicz introduced his m-valued logic $Ł_m$ with the truth domain $\{0, \frac{1}{m-1}, \frac{2}{m-1}, \dots, 1\}$. We can use it verbatim if we change the graded scale to become $G = \{0, \frac{1}{N-1}, \frac{2}{N-1}, \dots, 1\}$ in order to operate with N-soft sets. Anyhow since many examples have been published with the original notation, we can also adapt Łukasiewicz m-valued logic to act on the truth values $\{0, 1, \dots, m-1\}$ for convenience. Then the truth value of negation is computed by subtraction from $m-1$. This means $\neg 0 = m-1, \neg 1 = m-2, \dots, \neg(m-2) = 1, \neg(m-1) = 0$. The truth values of implication are defined by modified subtraction: the truth value of $a \to b$ is $m-1+b-a$ when $b \leqslant a$, $m-1$ otherwise. The usual formula is $a \to b = \min(m-1, m-1+b-a)$. Table 6 represents the truth values of implication and negation in $Ł_4$. There are four values of truth $\{0, 1, 2, 3\}$ and 3 represents the 'truest' whereas 0 represents the 'falsest' value. The other logical connectives are derived from these by rules inclusive of the following instances:

$a \vee b = (a \to b) \to b = \max(a, b)$

$a \wedge b = \neg(\neg a \vee \neg b) = \min(a, b)$

$a \leftrightarrow b = (a \to b) \wedge (b \to a) = m - 1 - |a - b|$

3.4. Discussion

Central to the success of fuzzy sets or probability theory is the existence of well understood, but multiple, semantical interpretation of their building blocks. It is in this context that we have presented three semantics of N-soft sets as an exploratory extension of Yang and Yao's [11] insights for the case of soft sets. The new 'values of truth' semantics

Table 6
A representation of the implication and negation connectives in Łukasiewicz 4-valued logic.

$\rightarrow$	0	1	2	3	$\neg$
0	3	3	3	3	0
1	2	3	3	3	1
2	1	2	3	3	2
3	0	1	2	3	3

reshapes our understanding of the grades in such way that logical considerations become apparent in the soft set model. Hence our study makes a convincing case that its potential is far from exhausted.

Nevertheless the three interpretations described above are not mutually exclusive. Some natural situations show that the 'values of truth' semantics is compatible with both the 'multi-context' and 'possible worlds' semantics, respectively:

1. When the attributes distribute the objects among categories, and (some of) these categories are vaguely defined. it may be the case that we can neither be sure that an object belongs to a category nor deny that it is a member of the category. Hence we need to go beyond the dichotomous structure of soft sets. Degrees of truth are a helpful complement and then N-soft sets arise.

   As an example, consider the case of age ratings and film classification. In UK, the British Board of Film Classification (BBFC) provides help for families to choose what films are right for them and their children. Several ratings are available, including 'Suitable for all', 'Parental guidance', 'Cinema release suitable for 12 years and over', or 'Suitable only for adults'. Their Compliance Officers make recommendations after extensive consultation, which are finally approved by Compliance Managers. Interestingly, BBFC explains what actions are taken "[i]f Compliance Officers are in any doubt, if a film is on the borderline between two categories, or if important policy issues are involved". As this example speaks for itself, it is often hard to be fully categorical, even for specialists with years of accumulated experience.

2. Regarding the compatibility of the 'values of truth' and 'possible world' semantics, the situation of Example 3, inspired by [11, Example 3], is a good case in point.

With the help of these ideas we can argue that the three semantics presented here are independent, inasmuch as the values of truth semantics overlaps with the other two, which were already different in the soft set framework. Notice that the original multi-context semantics has a special feature that makes it markedly different from the interpretation by values of truth, to wit, the grades may have different meanings for different contexts (v., e.g., any of the two cases in Example 1). In contrast, degrees of truth are an invariant of the model. They are a uniform representation of the values of the statements, including truth and falsity but possibly other values. This feature of the values of truth semantics makes it especially attractive, also from the mathematical point of view. First, because N-soft sets become a rightful extended form of incomplete soft sets (cf., section 3.3). Secondly, because logic provides a powerful lens to study N-soft sets under this interpretative semantics (and section 4 will take advantage of this essential quality). Likewise, the defining characteristic of the possible worlds semantics is that exactly one of them is the actual world defining the instances of a partially-known object. This is not necessarily the case of a values of truth interpretation.

In conclusion, we believe that the 'values of truth' semantics stands centrally in relation to the other two compatible semantics, and that these interpretations of N-soft sets are mutually independent. Our discussion has also shown that soft sets are capable of different levels of sophistication.

4. Applications of the new 'values of truth' semantics for N-soft sets

A semantical interpretation of N-soft sets in terms of 'values of truth' has connected them with many-valued logics. We shall take advantage of this new perspective in the next sections because in logical terms, aggregation is linked with the conjunction connective. Section 4.1 starts with incomplete soft sets, which have been described as a semantical interpretation of 3-soft sets. Interestingly, aggregation of incomplete soft sets has not been investigated so far. Section 4.2 extends the scope of the technique to N-soft sets. The key step is our argument that N-soft sets with $N \geqslant 3$ embrace and extend the idea of incompleteness in soft sets (cf., section 3.3).

Table 7
Tabular representation of the incomplete soft sets (F_1, T) and (F_2, T) in Example 4.

(F_1,T)	t_1	t_2	t_3	t_4	t_5	t_6
o_1	$*$	0	1	0	1	0
o_2	0	0	1	1	0	0
o_3	0	1	0	0	1	1
o_4	0	1	$*$	1	0	1
o_5	1	0	1	1	0	0

(F_2,T)	t_1	t_2	t_3	t_4	t_5	t_6
o_1	1	0	1	0	1	0
o_2	0	1	0	1	0	0
o_3	0	1	0	0	1	1
o_4	0	1	$*$	1	0	$*$
o_5	1	$*$	1	1	0	0

Table 8
Tabular representation of the 3-soft sets obtained from the incomplete soft sets (F_1, T) and (F_2, T) in Example 4.

$(F_1,T,3)$	t_1	t_2	t_3	t_4	t_5	t_6
o_1	1	0	2	0	2	0
o_2	0	0	2	2	0	0
o_3	0	2	0	0	2	2
o_4	0	2	1	2	0	2
o_5	2	0	2	2	0	0

$(F_2,T,3)$	t_1	t_2	t_3	t_4	t_5	t_6
o_1	2	0	2	0	2	0
o_2	0	2	0	2	0	0
o_3	0	2	0	0	2	2
o_4	0	2	1	2	0	1
o_5	2	1	2	2	0	0

Table 9
Tabular representation of the 3-soft set obtained from the aggregation of the 3-soft sets $(F_1, T, 3)$ and $(F_2, T, 3)$ in Example 4.

$(F_A,T,3)$	t_1	t_2	t_3	t_4	t_5	t_6
o_1	1	0	2	0	2	0
o_2	0	0	0	2	0	0
o_3	0	2	0	0	2	2
o_4	0	2	1	2	0	1
o_5	2	0	2	2	0	0

4.1. Aggregation of incomplete soft sets by an incomplete soft set

The 'values of truth' semantics of 3-soft sets allows us to define aggregation operators on incomplete soft sets. As incompleteness in soft set theory obtains by the inclusion of a third value, conjunction in Kleene's three-valued logic comes to mind naturally. This is similar to its use in knowledge retrieval from databases, by analogy with the analysis in section 2.4.1. Relatedly, this rational technique can be extended to the general case of aggregating N-soft sets. Then N-valued logics should replace Kleene's logic. We dwell on this issue in section 4.2.

To motivate the problem and its stepwise solution, let us pose an aggregation problem with this structure.

Example 4. *A company is recruiting new staff, and five persons apply for a job. Thus $O = \{o_1, \ldots, o_5\}$ is our universe of applicants. $T = \{t_1, \ldots, t_6\}$ is the set of attributes that a good candidate should meet. The recruiting committee receives two assessments of the candidates that should be combined to produce an evaluation of their capabilities. Both assessments received are incomplete. Thus the committee must make a judgement on the basis of the two incomplete soft sets (F_1, T) and (F_2, T) displayed in Table 7, that represent the respective opinions on the "adequacy of the applicants".*

The aggregation procedure that we have suggested uses 3-soft sets. Thus the first step adapts the input to the notation in N-soft set theory as explained in section 2.1. The output uses grades $0, 1, 2$ instead of $0, *, 1$, and it is displayed in Table 8. In the second step we apply Table 4 which represents the conjunction connective in Kleene's 3-valued logic. Section 3.3 explains that we must bear in mind that T holds for 2 (the 'truest' value), N holds for 1, and F holds for 0 (the 'falsest' value). A detailed description of this process ends with the formula $a \wedge b = \min(a, b)$ in Section 3.3. Table 9 shows the output. We can now return to the standard notation in the literature about incomplete soft sets easily. We only need to replace 1 with $*$, and 2 with 1 in the aggregate 3-soft set $(F_A, T, 3)$.

Table 10
Tabular representation of the 4-soft sets in Example 5, and their aggregate output.

$(F_1, T, 4)$	t_1	t_2	t_3
o_1	1	1	2
o_2	3	2	0
o_3	0	1	2
o_4	2	3	2
o_5	1	0	3

$(F_2, T, 4)$	t_1	t_2	t_3
o_1	1	0	3
o_2	2	3	0
o_3	0	0	3
o_4	2	1	2
o_5	2	0	2

Table 11
Tabular representation of the aggregate output of the two 4-soft sets in Example 5.

$(F, T, 4)$	t_1	t_2	t_3
o_1	1	0	2
o_2	2	2	0
o_3	0	0	2
o_4	2	1	2
o_5	1	0	2

4.2. Aggregation of N-soft sets

The rational procedure described in section 4.1 justifies a novel algorithm for the aggregation of N-soft sets with the aid of Łukasiewicz N-valued logic. Particular examples may call for the utilization of alternative logics like Belnap's logic in the case of aggregation of 4-soft sets.

A stylized example will show the application of this direct methodology:

Example 5. *The organizer of a special session of a conference receives two sets of reports on five articles submitted to her session. Thus $O = \{o_1, \dots, o_5\}$ is the universe of articles. $T = \{t_1, t_2, t_3\}$ is the set of attributes that a perfect candidate paper should meet. They stand for the parameters "scientific quality", "suitability for the special issue", and "quality of presentation". The assessments use 4 values of truth to declare whether it is 'true' that an article satisfies each of the desirable properties. Thus the organizer must make a judgement on the basis of the combined opinion of $(F_1, T, 4)$ and $(F_2, T, 4)$, two 4-soft sets represented in Table 10. The output is given by the resort to the conjunction connective in Łukasiewicz 4-valued logic, and it is shown in Table 11.*

5. Three-way decisions with N-soft sets

Yang and Yao [11] argued that both the 'multi-context' and the 'states of nature' semantics of soft sets are compatible with three-way decisions. We shall see that the same is true for N-soft sets, and that the claim is correct for the 'values of truth' semantics of N-soft sets too.

Guided by their semantical analysis, Yang and Yao devised strategies for trisecting the universe of alternatives on the basis of the information contained in a soft set. A qualitative model uses its core and support. A quantitative model proceeds by a successive transformation of a soft set into a fuzzy set and then of this resulting fuzzy set into a shadowed set. Two thresholds produce the desired trisection and a particular choice of the thresholds gives raise to their qualitative model of three-way decision. We summarize these findings in section 5.1. Afterwards we shall extend them to encompass N-soft sets. Yang and Yao's qualitative model is generalized in two successive steps. First we give a simple model that depends upon one 'threshold' value in section 5.2. This value establishes the level of truth above which we unquestionably accept the validity of a statement. One might argue that the choice of only one level of truth barely qualifies the model for the three-way frame of mind, since it imposes two-way decisions separately for each attribute. Lack of acceptance (of the fact that an object satisfies a property) forces automatic rejection. Therefore we expand this simple model to a more general version depending upon two 'threshold' values in section 5.3. Here a second, smaller value establishes the level of truth under which we definitely reject the validity of a statement. The adoption of two independent levels of truth is reflective of the spirit of three-way decisions: they let us separate acceptance and rejection from a third, non-commitment possibility. Finally, Yang and Yao's quantitative model is generalized in section 5.4.

Henceforth in this section (F, T, N) denotes an N-soft set over O, our set of options.

5.1. Three-way decisions with soft sets: Yang and Yao's qualitative and quantitative models

The constituents of the qualitative model in [11, Section 4] are the core and support of a soft set (F, T) over O:
$\text{core}(F, T) = \bigcap_{t \in T} F(t) = \{o \in O \mid o \in F(t), \forall t \in T\}$, and
$\text{supp}(F, T) = \bigcup_{t \in T} F(t) = \{o \in O \mid o \in F(t), \text{some } t \in T\}$.

With these elements, Yang and Yao defined:
$\text{POS}(F, T) = \text{core}(F, T)$,
$\text{BND}(F, T) = \text{supp}(F, T) \setminus \text{core}(F, T)$, and
$\text{NEG}(F, T) = O \setminus \text{supp}(F, T)$.

These regions are pairwise disjoint, and their union is O [11, Theorem 1]. They are a weak partition or a disjoint trisection of O [5], since the regions could be empty. Three-way decision rules are constructed as follows: when $o \in O$,
If $o \in \text{POS}(F, T)$ **then** o should be accepted.
If $o \in \text{NEG}(F, T)$ **then** o should be rejected.
If $o \in \text{BND}(F, T)$ **then** o can neither be accepted nor rejected.

That being established, Yang and Yao [11, Section 5] designed a more general quantitative model in the case that T is finite. It is summarized by [11, Definition 14]. It explains that for a fixed pair of thresholds (a, b) such that $0 \leqslant b < a \leqslant 1$, any soft set (F, T) over O produces the following positive, boundary and negative regions of O:
$\text{POS}_{(a,b)}(F, T) = \{o \in O \mid \frac{|\{t \in T \mid o \in F(t)\}|}{|T|} \geqslant a\}$,
$\text{BND}_{(a,b)}(F, T) = \{o \in O \mid b < \frac{|\{t \in T \mid o \in F(t)\}|}{|T|} < a\}$, and
$\text{NEG}_{(a,b)}(F, T) = \{o \in O \mid \frac{|\{t \in T \mid o \in F(t)\}|}{|T|} \leqslant b\}$.

The regions defined by the qualitative model explained above arise when $b = 0, a = 1$.

5.2. Three-way decisions with N-soft sets: a one-threshold qualitative model

Our presentation of this model closely follows Yang and Yao [11, Section 4] except for the introduction of a threshold. Thus for each $V \in \{1, \dots, N-1\} = G \setminus \{0\}$, we define the V-core and V-support of (F, T, N) as follows:
$\text{Vcore}(F, T, N) = \{o \in O \mid F(t)(o) \geqslant V, \forall t \in T\}$, and
$\text{Vsupp}(F, T, N) = \{o \in O \mid F(t)(o) \geqslant V, \text{some } t \in T\}$.

With these elements, we now define positive, boundary and negative regions of an N-soft set:

Definition 3. *Let (F, T, N) be an N-soft set. For each $V \in \{1, \dots, N-1\}$,*
$\text{POS}(F, T, N) = \text{Vcore}(F, T, N)$,
$\text{BND}(F, T, N) = \text{Vsupp}(F, T, N) \setminus \text{Vcore}(F, T, N)$, *and*
$\text{NEG}(F, T, N) = O \setminus \text{Vsupp}(F, T, N)$.

The behavior of the trisection given in Definition 3 reproduces [11, Theorem 1]:

Theorem 1. *Let (F, T, N) be an N-soft set, $V \in \{1, \dots, N-1\}$. Then* $\{\text{POS}(F, T, N), \text{BND}(F, T, N), \text{NEG}(F, T, N)\}$ *is a disjoint trisection of O.*

Three-way decision rules are constructed in the following manner: for a fixed $V \in \{1, \dots, N-1\}$, when $o \in O$:
If $o \in \text{POS}(F, T, N)$ **then** o should be accepted.
If $o \in \text{NEG}(F, T, N)$ **then** o should be rejected.
If $o \in \text{BND}(F, T, N)$ **then** o can neither be accepted nor rejected.

The first rule says that an object is accepted (as an instance of the concept under investigation) if it belongs to the approximations of the set of objects defined by all parameters, with a value of truth of at least V. We reject this hypothesis when it belongs to neither of these approximations at the required level of truth.

Let us examine the case $N = 2$ in order to prove that the approach described above extends Yang and Yao's qualitative model. We have explained in section 2.1 that $(F, T, 2)$ can be identified with a soft set. Let us denote it with the standard notation (F', T). Observe that $V = 1$ is the only available option for the computation of the V-core and V-support of $(F, T, 2)$. These subsets of O respectively coincide with the core and support of the soft set (F', T), i.e.,
$\text{Vcore}(F, T, 2) = \text{core}(F', T)$, and
$\text{Vsupp}(F, T, 2) = \text{supp}(F', T)$.

Now it is immediate to check that
$\text{POS}(F, T, N) = \text{POS}(F', T)$,

$\mathrm{BND}(F,T,N) = \mathrm{BND}(F',T)$, and
$\mathrm{NEG}(F,T,N) = \mathrm{NEG}(F',T)$.

Long story short, when a 2-soft set is identified with a soft set, both the qualitative model described in this section (applied to the 2-soft set) and Yang and Yao's qualitative model described in section 5.1 (applied to the soft set) produce the same trisection of O. Hence the rules derived from both qualitative models are the same too.

The next section produces a more general qualitative model by the resort to a second, smaller threshold value V' that discriminates the failure of a property to hold true. It is more general because it subsumes the three-way decision model presented in this section under well-defined circumstances.

5.3. Three-way decisions with N-soft sets: a two-thresholds qualitative model

This extension of the previous model requires a new concept. For $V' \in \{0, 1, \dots, N-1\}$ we define the V'-kernel of (F,T,N) as follows:

$\mathrm{V'ker}(F,T,N) = \{o \in O \mid F(t)(o) \leqslant V', \forall t \in T\}$.

We can now produce the positive, boundary and negative regions of an N-soft set that depend upon two thresholds:

Definition 4. *Let (F,T,N) be an N-soft set. For each $V' \in \{0,1,\dots,N-1\}$ and $V \in \{1,\dots,N-1\}$ with $V' < V$,*
$\mathrm{POS}(F,T,N) = \mathrm{Vcore}(F,T,N)$,
$\mathrm{NEG}^*(F,T,N) = \mathrm{V'ker}(F,T,N)$, *and*
$\mathrm{BND}^*(F,T,N) = O \setminus (\,\mathrm{POS}(F,T,N) \cup \mathrm{NEG}^*(F,T,N)\,) = \{o \in O \mid \exists t, t' \in T$ such that $F(t')(o) > V', F(t)(o) < V\}$.

If we compare this model with the proposal in section 5.2, the positive region is defined in the same terms. Importantly, when $V' = V - 1$, the set $\mathrm{NEG}^*(F,T,N)$ coincides with the definition of $\mathrm{NEG}(F,T,N)$ given in section 5.2. Hence if we fix $V' = V - 1$ the trisections of O produced here and in section 5.2 are the same. For this reason this section yields a more general version of the three-way decision model defined in section 5.2.

The behavior of this trisection reproduces Theorem 1. And both structures can be compared by inclusion:

Theorem 2. *Let (F,T,N) be an N-soft set, $V \in \{1,\dots,N-1\}$ and $V' \in \{0,1,\dots,N-1\}$, with $V' < V$. Then:*

1. $\{\mathrm{POS}(F,T,N), \mathrm{BND}^*(F,T,N), \mathrm{NEG}^*(F,T,N)\}$ is a disjoint trisection of O.

2. $\mathrm{NEG}^*(F,T,N) \subseteq \mathrm{NEG}(F,T,N)$, $\mathrm{BND}(F,T,N) \subseteq \mathrm{BND}^*(F,T,N)$, when the negative and boundary regions $\mathrm{NEG}(F,T,N)$, $\mathrm{BND}(F,T,N)$ are defined from V.

Proof. The proof of statement 1 is routine. As to claim 2, we observe

$$\mathrm{NEG}^*(F,T,N) = \mathrm{V'ker}(F,T,N) =$$
$$= \{o \in O \mid F(t)(o) \leqslant V', \forall t \in T\} \subseteq$$
$$\subseteq \{o \in O \mid F(t)(o) < V, \forall t \in T\} =$$
$$= O \setminus \mathrm{Vsupp}(F,T,N) = \mathrm{NEG}(F,T,N).$$

It is now obvious from statement 1 that $\mathrm{BND}(F,T,N) \subseteq \mathrm{BND}^*(F,T,N)$. □

Theorem 2 explains that the trisection produced from one value V is 'more decisive' than the trisection produced from two values $V' < V$ because its boundary region is smaller (or its negative region is wider). The choice of a smaller threshold value V' to certify that a property is not satisfied by an object, produces a more conservative attitude towards rejection.

Now three-way decision rules are constructed in the following manner. Let us fix $V \in \{1,\dots,N-1\}$ and $V' \in \{0,1,\dots,N-1\}$ with $V' < V$. When $o \in O$:

If $o \in \mathrm{POS}(F,T,N)$ **then** o should be accepted.
If $o \in \mathrm{NEG}^*(F,T,N)$ **then** o should be rejected.
If $o \in \mathrm{BND}^*(F,T,N)$ **then** o can neither be accepted nor rejected.

Theorem 2 renders the message that objects that are accepted, or rejected, under the two-thresholds qualitative model are also accepted under the one-threshold qualitative model associated with the larger threshold.

As happens with the model described in section 5.2, in the case $N = 2$ the approach presented in this section extends Yang and Yao's qualitative model as well. Observe that under this restriction, our requirements on V' and V

forcefully entail $V' = 0$ and $V = 1$ in order to define the V-core and V'-kernel of $(F, T, 2)$. But $V' = V - 1$, therefore the model coincides with that of section 5.2 as explained above. We have argued in section 5.2 that such a model is Yang and Yao's qualitative model because $N = 2$.

5.4. Three-way decisions with N-soft sets: a quantitative model

This section produces an even more general approach to N-soft set based three-way decision making, provided that the set of attributes is finite. Under this assumption we shall define a trisection that in the case $N = 2$, boils down to Yang and Yao's quantitative model presented in section 5.2. Therefore our procedure also depends upon a fixed pair of thresholds (a, b) such that $0 \leqslant b < a \leqslant 1$.

We have recalled in section 5.2 that for a soft set (F, T), $\text{POS}_{(a,b)}(F, T)$, $\text{BND}_{(a,b)}(F, T)$, and $\text{NEG}_{(a,b)}(F, T)$ are defined in terms of a ratio between the cardinalities of two sets: the subset of all the attributes that a fixed object possesses, and the total set of attributes. The case of an N-soft set (F, T, N) is more elaborate because the objects may possess a property in various degrees. To extend these expressions we resort to a proxy of the 'cardinality of the attributes that an object possesses' inspired by the sigma-count in fuzzy set theory. Zadeh [48] popularized this formula for the computation of the 'cardinality' of a fuzzy set μ on $O = \{o_1, \ldots, o_p\}$. Its sigma-count is

$$\sigma(\mu) = \sum_{k=1}^{p} \mu(o_k).$$

We next define the A-sigma-count (for *attribute sigma-count*) of an alternative in an N-soft set:

Definition 5. *The A-sigma-count of $o \in O$ for (F, T, N) is*

$$A\sigma_{(F,T,N)}(o) = \sum_{t \in T} F(t)(o).$$

The behavior of Definition 5 when $N = 2$ replicates in important feature of the sigma-count. When a fuzzy set is a crisp set, the sigma-count produces its ordinary cardinality. When an N-soft set becomes a soft set because $N = 2$, the formula $\sum_{t \in T} F(t)(o)$ simply counts the number of attributes that o satisfies, since $F(t)(o) \in \{0, 1\}$.

Thus we can rightfully use $A\sigma_{(F,T,N)}(o)$ as a proxy of the 'cardinality of the subset of all the attributes that o possesses'. To be consistent with this measurement for the purpose of comparison, the 'cardinality of the set of all the attributes' must be augmented to become $(N-1)|T|$, i.e., the maximum value that can be attained by the A-sigma-count. These expressions imprint the spirit of Zadeh's sigma-count in N-soft set inspired three way decision theory.

We are ready to produce the positive, boundary and negative regions of an N-soft set in a quantitative manner:

Definition 6. *Let (F, T, N) be an N-soft set. Suppose that T is finite. For each pair of thresholds (a, b) that satisfy $0 \leqslant b < a \leqslant 1$,*
$\text{POS}_{(a,b)}(F, T, N) = \{o \in O \mid \frac{A\sigma_{(F,T,N)}(o)}{(N-1)|T|} \geqslant a\} = \{o \in O \mid \frac{\sum_{t \in T} F(t)(o)}{(N-1)|T|} \geqslant a\}$,
$\text{BND}_{(a,b)}(F, T, N) = \{o \in O \mid b < \frac{A\sigma_{(F,T,N)}(o)}{(N-1)|T|} < a\} = \{o \in O \mid b < \frac{\sum_{t \in T} F(t)(o)}{(N-1)|T|} < a$*, and*
$\text{NEG}_{(a,b)}(F, T, N) = \{o \in O \mid \frac{A\sigma_{(F,T,N)}(o)}{(N-1)|T|} \leqslant b\} = \{o \in O \mid \frac{\sum_{t \in T} F(t)(o)}{(N-1)|T|} \leqslant b\}$.

Let us examine the case $N = 2$ in order to confirm that the approach described above extends Yang and Yao's quantitative model (which in turn, extends their qualitative model). To this purpose we denote the soft set formed by $(F, T, 2)$ with the standard notation (F', T), as in sections 2.1 or 5.2. Now we can easily check that

$\text{POS}_{(a,b)}(F, T, 2) = \text{POS}_{(a,b)}(F', T)$,
$\text{BND}_{(a,b)}(F, T, 2) = \text{BND}_{(a,b)}(F', T)$, and
$\text{NEG}_{(a,b)}(F, T, 2) = \text{NEG}_{(a,b)}(F', T)$.

Consider the first equality and recall that we have adopted the notation $o \in F'(t) \Leftrightarrow F(t)(o) = 1$. Then

$$\text{POS}_{(a,b)}(F, T, 2) = \{o \in O \mid \frac{\sum_{t \in T} F(t)(o)}{(2-1)|T|} \geqslant a\} =$$

$$= \{o \in O \mid \frac{|\{t \in T \mid F(t)(o) = 1\}|}{|T|} \geqslant a\} =$$

$$= \{o \in O \mid \frac{|o \in F'(t)|}{|T|} \geqslant a\} = \mathrm{POS}_{(a,b)}(F', T)$$

The other two equalities can be proved with similar arguments.

6. Conclusion

N-soft sets are a multinary version of the parameterized descriptions that define a soft set (on the set of objects). Despite their growing popularity, this paper has produced the first comprehensive analysis of their semantics. We should not discount the relevance of the principles that underlie this model, for the question of exactly what we mean to capture helps us establish what (and how) we can do with the data. Yang and Yao [11] were very aware of this issue when they first approached the semantics of soft sets and took advantage of their conclusions to derive three-way decision strategies bases on soft sets.

Our inspection has concluded that N-soft sets may be strongly linked with the idea of many-valued logics. Their semantical interpretation in terms of values of truth underpins both the multi-context and possible worlds semantics that originate in Yang and Yao's analysis. We have given arguments supporting the view that these three semantics are otherwise independent. Two further conclusions are particularly worthy of mention: three-valued logics and incomplete soft sets are closely entwined, and N-soft sets can be regarded as a generalized form of incomplete soft sets. The later use a unique value to represent anything that is neither 'falsest' nor 'truest'. The former considers a range of intermediate degrees of uncertainty.

Applications of the semantical interpretations of N-soft sets include aggregation. In the context of their 'values of truth' semantics, it is only natural that we do this by the resort to conjunctions from salient many-valued logics. Other applications might derive from this analysis in the future. For example, Feng *et al.* [46] among others have studied soft set based association rules. The deeper understanding of the semantics of N-soft sets that we have achieved should help to investigate association rule mining in this framework. In addition, there is potential to support reasoning and decision making mechanisms based on structures of opposition created with partitions, in light of the new semantic interpretation of truth values [49].

Three-way decision has been approached from increasingly complex positions. A simple qualitative model with one threshold already gives an extension of Yang and Yao's [11] qualitative model. However the choice of only one threshold makes for some cognitive dissonance because it forces two-way decisions when we look at each characteristic. Two-thresholds qualitative mechanisms implement trisections at each cross section of the problem. They generalize one-threshold qualitative three-way decision, and are generalized by quantitative three-way decision.

Zadeh's sigma-count has left its imprint in N-soft set inspired three-way decision theory thorugh the expressions that we have proposed for our quantitative model.

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The author has benefitted from valuable comments and suggestions by three perceptive referees.

Financial support from the Junta de Castilla y León and the European Regional Development Fund (Grant CLU-2019-03) to the Research Unit of Excellence "Economic Management for Sustainability" (GECOS) is gratefully acknowledged.

References

[1] D. Molodtsov, Soft set theory - first results, Computers & Mathematics with Applications 37 (1999) 19–31.

[2] F. Fatimah, D. Rosadi, R. B. F. Hakim, J. C. R. Alcantud, N-soft sets and their decision making algorithms, Soft Computing 22 (2018) 3829–3842.

[3] Y. Yao, Three-way decisions with probabilistic rough sets, Information Sciences 180 (2010) 341–353.

[4] Y. Yao, An outline of a theory of three-way decisions, in: J. Yao, Y. Yang, R. Słowiński, S. Greco, H. Li, S. Mitra, L. Polkowski (Eds.), Rough Sets and Current Trends in Computing, Springer Berlin Heidelberg, Berlin, Heidelberg, 2012, pp. 1–17.

[5] Y. Yao, Three-way decision and granular computing, International Journal of Approximate Reasoning 103 (2018) 107–123.
[6] Y. Yao, The geometry of three-way decision, Applied Intelligence 51 (2021) 6298–6325.
[7] Z. Pawlak, Rough sets, International Journal of Computer & Information Sciences 11 (1982) 341–356.
[8] W. Pedrycz, Shadowed sets: Representing and processing fuzzy sets, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics 28 (1998) 103–109.
[9] Y. Yao, Interval sets and three-way concept analysis in incomplete contexts, International Journal of Machine Learning and Cybernetics 8 (2017) 3–20.
[10] L. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[11] J. Yang, Y. Yao, Semantics of soft sets and three-way decision with soft sets, Knowledge-Based Systems 194 (2020) 105538.
[12] J. C. R. Alcantud, F. Feng, R. R. Yager, An N-soft set approach to rough sets, IEEE Trans Fuzzy Syst 28 (2020) 2996–3007.
[13] J. C. R. Alcantud, G. Santos-García, M. Akram, OWA aggregation operators and multi-agent decisions with N-soft sets, Expert Systems with Applications 230 (2022) 117430.
[14] A. Adeel, M. Akram, N. Yaqoob, W. Chammam, Detection and severity of tumor cells by graded decision-making methods under fuzzy N-soft model, Journal of Intelligent & Fuzzy Systems 39 (2020) 1303–1318.
[15] M. Akram, G. Ali, J. C. R. Alcantud, F. Fatimah, Parameter reductions in N-soft sets and their applications in decision-making, Expert Systems 38 (2021) e12601.
[16] J. C. R. Alcantud, G. Santos-García, A new criterion for soft set based decision making problems under incomplete information, International Journal of Computational Intelligence Systems 10 (2017) 394–404.
[17] G. A. Aranda-Corral, J. Borrego-Díaz, J. Galán-Páez, A model of three-way decisions for knowledge harnessing, International Journal of Approximate Reasoning 120 (2020) 184–202.
[18] S. M. Abbas, K. A. Alam, K.-M. Ko, A three-way classification with game-theoretic N-soft sets for handling missing ratings in context-aware recommender systems, in: 2020 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2020, pp. 1–8. doi:10.1109/FUZZ48607.2020.9177701.
[19] G. Lang, D. Miao, H. Fujita, Three-way group conflict analysis based on Pythagorean fuzzy set theory, IEEE Transactions on Fuzzy Systems 28 (2020) 447–461.
[20] Y. Zhang, D. Miao, J. Wang, Z. Zhang, A cost-sensitive three-way combination technique for ensemble learning in sentiment classification, International Journal of Approximate Reasoning 105 (2019) 85–97.
[21] K. Zhang, J. Dai, Z. Xu, The criteria-oriented three-way ranking and clustering strategies in fuzzy decision environments, IEEE Transactions on Fuzzy Systems (2021) 1–1.
[22] Q. Zhang, G. Lv, Y. Chen, G. Wang, A dynamic three-way decision model based on the updating of attribute values, Knowledge-Based Systems 142 (2018) 71–84.
[23] X. Huang, J. Zhan, TWD-R: a three-way decision approach based on regret theory in multi-scale decision information systems, Information Sciences 581 (2021) 711–739.
[24] X. Huang, J. Zhan, B. Sun, A three-way decision method with pre-order relations, Information Sciences 595 (2022) 231–256.
[25] Y. Li, C. Zhang, J. Swan, An information filtering model on the web and its application in jobagent, Knowledge-Based Systems 13 (2000) 285–296.
[26] K. Zhang, J. Dai, J. Zhan, A new classification and ranking decision method based on three-way decision theory and TOPSIS models, Information Sciences 568 (2021) 54–85.
[27] J. Ye, J. Zhan, W. Ding, H. Fujita, A novel three-way decision approach in decision information systems, Information Sciences 584 (2022) 1–30.
[28] J. Deng, J. Zhan, W.-Z. Wu, A three-way decision methodology to multi-attribute decision-making in multi-scale decision information systems, Information Sciences 568 (2021) 175–198.
[29] Q. Wang, J. Dai, Z. Xu, A new three-way multi-criteria decision-making method with fuzzy complementary preference relations based on additive consistency, Information Sciences (2022).
[30] J. Wang, X. Ma, J. Dai, J. Zhan, A novel three-way decision approach under hesitant fuzzy information, Information Sciences 578 (2021) 482–506.
[31] J. Wang, X. Ma, Z. Xu, J. Zhan, Three-way multi-attribute decision making under hesitant fuzzy environments, Information Sciences 552 (2021) 328–351.
[32] J. Wang, X. Ma, Z. Xu, J. Zhan, A three-way decision approach with risk strategies in hesitant fuzzy decision information systems, Information Sciences (2021).
[33] J. Qiao, Hesitant relations: Novel properties and applications in three-way decisions, Information Sciences 497 (2019) 165–188.
[34] F. Feng, Z. Wan, J. C. R. Alcantud, H. Garg, Three-way decision based on canonical soft sets of hesitant fuzzy sets, AIMS Mathematics 7 (2022) 2061–2083.
[35] W. Wang, J. Zhan, J. Mi, A three-way decision approach with probabilistic dominance relations under intuitionistic fuzzy information, Information Sciences 582 (2022) 114–145.
[36] A. Kharal, Soft approximations and uni-int decision making, The Scientific World Journal 2014 (2014) 327408.
[37] J. Łukasiewicz, O logice trójwartościowej, Ruch filozoficzny 5 (1920) 170–171.
[38] E. L. Post, Introduction to a general theory of elementary propositions, American Journal of Mathematics 43 (1921) 163–185.
[39] J. Łukasiewicz, A. Tarski, Untersuchungen über den aussagenkalkül, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie Classe III 23 (1930) 30–50.
[40] Y. Yao, Interval-set algebra for qualitative knowledge representation, in: Proceedings of ICCI'93: 5th International Conference on Computing and Information, IEEE Computer Society, 1993, pp. 370–374. doi:10.1109/ICCI.1993.315346.
[41] Y. Yao, X. Li, Comparison of rough-set and interval-set models for uncertain reasoning, Fundamenta Informaticae 27 (1996) 289–298.

[42] N. D. Belnap, How a computer should think, in: G. Ryle (Ed.), Contemporary Aspects of Philosophy, Oriel Press, 1977, pp. 30–56.

[43] N. D. Belnap, How a computer should think, in: H. Omori, H. Wansing (Eds.), New Essays on Belnap-Dunn Logic, number 418 in Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science), Springer, 2019, pp. 35–53. doi:10.1007/978-3-030-31136-0_4.

[44] Y. Shramko, H. Wansing, Some useful 16-valued logics: How a computer network should think, Journal of Philosophical Logic 34 (2005) 121–153.

[45] S. P. Odintsov, On axiomatizing Shramko-Wansing's logic, Studia Logica 91 (2009) 407–428.

[46] F. Feng, J. Cho, W. Pedrycz, H. Fujita, T. Herawan, Soft set based association rule mining, Knowledge-Based Systems 111 (2016) 268–282.

[47] L. J. Savage, The Foundations of Statistics, volume 11, Wiley Publications in Statistics, 1954.

[48] L. A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Computers & Mathematics with Applications 9 (1983) 149–184.

[49] D. Ciucci, D. Dubois, H. Prade, Structures of opposition induced by relations, Annals of Mathematics and Artificial Intelligence 76 (2016) 351–373.