Necessary and Possible Hesitant Fuzzy Sets: A Novel Model for Group Decision Making

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Abstract

We propose an extension of Torra's notion of hesitant fuzzy set, which appears to be well suited to group decision making. In our model, indecisiveness in judgements is described by two nested hesitant fuzzy sets: the smaller, called necessary, collects membership values determined according to a rigid evaluation, whereas the larger, called possible, comprises socially acceptable membership values. We provide several instances of application of our methodology, and accordingly design suitable individual and group decision procedures. This novel approach displays structural similarities with Atanassov's intuitionistic fuzzy set theory, but has rather different goals. Our source of inspiration comes from preference theory, where a bi-preference approach has proven to be a useful extension of the classical mono-preference modelization in the fields of decision theory and operations research.

Key words: Hesitant fuzzy set; Necessary and possible preference; Decision making; Score; Aggregation operator.

1. Introduction

Zadeh's [56] *fuzzy set theory* deals with impreciseness or vagueness of data and evaluations by imputing degrees to which objects belong to a set. The appearance of fuzzy set theory induced the rise of several related theories, which codify subjectivity, uncertainty, imprecision, or roughness of evaluations. The rationale of these theories is to create new and more flexible methodologies, which allow one to realistically model a variety of concrete decision problems.

In this direction, a recent and promising extension of fuzzy set theory is represented by Torra's [46] *hesitant fuzzy sets* (HFSs). As Bustince et al. [12] explain, this notion coincides with set-valued fuzzy sets in Grattan-Guinness [30]. The characterizing feature of HFSs is that they permit to model a type of imprecise human knowledge that cannot be captured by classical fuzzy sets: for instance, collective decision making is a natural outlet for hesitant fuzzy models [3].

The naturalness of HFSs is witnessed by the quickly growing literature in the field. For instance, Alcantud and Torra [5] define the notion of *uniformly typical* HFS, which appears to be a rather good fit for real applications. Among the several extension/variations of the theory of hesitant fuzzy sets, let us mention Zhu and Xu [58], who define *extended* HFSs, and Alcantud and Santos-García [6], who introduce *expanded* HFSs. Liao and Xu [37, section 1.2] give an account of earlier extensions of the HFS model, e.g., dual hesitant fuzzy sets (Zhu et al. [59]), hesitant fuzzy linguistic term sets (Rodríguez et al. [44]), and interval-valued hesitant fuzzy sets (Chen et al. [14, 15]).

Enlarging the scope of HFS is motivated by an attempt to make modelization more precise and adherent to reality, also designing transparent and flexible decision procedures. In this direction, here we propose a hybrid approach to the topic, which employs a nested pair of HFSs in place of a single one. Although this approach bears some resemblance with the ethos of Atanassov's *intuitionistic fuzzy sets* (IFSs), our goals are rather different, being mostly related to the creation of effective decision procedures that are suited to a collective/social setting.

The inspiration and motivation of our model stems from a recent and non-standard approach to preference modeling, based on "necessary and possible preference structures". Due to the generality of its underlying philosophy, this approach has already adapted to different fields of research. For instance, a very recent fuzzy modelization of the political system [4] is inspired by a necessary and possible approach. In this new setting, the private opinion of a generic citizen on each topic of the political campaign is modeled by a trapezoidal (or quasi-trapezoidal) fuzzy profile on the interval $[-1, 1]$ (where -1 is extreme left, and 1 is extreme right). On the other hand, the opinions of political candidates are modeled by two profiles on each topic of the campaign: a *private* (necessary) profile describing the politician's intimate beliefs, and a *public* (possible) profile containing the former and expressing his political tolerance. The result is a *PaP-profile* –private and public profile– for each topic of the political campaign. Political parties are then created on the basis of the similarity of politicians' PaP-profiles.

In order to better motivate our model, let us preliminarily review some recent developments in the fields of both IFSs and preference theory. It may be the case that a practitioner believes that an option belongs to a set to a certain degree –as fuzzy sets assume– but, at the same time, she is unsure about such an assessment. In order to model such a situation, Atanassov [7, 8, 9] introduced intuitionistic fuzzy sets, which associate to each element of the ground set a pair of numbers in the unit interval, representing membership and non-membership evaluations. (Notice that, as a consequence of the definition, the sum of the two membership degrees should not exceed one for each element of the ground set.) In a nutshell, intuitionistic fuzzy set theory models judgements from both a "positive" and a "negative" point of view, thus refining the classical approach.

Concerning preference modeling, the typical way to represent an economic agent's preference structure (in a non-stochastic setting) on a set of alternatives is by means of a binary relation, which is typically assumed to satisfy forms of completeness and/or transitivity.[1] Historically, preorders, semiorders [25, 38, 41], interval orders [19, 20], and tournaments [39] are the types of binary relations that have been employed for the modelization of preferences, due to the fact that they satisfy some desirable –yet not too demanding– properties: see, e.g., the books [10, 41] and references therein for an extensive list of possible applications of these natural types of preference structures.

A very recent approach to preference modeling departs from this traditional setting by employing a pair of suitably interconnected binary relations in place of a single binary relation. The advantage of this *bi-preference* approach lies in the possibility to more flexibly model preference structures (of a single agent or a set of agents) in several scenarios. More specifically, the two layers of information in a bi-preference allow one to single out the binary relations of "strict preference", "indifference", "incomparability", and "indecisiveness". This represents an enrichment of the descriptive power of the model, since the mono-preference approach usually identifies indecisiveness with incomparability.

Technically, the two components of a bi-preference are nested into each other, and are connected by properties that make this structure well suited to applications in economics and psychology. The two traditional tenets of economic rationality –completeness and transitivity– are "spread" over the two components of a bi-preference: the first (called *rigid*) is transitive but usually incomplete, whereas the second (called *soft*) is possibly complete but typically intransitive. Finally, the property of *transitive coherence* ensures that the soft component partially retains transitivity with respect to the rigid component.[2]

It is apparent that the theory of bi-preferences displays, *mutatis mutandis*, some structural similarities with Atanassov's IFSs. However, there are major theoretical differences between the two approaches, both methodologically and semantically. For instance, contrary to bi-preferences, there is no nesting relationship in the case of IFSs. Further, bi-preferences (and the model that we propose from their hybridization with HFSs) completely dismiss the property that certain sums should be bounded by 1.

A special class of bi-preferences is given by *necessary and possible preferences* (*NaP-preferences*), which satisfy additional structural properties. Originally introduced in the field of *Multiple Criteria Decision Aid* as an operating tool for the so-called *Robust Ordinal Regression* [32], in the last years NaP-preferences have proven to be useful in the modelization of a large amount of decision problems that require techniques of operations research: see the survey [31].[3] In a necessary and possible modelization of preferences,

[1]The weakest form of transitivity is *acyclicity* (i.e., the non-existence of cycles of strict preferences), which has been studied at large in the literature: see, e.g., [1, 40].

[2]For a detailed discussion on the topic, see the introductory sections of [13] and [23], and especially the extensive analysis of uniform bi-preferences in [29].

[3]The axiomatization of NaP-preferences is quite recent [23]. Further, NaP-preferences have been examined from several points of view: their relationship with preference relations satisfying (m, n)-*Ferrers properties* [24, 28], and, in particular, with the genesis of interval orders and semiorders [21]; asymmetric and normalized forms of NaP-preferences [22]; *well-graded families* –in the sense of Doignon and Falmagne [16]– of NaP-preferences [26]; symmetric counterparts of NaP-preferences, called *NaP-indifferences* [27]; bi-preference structures under uncertainty [13]; generalizations of NaP-preferences by *uniform* types of bi-preferences [29].

the total information provided by an economic agent is split in two parts: (i) "positive", codified by the necessary component; and (ii) "negative", codified by the *im*possible component (that is, the complement of the possible component). Then the complement of the overall information representes a hesitation in preferential judgements: the parallel with intuitionistic fuzzy set theory is now apparent.

The idea underlying this paper is to replicate the new approach to preference theory mentioned above within the realm of hesitant fuzzy sets, defining a notion of NaP-HFS (*necessary and possible HFS*) as a suitable pair of HFSs. Then the two components of a NaP-HFS are: (i) the *necessary* HFS, which describes the set of "individually preferred" membership values; and (ii) the *possible* HFS, which provides the set of "socially acceptable" membership values. In this way we obtain a new notion, which encompasses hesitant fuzzy sets, and is more flexible in capturing the subtleties of decision making, especially in a collective setting. Here we start developing the main properties of this novel construction, and define some useful tools to help the practitioner to analyze the concept. In particular, several examples of this paper emphasize its capability to model situations of collective decisions. Finally, we employ these tools (aggregation operators, scores, etc.) to effectively propose a flexible decision making procedure, which generalizes widely accepted procedures that employ hesitant fuzzy sets.

The paper is organized as follows. **Section 2** collects all preliminaries on hesitant fuzzy sets, as well as some variations and extensions. In particular, we dwell on the notions of aggregation operator and score, and propose comprehensive definitions in the general setting. The core of the paper is **Section 3**, where we describe our necessary and possible approach to hesitant fuzzy sets. To motivate our analysis, we provide several examples of possible applications. We also extend some typical operations on hesitant fuzzy sets to a necessary and possible setting, and introduce new types of operators that are specifically related to nested pairs of hesitant fuzzy sets. In **Section 4** we develop a flexible decision making procedure for alternatives defined by typical NaP-HFSs. We illustrate this procedure in some examples, and then discuss its relationship with earlier approaches by HFSs. We also put forward a variation that initiates group decision making in this context. **Section 5** summarizes the results of the paper and suggests future directions of research. The **Appendix** contains some remarks about a novel notion of monotonicity for scores, which makes them better suited for sound decision procedures.

2. Preliminaries on hesitant fuzzy sets

In this section we recall the basic terminology on hesitant fuzzy sets, and mention some interesting extensions of the original definition. Hereafter, we fix a nonempty set X of alternatives. Further, we use the following notation:
- $\mathcal{P}([0,1])$ is the family of all subsets of the unit interval $[0,1]$,
- $\mathcal{P}^*([0,1])$ is the family of all nonempty subsets of $[0,1]$,
- $\mathcal{F}^*([0,1])$ is the family of all nonempty finite subsets of $[0,1]$, and
- $\mathcal{F}^*_n([0,1])$ is the family of all nonempty subsets of $[0,1]$ with n or fewer elements.

We warn the reader that at times we shall employ a slightly non-standard terminology, which is notationally simpler than the one typically found in the specialized literature. The reason for our choice is that the approach proposed in this paper naturally requires the introduction of several new notions, which would make the presentation cumbersome, should we stick with standard notation.

2.1. Main definitions

The main objects of our analysis are hesitant fuzzy sets:

Definition 1 (Xia and Xu [50],Torra [46])**.** A *hesitant fuzzy element* (*HFE*) is a subset E of $[0,1]$. A *hesitant fuzzy set* (*HFS*) over X is a function $h\colon X \to \mathcal{P}([0,1])$. An HFS h over X is *proper* if $h(x) \neq \emptyset$ for all $x \in X$; otherwise, it is *improper*. In particular, the *trivial* HFS over X is the (improper) HFS defined by $h(x) = \emptyset$ for all $x \in X$.

In many practical applications, the employed HFSs are proper, and they map each element of X to a finite subset of $[0,1]$.

Definition 2 (Bedregal et al. [11])**.** A *typical hesitant fuzzy set* (*THFS*) over X is a function $h\colon X \to \mathcal{F}^*([0,1])$. A *typical hesitant fuzzy element* (*THFE*) is a nonempty finite subset of $[0,1]$.

Notice that the family of THFEs is equal to $\mathcal{F}^*([0,1])$. Every HFS over X associates a set of membership values to each element of X, which may be empty for some $x \in X$. However, if the HFS is typical, then any such set of membership values must always be finite and nonempty, that is, each $x \in X$ is mapped to a THFE. It is customary to list the elements of a THFE in ascending order, i.e., E is a set of the type $\{e_1, \ldots, e_k\} \subseteq [0,1]$ for some $k \geqslant 1$, where $e_1 < \ldots < e_k$. Noteworthy examples of THFEs are $E = \{1\}$ (*full HFE*) and $E = \{0\}$ (*empty HFE*).

As usually done, we often represent a hesitant fuzzy set h over X as a set of ordered pairs, i.e., $h = \{(x, h(x)) : x \in X\}$. For example, the *ideal* or *full* HFS over X is $\{(x, \{1\}) : x \in X\}$, and the *anti-ideal* or *empty* HFS over X is $\{(x, \{0\}) : x \in X\}$.

The following definition provides a strengthening of the notion of typical HFS:

Definition 3 (Alcantud and Torra [5])**.** A typical hesitant fuzzy set h over X is *uniformly typical* if there is $n \in \mathbb{N}$ such that $|h(x)| \leqslant n$ for each $x \in X$. Equivalently, a uniformly typical hesitant fuzzy set is a function $h\colon X \to \mathcal{F}^*_n([0,1])$ for some $n \geqslant 1$. We shall abbreviate "uniformly typical HFS" by UHFS.

Clearly, every UHFS is a THFS, but the converse fails to hold in general. However, if the ground set X is finite, then the two notions THFS and UHFS coincide. Alcantud and Torra [5] associate a positive integer to each UHFS, its *characteristic*. Updated surveys of HFSs and their applications include Rodríguez et al. [43], Rodríguez et al. [45], and Xu [51]: these contributions illustrate the importance of hesitant fuzzy sets for both theoretical and applied purposes. Further, the recent paper [2] provides up-to-date references on the ranking of projects evaluated by HFSs.

2.2. Basic operations on hesitant fuzzy sets

Some typical operators on HFSs are the following:

Definition 4 (Torra [46], Torra and Narukawa [47])**.** To any HFS $h = \{(x, h(x)) : x \in X\}$ over X, associate the following "derived" HFSs:

1. the *lower bound* h^- of h, defined by $h^-(x) := \{\inf\{h(x) : x \in X\}\}$ for all $x \in X$;
2. the *upper bound* h^+ of h, defined by $h^+(x) := \{\sup\{h(x) : x \in X\}\}$ for all $x \in X$;
3. for any $\alpha \in [0, 1]$, the α*-lower bound* h^-_α of h, defined by $h^-_\alpha(x) := \{e \in h(x) : e \leqslant \alpha\}$ for all $x \in X$;
4. for any $\alpha \in [0, 1]$, the α*-upper bound* h^+_α of h, defined by $h^+_\alpha(x) := \{e \in h(x) : e \geqslant \alpha\}$ for all $x \in X$;
5. the *complement* h^c of h, defined by $h^c(x) = \{1 - h(x) : x \in X\}$ for all $x \in X$.

Inspired by practical reasons, Torra [46] (see also Torra and Narukawa [47]) introduces suitable notions of union and intersection –here renamed "discrete union" and "discrete intersection"– between two HFSs. The next definition collects these notions, as well as some less standard notions of union and intersection, here named "indiscrete".

Definition 5. Let h_1 and h_2 be two HFSs over X.

1. The *indiscrete union* $h_1 \vee h_2$ of h_1 and h_2 is the HFS over X defined as follows for all $x \in X$: $(h_1 \vee h_2)(x) := h_1(x) \cup h_2(x)$.
2. The *discrete union* $h_1 \cup h_2$ of h_1 and h_2 is the HFS over X defined as follows for all $x \in X$:

$$(h_1 \cup h_2)(x) \ := \ \big\{e \in h_1(x) \cup h_2(x) : e \geqslant \max\{h_1^-(x), h_2^-(x)\}\big\} \ \subseteq \ (h_1 \vee h_2)(x).$$

3. The *indiscrete intersection* $h_1 \wedge h_2$ of h_1 and h_2 is the HFS over X defined as follows for all $x \in X$: $(h_1 \wedge h_2)(x) := h_1(x) \cap h_2(x)$.
4. The *discrete intersection* $h_1 \cap h_2$ of h_1 and h_2 is the HFS over X defined as follows for all $x \in X$:

$$(h_1 \cap h_2)(x) \ := \ \big\{e \in h_1(x) \cup h_2(x) : e \leqslant \min\{h_1^+(x), h_2^+(x))\big\} \ \subseteq \ (h_1 \vee h_2)(x).$$

We shall use indiscrete union and intersection to aggregate nested pairs of HFSs.

2.3. Aggregation of hesitant fuzzy elements

Xia and Xu [50] propose several aggregation operators on HFEs, which are then employed to describe effective decision making algorithms based on pieces of hesitant fuzzy information. Many other examples of aggregation operators have been defined in the literature, e.g., by Yu et al. [55], and Wei [49]. (See [45, section 4] for an extensive account on the topic.) However, to the best of our knowledge, the literature on HFSs lacks a formal definition of "aggregation operator". Thus, we propose the following general notion:

Definition 6. Let $\mathbb{P}^n([0,1])$ denote the Cartesian product of n copies of $\mathcal{P}([0,1])$, that is,

$$\mathbb{P}^n([0,1]) \;:=\; \underbrace{\mathcal{P}([0,1]) \times \ldots \times \mathcal{P}([0,1])}_{n}.$$

Further, let $\mathbb{P}([0,1])$ denote the union of all finite Cartesian powers of $\mathcal{P}([0,1])$, that is,

$$\mathbb{P}([0,1]) \;:=\; \bigcup_{n=1}^{\infty} \mathbb{P}^n([0,1]).$$

A *HFE-aggregation operator* is a map $\Psi\colon \mathbb{P}([0,1]) \to \mathcal{P}([0,1])$ satisfying the following properties:

1. (*Non-triviality*) for each $n \geqslant 1$ and $E_1, \ldots, E_n \in \mathcal{P}^*([0,1])$, we have $\Psi(E_1, \ldots, E_n) \neq \emptyset$;

2. (*Identity*) for each $E \in \mathcal{P}([0,1])$, we have $\Psi(E) = E$;

3. (*Inclusiveness*) for each $E \in \mathcal{P}([0,1])$ and $n \geqslant 1$, the inclusion $E \subseteq \Psi(\underbrace{E, \ldots, E}_{n})$ holds;

4. (*Monotonicity*) for each $n \geqslant 1$ and $E_1, \ldots, E_n, E'_1, \ldots, E'_n \in \mathcal{P}^*([0,1])$ such that $E_i \subseteq E'_i$ for all i, the inclusion $\Psi(E_1, \ldots, E_n) \subseteq \Psi(E'_1, \ldots, E'_n)$ holds;

5. (*Typicality*) for each $n \geqslant 1$ and $E_1, \ldots, E_n \in \mathcal{F}^*([0,1])$, $\Psi(E_1, \ldots, E_n)$ is a THFE.

Further, we say that Ψ is an *anonymous* HFE-aggregation operator if it satisfies the following additional property:

6. *(Anonymity)* for each $n \geqslant 1$ and $E_1, \ldots, E_n \in \mathcal{P}^*([0,1])$, if π is any permutation of the set $\{1, \ldots, n\}$, then the equality $\Psi(E_1, \ldots, E_n) = \Psi(E_{\pi(1)}, \ldots, E_{\pi(n)})$ holds.

A *THFE-aggregation operator* is an HFE-aggregation operator restricted to the union of all finite Cartesian powers of $\mathcal{F}^*([0,1])$, i.e., a map $\Phi\colon \mathbb{F}^*([0,1]) \to \mathcal{F}^*([0,1])$, where

$$\mathbb{F}^*([0,1]) = \bigcup_{n=1}^{\infty} \underbrace{\mathcal{F}^*([0,1]) \times \ldots \times \mathcal{F}^*([0,1])}_{n},$$

such that properties 2–4 hold (properties 1 and 5 are obviously satisfied). THFE-aggregation operators can also be anonymous in the obvious sense.

Non-triviality and Identity are very natural in this context: the former means that the aggregation of nonempty HFEs is a nonempty HFE, whereas the latter requires that the aggregation of a single HFE produces exactly the same HFE. Inclusiveness assures that when we aggregate several instances of a given HFE, every membership degree in such HFE is a membership degree in the aggregate. With Monotonicity we guarantee that larger HFEs produce larger aggregate results. Typicality is self-explanatory. Anonymity is standard, but it is not always desirable: it states that the order of presentation of HFEs has no bearing on their aggregate result.[4]

To aggregate evaluations and eventually obtain decision procedures, some authors have proposed several types of "averaging operators" over time. The following operators are defined in [50]:

Definition 7. For all $E_1, \ldots, E_n \in \mathcal{F}^*([0,1])$, define:

(i) the *hesitant fuzzy averaging* (*HFA*) operator by

$$\mathrm{HFA}(E_1, \ldots, E_n) = \bigcup_{e_1 \in E_1, \ldots, e_n \in E_n} \left\{ \left(1 - \prod_{i=1}^{n} (1 - e_i)^{1/n} \right) \right\};$$

(ii) the *hesitant fuzzy geometric* (*HFG*) operator by

$$\mathrm{HFG}(E_1, \ldots, E_n) = \bigcup_{e_1 \in E_1, \ldots, e_n \in E_n} \prod_{i=1}^{n} (e_i)^{1/n};$$

(iii) the *hesitant fuzzy weighted averaging* (*HFWA*) or *geometric* (*HFWG*) operators, which extend (i) and (ii) by introducing weights $w_i \in [0,1]$ with $\sum_{i=1}^{n} w_i = 1$, by

$$\mathrm{HFWA}(E_1, \ldots, E_n) = \bigcup_{e_1 \in E_1, \ldots, e_n \in E_n} \left\{ \left(1 - \prod_{i=1}^{n} (1 - e_i)^{w_i} \right) \right\}$$

$$\mathrm{HFWG}(E_1, \ldots, E_n) = \bigcup_{e_1 \in E_1, \ldots, e_n \in E_n} \prod_{i=1}^{n} (e_i)^{w_i}.$$

All operators listed in Definition 7 are indeed THFE-aggregation operators in the sense of Definition 6:

Proposition 1. *The HFA, HFG, HFWA, and HFWG operators are THFE-aggregation operators. Further, HFA and HFG are anonymous, whereas HFWA and HFWG are non-anonymous.*

Proof. Straightforward. □

[4]The order of presentations of HFEs may well influence the aggregate results in some circumstances: imagine a situation in which the first agent to express an opinion is the CEO of a corporation, whereas the other agents are VPs at a lower level of the hierarchy.

2.4. Scores of hesitant fuzzy elements

Scores are $[0,1]$-valued functions that evaluate hesitating fuzzy elements. Scores close to zero are typical of HFEs with very low membership values, whereas scores close to one characterize HFSs that contain very high membership values. Formally, scores can be defined as follows:[5]

Definition 8. Given a family $\mathcal{G} \subseteq \mathcal{P}([0,1])$, a *score on* $\mathcal{G}$ is a function $s\colon \mathcal{G} \to [0,1]$ with the following properties:[6]

1. $s(\emptyset) = 0$ whenever $\emptyset \in \mathcal{G}$;

2. (*Boundedness*) for all $E \in \mathcal{G}$, we have $\inf(E) \leqslant s(E) \leqslant \sup(E)$.

A score on $\mathcal{G} = \mathcal{P}([0,1])$ is called *total*, and a score on $\mathcal{G} = \mathcal{F}^*([0,1])$ is called *typical*.

Notice that Boundedness implies that $s(\{a\}) = a$ for each $a \in [0,1]$ such that $\{a\} \in \mathcal{G}$; in particular, we have $s(\{0\}) = 0$ and $s(\{1\}) = 1$. The most used scores are typical (in the sense of Definition 8), since they are defined on the family of all finite subsets of the unit interval. The next example collects some simple instances of this kind.

Example 1. The following functions $s\colon \mathcal{F}^*([0,1]) \to [0,1]$ are typical scores, where $E = \{e_1, \ldots, e_n\}$, with $e_1 < \ldots < e_n$, is an arbitrary element of $\mathcal{F}^*([0,1])$:

- (*min*) $s(E) = e_1$;
- (*max*) $s(E) = e_n$;
- (*second worst*) $s(E) = e_1$ if $|E| = 1$, and $s(E) = e_2$ otherwise;
- (*second best*) $s(E) = e_n$ if $|E| = 1$, and $s(E) = e_{n-1}$ otherwise;
- (*average*)[7] $s(E) = (e_1 + \ldots + e_n)/n$;
- (*cut average*) $s(E)$ is the average if $|E| \leqslant 2$, and $s(E) = (e_2 + \ldots + e_{n-1})/(n-2)$ otherwise.

The reason to employ typical scores such as "min" or "second worst" may be due to a pessimistic/cautious view of the given evaluations. For instance, if a THFE collects all membership values provided by experts in a certain field, the decision maker may want to adopt a very cautious attitude, and select either the minimum evaluation or the second

[5] The generic term "the score function for HFEs" has often been used to designate specific examples, as explained in [45]. Alcantud et al. [3] formally define score operators for typical HFEs: here we employ a slight weakening of their definition.

[6] Farhadinia [18, Section 2] defines scores for the special case of THFEs with the same number of (possibly repeated) elements. Besides this restriction, which we eschew, his definition imposes some unnecessary boundary conditions.

[7] This is the *Xia–Xu score*: see Definition 9(i).

worst. Dually, using scores such as "max" or "second best" reveals an optimistic/reckless view of the evaluations collected in a THFE. Finally, scores such as the average or the cut average are typical of situations that require to take the mean of evaluations (possibly eliminating the worst and the best evaluations, as for some competitions at the Olympics).

The literature on typical scores is abundant: see Xia and Xu [50], and Farhadinia [17, 18], as well as [45, section 2] for further examples. Three noteworthy examples of scores are the following:

Definition 9. For each $E = \{e_1, \ldots, e_n\}$, with $e_1 < \ldots < e_n$,

(i) the (typical) *Xia–Xu score* is the function $s_{\mathrm{XX}}(E) = \sum_{i=1}^{n} e_i/n$,

(ii) for a nondecreasing sequence $(\delta_n)_{n\geqslant 1}$ of positive numbers, the (typical) *Farhadinia score* is the function $s_{\mathrm{F}}(E) = \sum_{i=1}^{n}(\delta_i e_i)/\sum_{i=1}^{n}\delta_i$, and

(iii) the (typical) *geometric-mean score* is the function $s_{\mathrm{gm}}(E) = \big(\prod_{i=1}^{n} e_i\big)^{1/n}$.

Clearly, (ii) includes (i) as a special case, taking the constant sequence with value 1.

Scores were originally introduced because they permit to rank HFEs: the higher the score of a HFE, the higher its relevance. Notice that, however, this ranking fails to be fully discriminative in general, due to the possibility of ties between distinct HFSs. As a matter of fact, ties must always appear for total scores, because there is no injection from $\mathcal{P}(X)$ to X for any nonempty X.[8] Rodríguez et al. [45, section 2] attest this issue both with Xia–Xu's and Farhadinia's score, and examples can be produced to show that the geometric score displays the same problem. Thus, even if the scores given in Definition 9 are not fully satisfactory, we cannot discard them on account of this issue, because it affects them all in the same way.

In this paper, we shall only employ (an adaptation of) the averaging Xia–Xu score. The main reason for our choice is this score satisfies a (desirable) property of "strong monotonicity", which makes it better suited to sound decision procedures. On the contrary, scores (ii) and (iii) even fail to be "monotonic". For the notions of *strongly monotonic* and *monotonic*, see the Appendix.

3. Necessary and possible hesitant fuzzy sets

In this section we propose a new model of hesitant fuzzy sets, which generalizes some known approaches by considering a nested pair of hesitant fuzzy sets. As we shall see, the advantages of our model are not limited to a syntactic extension of the typical setting. Indeed, the main feature of this novel approach is the possibility to allow flexibility in applying it to concrete decision problems, especially when a group of agents is involved.

[8] In order to lessen this handicap, Alcantud and de Andrés [2] suggest a segment-based approach to the problem of prioritizing hesitant fuzzy sets.

3.1. Definition and semantics

Definition 10. A *necessary and possible hesitant fuzzy set* (*NaP-HFS*) over X is a pair $\mathcal{H} = (h_N, h_P)$ of hesitant fuzzy sets over X such that the inclusion $h_N(x) \subseteq h_P(x)$ holds for all $x \in X$. In this case, h_N and h_P are called, respectively, the *necessary HFS* and the *possible HFS* of $\mathcal{H}$. A NaP-HFS is *proper* if so is its possible component. Following standard conventions in HFSs, we sometimes denote a NaP-HFS by the corresponding set of triples, that is,

$$\mathcal{H} = \big\{(x, h_N(x), h_P(x)) : x \in X\big\}.$$

A pair (E_N, E_P), with $E_N \subseteq E_P \subseteq [0,1]$, is a *necessary and possible hesitant fuzzy element* (*NaP-HFE*); E_N and E_P are, respectively, its *necessary HFE* and its *possible HFE.*

Although some applications may require the possible HFS to be infinite-valued (e.g., it maps elements of the ground set on an interval, as in Example 5 below), in several cases a finite environment suffices to model a decision problem. This consideration motivates the following definition:

Definition 11. A NaP-HFS is *typical* (resp. *uniformly typical*) whenever its possible HFS is typical (resp. uniformly typical) and its necessary HFS is proper. A pair (E_N, E_P) such that E_P is a THFE and $\emptyset \neq E_N \subseteq E_P$ is a *NaP-THFE.*

Example 2. Let $X = \{x, y\}$ be the set of options. Then

$$\mathcal{H} = \big\{(x, \{0.2, 0.6\}, \{0.2, 0.3, 0.4, 0.5, 0.6\})\,,\,(y, \{0.9\}, \{0.8, 0.9, 1\})\big\}$$

is a uniformly typical NaP-HFS .

The semantics of a necessary and possible hesitant fuzzy set is quite natural in a social/collective context. To clarify the point, assume that we are given a set $\mathfrak{C}$ of economic agents, who have to deliberate on the set of membership degrees of each $x \in X$. Then the attitude of each agent $c \in \mathfrak{C}$ may be synthesised by a NaP-HFS (h_N^c, h_P^c), where the necessary HFS h_N^c represents the core part of his beliefs, and the possible HFS h_P^c enlarges the range of his evaluations by adding some compromise solutions.

More generally, in a social context we can identify two dual semantics of a NaP-HFS:

(1) from personal to social (*bottom-up*);

(2) from social to personal (*top-down*).

In semantics (1), the personal beliefs of each actors are primitive, and his social beliefs are constructed only successively according to the goal at hand. Conversely, in semantics (2), a social context is exogenously given (and shared by all actors involved in the decision process), and only successively agents identify those values that fit their intimate attitude. Let us further elaborate on these two semantics.

From point of view (1), each $h_N^c(x)$ is primitive, and comprises all membership values for the alternative x that the agent c intimately regards as suitable for the decision problem at hand. In a social context, however, some compromises on personal beliefs are often inevitable. Thus, each agent $c \in \mathfrak{C}$ may "enrich" his evaluations by adding new membership values for all $x \in X$, according to several factors (e.g., cultural/political opportunity, discussions with other members of the community, attempts to create a coalition, etc.). This process yields a (possibly larger) set $h_P^c(x)$, which contains all membership values for the element $x \in X$ that the agent c considers as socially admissible/acceptable.[9]

On the contrary, considering point of view (2), assume that a "social perspective" is given in the form of all admissible membership values for each alternative x: this is synthesised by a set $h^{\mathfrak{C}}(x)$ of membership values for all $x \in X$. Then each agent $c \in \mathfrak{C}$ selects within $h^{\mathfrak{C}}(x)$ those membership values that he considers somehow justifiable from his point of view, thus identifying a (possible) subset $h_P^c(x) \subseteq h^{\mathfrak{C}}(x)$ for all $x \in X$. At the last stage, the agent c creates the (necessary) sets $h_N^c(x) \subseteq h_P^c(x)$ for all $x \in X$, which comprise only those membership values that really fit his personal attitude.

In the above (simplified) discussion, we assumed that each hesitant fuzzy element that appears in the modelization is composed of all membership values provided by a single agent. However, in our necessary and possible setting, HFEs appear to be better suited to collect membership values expressed by subgroups of agents, who are a part of a larger group of decision makers. The next subsection is devoted to clarify this point, by presenting some practical applications of our approach.

3.2. Justification of the model and some motivating examples

The purpose of any novel model is typically manifold. It should be a faithful expression of natural situations. It should help explain or understand facts better than existing models. Ultimately, it should be applicable to make decisions.

Although individual agents are unlikely to employ apparently sophisticated principles like those underlying NaP-HFSs, it is precisely because of their complexity that NaP-HFSs may be successful in representing the intricacy of *social scenarios*. Some examples given in this section justify such a claim. At any rate, it does not come as a surprise that complicated models are needed to encapsulate collective interactions and beliefs, whence the proliferation of statistical, game-theoretic, and optimization models. Observe also the complexity of tax imposition, mortgages, insurance fees, investment decisions, etc.

As a first example supporting our claim, let us mention again the recent work in fuzzy politics [4], where the authors have successfully employed a non-standard approach to create a more faithful model of party formation in a fuzzy political setting. Its explanatory power derives from a natural revision of existing well-established models according to a necessary and possible logic, which improves widely accepted descriptions of the societal mechanisms of party formation.

To provide further motivating evidence for the usefulness of our approach, below we propose some decision making procedures, which are admittedly not simple to execute

[9]This point of view has already been employed in a recent modelization of the political system by means of fuzzy sets [4]. In fact, the so-called *private and public profiles* (*PaP-profiles*) of politicians, which are the prerequisite for the genesis of parties by the aggregation of similar profiles, are created for each topic of interest in the political campaign according to a bottom-up logic.

but, on the other hand, quite powerful. In terms of computational costs, they appear to be perfectly affordable for the current state of affairs in mathematical software and computers. Quite surprisingly, the fact that the agents may not be fully aware of the complications of the procedure can even be regarded as an advantage: in fact, it can deter them from using forbidden strategic types of behavior (e.g., cheating, since in a simple procedure they can easily observe the benefits of their false statements). At any rate, the agents' concern should be the correct expression of their beliefs, rather than their final implications with respect to decisions. In this respect, the procedures proposed below appear to be well suited even for complicated types of judgements. Notice also that the possible technicalities of the process end up attaching more value to the practicioner.

The two types of semantics of a NaP-HFS discussed in 3.1 ("bottom-up" and "top-down") provide the rationale for a meaningful extension of hesitant fuzzy set theory by a necessary and possible approach. In order to better illustrate the applicability of NaP-HFSs to concrete decision problems, as well as the differences between the two types of semantics, here we present three examples. The first one is concerned with a social scenario where necessary and possible expressions appear quite directly. Then we put forward two additional (more subtle) examples, which illustrate some realistic situations in a corporate setting. These last two examples suggest the adequacy of our modelization in a suitable corporate setting, especially for cases in which a business company has a rather complex hierarchical structure (that is, with a CEO, departments' directors, sub-departments' officers, etc.).

Example 3. The Dean of a School intends to launch one new Degree, which combines the skills of two existing Degrees (say, A and B). A committee has prepared two alternative degree programmes (say, D_1 and D_2). The Dean decides to incorporate opinions about their respective adequacies to the report.

Her corporate sense leads the Dean to seek the advice only from the academic faculty of the Departments with responsibilities in the existing Degrees. In this way, she receives several evaluations (whose internal production we do not describe here, but it could be itself a source of multiplicities). These evaluations are grouped into two categories, namely, those pertaining to faculty involved in Degree A, and those from faculty with responsibilities in Degree B. However, in an attempt to be politically correct, she also collects the feedback of the administrative staff and students of each of the current Degrees A and B, which she clearly separates from the other sources of information. In her report, the Dean insists that the opinion of the faculty must be taken into account, whereas the opinion of the School community is given for guidance. Table 1 below gives a graphical display of the information that the Dean collects.
According to our definitions, we have:
- $h_N^A(D_1) = \{0.7, 0.8\}$, $h_N^B(D_1) = \{0.5, 0.65\}$,
- $h_P^A(D_1) = \{0.7, 0.75, 0.8\}$, $h_P^B(D_1) = \{0.5, 0.6, 0.65\}$,
- $h_N^A(D_2) = \{0.4, 0.6\}$, $h_N^B(D_2) = \{0.4, 0.5, 0.55\}$,
- $h_P^A(D_2) = \{0.4, 0.6, 0.9\}$, $h_P^B(D_2) = \{0.4, 0.5, 0.55, 0.7\}$.

Thus, NaP-HFSs naturally arise in this scenario.

Table 1: A tabular representation of the information in Example 3.

	Current Degree A		Current Degree B	
	Faculty	Faculty+Staff+Students	Faculty	Faculty+Staff+Students
D_1	$\{0.7, 0.8\}$	$\{0.7, 0.75, 0.8\}$	$\{0.5, 0.65\}$	$\{0.5, 0.6, 0.65\}$
D_2	$\{0.4, 0.6\}$	$\{0.4, 0.6, 0.9\}$	$\{0.4, 0.5, 0.55\}$	$\{0.4, 0.5, 0.55, 0.7\}$

Let us emphasize that the process in Example 3 is transparent and simple. The users only need to submit the standard information –the belongingness to the characteristic of being the adequate Degree. It is the expert who obtains the NaP-HFS as the result of a quite natural "information fusion".

The next two examples are respectively devoted to provide instances of the semantics (1) and (2) of our approach, as described in Section 3.1. These examples will be also used to informally sketch two general procedures to prioritize evaluations, which may fit the two different settings. Nonetheless, in the present paper we do not pursue the goal of creating complete decision models that fit these procedures, since this would require a lengthy discussion on setting, motivation, and methodology. Future research on the topic shall be devoted to state and complete these (classes of) procedures in the form of sound algorithms for a decision.

We start with a setting that describes semantics (1) at work.

Example 4. Let $\mathfrak{C} = \{c_1, c_2, c_3\}$ be the executive committee of a corporation, whose members have to express a judgement about the convenience of undertaking some investments within a set X of possible projects. (For instance, c_1, c_2 and c_3 are the directors of three different departments of the corporation, whose evaluation is considered equally important by the CEO.) To that end, each member of $\mathfrak{C}$ is asked to provide membership values for all projects in X: each such value is an estimation of the likelihood that the respective investment will be successful. According to semantics (1), suppose the three members of $\mathfrak{C}$ do have a clear personal view of the membership values of each alternative in X, that is, their *necessary* HFSs $h_N^{c_1}$, $h_N^{c_2}$, $h_N^{c_3}$ are primitive. These necessary membership values may have quite variegated origins: distinct "subjective states of nature" of each member of the committee, suggestions/opinions coming from some key people in each department (e.g., the main officers of the sub-departments coordinated by each director), beliefs associated to different perspectives about the strategic actions undertaken by the CEO, etc. For a specific project $x \in X$, assume that we have, for instance, $h_N^{c_1}(x) = \{0.2, 0.6\}$, $h_N^{c_2}(x) = \{0.4, 0.5, 0.6\}$, and $h_N^{c_3}(x) = \{0.9, 1\}$. In summary, c_1 (or the key people of his department) judges that project x has either a small chance (0.2) or a reasonable chance (0.6) to be successful, c_2 thinks that there is roughly a 50-50 chance of success for project x, and c_3 believes that x is definitively a very good investment.

The global assessment of the committee $\mathfrak{C}$ is obtained by aggregating all points of view, and in this respect the CEO regards an agreement among departments as a rather desirable feature. To that end, the three agents are then asked to "expand their horizons", by additionally providing *possible* HFSs that express their compromise solutions (e.g., asking additional points of view from relevant people of the respective departments, averaging opinions of all of them, etc.). For instance, assume that this expansion process yields $h_P^{c_1}(x) = \{0.2, 0.3, 0.4, 0.5, 0.6\}$, $h_P^{c_2}(x) = \{0.3, 0.4, 0.5, 0.6, 0.7\}$, and $h_P^{c_3}(x) = \{0.9, 1\}$. In other words, c_1 and c_2 display some flexibility in their judgements, whereas c_3 turns out to be fully rigid in his very optimistic evaluation. Indeed, on one hand, c_1 decides to "fill the gap" between the two rather far membership values by some intermediate values, and c_2 slightly enlarges his 50-50 evaluation in both directions; on the other hand, c_3 decides to completely maintain his necessary judgement also at a social level.

At this point, by proceeding in a similar way for every project $y \in X$, the agents c_1, c_2, c_3 submit their respective NaP-HFS $\mathcal{H}_1 = (h_N^{c_1}, h_P^{c_1})$, $\mathcal{H}_2 = (h_N^{c_2}, h_P^{c_2})$, $\mathcal{H}_3 = (h_N^{c_3}, h_P^{c_3})$ over X for the final assessment. In order to get to a final assessment of the likelihood of successfulness of each investment, below we sketch a procedure that determines a "preferred" (by the CEO) set of membership values $V(x)$ for each project $x \in X$. This procedure is inspired by a logic of agreement/majority, however giving a higher weight to personal assessments rather than to social compromises. Obviously, a different attitude of the CEO would make a different evaluation procedure better suited to the task.

For each $k \in \{1, 2, 3\}$, define the k-th level NaP-HFS $\mathcal{H}^{(k)} = \left(h_N^{(k)}, h_P^{(k)}\right)$ over X as follows for each $x \in X$:

$$\begin{aligned} h_N^{(k)}(x) &:= \Big\{a \in [0,1] : a \in \textstyle\bigcap_{c \in \mathfrak{C}'} h_N^c(x) \text{ for some } \mathfrak{C}' \subseteq \mathfrak{C} \text{ such that } |\mathfrak{C}'| = k\Big\} \\ h_P^{(k)}(x) &:= \Big\{a \in [0,1] : a \in \textstyle\bigcap_{c \in \mathfrak{C}'} h_P^c(x) \text{ for some } \mathfrak{C}' \subseteq \mathfrak{C} \text{ such that } |\mathfrak{C}'| = k\Big\}. \end{aligned}$$

Then rank the following eight subsets of $[0, 1]$ in descending order of preferability (where we omit the argument x for the sake of clarity, e.g., $h_N^{(3)}$ stands for $h_N^{(3)}(x)$):

$$(1)\, h_N^{(3)},\ (2)\, h_N^{(2)} \cap h_P^{(3)},\ (3)\, h_N^{(2)},\ (4)\, h_N^{(1)} \cap h_P^{(2)},\ (5)\, h_N^{(1)},\ (6)\, h_P^{(3)},\ (7)\, h_P^{(2)},\ (8)\, h_P^{(1)}.$$

Finally, select the first nonempty set as output $V_j(x)$, where the index $j \in \{1, 2, \ldots, 8\}$ represents the stage at which the preferential evaluation is made (hence the higher the integer, the lower the "strength" of the evaluation). Going back to our numerical example, one can check that $h_N^{(3)}(x) = h_P^{(3)}(x) = \emptyset$ and $h_N^{(2)}(x) = \{0.6\}$. It follows that the final assessment on x is $V(x) = V_3(x) = h_N^{(2)}(x) = \{0.6\}$.

Example 4 employs *typical* NaP-HFSs to describe a corporate setting that fits semantics (1). The next example illustrates semantics (2) of our necessary and possible approach. However, also in order to display the flexibility of our methodology, we shall instead employ *non-typical* NaP-HFSs, in which both components are interval-valued HFSs.

Example 5. In the same setting as the previous example, assume that the corporation's CEO provides a cautious estimation of the successfulness of all projects in X, thus giving a large range of membership values for each possible investment. Specifically, the CEO establishes a primitive interval-valued HFS $h^{\mathfrak{C}}$, which acts as a sort of "corporate guidelines" for the evaluations to be given by the members of the executive committee $\mathfrak{C}$. Therefore, according to semantics (2), all members of $\mathfrak{C}$ have to abide by these guidelines.

For the sake of concreteness, suppose that for a fixed project $y \in X$, the CEO has decided that the membership values ought to be located within the interval $h^{\mathfrak{C}}(y) = [0.5, 1]$, because he believes that the investment has rather good chances of being successful. Further, let us assume that the three members of the committee $\mathfrak{C}$ select, respectively, $h_P^{c_1}(y) = [0.5, 0.7]$, $h_P^{c_2}(y) = [0.6, 0.9]$, and $h_P^{c_2}(y) = [0.75, 1]$ as those subintervals of $h^{\mathfrak{C}}(y) = [0.5, 1]$ that they deem *possible*. In other words, c_1 is not so optimistic (but he cannot go below 0.5, because the CEO does not allow that), c_2 basically agrees with the CEO's evaluation and simply restricts his range of possible membership values, and finally c_3 is quite confident that the investment will be successful. However, the three profiles $h_P^{c_1}(y)$, $h_P^{c_2}(y)$, and $h_P^{c_3}(y)$ only express the compromise solution of each agent. That is the reason why the CEO also requires the three members of $\mathfrak{C}$ to describe their inner beliefs (within the given boundaries imposed by him) in the form of three interval-valued *necessary* HFSs, which are sharper than their respective possible HFSs. For istance, suppose that, according to their personal attitudes described above, they provide $h_N^{c_1}(y) = [0.5, 0.55]$, $h_N^{c_2}(y) = [0.75, 0.8]$, and $h_N^{c_3}(y) = [0.85, 0.9]$.

At this point, by proceeding in a similar way for every project $x \in X$, the agents c_1, c_2, c_3 submit their respective $\mathsf{NaP\text{-}HFS}$s $\mathcal{H}_1 = (h_N^{c_1}, h_P^{c_1})$, $\mathcal{H}_2 = (h_N^{c_2}, h_P^{c_2})$, $\mathcal{H}_3 = (h_N^{c_3}, h_P^{c_3})$ over X for the final assessment. Again, for the sake of illustrating the flexibility of our necessary and possible approach, below we informally sketch a procedure that establishes a preferred set of evaluations.

Assume that the CEO decides to assess the membership values of each investment according to a logic that regards similar points of view among the members of $\mathfrak{C}$ as being a winning strategy. To that end, first the CEO determines the "level or agreement" among the three members of $\mathfrak{C}$, by using both components of their $\mathsf{NaP\text{-}HFS}$ as follows: declare that c_i *agrees with* c_j on option y whenever $h_N^{c_i}(y) \subseteq h_P^{c_j}(y)$ and $h_N^{c_j}(y) \subseteq h_P^{c_i}(y)$. In other words, the CEO regards an agreement between c_i and c_j to be in place whenever the personal evaluation of c_i lies within the social evaluation of c_j, and, vice versa, the personal evaluation of c_j lies within the social evaluation of c_i. These comparisons of evaluations on each project $y \in X$ yield an *adjacency graph* associated to y, having the members of $\mathfrak{C}$ as vertices and their agreement on y (if any) as edges. Finally, the CEO determines a finite set of numbers (possibly one), obtained by averaging the necessary judgements of the members of $\mathfrak{C}$ that belong to the longest *paths* (i.e., connected chains of vertices) in the graph. These numbers shall represent the preferred values of the successfulness likelihood of each project.[10]

In our example, the only agreement on the evaluation of y is between c_2 and c_3, as one can easily check. Therefore, the longest path in the graph associated to y is unique, and

[10] Exactly the same logic is employed in [4] to describe aggregation of politicians according to their personal and social views (cf. Footnote 9). Specifically, a *matching graph* of all of the candidates involved in a political campaign is obtained as follows: a politician x "agrees" with another politician y if both (i) the private profile of x is contained in the public profile of y, and (ii) the private profile of y is contained in the public profile of x. The matching graph is the starting point for an algorithm that describes the creation of political parties in our fuzzy setting.

is given by the sequence of vertices c_2c_3. As a consequence, the CEO takes the average of the two necessary HFS (evaluated at y) $h_N^{c_2}(y) = [0.75, 0.8]$ and $h_N^{c_3}(y) = [0.85, 0.9]$, obtaining the unique value 0.825: this will be the successfulness likelihood of project y, as derived from the evaluations of the committee $\mathfrak{C}$, subject to the constraints/preferences displayed by the top management.

3.3. Basic operations on NaP-HFSs

We extend to NaP-HFSs some standard operators defined for HFSs (see Section 2.2).

Definition 12. For each NaP-HFS $\mathcal{H} = \big\{(x, h_N(x), h_P(x)) : x \in X\big\}$ over X, we define the following "derived" NaP-HFSs over X:

1. the *lower bound* of $\mathcal{H}$ is the pair $\mathcal{H}^- = (h_N^-, \langle h_N|h_P\rangle^-)$, where h_N^- denotes the lower bound of the necessary HFS, and $\langle h_N|h_P\rangle^-$ is the HFS defined as follows for each $x \in X$:

$$\langle h_N|h_P\rangle^-(x) \;:=\; \big\{\inf\{h_N(x) : x \in X\}, \inf\{h_P(x) : x \in X\}\big\};$$

2. the *upper bound* of $\mathcal{H}$ is the pair $\mathcal{H}^+ = (h_N^+, \langle h_N|h_P\rangle^+)$, where h_N^+ denotes the upper bound of the necessary HFS, and $\langle h_N|h_P\rangle^+$ is the HFS defined as follows for each $x \in X$:

$$\langle h_N|h_P\rangle^+(x) \;:=\; \big\{\sup\{h_N(x) : x \in X\}, \sup\{h_P(x) : x \in X\}\big\};$$

3. for any $\alpha \in [0, 1]$, the *α-lower bound* of $\mathcal{H}$ is the pair $\mathcal{H}_\alpha^- = ((h_N)_\alpha^-, (h_P)_\alpha^-)$, where $(h_N)_\alpha^-$ (resp. $(h_P)_\alpha^-$) denotes the α-lower bound of the necessary (resp. possible) HFS;

4. for any $\alpha \in [0, 1]$, the *α-upper bound* of $\mathcal{H}$ is the pair $\mathcal{H}_\alpha^+ = ((h_N)_\alpha^+, (h_P)_\alpha^+)$, where $(h_N)_\alpha^+$ (resp. $(h_P)_\alpha^+$) denotes the α-upper bound of the necessary (resp. possible) HFS;

5. the *complement* of $\mathcal{H}$ is the pair $\mathcal{H}^c = ((h_N)^c, (h_P)^c)$, where $(h_N)^c$ (resp. $(h_P)^c$) denotes the complement of the necessary (resp. possible) HFS.

In addition, the *suplement* of $\mathcal{H} = \big\{(x, h_N(x), h_P(x)) : x \in X\big\}$ is the pair $\mathcal{H}^s = \big(\overline{h_P}, \overline{h_N}\big)$, where the bar denotes set-theoretic complement in the interval $[0, 1]$: thus, for each $x \in X$, we have $\overline{h_P}(x) = [0, 1] \setminus h_P(x)$ and $\overline{h_N}(x) = [0, 1] \setminus h_N(x)$.

As is Definition 4, there is a remarkable asymmetry between upper/lower bounds and α-upper/lower bound. In HFSs, the former are singletons, whereas the latter are subsets. Here, since we wish to retain this ethos together with the necessary and possible spirit, we adapt the definitions of upper and lower bounds in order to still obtain NaP-HFSs.

The following example exhibits some simple instances of the above notions.

Example 6. Let $\mathcal{H} = \{(x, \{0.2, 0.6\}, \{0.2, 0.3, 0.4, 0.5, 0.6\}), (y, \{0.9\}, \{0.8, 0.9, 1\})\}$ be the NaP-HFS already examined in Example 2. According to Definition 12, we can compute the following derived NaP-HFSs.

1. The lower bound of $\mathcal{H}$: this is the NaP-HFS defined as

$$\mathcal{H}^- = \big\{(x, \{0.2\}, \{0.2\}), (y, \{0.9\}, \{0.8, 0.9\})\big\}.$$

2. The upper bound of $\mathcal{H}$: this is the NaP-HFS defined as

$$\mathcal{H}^+ = \big\{(x, \{0.6\}, \{0.6\}), (y, \{0.9\}, \{0.9, 1\})\big\}.$$

3. For $\alpha = 0.3$, the α-lower bound of $\mathcal{H}$: this is the (improper) NaP-HFS defined as

$$\mathcal{H}^-_{0.3} = \big\{(x, \{0.2\}, \{0.2, 0.3\}), (y, \emptyset, \emptyset)\big\}.$$

4. For $\alpha = 0.6$, the α-upper bound of $\mathcal{H}$: this is the NaP-HFS defined as

$$\mathcal{H}^+_{0.6} = \big\{(x, \{0.6\}, \{0.6\}), (y, \{0.9\}, \{0.8, 0.9, 1\})\big\}.$$

5. The complement of $\mathcal{H}$: this is the NaP-HFS defined as

$$\mathcal{H}^c = \big\{(x, \{0.4, 0.8\}, \{0.4, 0.5, 0.6, 0.7, 0.8\}), (y, \{0.1\}, \{0, 0.1, 0.2\})\big\}.$$

3.4. Set-theoretic operations for NaP-HFSs

Now we proceed to define some operations for NaP-HFSs. Two of them, discrete union and intersection, extend the corresponding notions for HFSs introduced in [46, 47]. Another two are the indiscrete forms of union and intersections. The last one, called "merge", is peculiar to our approach.

Definition 13. Let $\mathcal{H} = \big\{(x, h_N(x), h_P(x)) : x \in X\big\}$ and $\mathcal{H}' = \big\{(x, h'_N(x), h'_P(x)) : x \in X\big\}$ be NaP-HFSs over X.

1. The *indiscrete union* $\mathcal{H} \vee \mathcal{H}'$ of $\mathcal{H}$ and $\mathcal{H}'$ is defined by $\mathcal{H} \vee \mathcal{H}' = (h_N \vee h'_N, h_P \vee h'_P)$.
2. The *discrete union* $\mathcal{H} \cup \mathcal{H}'$ of $\mathcal{H}$ and $\mathcal{H}'$ is defined by $\mathcal{H} \cup \mathcal{H}' = (h_N \cup h'_N, h_P \cup h'_P)$.
3. The *indiscrete intersection* $\mathcal{H} \wedge \mathcal{H}'$ of $\mathcal{H}$ and $\mathcal{H}'$ is defined by $\mathcal{H} \wedge \mathcal{H}' = (h_N \wedge h'_N, h_P \wedge h'_P)$.
4. The *discrete intersection* $\mathcal{H} \cap \mathcal{H}'$ of $\mathcal{H}$ and $\mathcal{H}'$ is defined by $\mathcal{H} \cap \mathcal{H}' = (h_N \cap h'_N, h_P \cap h'_P)$.
5. The *merge* $\mathcal{H} \circlearrowleft \mathcal{H}'$ of $\mathcal{H}$ and $\mathcal{H}'$ is defined by $\mathcal{H} \circlearrowleft \mathcal{H}' = (h_N \wedge h'_N, h_P \vee h'_P)$.

The above definitions are extended to arbitrary families of NaP-HFSs in an obvious way.

The notions of (discrete and indiscrete) union and intersection require no further explanation. On the contrary, the notion of merge needs some clarification, since it is motivated by the employment of two different principles in its two components. Indeed, the merge of a family of NaP-HFSs is inspired by a logic of "caution" (modeled by intersection/minimality) for the core part of the membership values, and by one of "tolerance" (modeled by union/maximality) for the socially acceptable membership values.[11]

Next, we show that the notions in Definition 13 are well-defined:

Lemma 1. *For any two NaP-HFSs over X, their indiscrete union and intersection, their discrete union and intersection, and their merge are proper NaP-HFSs over X.*

Proof. The result is immediate for indiscrete union and intersection, as well as for merge.

For discrete union, let $\mathcal{H} = (h_N, h_P)$ and $\mathcal{H}' = (h'_N, h'_P)$ be NaP-HFSs over X. Since $h_P \cup h'_P$ is a proper HFS, to prove the claim it suffices to show that the inclusion $(h_N \cup h'_N)(x) \subseteq (h_P \cup h'_P)(x)$ holds for any $x \in X$. Thus, fix $x \in X$, and pick an arbitrary $e \in (h_N \cup h'_N)(x)$. The hypothesis readily yields $e \in h_N(x) \cup h'_N(x) \subseteq h_P(x) \cup h'_P(x)$. Further, since by definition of discrete union we have $e \geqslant \max\{(h_N)^-(x), (h'_N)^-(x)\}$, and the latter is equivalent to the join of $e \geqslant \inf\{h_N(x) : x \in X\}$ and $e \geqslant \inf\{h'_N(x) : x \in X\}$, we deduce $e \geqslant \inf\{h_P(x) : x \in X\}$ plus $e \geqslant \inf\{h'_P(x) : x \in X\}$. In conclusion, $e \in (h_P \cup h'_P)(x)$ by the definition of discrete union.

The argument for discrete intersection is similar to that given for discrete union. □

As clarified in Sections 3.1 and 3.2, the main rationale of a necessary and possible approach to hesitant fuzzy set theory is that it allows one to naturally deal with decision making in a social/collective setting. Thus, it does not come as a surprise that the next definition introduces some notions that aim at capturing the inner coherence of a family of NaP-HFSs over a common set of alternatives.

Definition 14. Let $\mathcal{H} = (h_N, h_P)$ and $\mathcal{H}' = (h'_N, h'_P)$ be two proper NaP-HFSs over X. We say that $\mathcal{H}$ and $\mathcal{H}'$ are *weakly compatible* if the inclusions $h_N(x) \subseteq h'_P(x)$ and $h'_N(x) \subseteq h_P(x)$ hold for all $x \in X$. If, in addition, we also have $h_N(x) \cap h'_N(x) \neq \emptyset$ for all $x \in X$, then we say that $\mathcal{H}$ and $\mathcal{H}'$ are *strongly compatible*.

More generally, we have:

Definition 15. Let $\mathbb{H}$ be a nonempty family of proper NaP-HFSs over X. We say that $\mathbb{H}$ is a *weakly compatible family* if any two elements in it are weakly compatible. If, in addition, we also have $\bigcap_{(h_N,h_P)\in\mathbb{H}} h_N(x) \neq \emptyset$ for all $x \in X$, then we say that $\mathbb{H}$ is a *strongly compatible family*.

Obviously, two proper NaP-HFSs $\mathcal{H}$ and $\mathcal{H}'$ over X are weakly (resp. strongly) compatible if and only if $\{\mathcal{H}, \mathcal{H}'\}$ is a weakly (resp. strongly) compatible family. Notice that Definition 15 is asymmetric for the conditions of weak and strong compatibility of a family

[11] The underlying logic of the merge has been employed in [4] for the creation of the profiles of parties from the profiles of its members (cf. Footnotes 9 and 10). Specifically, for each topic of the political campaign, the PaP-profiles of the politicians who belong to a party are aggregated according to a logic of caution (i.e., minimal) for their private profiles, and one of tolerance (i.e., maximal) for their public profiles: this procedure of aggregation gives rise to the *bi-profile* of a party on each topic.

of NaP-HFSs over X: indeed, a family $\mathbb{H}$ is weakly compatible if and only if so are any two elements in it, but the same does not hold for strong compatibility. The next example exhibits a weakly (but not strongly) compatible family such that every two elements in it are strongly compatible.

Example 7. Assume that the three departments D_1, D_2, D_3 of a corporation are required to provide an evaluation of successfulness for the two projects in the set of investments $X = \{x, y\}$. The evaluations of the three departments are collected into a family $\mathbb{H} = \{\mathcal{H}^{D_1}, \mathcal{H}^{D_2}, \mathcal{H}^{D_3}\}$ of NaP-HFSs over X, where the pairs $\mathcal{H}^{D_1} = \left(h_N^{D_1}, h_P^{D_1}\right)$, $\mathcal{H}^{D_2} = \left(h_N^{D_2}, h_P^{D_2}\right)$, $\mathcal{H}^{D_3} = \left(h_N^{D_3}, h_P^{D_3}\right)$ represent the necessary and possible membership values assigned to the projects in X by D_1, D_2, D_3, respectively. Specifically, for each $i = 1, 2, 3$, the necessary HFS $h_N^{D_i}$ collects all values individually assigned by the members of the board of directors of department D_i, whereas the possible HFS $h_P^{D_i}$ is an acceptable compromise solution that is collectively reached by all top managers of department D_i (which of course must respect the evaluations of the directors of D_i). Assume that the following evaluations are given:

$$h_N^{D_1}(x) = \{0.3, 0.5, 0.6\}, \; h_N^{D_1}(y) = \{0.8, 0.9\}, \qquad h_P^{D_1}(x) = [0.2, 0.8], \; h_P^{D_1}(y) = [0.6, 1],$$

$$h_N^{D_2}(x) = \{0.4, 0.5\}, \qquad h_N^{D_2}(y) = \{0.7, 0.8, 0.9\}, \; h_P^{D_2}(x) = [0.1, 0.7], \; h_P^{D_2}(y) = [0.5, 0.9],$$

$$h_N^{D_3}(x) = \{0.2, 0.4, 0.6\}, \; h_N^{D_3}(y) = \{0.6, 0.7, 1\}, \quad h_P^{D_3}(x) = [0, 0.7], \quad h_P^{D_3}(y) = [0.5, 1].$$

One can readily check that the family $\mathbb{H}$ is weakly compatible. Furthermore, any two elements of $\mathbb{H}$ are strongly compatible (in the sense of Definition 14), and yet $\mathbb{H}$ fails to be a strongly compatible family (in the sense of Definition 15).

Example 7 shows that a family $\mathbb{H}$ of NaP-HFSs may not be strongly compatible although any two elements in it display a strong compatibility. However, for the types of NaP-HFSs described below, the property of strong compatibility extends from the binary case to an arbitrary one.

Definition 16. A proper NaP-HFS $\mathcal{H} = (h_N, h_P)$ over X is *convex* (resp. *compact*) if $h_N(x)$ is a nonempty convex (resp. compact) subset of $[0, 1]$ for each $x \in X$; equivalently, if its necessary component h_N maps all elements of X to nonempty convex (resp. compact) subsets of $[0, 1]$.

Obviously, a convex compact NaP-HFS over X maps each element of X to a nonempty closed subinterval of $[0, 1]$. Then, we have:

Theorem 1. *A nonempty family of convex compact NaP-HFSs over X is strongly compatible if and only if so are any two of its elements.*

Proof. Necessity is obvious. For sufficiency, let $\mathbb{H}$ be a nonempty family of convex compact NaP-HFSs over X, and assume that any two elements of $\mathbb{H}$ are strongly compatible. By Definitions 14 and 15, this means that, for any $\mathcal{H} = (h_N, h_P), \mathcal{H}' = (h'_N, h'_P) \in \mathbb{H}$, we have that $\mathcal{H}$ and $\mathcal{H}'$ are weakly compatible, and $h_N(x) \cap h'_N(x) \neq \emptyset$ for all $x \in X$. To prove that the $\mathbb{H}$ is a strongly compatible family, we need show that $\bigcap_{(h_N,h_P)\in\mathbb{H}} h_N(x) \neq \emptyset$ holds for all $x \in X$. Fix $x \in X$. We prove the claim in the following two steps:

(i) for each nonempty $\mathbb{K} \subseteq \mathbb{H}$ of finite size, $\bigcap_{(k_N,k_P)\in\mathbb{K}} k_N(x) \neq \emptyset$;

(ii) if (i) holds, then $\bigcap_{(h_N,h_P)\in\mathbb{H}} h_N(x) \neq \emptyset$.

For part (i), apply Lemma 2 stated below for the linearly ordered set $(L, \prec) = ([0, 1], <)$. Part (ii) readily follows from part (i) by Tychonoff theorem. This completes the proof of the theorem. □

The technical result needed to derive (i) in the proof of Theorem 1 is the following:

Lemma 2. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a family of $n \geqslant 2$ convex subsets of a linearly ordered set $(L, \prec)$.*[12] *If any two elements of $\mathcal{A}$ intersect, then $\bigcap \mathcal{A} \neq \emptyset$.*

Proof. The claim holds trivially for $n = 2$. Next, let $\mathcal{A} = \{A_1, A_2, A_3\}$ be a family of three nonempty convex subsets of $(L, \prec)$. Toward a contradiction, assume that we have

$$x \in A_1 \cap A_2 \neq \emptyset\,, \quad y \in A_1 \cap A_3 \neq \emptyset\,, \quad A_2 \cap A_3 \neq \emptyset\,, \quad A_1 \cap A_2 \cap A_3 = \emptyset\,.$$

It follows that $x \notin A_3$ and $y \notin A_2$. Without loss of generality, suppose $x \prec y$. Since A_3 is convex and $x \notin A_3$, we have $x \prec A_3$ (which means that $x \prec a$ for all $a \in A_3$). Similarly, we obtain $A_2 \prec y$ (which means that $b \prec y$ for all $b \in A_2$). Pick $z \in A_2 \cap A_3$. By what we have just proved, we get $x \prec z \prec y$. However, A_1 is convex, hence $x, y \in A_1$ implies $z \in A_1$. Thus, we can conclude $z \in A_1 \cap A_2 \cap A_3$, a contradiction. This proves that the claim holds for $n = 3$.

Proceeding by induction on n, assume that the claim holds for any family of pairwise intersecting convex subsets of $(L, \prec)$ having size n. We shall prove it for any such family of size $n + 1$. Let $\mathcal{A} = \{A_1, \ldots, A_{n+1}\}$ be a family of convex subsets of $(L, \prec)$ such that $A_i \cap A_j \neq \emptyset$ for all $i, j \in \{1, \ldots, n + 1\}$. For each $i = 1, \ldots, n$, set $B_i := A_i \cap A_{n+1}$.

Claim: *$\mathcal{B} = \{B_1, \ldots, B_n\}$ is a family of nonempty convex pairwise intersecting subsets of $(L, \prec)$.*

Proof of Claim. First of all, notice that each B_i is nonempty by hypothesis. Further, it is not difficult to show that each B_i is convex as well. Finally, to prove that $B_i \cap B_j \neq \emptyset$ for all $i, j \in \{1, \ldots, n\}$, simply observe that $B_i \cap B_j = A_i \cap A_j \cap A_{n+1}$, hence the result follows from what proved in the first paragraph for the case $n = 3$. This completes the proof of the claim. □

Now the Claim allows us to apply the induction hypothesis to the family $\mathcal{B}$, and conclude $\bigcap \mathcal{B} \neq \emptyset$. Since $\bigcap \mathcal{A} = \bigcap \mathcal{B}$, this proves that the lemma holds. □

[12] Recall that a *linearly ordered set* is a pair $(L, \prec)$ such that L is a nonempty set, and $\prec$ is a binary relation on L that is asymmetric, transitive, and complete. (The typical linear order is the real line $(\mathbb{R}, <)$.) A set $A \subseteq L$ is *convex* whenever, for all $x, y, z \in L$ such that $x \prec y \prec z$, if $x, z \in A$ then $y \in A$.

3.5. Aggregation of NaP-hesitant fuzzy elements

Aggregation operators for hesitant fuzzy elements enable us to produce aggregation of NaP-HFEs in a natural way. Thus, we propose the following general notion:

Definition 17. For every Ψ aggregation operator for HFEs, we define an *aggregation operator for NaP-hesitant fuzzy elements* by the expression

$$\Phi_\Psi((E_N^1, E_P^1), \ldots, (E_N^n, E_P^n)) = (\Psi(E_N^1, \ldots, E_N^n), \Psi(E_P^1, \ldots, E_P^n))$$

for any $n = 1, 2, \ldots$ and any list $(E_N^1, E_P^1), \ldots, (E_N^n, E_P^n)$ of n NaP-HFEs.

Observe that the properties of the aggregation operators for HFEs ensure that Definition 17 produces a NaP-HFE for each (E_N^i, E_P^i). Similarly, any aggregation operator for THFEs defines an aggregation operator for NaP-THFEs by the analogous expression.

Definition 18. Noteworthy aggregation operators for NaP-THFEs include:

(i) the *NaP-hesitant fuzzy averaging (NHFA)* operator, defined by

$$\text{NHFA}((E_N^1, E_P^1), \ldots, (E_N^n, E_P^n)) = (\text{HFA}(E_N^1, \ldots, E_N^n), \text{HFA}(E_P^1, \ldots, E_P^n))$$

i.e., Ψ = HFA in Definition 17;

(ii) the *NaP-hesitant fuzzy geometric (NHFG)* operator, defined by

$$\text{NHFG}((E_N^1, E_P^1), \ldots, (E_N^n, E_P^n)) = (\text{HFG}(E_N^1, \ldots, E_N^n), \text{HFG}(E_P^1, \ldots, E_P^n))$$

i.e., Ψ = HFG in Definition 17;

(iii) the *NaP-hesitant fuzzy weighted averaging (NHFWA)* and *geometric (NHFWG)* operators, which extend (i) and (ii) by using HFWA and HFWG, respectively.

NHFA and NHFG verify the obvious anonymity property, whereas NHFWA and NHFWG are their non-anonymous extensions.

The following example exhibits some simple instances of the above notions.

Example 8. Let $(\{0.2, 0.6\}, \{0.2, 0.3, 0.6\})$ and $(\{0.9\}, \{0.9, 1\})$ be NaP-HFEs. According to Definition 18, we can compute the following aggregate NaP-HFEs:

1. $\text{NHFA}((\{0.2, 0.6\}, \{0.2, 0.3, 0.6\}), (\{0.9\}, \{0.9, 1\})) =$
   $= (\text{HFA}(\{0.2, 0.6\}, \{0.9\}), \text{HFA}(\{0.2, 0.3, 0.6\}, \{0.9, 1\})) =$
   $= (\{1-\sqrt{0.08}, 0.8\}, \{1-\sqrt{0.08}, 1-\sqrt{0.07}, 0.8, 1\}) \approx (\{0.717, 0.8\}, \{0.717, 0.735, 0.8, 1\});$

2. $\text{NHFG}((\{0.2, 0.6\}, \{0.2, 0.3, 0.6\}), (\{0.9\}, \{0.9, 1\})) =$
   $= (\text{HFG}(\{0.2, 0.6\}, \{0.9\}), \text{HFG}(\{0.2, 0.3, 0.6\}, \{0.9, 1\})) =$
   $\approx (\{0.424, 0.735\}, \{0.424, 0.447, 0.520, 0.548, 0.735, 0.775\}).$

4. Individual and group decision making procedures

In this section we use the tools developed in the previous sections to describe a flexible decision making procedure for alternatives defined by typical NaP-HFSs. In a first step of the construction, we develop our idea in the individual setting and give an illustrative example; successively, we compare our proposal with other possible approaches, which are inspired by decision making with hesitant fuzzy sets. In a second step, we adapt the mechanism in a way that it can be applied to a society with a finite number of agents.

Be it an individual or a societal context, our decision making problem consists of selecting one among a finite number of alternatives. These alternatives are evaluated by means of a finite number of features. A NaP-THFE characterizes (a) to which extent each alternative verifies each characteristic, or (b) to which extent each alternative should be selected according to one of the agents' opinion. Interpretation (a) concerns a multi-criteria decision making formulation of the problem, whereas interpretation (b) refers to a multi-agent decision making formulation.

The data of our problem can be displayed in tabular form. Rows are associated to alternatives, and columns are associated to agents, properties, or attributes: we shall refer to it as a *NaP-hesitant fuzzy decision table*. Each cell contains a NaP-THFE, which captures how the corresponding alternative verifies the corresponding property (in the multi-criteria decision making formulation), or whether it should be selected according to the opinion of the corresponding agent (in the multi-agent decision making formulation). Notice that we have already used this natural representation of the informational input in Example 3 (see Table 1).

The discussion about our model assures that this decision making problem is an extension of the well-known decision making problem in which alternatives are described by typical HFSs. The latter model has been investigated by several authors: see, e.g., Alcantud and de Andrés [2], Alcantud *et al.* [3], Farhadinia [17], Xia and Xu [50], and Xu and Xia [52].

Algorithm 1 below explains our proposal for solving this decision making problem. In words, it first computes the aggregate of the information submitted by each agent, or associated with each attribute. This process results into a NaP-THFE describing each alternative. Then a global score is computed for each alternative: this score is defined as the average of the scores associated with each necessary and possible THFE of the aggregate information. Such average depends on a "balance factor", which weights the importance of the core/individual beliefs vs. the expanded/socially acceptable beliefs. Finally, the alternatives are ranked by their average scores: score-maximizer options are the optimal ones.

Notice that the presence of three types of variables (aggregation operator, score, and balance factor) makes the proposed algorithm flexible and adaptable to the needs of practitioners. As observed below in this section, our algorithm incorporates well-known mechanisms for decision making, which are employed in cases when the alternatives are characterized by standard (typical) HFSs.

Algorithm 1: Decision making with alternatives identified by NaP-HFSs

A priori elements: (I) aggregation operator on NaP-THFEs, (II) score on THFEs, and (III) balance factor $\alpha \in [0,1]$.

1: Input a NaP-hesitant fuzzy decision table. (Alternatively, input k typical NaP-HFSs, one for each of the k alternatives $x_1, \ldots, x_k$ that must be ranked.) There are l parameters. In tabular form, t_{ij} denotes cell (i,j), which is the NaP-THFE associated with alternative x_i and attribute $j \in \{1, \ldots, l\}$.

2: Calculate the aggregate NaP-THFE associated with each of the k alternatives by the selected aggregation operator on NaP-THFEs.

3: Calculate the score of each necessary and possible THFE of these aggregate NaP-THFEs. At this point, for each $i = 1, \ldots, k$, alternative x_i is associated with two scores s^i_N and s^i_P.

4: For $i = 1, \ldots, k$, compute $\alpha_i = \alpha s^i_N + (1-\alpha) s^i_P$.

5: The result of the decision is any alternative x_j such that $\alpha_j = \max_{i=1,\ldots,k} \alpha_i$.

4.1. A numerical example

The following illustrative example describes step by step how to apply Algorithm 1. This example is designed especially to show how the balance factor –which basically models the attitude of the decision maker in weighting strictly individual values against social evaluations of tolerance– may affect the final selection.

Example 9. Let $\mathfrak{C} = \{c_1, c_2\}$ be a set of two agents who express a judgement about the convenience of undertaking some investments within a set $X = \{x_1, x_2\}$ of possible projects. Assume that c_1 and c_2 submit their respective NaP-HFS $\mathcal{H}_1 = (h^{c_1}_N, h^{c_1}_P)$ and $\mathcal{H}_2 = (h^{c_2}_N, h^{c_2}_P)$ over X for the final decision. These evaluations are captured by Table 2, which is a NaP-hesitant fuzzy decision table. Below we solve the multi-agent decision problem by a routine application of Algorithm 1.

(i) Suppose that we select NHFA as an aggregation operator, and Xia and Xu's score. Table 3 shows the corresponding computations of the elements that are needed in order to apply Algorithm 1. It is apparent that option x_1 will be chosen over x_2 irrespective of the value of the balance factor, because $s^1_N > s^2_N$ and $s^1_P > s^2_P$ imply $\alpha_1 > \alpha_2$ for each $\alpha \in [0,1]$.

(ii) Now suppose that we select instead NHFG as an aggregation operator, whereas the chosen score function is still Xia and Xu's. Table 4 displays the corresponding computations. In this case the choice of the balance factor is far from being irrelevant. Indeed, for $\alpha = 0.20$, we can conclude that x_1 should be selected, because

$$\alpha_1 = \alpha\, 0.579 + (1-\alpha)\, 0.575 = 0.5758 > 0.5662 = \alpha\, 0.583 + (1-\alpha)\, 0.562 = \alpha_2.$$

On the contrary, for $\alpha = 0.85$, it is x_2 that should be selected, since now we have

$$\alpha_1 = \alpha\, 0.579 + (1-\alpha)\, 0.575 = 0.5786 < 0.5809 = \alpha\, 0.583 + (1-\alpha)\, 0.562 = \alpha_2.$$

Table 2: A representation of the NaP-HFSs associated with Example 9.

	Agent 1	Agent 2
x_1	({0.2, 0.6}, {0.2, 0.3, 0.6})	({0.9}, {0.9, 1})
x_2	({0.4}, {0.3, 0.4, 0.5})	({0.8, 0.9}, {0.7, 0.8, 0.9})

Table 3: Aggregate NaP-HFEs and scores associated with Example 9. The aggregation operator is NHFA. The score is Xia and Xu's. Values are rounded off to the three decimal digits.

		Xia-Xu score	
	Aggregate NaP-HFE by NHFA	Necessary HFE	Possible HFE
x_1	({0.717, 0.8}, {0.717, 0.735, 0.8, 1})	$s_N^1 = 0.758$	$s_P^1 = 0.813$
x_2	({0.654,0.755}, {0.542, 0.576, 0.613, 0.626, 0.654, 0.684, 0.735, 0.755, 0.776})	$s_N^2 = 0.704$	$s_P^2 = 0.662$

Table 4: Aggregate NaP-HFEs and scores associated with Example 9. The aggregation operator is NHFG. The score is Xia and Xu's. Values are rounded off to the three decimal places.

		Xia-Xu score	
	Aggregate NaP-HFE by NHFG	Necessary HFE	Possible HFE
x_1	({0.424,0.735}, {0.424,0.447,0.520,0.548,0.735,0.775})	$s_N^1 = 0.579$	$s_P^1 = 0.575$
x_2	({0.566, 0.6}, {0.458, 0.490, 0.520, 0.529, 0.566, 0.592, 0.6, 0.632, 0.671})	$s_N^2 = 0.583$	$s_P^2 = 0.562$

4.2. Comparative analysis

It is worth noticing that the adjustable decision procedure described above heavily relies on the fact that all agents' opinions are characterized by suitable pairs of typical HFSs (instead of a single HFS). However, we can also apply it to cases where (some of) the alternatives are characterized by standard THFEs. The argument goes as follows.

Algorithm 1 uses a balance factor α that affects the conclusion as to which alternatives can be chosen. The value of this factor is related to the decision maker's attitude towards the value of the private or social information provided by agents. The balance factor ranges over the interval $[0, 1]$. In particular, when α is 0 (resp. 1), Algorithm 1 discards the information on the necessary (resp. possible) THFE. In other words, in extreme circumstances, we only need the either the possible or the necessary THFE to reach a decision. In fact, in these cases the algorithm results into known procedures based on scores of HFEs (cf. Xia and Xu [50], as described in [3, Table 1]). As a consequence, our algorithm ends up extending Xia and Xu's proposal for making decisions about alternatives defined by typical HFSs to the more general case of alternatives defined by typical NaP-HFSs.

Put briefly, if our input is in the form of a typical HFS associated with each alternative, then we can replicate the information in order to get one typical NaP-HFS for each alternative. We can now apply Algorithm 1 to this setting. The results of the computations coincide with Xia and Xu's computations for the original information.

The level of sophistication expected from users determines the complexity of the solution to our decision making problem. Even if we keep the fundamental structure of our algorithm, we can incorporate more information –such as weights of the criteria or their degrees of importance– through the aggregation operator in the second step of the procedure. To that end, we just need to refine the aggregators of HFEs mentioned in Section 2.3 by taking inspiration from, e.g., Liao and Xu [35].

However, there are other approaches whose structure is rather different from this general procedure. For instance, HF-VIKOR is an approach that ascribes to the rationale of compromise solutions (see Yu [54]). Yu's idea intends to overcome the difficulty of finding an alternative that is optimal for all the criteria (see also Liao and Xu [34]).

A different category of solutions pertain to the so-called *outranking approach*, the most popular of which is probably the ELECTRE approach (Roy [42]). Some variations of ELECTRE have been adapted to the hesitant context by Liao and Xu [37, Sections 4.2.2,4.2.3]. However, these adaptations consist of a large number of steps (up to eleven in the case of HF-ELECTRE I and II), some of which are rather involved.

Hesitant preference relations (Liao and Xu [36]) and interval-valued hesitant preference relations (Chen, Xu, and Xia [15]) provide an additional degree of sophistication. They are at the core of automatic [37, Section 6.3.1] or interactive [37, section 6.3.2] consensus reaching processes, and group decision making algorithms [37, Section 6.4.3], respectively. In comparison with our approach, these types of situations are informationally more demanding, and computationally more costly.

4.3. Extension to a multi-person setting

In this section we explain how our Algorithm can be adapted to a multi-person setting with m agents. The natural step to be added is the construction of a collective NaP-HFS from the m individual NaP-HFSs. Afterwards the algorithm proceeds exactly as in the individual case, i.e., it acts as if the group is a fictitious individual agent.

Algorithm 2: Group decision making with alternatives identified by NaP-HFSs

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A priori elements: (I) two aggregation operators on NaP-THFEs, Φ_1 and Φ_2, (II) score on THFEs, and (III) balance factor $\alpha \in [0, 1]$.

1: Input m NaP-hesitant fuzzy decision tables $(t^h_{ij})_{k\times l}$, one for each of the m agents. Here t^h_{ij} denotes cell (i, j) of table h, for each $h = 1, \ldots, m$.

2: Compute the collective NaP-HFS $(T_{ij})_{k\times l}$ by the application of the first aggregation operator to cells $t^1_{ij}, \ldots, t^m_{ij}$, for each cell (i, j): $T_{ij} = \Phi_1(t^1_{ij}, \ldots, t^m_{ij})$

3: Calculate the aggregate NaP-THFE associated with alternative i, for each $i = 1, \ldots, k$, by the second aggregation operator on NaP-THFEs: $(h_N^i, h_P^i) = \Phi_2(T_{i1}, \ldots, T_{il})$.

4: Calculate for each $i = 1, \ldots, k$, the score s_N^i of the necessary THFE h_N^i, and the score s_P^i of the possible THFE h_P^i.

5: For $i = 1, \ldots, k$, compute $\alpha_i = \alpha s_N^i + (1 - \alpha) s_P^i$.

6: The result of the decision is any alternative x_j such that $\alpha_j = \max_{i=1,\ldots,k} \alpha_i$.

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5. Conclusions and future directions of research

The spirit of hesitancy has been adapted to many settings beyond the original statement of Torra [46]. For instance, we find hesitant fuzzy linguistic term sets in Rodríguez et al. [44], hesitant fuzzy soft sets in Wang et al. [48], interval-valued hesitant fuzzy sets in Chen et al. [14, 15], hesitant intuitionistic fuzzy sets in Yao and Wang [53], which in turn refine dual hesitant fuzzy sets as in Zhu et al. [59], probabilistic hesitant fuzzy sets in Zhang et al. [57], probabilistic dual hesitant fuzzy sets in Hao et al. [33], among others. There are no doubts that, in general, hybridization usually produces interesting models and decision making mechanisms.

In this paper we have hybridized the hesitant model with a successful approach in preference theory, namely, bi-preferences of a suitable kind, called *necessary and possible preferences*: see Cerreia-Vioglio et al. [13], Giarlotta and Greco [23], Giarlotta [21, 22], Giarlotta and Watson [24, 25, 26, 27], and references therein for other similar approaches. In this way we have obtained a new notion, which encompasses hesitant fuzzy sets, and is more flexible in capturing the subtleties of decision making. We have started developing the main properties of this novel construction, and defined some helpful tools for the practitioner to analyze the concept. In particular, several examples of this paper emphasize its natural ability to model situations of collective decisions. Finally, we have employed these tools (aggregation operators, scores) to effectively propose a very flexible group decision making procedure, which generalizes widely accepted procedures in the setting of hesitant fuzzy sets.

In order to keep this first paper on the topic as intuitive as possible, we only imposed the condition that the codomain of the possible component enlarges that of the necessary component. In future research we shall aim at exploring further properties of connection between the two components of a NaP-HFS, and discuss at large the semantics of their implications, as well as relationships with other models.

Additional future research on the topic mainly goes in three directions. First of all, we aim at giving correlation coefficients, other aggregation operators or scores, and of course new decision making mechanisms to suit the needs of different frameworks. Furthermore, preference relations are an additional remarkable tool that we can adapt to this setting. In particular, we plan on designing flexible algorithms that are adequate to a corporate or a public setting, which should complete the evaluation procedures informally described in Examples 4 and 5.

Second, we wish to explore the relationship existing between our necessary and possible approach and some variations of hesitant fuzzy sets that have recently been introduced in the literature. In particular, the notion of *expanded* hesitant fuzzy set (XHFS), as introduced by Alcantud and Santos-García [6], appear to have strong connections with the approach undertaken in this paper. To that end, it appears feasible to develop a notion of *representability* of $\mathsf{NaP\text{-}HFS}$s, which is qualified by the existence of a suitable group of agents $\mathfrak{C}$ and a certain type of XHFS related to $\mathfrak{C}$.

Finally, we are in the process of formally defining a notion of *ranking procedure*. This novel notion should naturally resolve all *ex-equo* situations, which are a typical problem of all existing ranking procedures. A necessary and possible setting makes the design of such procedures particularly effective.

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Appendix: Monotonicity of scores

The general notion of score (Definition 8 in Section 2.4) imposes very few restrictions. However, typical scores may satisfy additional properties, which make them better suited to sound decision procedures. Below, we introduce forms of monotonicity, since they appear to be among the most natural conditions that should hold for any typical score. The next example illustrates the rationale of these properties, which are going to be formally introduced in Definition 19.

Example 10. Assume that $s\colon \mathcal{F}^*([0,1]) \to [0,1]$ is a "well-behaved" typical score. Let F, F', F'' be the following typical fuzzy elements:

$$F = \{0.1, 0.3, 0.4\}\,, \quad F' = \{0.1, 0.5, 0.7\}\,, \quad F'' = \{0.3, 0.6, 0.7\}\,.$$

Due to obvious considerations of monotonicity, one would naturally expect the inequalities $s(F) \leqslant s(F') \leqslant s(F'')$ to hold, possibly in a strict sense; and, most likely, $s(F) < s(F'')$ should hold as well, since F is component-wise dominated by F''. In this case the comparison appears rather easy, since the fuzzy elements F, F', F'' do have the same character.

What about the case of hesitant fuzzy elements having a different size? In order to derive some reasonable guidelines that may qualify the behavior of a score as "good", let us consider the fuzzy elements E, E', E'' defined as follows:

$$E = \{0.1, 0.3, 0.5\}\,, \quad E' = \{0.3, 0.6\}\,, \quad E'' = \{0.3, 0.7, 0.8, 1\}\,.$$

The fuzzy elements $E = \{e_1, e_2, e_3\}$ and $E' = \{e'_1, e'_2\}$ can be easily compared by looking at them in "descending order", that is, from right to left. Indeed, since $e_3 = 0.5 \leqslant 0.6 = e'_2$ and $e_2 = 0.3 \leqslant 0.3 = e'_1$, one ought to conclude that $s(E) \leqslant s(E')$ – and, most likely, $s(E) < s(E')$ – due to the fact that the score of E should be further lowered by its least value $e_1 = 0.1$. For dual reasons, the comparison of the fuzzy elements $E' = \{e'_1, e'_2\}$ and $E'' = \{e''_1, e''_2, e''_3, e''_4\}$ can be easily done if we read them in "ascending order", that is, from left to right. Indeed, since $e'_1 = 0.3 \leqslant 0.3 = e''_1$ and $e'_2 = 0.6 \leqslant 0.7 = e''_2$, one ought to conclude that $s(E') \leqslant s(E'')$ – and, most likely, $s(E') < s(E''))$ – due to the fact that the score of E'' should be further increased by its largest values $e''_3 = 0.8$ and $e''_4 = 1$. Finally, based on considerations similar to the ones above, one may naturally deduce that the strict inequality $s(E) < s(E'')$ should hold, since comparing E and E'' in any possible ordered way –that is, from right to left, or from left to right– always produces a strict domination of the latter over the former.

The next definition makes the above ideas into a formal notion.

Definition 19. A typical score $s\colon \mathcal{F}^*([0,1]) \longrightarrow [0,1]$ is *monotonic* if it satisfies the following two properties for all $m, n \in \mathbb{N}$ such that $1 \leqslant m \leqslant n$, and all $E, E' \in \mathcal{F}^*([0,1])$ such that $E = \{e_1, \ldots, e_m\}$ and $E' = \{e'_1, \ldots, e'_m, \ldots, e'_n\}$ (both written in ascending order):

(UM) *(Upper Monotonicity)* if $e_i \leqslant e'_i$ for each $i = 1, \ldots, m$, then $s(E) \leqslant s(E')$;

(LM) *(Lower Monotonicity)* if $e'_{n-m+i} \leqslant e_i$ for each $i = 1, \ldots, m$, then $s(E') \leqslant s(E)$.

Further, a monotonic typical score $s\colon \mathcal{F}^*([0,1]) \longrightarrow [0,1]$ is said to be *strongly monotonic* if it satisfies the following two additional properties:

(SUM) *(Strict Upper Monotonicity)* if $e_i \leqslant e'_i$ for each $i = 1, \ldots, m$, and either $e_j < e'_j$ for some j or $m < n$, then $s(E) < s(E')$;

(SLM) *(Strict Lower Monotonicity)* if $e'_{n-m+i} \leqslant e_i$ for each $i = 1, \ldots, m$, and either $e'_{n-m+j} < e_j$ for some j or $m < n$, then $s(E') < s(E)$.

Before dwelling on the semantics of Definition 19, let us consider a special case.

Remark 1. Assume that $s\colon \mathcal{F}^*([0,1]) \longrightarrow [0,1]$ is a monotonic score. Let $E = \{e_1, \ldots, e_n\}$ and $E' = \{e'_1, \ldots, e'_n\}$ be THFEs (listed, as usual, in ascending order) having the same character $n \geqslant 1$. If the inequality $e_i \leqslant e'_i$ holds for each $i = 1, \ldots, n$, then either one between (UM) and (LM) readily yields the inequality $s(E) \leqslant s(E')$; in particular, whenever s is strongly monotonic, either one between (SUM) and (SLM) implies $s(E) < s(E')$ as long as $e_j < e'_j$ holds for some j.

Although the properties of monotonicity given in Definition 19 appear a bit technical at a first sight, their semantics is absolutely natural. Specifically, Upper Monotonicity ensures that for typical hesitant fuzzy elements E and E' such that $|E| = m \leqslant n = |E'|$, if the lowest m elements of E' are greater than or equal to the respective elements of E, then the score of E' is greater than or equal to that of E. Strict Upper Monotonicity adds the requirement that in the particular case that either some inequality holds in a strict way or the sizes of E and E' are different, the score of E' is strictly larger than the one of E. For instance, in Example 10, property (UM) allows one to derive $s(E') \leqslant s(E'')$, whereas property (SUM) delivers $s(E') < s(E'')$. The property of (Strict) Lower Monotonicity is dual to that of (Strict) Upper Monotonicity. For instance, in Example 10, property (LM) allows one to derive $s(E) \leqslant s(E')$, whereas property (SLM) delivers $s(E) < s(E')$.

Although the typical scores introduced in Definition 9 appear to be somehow natural, regrettably only one of them is monotonic:

Proposition 2. *The Xia–Xu score is strongly monotonic. The Farhadinia score and the geometric-mean score fail to be monotonic.*

Proof. The proof that the Xia–Xu averaging score satisfies the four properties (UM), (LM), (SUM) and (SLM) is straightforward, and is left to the reader.

Next, we exhibit a pair of THFEs for which Lower Monotonicity fails upon using the Farhadinia score. Let $E = \{0.4, 1\}$ and $E' = \{0.3, 0.4, 1\}$. By (LM), any monotonic score s must satisfy $s(E') \leqslant s(E)$. Let $(\delta_n)_{n\geqslant 1}$ be any nondecreasing sequence of positive numbers such that $\delta_1 = 1$, $\delta_2 = 3$, and $\delta_3 = 13$. Then we have:

$$s_{\text{F}}(E) = \frac{1 \cdot 0.4 + 3 \cdot 1}{1+3} = 0.85 < \frac{14.5}{17} = \frac{1 \cdot 0.3 + 3 \cdot 0.4 + 13 \cdot 1}{1+3+13} = s_{\text{F}}(E')$$

which contradicts (LM).

Finally, we show that the geometric-mean score does not satisfy Upper Monotonicity. Let $E = \{0.5\}$ and $E' = \{0.5, 0.6\}$. If s is any score satisfying (UM), then $s(E) \leqslant s(E')$ holds. Since

$$s_{\text{gm}}(E) = \sqrt{0.5} > 0.7 > \sqrt[3]{0.3} = s_{\text{gm}}(E')\,,$$

we conclude that the geometric-mean score does not satisfy (UM). □

Proposition 2 explains why in this paper we have only employed the Xia–Xu score to rank HFEs.

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