Complemental fuzzy sets: A semantic justification of q-rung orthopair fuzzy sets

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***Abstract*—This paper introduces complemental fuzzy sets, explains their semantics, and presents a subclass of this model that generalizes intuitionistic fuzzy sets in a novel manner. It also provides practical results that will facilitate their implementation in real situations. At the theoretical level, we define a family of $\mathfrak{c}$-complemental fuzzy sets from each fuzzy negation $\mathfrak{c}$. We argue that this construction provides semantic justification for all subfamilies of complemental fuzzy sets, which include q-rung orthopair fuzzy sets (when $\mathfrak{c}$ is a Yager's fuzzy complement) and the new family of Sugeno intuitionistic fuzzy sets (when $\mathfrak{c}$ belongs to the class of Sugeno's fuzzy complements). We study fundamental operations and a general methodology for the aggregation of complemental fuzzy sets. Then we give some specific examples of aggregation operators to illustrate their applicability. On a more practical level, constructive proofs demonstrate that all orthopair fuzzy sets on finite sets that satisfy a mild restriction are Sugeno intuitionistic fuzzy sets, and they are q-rung orthopair fuzzy sets for some q too. These contributions produce a new operational model that semantically justifies, and mathematically contains, "almost all" orthopair fuzzy sets on finite sets.**

***Index Terms*—Yager's fuzzy complement; Sugeno's fuzzy complement; intuitionistic fuzzy set; q-rung orthopair fuzzy set; aggregation.**

I. Introduction

THE core principle of intuitionistic fuzziness is entrenched in many subsequent models. Its assumption that non-membership should be evaluated separately from membership has evolved to very general models (like q-rung orthopair fuzzy sets), where the original restriction that membership and non-membership should sum up to at most 1 has been weakened.

In this paper we shall lay out a very general construction (providing what we shall call "complemental fuzzy sets") which at the same time subsumes these models, and supplies a sound semantic justification for them.

Before we concentrate on the theoretical and practical aspects of this work, a few preliminary considerations are in order.

A. Intuitionistic fuzzy sets and their extensions

Atanassov [1] pioneered the utilization of "orthopairs", whereby a pair of values in the unit interval give the support for and against membership of an element to the "orthopair fuzzy set". He called his extension "intuitionistic fuzzy set" (abbreviated IFS). In an intuitionistic fuzzy set, at each orthopair one must observe that the sum of both values is bounded by 1. Without this restriction, one has an orthopair fuzzy set [2].

The q-rung orthopair fuzzy sets (Yager [2], [3]) are a subfamily of orthopair fuzzy sets that mitigate the constraint required by [1]. The condition imposed by q-rung orthopair fuzzy sets is that at each orthopair, the sum *of the q power* of both membership and non-membership values must be bounded by 1. Atanassov's intuitionistic fuzzy sets form the family of 1-rung orthopair fuzzy sets. The 2-rung orthopair fuzzy sets are the *intuitionistic fuzzy sets of second type* (Atanassov [4], reprinted in [5]), popularized as *Pythagorean fuzzy sets* [6]. To put some examples of the types of applications to decision making that have been presented, q-rung orthopair fuzzy sets have been used to evaluate strategies against COVID-19 [7] and in green supplier selection problems [8] with the help of the TOPSIS (for Technique for Order of Preference by Similarity to Ideal Solution) methodology.

Once the utilization of more than one evaluation went into the mainstream, we have witnessed the progress of many other types of extensions of the idea of a fuzzy set. Interval valued q-rung orthopair fuzzy sets [9] have allowed the experts to provide interval valuations to the support for and against membership of the elements. The addition of a third evaluation capturing indeterminacy to orthopair fuzzy sets produced further models like picture fuzzy sets [10] and spherical fuzzy sets [11]. Multi-fuzzy sets [12] are a more general framework: they simply gather several fuzzy sets in a non-structured manner. Recently, constrained Pythagorean fuzzy sets [13] have given a unified description of probabilistic and Pythagorean fuzzy information. We shall not be concerned here with these models.

B. Motivation and research goals

Atanassov made a convincing case that two figures are a better reflection of partial knowledge than simply one. Then the problem that arises is, what semantic interpretation justifies the acceptance of membership and non-membership degrees *summing up to more than* 1? This is the first challenge we set ourselves.

For inspiration, remember that a classical semantic interpretation of fuzzy sets explains that they provide a truth value for each clause of the type *an element belongs to the 'set'* [14]. This is the presentation that underpins fuzzy logic. One can imagine a model where a second figure yields the truth value for the associated clause *an element does not belong to the*

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'set'. Both truth values are related by a natural link from fuzzy logic: a fuzzy complement (also called fuzzy negation). When the second figure is the result of the application of the standard fuzzy negation to the first one, the model is tantamount to a fuzzy set. Put shortly, fuzzy sets 'are' orthopair fuzzy sets for which the standard fuzzy complement yields logical negation. If we still use the standard fuzzy complement but the second figure is just *bounded* by the first one, then we have an intuitionistic fuzzy set. When the second figure is bounded by the first one but we use a member of the Yager's family of fuzzy complements, then we obtain a q-rung orthopair fuzzy set [2, Section IV]. However when the second figure is bounded by the first one and we are allowed to use any fuzzy complement, then we obtain a new general model consisting of what we will call *complemental fuzzy sets*.

Motivated by these considerations, this paper formalises the aforementioned novel perspective of intuitionistic-type fuzzy models and explains the semantic interpretation associated with complemental fuzzy sets. Then we investigate their fundamental properties, inclusive of the set-theoretic operations that are the basis of their algebra. We design a general methodology that produces aggregation operators for complemental fuzzy sets. Some specific operators are defined and illustrated with synthetic examples to guarantee the applicability of this generic technique.

We also explore the relationship of our general model with the literature. We explain how complemental fuzzy sets feed into a narrative that showcases many popular models in a different perspective. In this respect, we recall that the complemental fuzzy set model encompasses many popular extensions of intuitionistic fuzzy sets, remarkably, the q-rung orthopair fuzzy sets. And we show that it allows us to define new parametric models, especially what we will call "Sugeno intuitionistic fuzzy sets". Thus one theoretical consequence of these achievements is that complemental fuzzy sets offer a unified semantic justification of q-rung orthopair fuzzy sets and other extensions of intuitionistic fuzzy sets (like the new Sugeno intuitionistic fuzzy sets).

Yet another applicable contribution of this paper concerns the clarification of certain relationships between orthopair fuzzy sets and q-rung orthopair fuzzy sets. Not only is (obviously) true that q-rung orthopair fuzzy sets are orthopair fuzzy sets. Here we prove that the parametric family of q-rung orthopair fuzzy sets has enough explanatory ability to encompass all orthopair fuzzy sets on a finite set of alternatives, as long as they satisfy a very mild structural property. Then we replicate this pioneering "goodness-of-fit" exercise with respect to the newly defined Sugeno intuitionistic fuzzy sets. These findings have practical consequences. In the presence of a series of data which are orthopairs, we shall be able to find an "optimal" representation by either q-rung orthopair fuzzy sets, or by Sugeno intuitionistic fuzzy sets, provided that the data do not contradict a very simple restriction. It is worth noting that our proofs are constructive. Examples will be given to show how this fitting can be made effectively.

C. Outline of this paper

We organize the rest of this paper as follows.

Section II recalls known concepts, defines new auxiliary ideas, and proves fundamental facts about the interaction of fuzzy complements and aggregation operators. Section III introduces the new framework called complemental fuzzy sets. We discuss their semantics in Section III-A. To prove the novelty of complemental fuzzy sets, a new model (the Sugeno intuitionistic fuzzy sets) is described in Section III-B that is a particular type of this concept. Section III-C defines algebraic operations for complemental fuzzy sets. Section III-D performs an original "goodness-of-fit" exercise proving that both q-rung orthopair fuzzy sets and Sugeno intuitionistic fuzzy sets (hence complemental fuzzy sets) encompass all orthopair fuzzy sets on finite sets that satisfy a very mild structural property. The main results are constructive, and respective examples clarify their practical implementation. Section IV is concerned with a general theory of aggregation in the setting of complemental fuzzy sets. Examples are provided that illustrate noteworthy cases of aggregation operators constructed from arithmetic and geometric averages, and from an OWA operator. Section V concludes this paper with a graphical summary of relationships.

II. Basic concepts and results

This section contains both known and new concepts. The later are defined in Section II-B. A fundamental result concerning these new ideas is proven in Section II-C.

For convenience, we shall write the unit interval $[0, 1]$ as I.

Suppose $(a_1, \ldots, a_m), (b_1, \ldots, b_m) \in I \times \overset{m}{\ldots} \times I$. The vector inequality $(a_1, \ldots, a_m) \geqslant (b_1, \ldots, b_m)$ is a shorthand for $a_i \geqslant b_i$ for each $i = 1, \ldots, m$.

In this paper X denotes a set of alternatives.

A. Review of concepts

This section recalls standard notions from the literature.

We shall often refer to the following general model and various of its particular cases:

Definition II.1 (Yager [2]). A *q-rung orthopair fuzzy set* (qROFS) A over X is

$$A = \{\langle x, (\mu_A(x), \nu_A(x))\rangle | \, x \in X\}$$

where the mappings $\mu_A, \nu_A : X \to [0, 1]$ respectively encode the degrees of membership and non-membership of $x \in X$ to the q-rung orthopair fuzzy set A, and they satisfy

$$(\mu_A(x))^q + (\nu_A(x))^q \leq 1 \text{ for all } x \in X.$$

Alternatively, we identify A with the triple (X, μ_A, ν_A). We also write $A(x) = (\mu_A(x), \nu_A(x))$ for each $x \in X$.

When $q = 1$, Definition II.1 produces an Atanassov's [1] *intuitionistic fuzzy set* (IFS). The case $q = 2$ produces a *Pythagorean fuzzy set* (PyFS), a concept had been presented earlier by Atanassov [4], [5] under the name Intuitionistic Fuzzy Set of Second Type (see also [15]). The case $q = 3$ has been labelled as *Fermatean fuzzy set*.

A multi-fuzzy set M over X of dimension k is $M = \{\langle x, (\mu_1(x), \ldots, \mu_k(x))\rangle | \, x \in X\}$ such that the mappings $\mu_1(x), \ldots, \mu_k(x) : X \to [0, 1]$ are otherwise unrestricted [12].

A multi-fuzzy set of dimension 2 is an *orthopair fuzzy set* [2, Section I]. Definition II.1 is a structured orthopair fuzzy set.

For further use, we say that the orthopair fuzzy set $M = \{\langle x, (\mu_1(x), \mu_2(x))\rangle | \, x \in X\}$ is *never-full* when $\mu_1(x) \neq 0$ implies $\mu_2(x) \neq 1$, and $\mu_2(x) \neq 0$ implies $\mu_1(x) \neq 1$, for each $x \in X$. Equivalently: $\mu_2(x) = 1$ implies $\mu_1(x) = 0$, and $\mu_1(x) = 1$ implies $\mu_2(x) = 0$, for each $x \in X$.

An *orthopair* is (μ, ν) where $0 \leq \mu, \nu \leq 1$. The orthopair is called a *q-rung orthopair* (qRO) when $0 \leq \mu^q + \nu^q \leq 1$.

Definition II.2. *A mapping* $\mathfrak{c} : I \to I$ *is called a fuzzy complement when it satisfies:*

- *Boundary condition:* $\mathfrak{c}(0) = 1$, $\mathfrak{c}(1) = 0$.
- *Monotonicity:* $\mathfrak{c}(a) \geqslant \mathfrak{c}(b)$ *if* $a, b \in [0, 1]$ *and* $a \leqslant b$.
- $\mathfrak{c}$ *is involutive:* $\mathfrak{c}(\mathfrak{c}(a)) = a$ *for all* $a \in [0, 1]$.

It is customary to add the continuity axiom to a fuzzy complement. We shall not make use of this property, hence we avoid it in the axiomatics.

Remark II.3. *Sometimes we shall use the notation* $a^{\mathfrak{c}}$ *as a replacement for* $\mathfrak{c}(a)$. *Besides, observe that 'monotonicity' is also referred to as 'inverse monotonicity' by many authors.*

Definition II.4. *A mapping* $\mathcal{A} : I \times \overset{m}{\cdots} \times I \to I$ *is called an aggregation operator when it satisfies:*

- *Boundary condition:* $\mathcal{A}(0, \overset{m}{\ldots}, 0) = 0$, $\mathcal{A}(1, \overset{m}{\ldots}, 1) = 1$.
- *Monotonicity:* $\mathcal{A}(a_1, \ldots, a_m) \geqslant \mathcal{A}(b_1, \ldots, b_m)$ *if* $a_i, b_i \in [0, 1]$ *for each* $i = 1, \ldots, m$ *and* $(a_1, \ldots, a_m) \geqslant (b_1, \ldots, b_m)$.

The next aggregation operator will be used in subsequent sections:

Definition II.5 (Yager [16])**.** *Let* $\boldsymbol{w} = (w_1, \ldots, w_m) \in [0, 1]^m$ *be a weighting vector such that* $\sum_{i=1}^m w_i = 1$. *The ordered weighted averaging (OWA) operator associated with* $\boldsymbol{w}$ *is the function* $\mathcal{A}^{\boldsymbol{w}} : I^m \longrightarrow I$ *defined by* $\mathcal{A}^{\boldsymbol{w}}(a_1, \ldots, a_m) = \sum_{i=1}^m w_i \, b_i$ for each $(a_1, \ldots, a_m) \in I^m$, *where* b_i *is the i-th largest element in the collection of (possibly repeated) values* $\{a_1, \ldots, a_m\}$.

B. New concepts

We proceed to present some auxiliary concepts that will be useful later on in this paper.

Definition II.6. *Let us fix* $\mathfrak{c}$, *a fuzzy complement.*

We say that $(a, b) \in I \times I$ *is a* $\mathfrak{c}$*-pair when* $b \leqslant \mathfrak{c}(a)$.

The set of all $\mathfrak{c}$*-pairs is denoted by* $S_{\mathfrak{c}}$. *It is a subset of the unit square* $I \times I$.

When $(a, b), (a', b') \in S_{\mathfrak{c}}$, *we let* $(a, b) \succcurlyeq (a', b')$ *stand for the property:* $a \geqslant a'$ *and* $b' \geqslant b$.

Remark II.7. *Notice* $(1, 0), (0, 1) \in S_{\mathfrak{c}}$ *independently of* $\mathfrak{c}$, *because both* $0 \leqslant \mathfrak{c}(1) = 0$ *and* $1 \leqslant \mathfrak{c}(0) = 1$ *hold true due to the boundary condition of a fuzzy complement.*

Besides, if $(a, b) \in I \times I$ *is a* $\mathfrak{c}$*-pair then* $\mathfrak{c}(b) \geqslant \mathfrak{c}(\mathfrak{c}(a)) = a$ *using the fact that* $\mathfrak{c}$ *is (inverse) monotonic and involutive. Thus we could use this inequality to define a* $\mathfrak{c}$*-pair as a replacement for the condition* $b \leqslant \mathfrak{c}(a)$ *throughout.*

Example II.8. *Let us consider the fuzzy complement* $\mathfrak{c}(a) = (1-a^q)^{1/q}$ *with* $q > 0$ *introduced by Yager in [17], [18]. When* $q \geqslant 1$, *the* $\mathfrak{c}$*-pairs defined here are the q-rung orthopairs that take part in the definition of qROFSs [2, Section V].*

Yager [2, Section IV] recalls that IFSs 'use' $\mathfrak{c}(a) = (1 - a^q)^{1/q}$ *with* $q = 1$, *which is the classic complement* $\mathfrak{c}(a) = 1 - a$, *whereas PyFSs 'use' the Pythagorean complement* $\mathfrak{c}(a) = (1-a^q)^{1/q}$ *with* $q = 2$, *i.e.,* $\mathfrak{c}(a) = \sqrt{1 - a^2}$. *Formally speaking: an IFS on* X *is* $A = \{\langle x, (\mu_A(x), \nu_A(x))\rangle | \, x \in X\}$ *with* $(\mu_A(x), \nu_A(x)) \in S_{\mathfrak{c}}$ *for all* $x \in X$ *when* $\mathfrak{c}$ *is the classic complement, whereas a PyFS on* X *is* $P = \{\langle x, (\mu_P(x), \nu_P(x))\rangle | \, x \in X\}$ *with* $(\mu_P(x), \nu_P(x)) \in S_{\mathfrak{c}}$ *for all* $x \in X$ *when* $\mathfrak{c}$ *is the Pythagorean complement. The case* $q = 3$ *produces Fermatean fuzzy sets.*

Definition II.9. *Let us fix* $\mathfrak{c}$, *a fuzzy complement.*

A mapping $\mathbf{A} : S_{\mathfrak{c}} \times \overset{m}{\cdots} \times S_{\mathfrak{c}} \to S_{\mathfrak{c}}$ *is called a* $\mathfrak{c}$*-aggregation operator when it satisfies:*

- *Boundary condition:* $\mathbf{A}((1, 0), \overset{m}{\ldots}, (1, 0)) = (1, 0)$, *and* $\mathbf{A}((0, 1), \overset{m}{\ldots}, (0, 1)) = (0, 1)$.
- *Monotonicity: if for every* $i = 1, \ldots, m$, $(a_i, b_i), (c_i, d_i) \in S_{\mathfrak{c}}$ *satisfy* $(a_i, b_i) \succcurlyeq (c_i, d_i)$, *then* $\mathbf{A}((a_1, b_1), \ldots, (a_m, b_m)) \succcurlyeq \mathbf{A}((c_1, d_1), \ldots, (c_m, d_m))$.

C. A fundamental result about $\mathfrak{c}$-aggregation operators

The next general construction produces $\mathfrak{c}$-aggregation operators with the help of standard aggregation operators:

Proposition II.10. *Let us fix* $\mathfrak{c}$, *a fuzzy complement, and let* $\mathcal{A} : I \times \overset{m}{\cdots} \times I \to I$ *be an aggregation operator.*

The mapping $\mathcal{A}_{\mathfrak{c}} : S_{\mathfrak{c}} \times \overset{m}{\cdots} \times S_{\mathfrak{c}} \to S_{\mathfrak{c}}$ *defined by*

$$\mathcal{A}_{\mathfrak{c}}((a_1, b_1), \ldots, (a_m, b_m)) = (\mathcal{A}(a_1, \ldots, a_m), \mathfrak{c}(\mathcal{A}(b_1^{\mathfrak{c}}, \ldots, b_m^{\mathfrak{c}})))$$

when $(a_i, b_i) \in S_{\mathfrak{c}}$ $(i = 1, \ldots, m)$, *is a* $\mathfrak{c}$*-aggregation operator.*

Proof. First we need to confirm that $\mathcal{A}_{\mathfrak{c}}$ is well-defined, i.e., that $(a_i, b_i) \in S_{\mathfrak{c}}$ for all $i = 1, \ldots, m$ implies $\mathcal{A}_{\mathfrak{c}}((a_1, b_1), \ldots, (a_m, b_m)) \in S_{\mathfrak{c}}$. To prove it, observe that $(b_1^{\mathfrak{c}}, \ldots, b_m^{\mathfrak{c}}) \geqslant (a_1, \ldots, a_m)$ by definition of $S_{\mathfrak{c}}$ and Remark II.7. Since $\mathcal{A}$ is an aggregation operator, $\mathcal{A}(b_1^{\mathfrak{c}}, \ldots, b_m^{\mathfrak{c}}) \geqslant \mathcal{A}(a_1, \ldots, a_m)$. Using (inverse) monotonicity of $\mathfrak{c}$, we obtain $\mathfrak{c}(\mathcal{A}(b_1^{\mathfrak{c}}, \ldots, b_m^{\mathfrak{c}})) \leqslant \mathfrak{c}(\mathcal{A}(a_1, \ldots, a_m))$. Therefore we have proven that $(\mathcal{A}(a_1, \ldots, a_m), \mathfrak{c}(\mathcal{A}(b_1^{\mathfrak{c}}, \ldots, b_m^{\mathfrak{c}})))$ is a $\mathfrak{c}$-pair, which is equivalent to $\mathcal{A}_{\mathfrak{c}}((a_1, b_1), \ldots, (a_m, b_m)) \in S_{\mathfrak{c}}$.

Let us now check that $\mathcal{A}_{\mathfrak{c}}$ satisfies both the boundary condition and monotonicity:

- Boundary condition. First we observe the equalities
$$\begin{aligned}\mathcal{A}_{\mathfrak{c}}((0, 1), \overset{m}{\ldots}, (0, 1)) =& (\mathcal{A}(0, \ldots, 0), \mathfrak{c}(\mathcal{A}(1^{\mathfrak{c}}, \ldots, 1^{\mathfrak{c}}))) = \\ & (0, \mathfrak{c}(\mathcal{A}(0, \ldots, 0))) = (0, \mathfrak{c}(0)) = (0, 1).\end{aligned}$$
In conclusion, we have shown $\mathcal{A}_{\mathfrak{c}}((0, 1), \overset{m}{\ldots}, (0, 1)) = (0, 1)$.
The proof that $\mathcal{A}_{\mathfrak{c}}((1, 0), \overset{m}{\ldots}, (1, 0)) = (1, 0)$ holds true is analogous.
- Monotonicity. Let us fix $(a_i, b_i), (c_i, d_i) \in S_{\mathfrak{c}}$ with $(a_i, b_i) \succcurlyeq (c_i, d_i)$ for $i = 1, \ldots, m$. Hence $a_i \geqslant c_i$ and $d_i \geqslant b_i$ for $i = 1, \ldots, m$. In order to prove
$$\mathcal{A}_{\mathfrak{c}}((a_1, b_1), \ldots (a_m, b_m)) \succcurlyeq \mathcal{A}_{\mathfrak{c}}((c_1, d_1), \ldots (c_m, d_m)),$$

or equivalently,

$$(\mathcal{A}(a_1,\dots a_m),\mathfrak{c}(\mathcal{A}(b_1^{\mathfrak{c}},\dots b_m^{\mathfrak{c}}))) \succcurlyeq (\mathcal{A}(c_1,\dots c_m),\mathfrak{c}(\mathcal{A}(d_1^{\mathfrak{c}},\dots d_m^{\mathfrak{c}}))),$$

two things need to be checked (cf., Definition II.6).
$\mathcal{A}(a_1,\dots a_m) \geqslant \mathcal{A}(c_1,\dots c_m)$ follows from the assumption $(a_1,\dots a_m) \geqslant (c_1,\dots c_m)$ and the monotonicity of $\mathcal{A}$.
Likewise, $(d_1,\dots d_m) \geqslant (b_1,\dots b_m)$ and the (inverse) monotonicity of $\mathfrak{c}$ imply $(d_1^{\mathfrak{c}},\dots d_m^{\mathfrak{c}}) \leqslant (b_1^{\mathfrak{c}},\dots b_m^{\mathfrak{c}})$. By the monotonicity of $\mathcal{A}$, $\mathcal{A}(d_1^{\mathfrak{c}},\dots d_m^{\mathfrak{c}}) \leqslant \mathcal{A}(b_1^{\mathfrak{c}},\dots b_m^{\mathfrak{c}})$. Using again the (inverse) monotonicity of $\mathfrak{c}$, we conclude $\mathfrak{c}(\mathcal{A}(d_1^{\mathfrak{c}},\dots d_m^{\mathfrak{c}})) \geqslant \mathfrak{c}(\mathcal{A}(b_1^{\mathfrak{c}},\dots b_m^{\mathfrak{c}}))$.
Both inequalities amount to
$\mathcal{A}_{\mathfrak{c}}((a_1,b_1),\dots,(a_m,b_m)) \succcurlyeq \mathcal{A}_{\mathfrak{c}}((c_1,d_1),\dots,(c_m,d_m))$. □

The first theorem stated in Yager [2, Section VII] is a simple Corollary to Proposition II.10. That theorem states that when $\mathfrak{c}(a) = (1-a^q)^{1/q}$ for a fixed $q \geqslant 1$, $\mathcal{A}_{\mathfrak{c}}$ defined as above is "closed". In other words, when $\mathcal{A}_{\mathfrak{c}}$ aggregates q-rung orthopairs (the $\mathfrak{c}$-pairs defined in this framework: see Example II.8) its output is a q-rung orthopair. This fact is the particular case of the first claim proven in Proposition II.10, when it applies to the complement defined by $\mathfrak{c}(a) = (1-a^q)^{1/q}$. Our Proposition II.10 not only gives a more general framework derived from any fuzzy complement and aggregation operator, but it also proves additional features of this construction.

Example II.11. *Yager [2, Section VIII] explores the OWA operator (Definition II.5) for q-rung orthopair fuzzy sets. His construction refers to an aggregation operator of q-rung orthopairs that coincides with $\mathcal{A}_{\mathfrak{c}}$ in Proposition II.10, when $\mathcal{A}$ is an OWA operator and $\mathfrak{c}$ is Yager's fuzzy complement.*

To finalize this section, we observe that the construction in Proposition II.10 is monotonic with respect to the aggregation operator. The reader can easily check that the following property holds true:

Lemma II.12. *If $\mathcal{A},\mathcal{A}'$ are aggregation operators such that $\mathcal{A} \geqslant \mathcal{A}'$, then $\mathcal{A}_{\mathfrak{c}} \succcurlyeq \mathcal{A}'_{\mathfrak{c}}$ for each fuzzy complement $\mathfrak{c}$.*

Here $\mathcal{A} \geqslant \mathcal{A}'$ stands for $\mathcal{A}(a_1,\dots a_m) \geqslant \mathcal{A}'(a_1,\dots a_m)$ for each $(a_1,\dots a_m) \in I\times \overset{m}{\cdots} \times I$, whereas $\mathcal{A}_{\mathfrak{c}} \succcurlyeq \mathcal{A}'_{\mathfrak{c}}$ stands for $\mathcal{A}_{\mathfrak{c}}((a_1,b_1),\dots (a_m,b_m)) \succcurlyeq \mathcal{A}'_{\mathfrak{c}}((a_1,b_1),\dots (a_m,b_m))$ for each $((a_1,b_1),\dots (a_m,b_m)) \in S_{\mathfrak{c}}\times \overset{m}{\cdots} \times S_{\mathfrak{c}}$.

III. The model: complemental fuzzy sets

This section introduces complemental fuzzy sets and dwells on its implications. The model is defined in section III-A, and here its semantics is discussed in detail. This section also shows that well-known models are embedded into this framework. Then in section III-B we explain that an original model stems from our construction too. Section III-C presents some fundamental operations on complemental fuzzy sets. Importantly, Section III-D proves that (families of) complemental fuzzy sets justify all never-full orthopair fuzzy sets over a finite set.

A. Complemental fuzzy sets

The main contribution of this paper is the next concept:

Definition III.1. *Every fuzzy complement $\mathfrak{c}$ defines a family of* complemental fuzzy sets *over X. A member of the collection of $\mathfrak{c}$-complemental fuzzy sets over X has the form*

$$F = \{\langle x,(\mu_F(x),\nu_F(x))\rangle \,|\, x \in X\}$$

with $F(x) = (\mu_F(x),\nu_F(x)) \in S_{\mathfrak{c}}$ for all $x \in X$.

So any fuzzy complement $\mathfrak{c}$ defines the collection of $\mathfrak{c}$-complemental fuzzy sets over X, that we shall denote by $\mathfrak{c}\mathbf{CFS}(X)$. Complemental fuzzy sets are the family of all $\mathfrak{c}$-complemental fuzzy sets over a common X, when $\mathfrak{c}$ varies across all fuzzy complements.

Semantics. The interpretation of a complemental fuzzy set goes as follows. For each element x of X, F captures two truth values in the range $[0,1]$. The first, $\mu_F(x)$, gauges the truth value of the clause "x belongs to F". If our fuzzy logic negation is fixed to be $\mathfrak{c}$, the truth value of the negation of that clause ("x does not belong to F") must be bounded by $\mathfrak{c}(\mu_F(x))$. This logical constraint is equivalent to the restriction $(\mu_F(x),\nu_F(x)) \in S_{\mathfrak{c}}$, and at the same time, provides a semantic justification for it.

It should now be obvious that a semantic justification has been proposed that concerns all particular cases defined by specific fuzzy negations. In the next example we take advantage of this fact to show that our construction provides a rationale for some popular models:

Example III.2. *Let us examine the versions of complemental fuzzy sets that stem from some benchmark fuzzy complements. Section III-B is in continuation of this analysis.*

1) *When $\mathfrak{c}$ is the standard or classic complement, the restriction $\nu_F(x) \leqslant \mathfrak{c}(\mu_F(x))$ is equivalent to $\nu_F(x) \leqslant 1-\mu_F(x)$. Therefore we obtain an intuitionistic fuzzy set.*
2) *If in addition to this, the restriction $\nu_F(x) = \mathfrak{c}(\mu_F(x))$ applies for all $x \in X$, then we obtain a complemental fuzzy set that can be identified with a fuzzy set.*
3) *Example II.8 has shown that when we use the parametric family of Yager's fuzzy complements, $\mathfrak{c}$-complemental fuzzy sets boil down to q-rung orthopair fuzzy sets. In particular, it has been established that Yager's fuzzy complements with $q = 1$, respectively, $q = 2$, $q = 3$, produce intuitionistic fuzzy sets, Pythagorean fuzzy sets, and Fermatean fuzzy sets.*

Example III.2 does not exhaust all possible types of complemental fuzzy sets. As a matter of fact, we proceed to define a model that is both new and embedded into the new family introduced in section III-A.

B. A new model: the Sugeno intuitionistic fuzzy sets

The Sugeno family of fuzzy complements (cf., Definition II.2) is quite popular. It is dependent on a parameter $\lambda > -1$. Then for any $\lambda > -1$, $\mathfrak{c}_\lambda : I \to I$ is defined by $\mathfrak{c}_\lambda(x) = \frac{1-x}{1+\lambda x}$ for each $x \in I$. The case $\lambda = 0$ produces the standard or classic complement, which therefore becomes a particular case of the Sugeno family of fuzzy complements. The application of Definition III.1 to this family produces the next novel model:

Definition III.3. *Let us fix $\lambda > -1$. The Sugeno fuzzy complement $\mathfrak{c}_\lambda$ defines the $\mathfrak{c}_\lambda$-complemental fuzzy sets, which therefore have the form*

$$G = \{\langle x, (\mu_G(x), \nu_G(x))\rangle | \, x \in X\}$$

with $G(x) = (\mu_G(x), \nu_G(x)) \in S_{\mathfrak{c}_\lambda}$ for all $x \in X$. The mappings $\mu_G, \nu_G : X \to [0,1]$ are such that

$$\nu_G(x) \leqslant \frac{1-\mu_G(x)}{1+\lambda\cdot\mu_G(x)} \quad \text{for each} \quad x \in X. \qquad (1)$$

We call this family the λ-Sugeno intuitionistic fuzzy sets *over X.*

The collection of all λ-Sugeno intuitionistic fuzzy sets with $\lambda > -1$ defines the Sugeno intuitionistic fuzzy sets over X.

It is easy to check that when $\lambda \geqslant 0$, λ-Sugeno intuitionistic fuzzy sets are intuitionistic fuzzy sets. We just need to observe that in this case, Eq. (1) guarantees

$$\nu_G(x) \leqslant \frac{1-\mu_G(x)}{1+\lambda\cdot\mu_G(x)} \leqslant 1-\mu_G(x) \quad \text{for each} \quad x \in X.$$

because $1 \leqslant 1+\lambda\cdot\mu_G(x)$ when $x \in X$. Hence the novelty of Definition III.3 reduces to the cases produced by $-1 < \lambda < 0$.

Observe that Definition III.3 provides a swift transition from novel models (λ-Sugeno IFSs with $-1 < \lambda < 0$) to fuzzy sets (0-Sugeno IFSs) and specific types of intuitionistic fuzzy sets (λ-Sugeno IFSs with $\lambda > 0$). Figure 1 illustrates the boundaries of the acceptable evaluations in an intuitionistic fuzzy set, a Pythagorean fuzzy set, a Fermatean fuzzy set, and the λ-Sugeno intuitionistic fuzzy sets with $\lambda = -0.9$ and $\lambda = -0.95$. This figure shows that the new class λ-Sugeno intuitionistic fuzzy sets contains cases not covered by the existing q-rung orthopair fuzzy model.

C. Algebraic operations on complemental fuzzy sets

This section contributes to enhance the theoretical performance of the model described in section III-A with some fundamental operations that lay the foundations of its algebra.

Definition III.4. *Let us fix a fuzzy complement $\mathfrak{c}$. Consider the $\mathfrak{c}$-complemental fuzzy sets*

$$F = \{\, \langle x, (\mu_F(x), \nu_F(x))\rangle \mid x \in X \,\} \text{ and}$$
$$G = \{\, \langle x, (\mu_G(x), \nu_G(x))\rangle \mid x \in X \,\}.$$

Their intersection is the $\mathfrak{c}$-complemental fuzzy set $F \cap G =$

$$\{\langle x, (\min(\mu_F(x), \mu_G(x)), \max(\nu_F(x), \nu_G(x)))\rangle | \, x \in X\}.$$

Their union is the $\mathfrak{c}$-complemental fuzzy set $F \cup G =$

$$\{\langle x, (\max(\mu_F(x), \mu_G(x)), \min(\nu_F(x), \nu_G(x)))\rangle | \, x \in X\}.$$

The complement of F is the $\mathfrak{c}$-complemental fuzzy set

$$F^c = \{\langle x, (\nu_F(x), \mu_F(x))\rangle | \, x \in X\}.$$

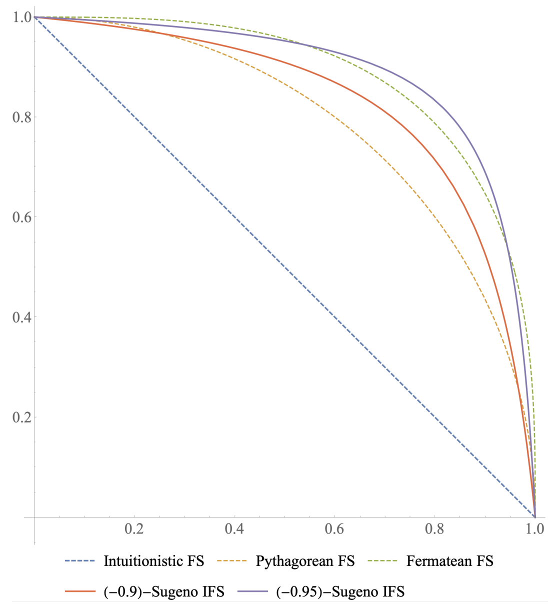

Figure 1. A graphical comparison of five models, inclusive of the λ-Sugeno IFSs with $\lambda = -0.9$ and $\lambda = -0.95$ (cf., Definitions III.3).

Proposition III.5. *The operations in Definition III.4 are well defined, i.e., they always produce a $\mathfrak{c}$-complemental fuzzy set.*

Proof. Consider intersection. By assumption, $(\mu_F(x), \nu_F(x)) \in S_\mathfrak{c}$ and $(\mu_G(x), \nu_G(x)) \in S_\mathfrak{c}$, i.e., $\nu_F(x) \leqslant \mathfrak{c}(\mu_F(x))$ and $\nu_G(x) \leqslant \mathfrak{c}(\mu_G(x))$, for all $x \in X$. Let us fix an arbitrary $x \in X$. We do not lose generality if we assume $\mu_F(x) \leqslant \mu_G(x)$. Then $\mathfrak{c}(\mu_G(x)) \leqslant \mathfrak{c}(\mu_F(x))$, hence $\nu_G(x) \leqslant \mathfrak{c}(\mu_F(x))$. In conclusion, $\max(\nu_F(x), \nu_G(x)) \leqslant \mathfrak{c}(\mu_F(x)) = \mathfrak{c}(\min(\mu_F(x), \mu_G(x)))$.

A symmetrical argument proves that union is well defined.

Let us show that the complement of the $\mathfrak{c}$-complemental fuzzy set F is also a $\mathfrak{c}$-complemental fuzzy set. Fix $x \in X$. From $\nu_F(x) \leqslant \mathfrak{c}(\mu_F(x))$ we get $\mathfrak{c}(\nu_F(x)) \geqslant \mathfrak{c}(\mathfrak{c}(\nu_F(x))) = \nu_F(x)$ since $\mathfrak{c}$ is (inverse) monotonic and involutive. This ends the proof. □

A direct consequence of these algebraic constructions is their application to particular cases. Two instances are worth emphasizing: q-rung orthopair fuzzy sets (when $\mathfrak{c}$ is Yager's fuzzy complement, cf., Example III.2) and Sugeno intuitionistic fuzzy sets (cf., section III-B). Admittedly, the first case is well known:

Corollary III.6. *[2, Section VI] The standard default operations for intersection, union, and complement of q-rung orthopair fuzzy sets are well defined. Hence the same holds true for Fermatean, Pythagorean, and intuitionistic fuzzy sets.*

We can also extend the concept of inclusion or containment defined in [2, Section VI] for q-rung orthopair fuzzy sets, to the more general case of complemental fuzzy sets:

Definition III.7. *Let us fix a fuzzy complement $\mathfrak{c}$. Consider the $\mathfrak{c}$-complemental fuzzy sets*

$$F = \{\, \langle x, (\mu_F(x), \nu_F(x))\rangle \mid x \in X \,\} \text{ and}$$
$$G = \{\, \langle x, (\mu_G(x), \nu_G(x))\rangle \mid x \in X \,\}.$$

We say that F is contained in G, $F \subseteq G$, when $(\mu_G(x), \nu_G(x)) \succcurlyeq (\mu_F(x), \nu_F(x))$ for all $x \in X$.

D. Complemental fuzzy sets as a semantic justification of never-full orthopair fuzzy sets

Now we shall prove a remarkable property of complemental fuzzy sets with a consequence in terms of the interpretation of orthopair fuzzy sets. We proceed to prove a mathematical claim: selected families of $\mathfrak{c}$-complemental fuzzy sets (i.e., with $\mathfrak{c}$ adhering to a certain parametric form) constitute a universal explanation of the whole family of never-full orthopair fuzzy sets over a fixed *finite* set X. Thus as a consequence, this achievement demonstrates that in the case of finite sets of alternatives, complemental fuzzy sets are a common semantic justification of never-full orthopair fuzzy sets. First we shall prove the mathematical claim for λ-Sugeno IFSs. Then a similar argument will show that the same is true for qROFSs.

Both analyses are reminiscent of a "goodness-of-fit" exercise whereby we find the thresholds below/above which a given orthopair fuzzy sets belongs to the parametric family under inspection.

Theorem III.8. *Let $M = \{\langle x_i, (\mu_1^i, \mu_2^i)\rangle \mid i = 1, \ldots, n\}$ be a never-full orthopair fuzzy set over $X = \{x_1, \ldots, x_n\}$.*

Then there is $\lambda_0 > -1$, the largest number such that M is a λ-Sugeno IFS for each $\lambda_0 \geqslant \lambda > -1$.

Proof. Consider the family of indices J formed by all $i \in \{1, \ldots, n\}$ for which both $\mu_1^i \neq 0$ and $\mu_2^i \neq 0$ are true. Observe that $\mu_1^i \neq 1$ and $\mu_2^i \neq 1$ when $i \in J$, because M is never-full. Then we claim that $\lambda_0 = \min\{\frac{1-(\mu_1^i+\mu_2^i)}{\mu_1^i \cdot \mu_2^i} \mid i \in J\}$ fulfils the thesis.

Observe that the fact that X is finite guarantees that the minimum is attained at some $i \in J$. Besides, we claim that $\frac{1-(\mu_1^i+\mu_2^i)}{\mu_1^i \cdot \mu_2^i} > -1$. Notice that

$$\frac{1 - (\mu_1^i + \mu_2^i)}{\mu_1^i \cdot \mu_2^i} > -1 \Leftrightarrow (1 - \mu_2^i) + \mu_1^i(\mu_2^i - 1) > 0.$$

To see why this inequality holds true, observe that $f(\mu_1^i) = 1 + \mu_1^i(\mu_2^i - 1)$ is an affine function of μ_1^i with a negative slope $\mu_2^i - 1$ (because $i \in J$ implies $\mu_2^i < 1$). When $\mu_1^i = 0$, $f(\mu_1^i) = 1 - \mu_2^i > 0$. When $\mu_1^i = 1$, one has $f(\mu_1^i) = 0$. In our situation we know $\mu_1^i \in (0, 1)$, therefore both evaluations prove our claim, i.e., $f(\mu_1^i) > 0$ when $\mu_1^i \in (0, 1)$.

Thus the parameter λ_0 is well defined.

Consider a parameter λ such that $\lambda_0 \geqslant \lambda > -1$. To prove that M is a λ-Sugeno IFS, we need to show $\mu_2^i \leqslant \mathfrak{c}_\lambda(\mu_1^i)$ for $i = 1, \ldots, n$. The inequality is trivially true when either $\mu_1^i = 0$ or $\mu_2^i = 0$ by the definition of complement. Now suppose i is such that $\mu_1^i \neq 0$ and $\mu_2^i \neq 0$. This means $i \in J$. Observe that

$$\mu_2^i \leqslant \mathfrak{c}_\lambda(\mu_1^i) = \frac{1 - \mu_1^i}{1 + \lambda \cdot \mu_1^i} \Leftrightarrow \lambda \leqslant \frac{1 - (\mu_1^i + \mu_2^i)}{\mu_1^i \cdot \mu_2^i}. \qquad (2)$$

The latter inequality holds true by the requirement $\lambda \leqslant \lambda_0$ and the definition of λ_0, which entails $\lambda_0 \leqslant \frac{1-(\mu_1^i+\mu_2^i)}{\mu_1^i \cdot \mu_2^i}$.

Finally, the definition of $\lambda_0 = \min\{\frac{1-(\mu_1^i+\mu_2^i)}{\mu_1^i \cdot \mu_2^i} \mid i \in J\}$ assures that λ_0 is the largest number with the property described in the statement. When $\lambda_0 < \lambda$, there must be $i \in J$ with $\frac{1-(\mu_1^i+\mu_2^i)}{\mu_1^i \cdot \mu_2^i} < \lambda$, thus $\mu_2^i \leqslant \mathfrak{c}_\lambda(\mu_1^i)$ is false by Eq. (2), and M is not a λ-Sugeno IFS. □

Theorem III.9. *In the conditions of Theorem III.8: there is $q_0 \geqslant 1$, the smallest index such that M is a qROFS for each $q \geqslant q_0$.*

Proof. We can assume $\mu_1^i \neq 1$ for each $i = 1, \ldots, n$ without loss of generality, because $\mu_1^i = 1$ means that x_i is evaluated by the orthopair $(1, 0)$ due to the fact that M is never-full. The orthopair $(1, 0)$ is a valid evaluation in all qROFSs with $q = 1, 2, \ldots$.

The key fact is that for each $i = 1, \ldots, n$,

$$\lim_{q\to\infty} \sqrt[q]{1 - (\mu_1^i)^q} = 1$$

(due to the restriction $\mu_1^i \neq 1$ which derives from our assumption), hence there is a smallest index q_i with $\mu_2^i \leqslant \sqrt[q]{1 - (\mu_1^i)^q}$ (because $\mu_2^i < 1$) when $q \geqslant q_i$. Here we are using the fact that the sequence $\{\sqrt[q]{1 - (\mu_1^i)^q}\}_{q=1}^{\infty}$ is increasing.

Then we can check that $q_0 = \max\{q_1, \ldots, q_n\}$ is the desired index directly.

Let us fix $q \geqslant q_0$. To prove that M is a qROFS, we need to show $\mu_2^i \leqslant \sqrt[q]{1 - (\mu_1^i)^q}$ for $i = 1, \ldots, n$. As $q \geqslant q_0 \geqslant q_i$, this inequality is guaranteed by the choice of q_i.

To finalize the argument, we need to show that if $q < q_0$, then M is not a qROFS, There must be $i \in \{1, \ldots, n\}$ with $q < q_i$. Using the fact that q_i is the smallest index with the property $\mu_2^i \leqslant \sqrt[q_i]{1 - (\mu_1^i)^{q_i}}$, it is compulsory that $\mu_2^i > \sqrt[q]{1 - (\mu_1^i)^q}$. This proves the claim. □

Both Theorems III.8 and III.9 are constructive, i.e., they give the corresponding explicit solution to our fitting problems. This is a very important and practical feature to have. The next examples demonstrate how we can use both constructions to produce the 'optimal' models that embed a given orthopair satisfying the mild restrictions of the theorems:

Example III.10. *Consider the following never-full orthopair fuzzy set on $X = \{x_1, \ldots, x_7\}$:*

$$\begin{aligned} M =\{&< x_1, (0.1, 0.91) >, < x_2, (0.3, 0.97) >, \\ &< x_3, (0.5, 0.91) >, < x_4, (0.6, 0.86) >, \\ &< x_5, (0.7, 0.89) >, < x_6, (0.8, 0.82) >, \\ &< x_7, (0.9, 0.71) >\}. \end{aligned}$$

To find the smallest λ-Sugeno IFS such that M is a λ-Sugeno IFS, we follow the steps in the proof of Theorem III.8. Here $J = \{1, \ldots, 7\}$ and

$$\begin{aligned} \lambda_0 = \min\{ &-0.10989, -0.927835, -0.901099, -0.891473, \\ &-0.94703, -0.945122, -0.954617\} \end{aligned}$$

hence $\lambda_0 = -0.954617$. For illustration, the first of these seven figures is associated with the evaluation provided for x_1, which is $\frac{1-(0.1+0.91)}{0.1 \cdot 0.91} \approx -0.10989$. The other figures *are calculated similarly with the evaluations given for the remaining elements of X.*

Figure 2 illustrates the conclusions of this example.

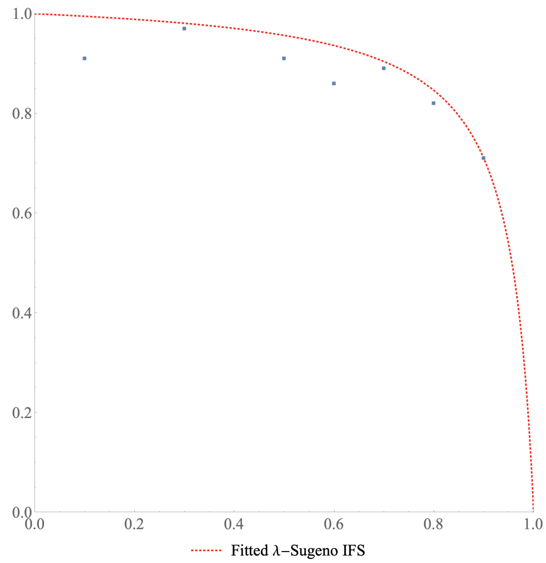

Figure 2. A graphical presentation of the smallest λ-Sugeno IFS that fits the orthopair fuzzy set M in Example III.10 (cf., Theorem III.8).

Example III.11. *We consider the orthopair fuzzy set M defined in Example III.10. In order to find the smallest qROFS such that M is a qROFS, we follow the steps in the proof of Theorem III.9. Now*

$$q_0 = \max\{q_1, \ldots, q_7\} = \max\{2, 3, 3, 3, 4, 4, 4\} = 4.$$

For illustration, we compute $q_7 = 4$ because $0.71 > \sqrt[3]{1 - 0.9^3} = 0.647127$ but $0.71 \leqslant \sqrt[4]{1 - 0.9^4} = 0.765787$.

Figure 3 illustrates the conclusions of this example. It shows how the orthopair fuzzy set M is a 4ROFS, but it is not a 3ROFS. Each point for which its q_i figure in the constructive proof is strictly smaller than 4 does not satisfy the condition determining a 3ROFSs on X.

Figure 4 summarizes the relationships among models that have been presented in this paper.

IV. AGGREGATION OF $\mathfrak{c}$-COMPLEMENTAL FUZZY SETS

Section II has prepared the ground for a general theory of aggregation of $\mathfrak{c}$-complemental fuzzy sets. Proposition II.10 is the touchstone result. The general construction that it has presented produces remarkable aggregation operators on $\mathfrak{c}$-pairs, like the OWA case (cf., Example II.11). Now we take advantage of both the general construction and the specific example provided by OWA operations in order to implement a general aggregation methodology for $\mathfrak{c}$-complemental fuzzy sets. An example will illustrate its application.

Definition IV.1. *Let us fix $\mathfrak{c}$, a fuzzy complement.*

A mapping $\mathbb{A} : \mathfrak{c}\boldsymbol{CFS}(X) \times \overset{m}{\cdots} \times \mathfrak{c}\boldsymbol{CFS}(X) \to \mathfrak{c}\boldsymbol{CFS}(X)$ is called a $\mathfrak{c}\boldsymbol{CFS}(X)$-aggregation operator when it satisfies: for each $(F_1, \ldots, F_m) \in \mathfrak{c}\boldsymbol{CFS}(X) \times \overset{m}{\cdots} \times \mathfrak{c}\boldsymbol{CFS}(X)$, if $\mathbb{A}(F_1, \ldots, F_m) = \{\langle x, (\mu_F(x), \nu_F(x))\rangle | x \in X\}$ then

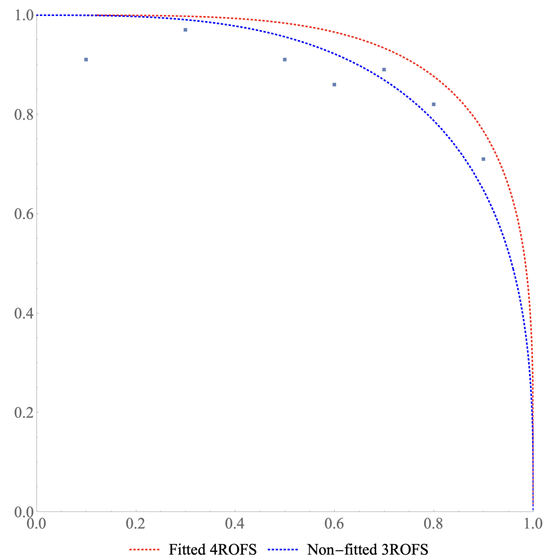

Figure 3. A representation of the smallest qROFS that fits the orthopair fuzzy set in Example III.10: v., Example III.11.

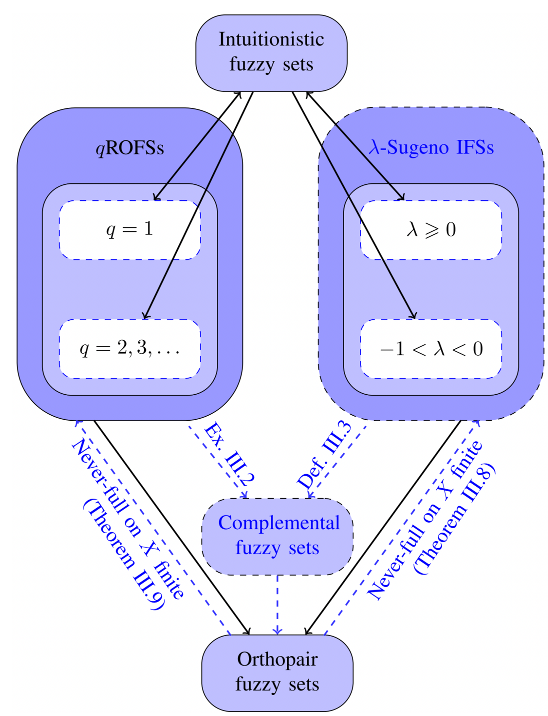

Figure 4. A graphical summary of relationships among models. The dashed blue arrows and models represent original results.

- *Boundary condition: $(\mu_F(x), \nu_F(x)) = (1, 0)$ when $F_i(x) = (1, 0)$ for all i; and $(\mu_F(x), \nu_F(x)) = (0, 1)$ when $F_i(x) = (0, 1)$ for all i.*
- *Monotonicity: if $F_i \subseteq G_i$ for every $i = 1, \ldots, m$ with $(G_1, \ldots, G_m) \in \mathfrak{c}\boldsymbol{CFS}(X) \times \overset{m}{\cdots} \times \mathfrak{c}\boldsymbol{CFS}(X)$, then $\mathbb{A}(G_1, \ldots, G_m) \succcurlyeq \mathbb{A}(F_1, \ldots, F_m)$.*

Theorem IV.2. *Let us fix $\mathfrak{c}$, a fuzzy complement, and an aggregation operator $\mathcal{A} : I \times \overset{m}{\cdots} \times I \to I$.*

Define a mapping $\bar{\mathcal{A}}$ as follows: if for each $i = 1, \ldots, m$, $F_i = \{\langle x, (\mu_{F_i}(x), \nu_{F_i}(x))\rangle | x \in X\}$ is a $\mathfrak{c}$-complemental *fuzzy set, we let*

$$\bar{\mathcal{A}}(F_1,\ldots,F_m) = F = \{\langle x, (\mu_F(x), \nu_F(x))\rangle | \, x \in X\}$$

where for each $x \in X$*,*

$$(\mu_F(x), \nu_F(x)) = \mathcal{A}_{\mathfrak{c}}\left((\mu_{F_1}(x), \nu_{F_1}(x)), \ldots, (\mu_{F_m}(x), \nu_{F_m}(x))\right)$$
$$= (\mathcal{A}(\mu_{F_1}(x), \ldots, \mu_{F_m}(x)), \mathfrak{c}(\mathcal{A}((\nu_{F_1}(x))^{\mathfrak{c}}, \ldots, (\nu_{F_m}(x))^{\mathfrak{c}}))) .$$

Then $\bar{\mathcal{A}} : \mathfrak{c}\boldsymbol{CFS}(X) \times \overset{m}{\cdots} \times \mathfrak{c}\boldsymbol{CFS}(X) \to \mathfrak{c}\boldsymbol{CFS}(X)$ *is a* $\mathfrak{c}\boldsymbol{CFS}(X)$*-aggregation operator.*

Proof. We prove that $\bar{\mathcal{A}}$ satisfies Definition IV.1 with the help of Proposition II.10.

• Boundary condition. Suppose first that $x \in X$ is such that $(\mu_{F_i}(x), \nu_{F_i}(x)) = (1, 0)$ for all i. Then $\bar{\mathcal{A}}(F_1,\ldots,F_m)(x) = \mathcal{A}_{\mathfrak{c}}((\mu_{F_1}(x), \nu_{F_1}(x)), \ldots, (\mu_{F_m}(x), \nu_{F_m}(x))) = (1, 0)$, because $\mathcal{A}_{\mathfrak{c}}$ is a $\mathfrak{c}$-aggregation operator by Proposition II.10. We obtain $\bar{\mathcal{A}}(F_1,\ldots,F_m)(x) = (0, 1)$ when $F_i(x) = (0, 1)$ for all i using an analogous argument.

• Monotonicity. Suppose $F_i \subseteq G_i$ for every $i = 1,\ldots,m$, $(F_1,\ldots,F_m), (G_1,\ldots,G_m) \in \mathfrak{c}\mathbf{CFS}(X) \times \overset{m}{\cdots} \times \mathfrak{c}\mathbf{CFS}(X)$. Let us write $F_i = \{\langle x, (\mu_{F_i}(x), \nu_{F_i}(x))\rangle | x \in X\}$ and $G_i = \{\langle x, (\mu_{G_i}(x), \nu_{G_i}(x))\rangle | x \in X\}$ for all $i = 1,\ldots,m$. The assumption means that for each $x \in X$ and $i = 1,\ldots,m$, $(\mu_{G_i}(x), \nu_{G_i}(x)) \succcurlyeq (\mu_{F_i}(x), \nu_{F_i}(x))$, cf. Definition III.7. Using the definition for a fixed $x \in X$:

$\bar{\mathcal{A}}(G_1,\ldots,G_m)(x) = \mathcal{A}_{\mathfrak{c}}((\mu_{G_1}(x), \nu_{G_1}(x)), \ldots, (\mu_{G_m}(x), \nu_{G_m}(x)))$

and

$\bar{\mathcal{A}}(F_1,\ldots,F_m)(x) = \mathcal{A}_{\mathfrak{c}}((\mu_{F_1}(x), \nu_{F_1}(x)), \ldots, (\mu_{F_m}(x), \nu_{F_m}(x)))$.

We conclude $\bar{\mathcal{A}}(G_1,\ldots,G_m)(x) \succcurlyeq \bar{\mathcal{A}}(F_1,\ldots,F_m)(x)$ because $\mathcal{A}_{\mathfrak{c}}$ is a $\mathfrak{c}$-aggregation operator. Since this property holds true for all $x \in X$, we deduce $\bar{\mathcal{A}}(G_1,\ldots,G_m) \succcurlyeq \bar{\mathcal{A}}(F_1,\ldots,F_m)$. □

Theorem IV.2 can be used to produce specific formulas for IFSs when $\mathfrak{c}$ is the classic complement $\mathfrak{c}(a) = 1 - a$, and also for qROFSs when $\mathfrak{c}$ is Yager's fuzzy complement. Then any choice of $\mathcal{A}$ yields a particular case of aggregation operator. Thus for example, when we let $\mathfrak{c}$ be the classic complement, we can use weighted geometric or OWA operators to obtain the IFWG and IFOWG operators defined (on intuitionistic fuzzy values only) in [19]. Or we can use the Bonferroni mean to develop the IFBM operator (also on intuitionistic fuzzy values or numbers) presented in [20]. Einstein operators can be utilized like in [21]. In the case where $\mathfrak{c}$ is Yager's fuzzy complement, fixing $\mathcal{A}$ as an OWA operator produces the aggregation operator on qROFSs studied by Yager in [2, Section VIII] (cf., Example II.11).

Needless to say, aggregation operators for the λ-Sugeno intuitionistic fuzzy sets will be defined if we let $\mathfrak{c}$ belong to the Sugeno family of fuzzy complements, when $\mathcal{A}$ is an aggregation operator like those mentioned above (weighted arithmetic/geometric, OWA, Bonferroni, or others). The next example shows this practical application in a concrete situation.

Example IV.3. *For illustration, let us examine the* $\mathfrak{c}\boldsymbol{CFS}(X)$*-aggregation operators that stem from some particular cases of aggregation operators when the fuzzy complement is* $\mathfrak{c}_{-\frac{1}{2}} = \frac{2-2x}{2-x}$*. To avoid making the notation too cumbersome, in this example we shall simply denote* $\mathfrak{c}_{-\frac{1}{2}}$ *as* $\mathfrak{c}$*. It will suffice to resort to* $X = \{x_1, x_2\}$ *for our purposes. We shall aggregate the following three* $(-\frac{1}{2})$*-Sugeno IFSs on* X*:*

$$\begin{aligned} F_1 =& \{\langle x_1, (0.1, 0.94)\rangle, \langle x_2, (0.2, 0.85)\rangle\}, \\ F_2 =& \{\langle x_1, (0.4, 0.75)\rangle, \langle x_2, (0.2, 0.1)\rangle\}, \text{ and} \\ F_3 =& \{\langle x_1, (0.7, 0.4)\rangle, \langle x_2, (0.8, 0.3)\rangle\} \end{aligned}$$

with the help of the construction in Theorem IV.2. The output depends upon the choice of the aggregation operator $\mathcal{A}$*.*

(1) Suppose that $\mathcal{A}$ *is the arithmetic average. Then*

$$\bar{\mathcal{A}}(F_1, F_2, F_3) = \{\langle x_1, (0.4, 0.73332)\rangle, \langle x_2, (0.4, 0.487992)\rangle\}$$

because $\mathcal{A}(0.1, 0.4, 0.7) = \frac{0.1+0.4+0.7}{3} = 0.4$*, and* $\mathfrak{c}(\mathcal{A}(0.94^{\mathfrak{c}}, 0.75^{\mathfrak{c}}, 0.4^{\mathfrak{c}})) = \mathfrak{c}(\frac{\mathfrak{c}(0.94)+\mathfrak{c}(0.75)+\mathfrak{c}(0.4)}{3}) = 0.73332$*. Also,* $\mathcal{A}(0.2, 0.2, 0.8) = \frac{0.2+0.2+0.8}{3} = 0.4$*, and* $\mathfrak{c}(\mathcal{A}(0.85^{\mathfrak{c}}, 0.1^{\mathfrak{c}}, 0.3^{\mathfrak{c}})) = \mathfrak{c}(\frac{\mathfrak{c}(0.85)+\mathfrak{c}(0.1)+\mathfrak{c}(0.3)}{3}) = 0.487992$*.*

(2) Suppose that $\mathcal{A}$ *is the geometric average. Then*

$$\begin{aligned} \bar{\mathcal{A}}(F_1, F_2, F_3) =& \{\langle x_1, (0.303659, 0.806796)\rangle, \\ & \langle x_2, (0.31748, 0.583348)\rangle\} \end{aligned}$$

because $\mathcal{A}(0.1, 0.4, 0.7) = \sqrt[3]{0.1 \cdot 0.4 \cdot 0.7} = 0.303659$*, and* $\mathfrak{c}(\mathcal{A}(0.94^{\mathfrak{c}}, 0.75^{\mathfrak{c}}, 0.4^{\mathfrak{c}})) = \mathfrak{c}(\sqrt[3]{\mathfrak{c}(0.94) \cdot \mathfrak{c}(0.75) \cdot \mathfrak{c}(0.4)}) = 0.806796$*. Also,* $\mathcal{A}(0.2, 0.2, 0.8) = \sqrt[3]{0.2 \cdot 0.2 \cdot 0.8} = 0.31748$*, and* $\mathfrak{c}(\mathcal{A}(0.85^{\mathfrak{c}}, 0.1^{\mathfrak{c}}, 0.3^{\mathfrak{c}})) = \mathfrak{c}(\sqrt[3]{\mathfrak{c}(0.85) \cdot \mathfrak{c}(0.1) \cdot \mathfrak{c}(0.3)}) = 0.583348$*.*

(3) Suppose that $\mathcal{A}$ *is an OWA operator (cf., Definition II.5). Example II.11 has referred to this case in the framework of qROFSs. Here we shall use the operator with weights* $w = (0.5, 0.3, 0.2)$*. Then*

$$\bar{\mathcal{A}}(F_1, F_2, F_3) = \{\langle x_1, (0.49, 0.653199)\rangle, \langle x_2, (0.5, 0.597786)\rangle\}$$

because $\mathcal{A}(0.1, 0.4, 0.7) = 0.5 \cdot 0.7 + 0.3 \cdot 0.4 + 0.2 \cdot 0.1 = 0.49$*,* $\mathcal{A}(0.94^{\mathfrak{c}}, 0.75^{\mathfrak{c}}, 0.4^{\mathfrak{c}}) = \mathcal{A}(0.113208, 0.4, 0.75) = 0.515$*, and* $\mathfrak{c}(\mathcal{A}(0.94^{\mathfrak{c}}, 0.75^{\mathfrak{c}}, 0.4^{\mathfrak{c}})) = \mathfrak{c}(0.515) = 0.653199$*. Also,* $\mathcal{A}(0.2, 0.2, 0.8) = 0.5 \cdot 0.8 + 0.3 \cdot 0.2 + 0.2 \cdot 0.2 = 0.5$*,* $\mathcal{A}(0.85^{\mathfrak{c}}, 0.1^{\mathfrak{c}}, 0.3^{\mathfrak{c}}) = \mathcal{A}(0.26087, 0.947368, 0.823529) = 0.573684$*, and* $\mathfrak{c}(\mathcal{A}(0.85^{\mathfrak{c}}, 0.1^{\mathfrak{c}}, 0.3^{\mathfrak{c}})) = \mathfrak{c}(0.573684) = 0.597786$*.*

V. Concluding Remarks

Intense research was set in motion after Atanassov's intuitionistic fuzzy sets were launched. This paper presents complemental fuzzy sets as a unified framework for the advancement of the research about various theories studying orthopair fuzzy sets. An immediate benefit of this novel presentation is that it provides a common semantic justification for the theories embedded into complemental fuzzy sets. Remarkable examples include q-rung orthopair fuzzy sets, a generalization of Atanassov's intuitionistic fuzzy sets.

In passing, we have inaugurated the analysis of λ-Sugeno intuitionistic fuzzy sets. But new practical avenues of research have also been explored. Particularly, respective "goodness-of-fit" exercises have found *the largest* λ for which a never-full orthopair fuzzy set is a λ-Sugeno intuitionistic fuzzy set, and *the smallest* q for which a never-full orthopair fuzzy set is a q-rung orthopair fuzzy set. These are novel exercises that should help to put both λ-Sugeno intuitionistic fuzzy sets and q-rung orthopair fuzzy sets to good use in practice.

Other future lines of research come to mind easily. A similarity measure for constrained Pythagorean fuzzy sets has been applied to medical diagnosis [13, Section VI]. The hybridization between probabilistic and fuzzy information that produced that model can be extended to complemental fuzzy sets in the future. And then on a practical level, analogous applications should follow. It is worthy of note that similarity measures (of IFSs, and more general models) have applications to pattern recognition too [22, Section V]. Aggregation operators that incorporate the familiarity degree of the experts with the objects can be studied by inspiration of e.g., [23]. Additionally, the integration of the new Sugeno intuitionistic fuzzy sets with multi-criteria decision making methodologies is worth investigating. These methodologies include TOPSIS, CODAS (for Combinative Distance based ASsesment), ARAS (for Additive Ratio ASsessment), and many others. In point of fact, applications that use these methodologies in the qROFS setting abound in recent times, e.g., [7], [8], [24], [25].

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