ON THE MODULAR REPRESENTATION THEORY OF ALGEBRAIC CHEVALLEY GROUPS

by PAUL WILLIAM WINTER

A thesis submitted for the degree of Doctor of Philosophy at the University of Warwick

January 1976

CONTENTS

Page No.

References. (i)

Glossary of Symbols. (iii)

Introduction. (iv)

§1. The Modular Representation Theory of Algebraic Chevalley Groups.

1.1 Preliminaries.

(a) The Affine Ring R (of a linear algebraic group). 1

(b) R-modules. 1

(c) The Coefficient Space. 3

(d) The Socle of R. 4

(e) Injective R-modules. 4

1.2 The Chevalley Group, G_K. 6

1.3 The Socle of R and the Irreducible R-modules, $R = K[G_K]$. 7

1.4 Reduction Modulo p. 10

1.5 The Decomposition Numbers. 11

1.6 The Injective Indecomposables. 13

1.7 Cartan Invariants. 13

1.8 Blocks. 14

1.9 Characters. 15

§2. On the Decomposition Matrix.

2.1 The Pseudo - Decomposition. 18

2.2 Determination of D'. 20

2.3 Application to Types A_1, A_2 and B_2. 23

§3. The Modular Representation Theory of SL(2,K).

3.1 The Group SL(2,K). 37

3.2 The structure of the Weyl module reduced mod p. 37

3.3 The Affine Ring ; Submodules and Decompositions. 45

3.4 Restriction to the Borel subgroup. 51

3.5 The Injective Indecomposables. 55

CONTENTS (ctd.)

Page No.

3.6 Cartan Invariants and Blocks. 61

Conjectures. 63

Illustrations.

FIG 1. Configurations for A_2. 33

FIG 2. Example in A_2 34

FIG 3. Example in A_2. 35

FIG 4. Example in B_2. 36

The Injective Indecomposables. 60

References.

[1] A. Borel : Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Math. 131, Springer Verlag 1970.

[2] N. Bourbaki : Groupes et algèbres de Lie, Chaps. IV-VI, Hermann, Paris 1969.

[3] B. Braden : Restricted Representations of Classical Lie Algebras of type A_2 and B_2. Bull. A.M.S. 73 (1967), 482-486.

[4] R.W. Carter, G. Lustzig : On the Modular Representations of the General Linear and Symmetric Groups, Math. Zeit. 136, 193-242 (1974).

[5] E. Cline, R. Carter : Proc. of the Conference on Finite Groups. Park City, Utah (1975).

[6] J.A. Green : Locally Finite Representations, Preprint Univ. of Warwick (1975).

[7] J.E. Humphreys : Modular Representations of Classical Lie Algebras and Semisimple Groups, J. Alg. 19, 57-79 (1971).

[8] ____________ : Projective Modules for SL(2,q), J. of Alg., 25, 513-518 (1973).

[9] N. Jacobson : Lie Algebras, New-York, London. Wiley-Interscience 1962.

[10] J.C. Jantzen : Zur Charackterformel gewisser Darstellungen Halbeinfacher Gruppen und Lie-Algebren, Math. Zeit 140, 127-149, (1974).

[11] ____________ : Üben das Dekompositionsverhalten gewisser Modularen Darstellungen Halbeinfachen Gruppen. Universität Bonn (1975).

[12] A.V. Jeyakumar : Principal Indecomposable Representations for the Group SL(2,q), unpublished manuscript.

[13] D.E. Littlewood : The Theory of Group Characters.

[14] B. Srinivasan : On the Modular Characters of the Special Linear Group $SL(2,p^n)$, Proc. London Math. Soc. 14 (1964) 101-114.

[15] R. Steinberg : Lectures on Chevalley Groups, Yale Univ. Math. Dept.(1968).

[16] __________ : Representations of Algebraic Groups, Nagoya Math. J. 22 (1963) 33-56.

[17] D.N. Verma : Role of Affine Weyl Groups in the Representation Theory of Algebraic Chevalley Groups and their Lie-Algebras, Proc. of the Budapest Summer School on Group Representations 1971.

[18] P.W. Winter : Locally Finite Representations, M.Sc. Thesis Univ. of Warwick.

Glossary of Symbols.

	Page.
$K[G]$.	1
$cf(V)$.	3
$\sigma(V)$.	4
$\Phi, W, \Delta, \beta_i, X, \geqslant$.	6
$G_K, U^-, H, U, B, X(H), H^*_{\mathbb{R}}, K_0, K_p, R_0, R_p$.	7
$V^\lambda, \Pi_V, V(\lambda), M(\lambda), V_{\lambda,K}$.	8
V^{Fr}.	9
$\bar{V}, D = (d_{\lambda\mu})$.	11
$\Gamma_p, H_{\alpha,np}, s_{\alpha,np}, W_p, X_p^+$, p-alcove, ρ.	12
$C = (c_{\lambda\mu}), I(\lambda)$.	13
$\chi_V, Z[X], [\lambda]$.	15
$\chi(\lambda), \varphi(\lambda), \Phi(\lambda)$.	16
$w.\lambda$.	17
$\psi(\lambda), T = (t_{\kappa\mu}), D' = (\Delta_{\lambda\kappa})$	18
$\Omega(\lambda)$.	19
$\mathfrak{A}, \hat{C}, \lambda_{(i)}, \lambda^{(i)}, \chi_i$, generating equation.	20
${}_i\lambda, \sigma_i$.	21
$\rho_i^{(w)}$.	22
$\bar{\Omega}(\lambda)$.	23
$\rho_i(A_1)$.	25
$\rho_i(A_2), \bar{\rho}_i$.	27
A	37
S_I	39
$\prec, D_\lambda$.	40
$T_\mu, \hat{I}(\mu)$.	41
I_μ.	42

	Page.
$\Pi_r, \lambda_r, \Pi(n), A(n)$.	46
$N(r,n), \iota_s, H(r)$.	47
$T(n), c_{r,n}, M(r,n), I(r,n)$.	49
$\Pi_r(n)$.	52
$B(r)$.	54
$J(r,n), r'$.	56
$J_m(r,n), B(r)_m, \bar{\rho}_m^{-1}.r$.	57
$T(n)_m$.	58

Acknowledgements

I would like to thank Prof. J.A. Green for supervising me during the past three years with patience and application. Without his generous help sections (3.3) - (3.5) of this work would probably not have been written. Thanks also to Prof. R.W. Carter for providing information on the material of §2.

Introduction

This thesis aims to provide an introduction to the modular representation theory of algebraic Chevalley groups. Chapter §1 contains the general theory so far known, most of which is due to Green [6] who sets up the modular theory in the more general context of co-algebras. In §2 the decomposition matrix is discussed. In particular, its reliance on the p-restricted part is made as explicit as possible. The general results obtained are applied to the A_1, A_2 and B_2 cases. Chapter §3 provides the simplest example of the theory, that of the group SL(2,K), K an algebraically closed field of char. $p \neq 0$. The structure of the Weyl module reduced modulo p is given in (3.2). This was done independently of Cline [5]. In (3.3) the structure of the affine ring K[SL(2,K)] is analysed, which provides the setting for (3.5) where the injective indecomposable modules are found. Section (3.6) gives the Cartan invariants and blocks, their nature in general being conjectured at the end of the thesis.

§1 THE MODULAR REPRESENTATION THEORY OF ALGEBRAIC CHEVALLEY GROUPS.

(1.1) Preliminaries.

Let K be an algebraically closed field and G a linear algebraic group. Then G is a closed subgroup of $GL_n = GL(n,K)$ for some n. Let $\mathfrak{F} = \mathfrak{F}(G,K)$ be the commutative K-algebra with identity of all functions $G \rightarrow K$ under pointwise operations.

(1.1a) The Affine Ring R

G possesses an affine (co-ordinate) ring $R = K[G]$ of regular functions. R is a finitely generated K-subalgebra of $\mathfrak{F}$ and arises in the following way.

Let $K[GL_n] = K[\gamma_{ij}, \delta]$ where $\gamma_{ij}, \delta \in \mathfrak{F}$; $1 \leq i,j \leq n$, are defined by $\gamma_{ij} : x \mapsto x_{ij}$, $\delta : x \mapsto (\det x)^{-1}$ for all $x = (x_{ij}) \in GL_n$.
Then there exists an exact sequence,

$$0 \longrightarrow \mathfrak{J}(G) \longrightarrow K[GL_n] \xrightarrow{\tau} K[G] \longrightarrow 0$$

where $\tau : f \mapsto f|_G$ restricts functions to G, $\ker \tau = \mathfrak{J}(G)$ and $G = \{x \in GL_n : f(x) = 0 \ \forall f \in \mathfrak{J}(G)\}$. Regular functions can be described as follows. Define $f \in \mathfrak{F}$ to be finitary if it satisfies the property,

(1.1.1) $\quad f(xy) = \sum_{i \in I} f_i(x) f_i'(y)$ all $x,y \in G$ where $f_i, f_i' \in R$ and I is finite. Then a function is regular if and only if it is finitary. For suppose f is finitary. Then from (1.1.1) $f = \sum_{i \in I} f_i(1) f_i' \in R$.
Conversely, if f is regular then it must be finitary, since γ_{ij}, δ are clearly so. This, of course, is a restatement of the fact that group multiplication is a morphism.

(1.1.1) also shows that R is closed under left and right translations. For if $f \in R$ then $R_x f = \sum f_i'(x) f_i \in R$, and $L_x f = \sum f_i(x) f_i' \in R$. Hence R may be regarded as a 2-sided KG-module with right translation as left G-action and left translation as right G-action.

(1.1b) R-modules

Suppose that V is a (left) rational KG-module having a K-basis $(v_i)_{i \in I}$. Then, by definition, the functions $a_{ij} : G \rightarrow K$ in the equations

(1.1.2) $$xv_j = \sum_{i \in I} a_{ij}(x)v_i \qquad (i \in I,\ x \in G)$$

all belong to R. The matrix $A = (a_{ij})$ is called the invariant matrix of the representation. V is said to be locally finite if A is column finite and it is easy to show that this definition is independent of the basis chosen for V and is equivalent to the statement t hat every cyclic submodule of V is finite dimensional. By a (left, right, 2sided) R-module we understand a locally finite rational (left, right, 2-sided) KG-module. A R-homomorphism is simply a KG-homomorphism. It is immediate from local finiteness that every irreducible R-module must be finite dimensional.

(1.1.3) Proposition R is a 2-sided R-module.

Proof We prove R is a (left) R-module. Let $f \in R$ and m be minimal such that, in the above notation,

$$R_x f = \sum_{i=1}^{m} f'_i(x) f_i \qquad \text{all } x \in G.$$

Clearly it suffices to prove that KGf is finite dimensional (i.e. R is locally finite). In fact we show that $(f_i)_{i=1\ldots m}$ is a basis. By minimality of m, (f_i) is a linearly independent set. Similarly (f'_i) is a linearly independent set and hence there exist $y_1 \ldots y_m \in G$ such that $(f'_i(y_j))$ is non-singular. This means that f_i may be expressed as a linear combination of the $R_{y_j} f$ and so belongs to KGf. //

The category of R-modules and R-homomorphisms is closed to taking submodules, quotients and sums. Also the Krull-Schmidt and Jordan-Hölder theorems hold in this category, as do standard theorems on complete reducibility.

Henceforth we assume R-modules to be left unless otherwise stated.

(1.1c) The Coefficient Space

Let V be an R-module with functions $a_{ij} \in R$ defined as in (1.1.2). Then the coefficient space $cf(V)$ of V is the K-subspace of R spanned by the a_{ij} $(i,j \in I)$. This is independent of the basis $(v_i)_{i \in I}$ chosen for V. Since

$$(1.1.4) \qquad a_{ij}(xy) = \sum_{k \in I} a_{ik}(x)a_{kj}(y) \qquad x,y \in G,$$

we see that $cf(V)$ is a 2-sided R-submodule of R. In fact,

(1.1.5) Proposision Let V be an R-submodule of R. Then $cf(V)$ is the least 2-sided R-submodule of R containing V.

Proof Evaluating $xv_j = \sum_{i \in I} a_{ij}(x)v_i$ at the identity gives $v_j(1.x) = \sum a_{ij}(x)v_j(1)$. That is $v_j = \sum v_i(1)a_{ij} \in cf(V)$.

Hence $V \subseteq cf(V)$.

Now let W be any 2-sided R-submodule of R containing V.

Then $L_{x^{-1}} v_j = \sum_{i \in I} v_i(x)a_{ij} \qquad \forall\, j \in I,\ x \in G.$

As in the proof of (1.1.3) we find that all $a_{ij} \in W$ since W is a right R-module. Therefore $cf(V) \subseteq W$. //

More generally we remark that any R-module V may be embedded in a direct sum of copies of $cf(V)$ via the map $v_i \longmapsto (a_{ir})_{r \in I}$.

(1.1.6) Theorem (Burnside)

Let V be an irreducible R-module of K-dimension d.

Then, (i) V is isomorphic to a submodule of R.

(ii) $cf(V) = \bigoplus_{i=1}^{d} V_i$ where each $V_i \cong V$

(iii) $cf(V)$ contains all copies of V in R.

(In fact (ii) is the classical Burnside theorem. Since (iii) follows immediately from (1.1.5) we prove part (i) only.)

Proof of (i) Let V be as in (1.1.2) with $|I| = d$.

For each $r = 1,\ldots,d$ the K-map $\theta_r : V \to R$ defined by $\theta_r(v_j) = a_{rj}$ $(1 \leq j \leq d)$, is a R-homomorphism.

Since V is irreducible θ_r is either 0 or a monomorphism. But $\theta_r = 0$ implies that $a_{rj} = 0$ $(1 \leq j \leq d)$, contradicting the non-singularity of (a_{ij}). Hence θ_r is a R-monomorphism.

(1.1d) <u>The Socle of R</u>

If V is an R-module, then the <u>socle</u> $\sigma(V)$ of V is the sum of all the irreducible submodules of V. Hence $\sigma(V)$ is the unique maximal completely reducible (c.r.) submodule of V.

(1.1.7) <u>Theorem</u> (i) Let $\{V_\lambda\}_{\lambda \in \Lambda}$ be a full set of irreducible R-modules. Then $\sigma(R) = \bigoplus_{\lambda \in \Lambda} cf(V_\lambda)$.

(ii) If $\sigma(R) = \bigoplus_{\alpha \in A} W_\alpha$ with $\{W_\alpha\}_{\alpha \in A}$ a set of irreducible R-modules, then for each $\lambda \in \Lambda$, the set $A_\lambda = \{\alpha \in A : W_\alpha \cong V_\lambda\}$ contains exactly $\dim V_\lambda$ elements, and $\sum_{\alpha \in A_\lambda} W_\alpha = cf(V_\lambda)$.

<u>Proof</u> (i) Clearly $\sigma(R) \subseteq \sum_{\lambda \in \Lambda} cf(V_\lambda)$ using (1.1.5). But by (1.1.6(ii)), $cf(V_\lambda)$ is a sum of irreducibles. Hence $\sigma(R) = \sum_{\lambda \in \Lambda} cf(V_\lambda)$. Directness follows from the Jordan-Holder theorem.

(ii) Let $C_\lambda = \sum_{\alpha \in A_\lambda} W_\alpha$, then $\sigma(R) = \bigoplus_\lambda C_\lambda$. Now by (1.1.6), $C_\lambda \subseteq cf(V_\lambda)$ for all $\lambda \in \Lambda$. Comparison with (i) gives $C_\lambda = cf(V_\lambda)$. Hence the result.//

Finally a result on c.r. modules.

(1.1.8) <u>Proposition</u> V is a c.r. R-module if and only if $cf(V) \subseteq \sigma(R)$.

<u>Proof</u> Let V be c.r., then $V = \sum V_\alpha$ with V_α irreducible. Hence $cf(V) = \sum cf(V_\alpha) \subseteq \sigma(R)$. Conversly if $cf(V) \subseteq \sigma(R)$ then $cf(V)$ is c.r. But as already pointed out in (1.1c), V can be embedded in a direct sum of copies of $cf(V)$. Hence V is c.r.

R is said to be <u>semisimple</u> if $\sigma(R) = R$. By (1.1.5), $cf(R) = R$. Therefore (1.1.8) gives R semisimple if and only if every R-module is c.r.

(1.1e) <u>Injective R-modules</u> (Green [6])

An R-module I is said to be <u>injective</u> if whenever V, W are R-modules with $V \subseteq W$ and $\theta : V \rightarrow I$ a R-homomorphism, then there exists an extension $\theta^* : W \rightarrow I$ of θ.

It is routine to show that finite direct sums and direct summands of injective R-modules are injective, and that an injective submodule of an R-module V is complemented in V. What is not quite so obvious is the fact that a direct sum of injectives is injective, but we do not prove this here. The following proposition shows that injectives are characterised by their socles.

(1.1.9) Proposition Let I, I' be injective R-modules. Then every hom. (isom.) $\theta : \sigma(I) \to \sigma(I')$ extends to a hom. (isom.) $\theta^* : I \to I'$.

Proof The R-map $\theta : \sigma(I) \to \sigma(I')$ extends to $\theta^* : I \to I'$ by injectivity of I. Suppose θ is an isomorphism. Then θ^* is monomorphic since $(\ker\theta^*) = \ker\theta^* \cap \sigma(I) = 0$.

Hence $I_1 = \theta^*(I)$ has a complement I_2 in I'.

i.e. $I' = I_1 \oplus I_2$ and so $\sigma(I') = \sigma(I_1) \oplus \sigma(I_2)$.

But $\sigma(I') = \theta^*(\sigma(I)) \subseteq \sigma(I_1)$. Hence $\sigma(I_2) = 0$ and so $I_2 = 0$.

i.e. θ^* is injective. //

Let V be an R-module and I an injective R-module. Then I is said to be an injective cover of V if there exists an R-map $\theta : V \to I$ which induces an isomorphism $\sigma(V) \to \sigma(I)$. The map θ is necessarily an embedding, and it is an easy consequence of (1.1.9) that I is unique up to isomorphism.

(1.1.10) Theorem [6] Every R-module has an injective cover.

The proof of this important result rests on Brauer's idempotent lifting process and will be omitted.

It is immediate that an injective cover of an irreducible module must be indecomposable. In fact,

(1.1.11) Proposition There is a 1-1 correspondence between the isomorphism classes of injective indecomposable R-modules and the isomorphism classes of irreducible R-modules given by $I \leftrightarrow \sigma(I)$.

Proof Because of (1.1.9) and the above remark we need only show that the socle of an injective indecomposable I is irreducible.

Suppose not, then $\sigma(I) = V_1 \oplus V_2$ where V_1, V_2 are non-zero. Let I_1, I_2 be injective covers of V_1, V_2 resp. Then I is isomorphic to $I_1 \oplus I_2$ since they are both injective covers of $\sigma(I)$. This is a contradiction.

In a manner to be elucidated in (1.6.1) R may be decomposed uniquely into a direct sum of injective indecomposables, and the proof of this depends crucially on the following,

(1.1.12) Proposition R is an injective R-module.

Proof Let V be an R-module. Then it is easy to verify that the map $\xi^* : \mathrm{Hom}_R(V,R) \to \mathrm{Hom}_R(V,K)$ defined by $\xi^*(f) = \xi \circ f$, where $\xi : R \to K$ is given by $\xi(f) = f(1)$, is a K-linear isomorphism.

Let V,W be R-modules with $V \subseteq W$ and $\theta : V \to R$ any R-map. Extend $\xi\theta$ in any way to a K-linear map $\alpha : W \to K$.

Let $\theta' : W \to R$ be the unique R-map such that $\xi\theta' = \alpha$. Then $\theta'|_V$ is an R-map $V \to R$, and $\xi(\theta'|_V) = (\xi\theta')|_V = \alpha|_V = \xi\theta$

Hence $\theta'|_V = \theta$ .//

This completes (1.1) and attention will now be restricted to G a Chevalley group.

(1.2) The Chevalley Group [1] , [15]

Let $\underline{g}$ be a simple complex Lie algebra with Cartan subalgebra $\underline{h}$ and associated root system Φ with Weyl group W. Φ contains a fundamental sysem Δ and a set Φ^+ of positive roots. $\underline{h}^*_{\mathbb{R}}$, the $\mathbb{R}$-space generated by Δ, becomes a Euclidean space of dimension $l = |\Delta|$ (the rank of $\underline{g}$) when equipped with the positive definite inner product (,) dual to the Killing form. Let X denote the full lattice of weights in $\underline{h}^*_{\mathbb{R}}$ and X^+ the subset of dominant weights. Then X has a basis consisting of the fundamental dominant weights $\{\beta_i\}_{i=1,\dots,l}$ (relative to Δ) and is endowed with a partial order $\leq$, where $\lambda \geq \mu$ whenever $\lambda - \mu$ is a non-negative integral combination of fundamental roots.

Let π be a representation of $\underline{g}$ on a $\mathbb{C}$-space V of dimension n. Then given an algebraically closed field K and an admissible $\mathbb{Z}$-form $V_{\mathbb{Z}}$ of V, a Chevalley group $G_K = G_{V,K}$ can be constructed as follows. Let $\{x_r, h_s : r \in \Phi, s \in \Delta\}$ be a Chevalley basis of $\underline{g}$ and set $x_r(t) = \exp t\pi(x_r)$, $V_K = V_{\mathbb{Z}} \otimes K$. Then $G_K = \langle x_r(t) : r \in \Phi, t \in K \rangle$ and is a closed subgroup of $GL(V_K) = GL(n,K)$. Thus G_K may be viewed as a (semi-simple) algebraic group so the results in (1.1) apply.

Certain subgroups of G_K turn out to very important in the representation theory. Let $u_r : SL(2,K) \to G_K$, $r \in \Phi$, be the homomorphism which maps $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ onto $x_r(t)$, $x_{-r}(t)$ resp., and let $h_r(t)$ be the image of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Then we define subgroups

$U = \langle x_r(t) : r \in \Phi^+, t \in K \rangle$, $H = \langle h_s(t) : s \in \Delta, t \in K \rangle$ and

$U^- = \langle x_r(t) : r \in \Phi^-, t \in K \rangle$ where $\Phi = \Phi^+ \dot{\cup} \Phi^-$

U, U^- are unipotent groups and H is a torus in G_K. $B = HU$ is a Borel subgroup and $G_K = (B, N = N_G(H))$ is a $B.N$ pair with Weyl group $W = {}^N\!/_H$ and root system Φ.

<u>Henceforth we assume that G_K is a universal Chevalley group of type $\underline{g}$ over K.</u> This means precisely that X may be identified with $X(H)$, the group of rational characters of H. Hence $\underline{h}^*_{\mathbb{R}}$ may be identified with $H^*_{\mathbb{R}} = X(H) \otimes \mathbb{R}$. Some implications of this restriction are that H is isomorphic to a direct product of l copies of K^* and that $\underline{g}, \underline{h}$ are the Lie algebras of G_K, H resp.

(1.3) <u>The Socle of R and the irreducible R-modules, $R = K[G_K]$</u>.

Here and throughout the remainder of §1, $R_o = K_o[G_{K_o}]$ and $R_p = K_p[G_{K_p}]$ where K_o, K_p are algebraically closed fields of characteristic 0 and char $p \neq 0$ resp.

Let V be an R-module, and for $\lambda \in X(H)$ define

(1.3.1) $V^{\lambda} = \{v \in V: hv = \lambda(h)v, \text{ all } h \in H\}$. Then $V = \oplus V^{\lambda}$.

If λ is such that $V^{\lambda} \neq 0$ then it is called a <u>weight</u> of V and we let Π_V denote the set of weights of V. Now W acts on $X(H)$ thus: for $w \in W$, $w(\lambda)(h) = \lambda(w^{-1}h\,w)$. Hence $wV^{\lambda} = V^{w(\lambda)}$ and $\dim V^{\lambda} = \dim V^{w(\lambda)}$. In particular Π_V is W-invariant.

(1.3.2) <u>Theorem</u> [15] R_0 is semisimple.

Thus $\sigma(R_0) = R_0$ but we point out that this is not true for R_p.

<u>The Irreducible R-modules</u>

The irreducibles are known completely and we state the fundamental result at once.

(1.3.3) <u>Theorem</u> [15] Let V be an irreducible R-module. Then,

(i) There exists a unique B-fixed line L in V with corresponding character λ which is uniquely determined and dominant. All other weights of V are of the form $\mu < \lambda$. Any non-zero vector in L is called a <u>highest weight vector</u>.

(ii) $V \cong V'$ if and only if their corresponding dominant characters, called <u>highest weights</u>, are equal.

(iii) Every $\lambda \in X^+$ is the highest weight of some V.

We remark that a similar theorem holds for irreducible $\mathfrak{g}$-modules.

(1.3.4) <u>Corollary</u> Let V be an irreducible R-module. Then

$$\mathrm{End}_R(V) = K\iota_V, \text{ where } \iota_V \text{ is the identity map on } V.$$

<u>Proof</u> Clear, since V is absolutely irreducible by the theorem.

Let $\{V(\lambda)\}_{\lambda \in X^+}$ denote a full set of irreducible R_0-modules and $\{M(\lambda)\}_{\lambda \in X^+}$ a full set of irreducible R_p-modules. Then they may be realised as follows. Let V be an irreducible $\mathfrak{g}$-module of highest weight λ with admissible $\mathbb{Z}$-form $V_{\lambda,\mathbb{Z}}$. Set $V_{\lambda,K} = V_{\lambda,\mathbb{Z}} \otimes K$. Then $V(\lambda) \cong V_{\lambda,K_0}$. Suppose further that $V_{\lambda,\mathbb{Z}}$ is minimal, then V_{λ,K_p} has a unique maximal R_p-submodule the quotient by which is $M(\lambda)$.

Let V be an irreducible R-module. Then V is finite dimensional (see (1.1b)) and may be embedded in R by (1.1.6(i)). Wenow describe the most natural copy of V in R. Let (v_i) be a basis of V adapted so that v_1 is a highest weight vector, and let (a_{ij}) be the corresponding invariant matrix. Then,

(1.3.5) <u>Proposition</u> Let V be as above with highest weight λ. Then there is an embedding of V in $A_{\lambda,K} = \{ f \in R : L_{b^-} f = \lambda(b^-)f, \text{ all } b^- \in B^- \}$, where $B^- = U^- H.$, with image KGa_{11}.

As a corollary we see that $V(\lambda)$ and A_{λ,K_0} are isomorphic as R_0-modules by complete reducibility.

<u>Steinberg's Tensor Product Theorem</u>

The nature of the irreducible R_p-modules is known in more detail. In order to give a description we mention that R_p admits a ring endomorphism Fr, called the Frobenius endomorphism, which takes each co-ordinate function to its p^{th} power. If (V,ρ) is a rational representation defined over the prime field then V^{Fr} denotes the R_p-module affording $\rho \circ Fr$. V^{Fr} has invariant matrix $A^{Fr} = (a_{ij}^p)$ where $A = (a_{ij})$ is the invariant matrix of V.

(1.3.6) <u>Lemma</u> (i) If V is an irreducible R_p-module of highest weight λ then V^{Fr} is irreducible of highest weight $p\lambda$.

(ii) If V is an R_p-submodule of R_p, then $Fr(V) = V^{Fr}$.

In view of (ii) we always let Fr act exponentially.

(1.3.7) <u>Theorem</u> (Steinberg[16])

Let $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$ where $0 \leq (\lambda_i, \alpha^\vee) < p$ for all $\alpha \in \Delta$

($\alpha^\vee$ is the co-root of α). Then ,

$M(\lambda) = \bigotimes_{i=0}^{n-1} M(\lambda_i)^{Fr^i}$ is the irreducible R_p-module of highest weight λ.

Hence if σ_λ is the invariant matrix of $M(\lambda)$, then

$$\sigma_\lambda = \sigma_{\lambda_0} \times \sigma_{\lambda_1}^{Fr} \times \cdots \times \sigma_{\lambda_{n-1}}^{Fr^{n-1}} \quad \text{(Kronecker product).}$$

<u>The Socle of R</u>

Combining the results so far in (1.3) with (1.1c) and (1.1d) we have,

(1.3.8) <u>Theorem</u> (i) Let V be an irreducible R-module of highest weight λ as in (1.3.4) and dimension $d_{\lambda,K}$. Then $cf(V)$ is the 2-sided R-module generated by a_{11}, and is isomorphic to a direct sum of $d_{\lambda,K}$ copies of V. Moreover $cf(V)$ contains all copies of V in R.

(ii) $R_0 = \sigma(R_0) = \bigoplus_{\lambda \in X^+} cf(V(\lambda))$,

$\sigma(R_p) = \bigoplus_{\lambda \in X^+} cf(M(\lambda))$ and these decompositions are unique in the sense of (1.1.7(ii)).

1.4 <u>Reduction Modulo p</u>

We retain the notation of (1.1a) and put $\tau_{ij} = \tau(\gamma_{ij})$. G_K is generated by unipotent elements (see 1.2), hence $\tau(\delta) = \underline{1}$ the identity of R, and $R = K[\tau_{ij}]$. Define a ring homomorphism $\mu : R \to R \otimes R$ by $\mu f(x,y) = f(xy)$. (i.e. μ is the co-morphism to multiplication in G_K). Note that from (1.1.1), $\mu f = \sum_{i \in I} f_i \otimes f_i'$ with $f_i, f_i' \in R$.

Also by (1.1.4) $\mu \tau_{ij} = \sum_{r=1}^{n} \tau_{ir} \otimes \tau_{rj}$ $\qquad (\psi)$

Let $K = K_0$, and set $L_0 = Z[\tau_{ij}]$. Then by (ψ) $\mu_0 L_0 \subset L_0 \otimes L_0$. Also L_0 is free as a Z-module, since it may be identified with a Z-subalgebra of the free Z-module $Z[U^-] \otimes Z[H] \otimes Z[U]$.

Now let $K = K_p$ and $L_p = F_p[\tau_{ij}]$, F_p the Galois field of p elements. Then $\mu_p L_p \subset L_p \otimes L_p$ and the obvious ring epimorphism $\theta : L_0 \to L_p$ makes the following diagram commute.

$$\begin{array}{ccc} L_0 & \xrightarrow{\mu_o} & L_0 \otimes L_0 \\ \theta \downarrow & & \downarrow \theta\otimes\theta \\ L_p & \xrightarrow[\mu_p]{} & L_p \otimes L_p \end{array}$$

Hence if $A = (a_{ij})$ is an invariant matrix for R_0 (i.e. $\mu_0 A = A \otimes A$), such that $a_{ij} \in L_0$, then $\theta(A) = (\theta(a_{ij})) = (\overline{a}_{ij}) = \overline{A}$ is an invariant matrix for L_p.

Now let V be an R_0-module. Then there is a free Z-submodule $L_0(V)$ of V, called an L_0-lattice, which has a Z-basis $(v_i)_{i\in I}$ such that (i) (v_i) is also a K_0-basis of V, and (ii) the invariant matrix $A = (a_{ij})$ afforded by (v_i) has coefficients $a_{ij} \in L_0$. Define $\overline{V} = L_0(V) \otimes K_p$. Then $\overline{V}$ is an R_p-module with invariant matrix $\overline{A} = (\overline{a}_{ij})$ afforded by the basis $\overline{v}_i = v_i \otimes 1_{K_p}$.
$\overline{V}$ is called the 'reduction mod p' of V. The structure of $\overline{V}$ depends upon the choice of L_0-lattice but we do have $\Pi_V = \Pi_{\overline{V}}$, multiplicities counted.

<u>Remark</u> As the notation indicates, we do not specify the choice of L_0-lattice in the reduction unless it is essential to the argument.
With notation as in (1.3), let $V^{min}_{\lambda,Z}$ have basis $\{v_i : 1 \leq i \leq d_{\lambda,K_0}\}$
Then $L_0(V(\lambda)) = \sum_{i=0}^{d_{\lambda,K_0}} Z(v_i \otimes 1_{K_0})$ is an L_0-lattice and $V_{\lambda,K_p} = \overline{V(\lambda)}$ has a unique top composition factor $M(\lambda)$. $\overline{V(\lambda)}$ is then cyclic, generated by $v_0 \otimes 1_{K_p}$ where v_0 is a highest weight vector of weight λ in $V(\lambda)$, and also indecomposable. In the next section we discuss the composition factors of $\overline{V(\lambda)}$.

1.5 <u>The Decomposition Numbers</u>

Define $d_{\lambda\mu}$ to be the composition multiplicity of $M(\mu)$ in $\overline{V(\lambda)}$. The integers $d_{\lambda\mu}$ are well defined (since $\Pi_V = \Pi_{\overline{V}}$) and are called the <u>decomposition numbers</u> of R.

Since $\overline{V(\lambda)} = \sum_{\mu\leq\lambda} d_{\lambda\mu} M(\mu)$ in the Grothendieck ring of the category of

R_p-modules we see at once that the decomposition matrix $D = (d_{\lambda\mu})_{\lambda,\mu \in X^+}$ can be put in unitriangular form. The relationship between those λ, μ satisfying $d_{\lambda\mu} \neq 0$ can be described in terms of the geometry $(H^*_{\mathbb{R}}, W_p)$ where W_p is a certain subgroup of the affine Weyl group to be defined presently.

To this end let p be a positive integer, usually taken to be the 'reduction prime'. Define the p-diagram Γ_p to be the union of all hyperplanes of the form $H_{\alpha,np} = \{ x \in H^*_{\mathbb{R}} : (x,\alpha^\vee) = np, \alpha \in \Phi, n \in Z \}$. A p-alcove is defined to be a component of $H^*_{\mathbb{R}} \setminus \Gamma_p$. Let $s_{\alpha,np}$ denote the reflection $s_{\alpha,np}(x) = x - ((x,\alpha^\vee) - np)$ in the hyperplane $H_{\alpha,np}$. Then W_p is defined to be the group generated by all such reflections. In particular Γ_1 is the Cartan-Stiefel diagram and W_1 the affine Weyl group; Γ_0 is the infinitesimal diagram and $W_0 = W$. In Γ_0 the alcoves which are unbounded, are the Weyl chambers. X^+ is contained in the closure of the fundamental Weyl chamber and the fundamental dominant weights lie on its walls. We call $X^+_p = \{\lambda \in X^+ : 0 \leq (\lambda,\alpha^\vee) < p, \alpha \in \Delta\}$ the p-restricted region of X^+. It is well known that X^+_p contains $\frac{|W|}{[X:X']}$ p-alcoves where X' is the root lattice. Finally we note that W_p is simply transitive on p-alcoves.

Now the main general results known about D are two conjectures of Verma [17]. The first asserts that the so called Harish-Chandra principle is true, i.e. $d_{\lambda\mu} \neq 0$ only if

(1.5.1) $\mu + \rho = w(\lambda + \rho)$ for some $w \in W_p$, where ρ is the half sum of positive roots. This was proved by Humphreys [7] for $p > h$, the Coxeter number of Φ, and for general p in type A_n ($G = SL_{n+1}$) by Carter-Lustzig. [4]. In fact Humphreys proved that the highest weights of the composition factors of an indecomposable R_p-module are related in the above way, but we know that $\overline{V(\lambda)}$ is indecomposable. The second result, a refinement of the first, states that if $d_{\lambda\mu} \neq 0$ then there exist reflections $w_1, \ldots, w_k \in W_p$ such that

(1.5.2) $\lambda + \rho \geq w_1(\lambda + \rho) \geq w_2 w_1(\lambda + \rho) \geq \ldots\ldots \geq w_k \ldots w_1(\lambda + \rho) = \mu + \rho$.

This was proved by Jantzen[10] for Φ of type A_n and a few other cases of small rank, again for $p > h$. We attempt to gain further information about D in §2.

1.6 The Injective Indecomposables

Let $I(\lambda)$ be an injective cover of $M(\lambda)$. Then by (1.1.11) $\{I(\lambda)\}_{\lambda \in X^+}$ is a full set of injective indecomposable R_p-modules. Copies of these injectives are distributed in R_p in a way given by the following.

(1.6.1) Theorem [6] (i) There exist R_p-submodules $I(\lambda)_i$ of R_p, $1 \leq i \leq d_\lambda = \dim M(\lambda)$, such that $R_p = \sum^{\oplus}_{\lambda \in X^+} (I(\lambda)_1 \oplus \ldots \oplus I(\lambda)_{d_\lambda})$. This decomposition extends that of $\sigma(R_p)$ in (1.3.8(ii)), $\sigma(I(\lambda)_i) = M(\lambda)_i$ and $I(\lambda)_i \cong I(\lambda)$.

(ii) If $R_p = \bigoplus_{\alpha \in A} J_\alpha$ is any decomposition of R_p as a direct sum of indecomposable R_p-submodules J_α of R_p, then for each $\lambda \in X^+$ the set $A_\lambda = \{\alpha \in A : J_\alpha \cong I(\lambda)\}$ contains exacly d_λ elements.

Proof (i) Recall that $cf(M(\lambda)) = \bigoplus_{i=1}^{d_\lambda} M(\lambda)_i$ in (1.3.8(ii)), with $M(\lambda)_i \cong M(\lambda)$. Let $I(\lambda)_i'$ be an injective cover of $M(\lambda)_i$. Then since R_p is injective (1.1.12) we have 2 injective covers of $\sigma(R_p)$ viz : R_p and $\bigoplus_{\lambda,i} I(\lambda)_i'$. Hence there is an R_p-isomorphism $\phi : \bigoplus I(\lambda)_i' \longrightarrow R_p$. Defining $I(\lambda)_i = \phi(I(\lambda)_i')$ proves (i).

(ii) It is easy to show that $\sigma(R_p) = \bigoplus \sigma(J_\alpha)$. But J_α is injective indecomposable and so $\sigma(J_\alpha)$ is irreducible. An application of (1.1.7(ii)) gives the result.

Remark The theorem holds trivially for R_0 being identical with (1.1.7).

1.7 Cartan Invariants

Define the Cartan invariants $c_{\lambda\mu}$, for $\lambda, \mu \in X^+$, to be the composition multiplicity of $M(\mu)$ in $I(\lambda)$. We show that these integers depend solely on the decomposition numbers.

For R_p-modules V, W let $(V,W)_p = \dim_{K_p} \operatorname{Hom}_{R_p}(V,W)$.

(1.7.1) <u>Theorem</u> [6] (i) Every injective R_p-module arises as the reduction mod p of an injective R_0-module.

(ii) Let V be an R_p-module. Then for $\mu \in X^+$, $(V, I(\mu))_p$ is the multiplicity of $M(\mu)$ in V.

(iii) Let V be an R_0-module and I an injective R_0-module. Then $(V,I)_0 = (\overline{V},\overline{I})_p$.

We suppress the proof, but use the theorem to reach our objective. By (i) there exists an injective R_0-module I_μ such that $\overline{I}_\mu \cong I(\mu)$. From (ii) and (iii), $d_{\lambda\mu} = (\overline{V(\lambda)}, I(\mu))_p = (V(\lambda), I_\mu)_0$.

But by complete reducibility this is just the multiplicity of $V(\lambda)$ in I_μ. It follows that $d_{\lambda\mu}$ is the multiplicity of $\overline{V(\lambda)}$ in $I(\mu)$ and in turn,

(1.7.2) $$c_{\lambda\mu} = \sum_{\nu \in X^+} d_{\nu\lambda}\, d_{\nu\mu} \quad (= (I(\lambda), I(\mu))_p \text{ by (ii) }).$$

In general the Cartan invariants will be infinite. In fact, evidence in cases of low rank suggests that they are either zero or infinite. Equation (1.6.3) is then interpreted to mean that $c_{\lambda\mu} \neq 0$ if and only if there exists a $\nu \in X^+$ such that $d_{\nu\lambda} \neq 0$, $d_{\nu\mu} \neq 0$. With this in mind, the <u>Cartan matrix</u> $C = (c_{\lambda\mu})$ has the form $C = {}^tD.D$.

1.8 <u>Blocks</u>

Weights $\lambda, \mu \in X^+$ are said to be <u>adjacent</u> if either $(I(\lambda), I(\mu))_p \neq 0$ or $(I(\mu), I(\lambda))_p \neq 0$ or both. Hence by (1.6.3),

(1.8.1) λ, μ are adjacent $\Leftrightarrow c_{\lambda\mu} \neq 0$.

Therefore, since $d_{\lambda\lambda} = 1$, $d_{\lambda\mu} \neq 0$ implies the adjacency of λ and μ. An equivalence relation $\leftrightarrow$ on X^+ is then defined by $\lambda \leftrightarrow \mu$ if there exists a finite sequence $\lambda = \mu_0, \mu_1, \dots, \mu_n = \mu$ in X^+ such that μ_i, μ_{i-1} are adjacent $1 \leq i \leq n$. Hence we have a partition,

(1.8.2) $X^+ = \dot{\bigcup}_{\lambda \in B} B_\lambda$, where $\{B_\lambda\}_{\lambda \in B}$ is the set of equivalence classes.

under $\leftrightarrow$ and B is some set of class representatives in X^+. We call

(1.8.2) the <u>block partition</u> of X^+ and the classes B_λ <u>blocks</u>.

For a conjecture on blocks see P63. It is evident from (1.6.3) and (1.8.1) that the following is true.

(1.8.3) <u>Proposition</u> The Cartan matrix C may be put in the form

$C = \bigoplus_{\lambda \in B} C_\lambda$ where C_λ are 'indecomposable' matrices. Also

$D = \bigoplus_{\lambda \in B} D_\lambda$ where D_λ are 'indecomposable' and $C_\lambda = {}^tD_\lambda . D_\lambda$.

The rows and columns of C_λ, D_λ are indexed by B_λ.

Now for each $\lambda \in B$, define the <u>block components</u> R_λ of R_p by

$R_\lambda = \sum^{\oplus}_{\mu \in B_\lambda} (\bigoplus_{i=1}^{d_\mu} I(\mu)_i)$ c.f. (1.6.1). Then

(1.8.4) $R_p = \bigoplus_{\lambda \in B} R_\lambda$.

The main result on block components runs as follows.

(1.8.5) <u>Theorem</u> [6] Let $\lambda \in B$.

(i) R_λ is an indecomposable 2-sided R_p-submodule of R_p.

(ii) The decomposition (1.8.4) is a refinement of any decomposition of R_p as a direct sum of 2-sided R_p-submodules.

(iii) $R_\lambda = \sum_{\mu \in B_\lambda} cf(I(\mu))$.

We leave the proof except to say that (iii) is an easy consequence of (i) and (1.1.5).

<u>Remark</u> For $p = 0$, equation (1.8.4) is precisely that of (1.3.7(ii)).

Finally we mention that any R_p-module V has a decomposition

$V = \bigoplus_{\lambda \in B} W_\lambda$ with $cf(W_\lambda) \subset R_\lambda$. V is said to belong to the block B_λ if this decomposition is trivial. i.e. if $cf(V) \subset R_\lambda$.

1.9 <u>Characters</u>

Let V be a finite dimensional R-module and define the character χ_V of V by $\chi_V(x) = Tr_V(x)$, all $x \in G_K$. Clearly $\chi_V \in R$. By virtue of the fact that $\chi_V(x) = \chi_V(x_s)$ where x_s is the semisimple part of x, χ_V is determined by its values on H. Denote by $Z[X]$ the integral group ring of X with multiplication given by $[\lambda] . [\mu] = [\lambda + \mu]$.

Then from (1.3.1) we see that χ_V may be represented formally

$$\chi_V = \sum_{\mu \in X} m_V(\mu)\,[\mu] \qquad \text{where} \quad m_V(\mu) = \dim V^{\mu}.$$

The Weyl group W acts on X, and so on $Z[X]$, this action extending Z-linearly to ZW. Thus $w[\lambda] = [w\lambda]$ for $w \in W$. Hence from the remarks following (1.3.1) and the fact that $\Pi_V = \Pi_{\bar{V}}$, the following is clear:-

(1.9.1) Proposition (i) $\chi_V = \sum_{w \in W} w\,\chi_V^+ \in Z[X]^W$ where

$$\chi_V^+ = \sum_{\mu \in \Pi_V \cap X^+} m_V(\mu)\,[\mu].$$

(ii) $\chi_V = \chi_{\bar{V}}$.

(iii) $\chi_{V \otimes W} = \chi_V . \chi_W$.

(Note : strictly speaking, in (i), $m_V(\mu)$ should be multiplied by the factor $|\mathrm{Stab}_W(\mu)|^{-1}$.)

Let $\chi(\lambda) = \chi_{V(\lambda)}$. Then Weyl [9] showed that $\chi(\lambda)$ could be written as a quotient of alternating elements,

(1.9.2) $\chi(\lambda) = G(\lambda)/G(0)$ where $G(\lambda) = \sum_{w \in W} \det w\,[w(\lambda+\rho)]$, and that

(1.9.3) $d_{\lambda, K_0} = \dim V(\lambda) = \prod_{\alpha \in \Phi^+} \dfrac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}$

The multiplicities $m(\lambda,\mu)$ of V^{μ} in $V(\lambda)$ may be calculated from the formula of Kostant or by the method of Freudenthal [9].

The action of Fr on $Z[X]$ is given by $[\lambda]^{Fr} = [\lambda]^p = [p\lambda]$. Hence for an R_p-module V, $\chi_{V^{Fr}} = \chi_V^{Fr} = \sum_{\mu \in X} m_V(\mu)\,[p\mu]$.

Now set $\varphi(\lambda) = \chi_{M(\lambda)}$ and suppose that $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$ with $\lambda_i \in X_p^+$.

Then from Steinberg's theorem (1.3.7),

$$\varphi(\lambda) = \prod_{i=0}^{n-1} \varphi(\lambda_i)^{Fr^i}.$$

Letting $\Phi(\lambda) = \chi_{I(\lambda)}$, we have the following set of equations,

(1.9.4)
$$\chi(\lambda) = \sum_{\mu \leq \lambda} d_{\lambda\mu}\,\varphi(\mu)$$
$$\Phi(\lambda) = \sum_{\mu \geq \lambda} d_{\mu\lambda}\,\chi(\mu)$$
$$\Phi(\lambda) = \sum_{\mu \in X^+} c_{\lambda\mu}\,\varphi(\mu)$$

where $d_{\lambda\mu}$, $c_{\lambda\mu}$ are defined as in (1.5) and (1.7).

Let $X(G_K)$ denote the character ring of G_K with multiplication as in (1.9.1(iii)). Then,

(1.9.5) <u>Proposition</u> The characters of the irreducible R-modules form a Z-free basis of $X(G_K)$.

<u>Proof</u> $\{\chi(\lambda)\}_{\lambda \in X^+}$ is a Z-free basis of $Z[X]^W$. ([2] Ch VI)
Hence if $K = K_0$ we are done. But since D is unitriangular,
$\varphi(\lambda) = \sum_{\mu \leq \lambda} \gamma_{\lambda\mu} \chi(\mu)$ where $(\gamma_{\lambda\mu}) = D^{-1}$. This means that $\{\varphi(\lambda)\}_{\lambda \in X^+}$ is a Z-free basis of $X(G_{K_p})$. //

The characters, indeed the dimensions, of the modules $M(\lambda)$ are not known in general. However, as in (1.5), if
$\varphi(\lambda) = \sum_{\mu} a(\lambda,\mu)\, \chi(\mu)$ with $a(\lambda,\mu) \in Z$, then $a(\lambda,\mu) \neq 0$ only if
$\mu \in W_p . \lambda$, where $w.\lambda = w(\lambda+\rho) - \rho$, $w \in W_p$, $\lambda \in X$.

For more information on the character formula see Jantzen [10].

§2. On The Decomposition Matrix

In this section we are primarily concerned with the essentially combinatoric problem of finding the decomposition matrix D given its p-restricted part $(d_{\lambda\mu})_{\lambda \in X_p^+, \mu \in X^+}$. That this is possible was observed by Verma [17]. We begin by discussing a certain matrix fundamental to our method.

2.1 The Pseudo - Decomposition

All weights appearing in this section will be dominant. From (1.9.4) we have $\chi(\lambda) = \sum_{\mu \le \lambda} d_{\lambda\mu} \varphi(\mu)$, and since D is unitriangular, $\varphi(\lambda) = \sum_{\mu \le \lambda} \gamma_{\lambda\mu} \chi(\mu)$ where $(\gamma_{\lambda\mu}) = D^{-1}$.

Let $\kappa = \sum_{i=0}^{n-1} \kappa_i p^i$, $\kappa_i \in X_p^+$. The we form the R-module $N(\kappa) = \overline{V(\kappa_0)} \otimes \overline{V(\kappa_1)}^{Fr} \otimes \ldots\ldots \otimes \overline{V(\kappa_{n-1})}^{Fr^{n-1}}$ with character $\psi(\kappa) = \prod_{i=0}^{n-1} \chi^{Fr^i}(\kappa_i)$. It is not irreducible in general.

(2.1.1) Proposition $\psi(\kappa) = \sum_{\mu \le \kappa} t_{\kappa\mu} \varphi(\mu)$ where $T = (t_{\kappa\mu})$ is unitriangular. Hence $\chi(\lambda) = \sum_{\kappa \le \lambda} \Delta_{\lambda\kappa} \psi(\kappa)$ where $D' = (\Delta_{\lambda\kappa})$ is unitriangular and $D = D'.T$.

We call D' the pseudo - decomposition matrix. Before we can prove (2.1.1) we need a

Lemma $\varphi(\lambda)\, \varphi(\mu) = \sum_{\nu \le \lambda+\mu} b_{\lambda\mu}(\nu)\, \varphi(\nu)$ with $b_{\lambda\mu}(\nu) \in \mathbb{N}$, $b_{\lambda\mu}(\lambda+\mu) = 1$.

Proof $\varphi(\lambda)\, \varphi(\mu) = \sum_{\substack{\kappa \le \lambda \\ \tau \le \mu}} \gamma_{\lambda\kappa} \gamma_{\mu\tau} \chi(\kappa).\chi(\tau) = \sum_{\substack{\kappa \le \lambda \\ \tau \le \mu}} \gamma_{\lambda\kappa} \gamma_{\mu\tau} n_{\kappa\tau}(\alpha)\, \chi(\alpha)$

where the coefficient $n_{\kappa\tau}(\alpha)$ of $\chi(\alpha)$ in $\chi(\kappa).\chi(\tau)$ is zero unless $\alpha \le \kappa + \tau$, and $n_{\tau\kappa}(\kappa+\tau) = 1$.

$= \sum \gamma_{\lambda\kappa} \gamma_{\mu\tau} n_{\kappa\tau}(\alpha)\, d_{\alpha\nu}\, \varphi(\nu)$ where the coefficient of $\varphi(\nu)$ is zero unless $\nu \le \alpha \le \kappa+\tau \le \lambda+\mu$.

The coefficient of $\varphi(\lambda+\mu)$ is $\gamma_{\lambda\lambda} \gamma_{\mu\mu} n_{\lambda\mu}(\lambda+\mu)\, d_{\lambda+\mu,\lambda+\mu} = 1$. //

Proof of (2.1.1) If $\kappa = \sum_{i=0}^{n-1} \kappa_i p^i$, $\kappa_i \in X_p^+$, then

$$\psi(\kappa) = \sum_{\nu_i \leq \kappa_i} \prod_{i=0}^{n-1} d_{\kappa_i \nu_i} \varphi^{Fr^i}(\nu_i) . \qquad (\Psi)$$

If each $\nu_i \in X_p^+$, then Steinberg's theorem gives $\prod_{i=0}^{n-1} \varphi^{Fr^i}(\nu_i) = \varphi(\nu)$ where $\nu = \sum_{i=0}^{n-1} \nu_i p^i \leq \kappa$ and we are done. But it is possible to have $\nu_i \in X^+$, $\nu_i \leq \kappa_i \in X_p^+$ and yet $\nu_i \notin X_p^+$. Let t be the least i such that this happens.

If $t = n-1$, write $\nu_{n-1} = \sum_{i=0}^{k} \nu_{n-1,i} p^i$, $\nu_{n-1,i} \in X_p^+$.

Then $\varphi(\nu_{n-1}) = \prod_{i=0}^{k} \varphi^{Fr^i}(\nu_{n-1,i})$ and again we have $\prod_{i=0}^{n-1} \varphi^{Fr^i}(\nu_i) = \varphi(\nu)$.

If $t < n-1$, write $\nu_t = \nu_{t,0} + \nu_{t,1} p$, $\nu_{t,0} \in X_p^+$.

Then $\varphi^{Fr^t}(\nu_t)\, \varphi^{Fr^{t+1}}(\nu_{t+1}) = \varphi^{Fr^t}(\nu_{t,0})\, (\varphi(\nu_{t,1})\, \varphi(\nu_{t+1}))^{Fr^{t+1}}$.

Now $\varphi(\nu_{t,1})\, \varphi(\nu_{t+1}) = \sum_{\xi \leq \nu_{t,1} + \nu_{t+1}} b(\xi)\, \varphi(\xi)$ by the lemma.

Hence $\prod_{i=0}^{n-1} \varphi^{Fr^i}(\nu_i) = \sum_{\mu \leq \nu} c(\mu) \prod_{i=0}^{n-1} \varphi^{Fr^i}(\mu_i)$, $\mu = \sum_{i=0}^{n-1} \mu_i p^i$, $c(\nu) = 1$.

Now $t+1$ is least such that $\mu_{t+1} \notin X_p^+$, and $\mu \leq \nu \leq \kappa$.

Continuing this process gives the desired result. //

(2.1.2) <u>Corollary</u> (of the above proof).

If Φ is of type A_1, A_2, or B_2 then $t_{\kappa\mu} = \prod_{i=0}^{n-1} d_{\kappa_i \mu_i}$

<u>Proof</u> This follows immediately from (Ψ) and the fact that in these cases if $\lambda \in X_p^+$ and $d_{\lambda\mu} \neq 0$ then $\mu \in X_p^+$. //

We now focus attention on the pseudo - decomposition and first demonstrate that for this matrix the Harish - Chandra property holds for all primes p. Write $\alpha \sim \beta$ if α and β lie in the same orbit of W_p. This relation is clearly an equivalence relation.

(2.1.3) <u>Proposition</u> $\Delta_{\lambda\kappa} \neq 0$ implies $\lambda + \rho \sim \kappa + \rho$.

<u>Proof</u> Let $\Omega(\lambda) = \{\kappa : \Delta_{\lambda\kappa} \neq 0\}$. Then $\chi(\lambda) = \sum_{\kappa \in \Omega(\lambda)} \Delta_{\lambda\kappa}\, \psi(\kappa)$, and on substituting the Weyl character formula and mmltiplying through by the denominator in the right hand side containing the highest power of p we obtain,

$$\sum_{S\in W} \det S\,[S(\lambda+\rho)]\,.\,E_1 = \sum_{\kappa\in\Omega(\lambda)} \Delta_{\lambda\kappa} \sum_{T\in W} \det T\,[T(\kappa_0+\rho)]\,.\,E_2^{\kappa}$$

where $E_1, E_2^{\kappa} \in Z[pX]$ and $E_2^{\kappa}/E_1 = \psi(\kappa-\kappa_0)$.

Hence for cancellation to occur there must exist a sequence $\kappa^{(1)},\ldots,\kappa^{(s)} \in \Omega(\lambda)$, not necessarily unique, such that $\kappa_0+\rho \sim \kappa_0^{(1)}+\rho \sim \ldots \sim \kappa_0^{(s)}+\rho \sim \lambda_0+\rho$ for each $\kappa\in\Omega(\lambda)$. This implies that $\kappa+\rho \sim \lambda+\rho$ from the very definition of W_p.

If we now assume that $d_{\lambda\mu} \neq 0$ implies $\lambda+\rho \sim \mu+\rho$ for $\lambda \in X_p^+$, then $t_{\kappa\nu} \neq 0$ implies $\kappa+\rho \sim \nu+\rho$. For from (Ψ) in the proof of (2.1.1) we have $t_{\kappa\nu} \neq 0$ if all $d_{\kappa_i\nu_i} \neq 0$. In particular $d_{\kappa_0\nu_0} \neq 0$ and hence $\kappa_0+\rho \sim \nu_0+\rho$, giving $\kappa+\rho \sim \nu+\rho$. //

This, together with (2.1.3), extends the Harish - Chandra property to D.

2.2 Determination of D'.

We retain the notation of (1.5) and add to it the following. Let $\mathcal{Q} = \{$ p-alcoves in $X_p^+\}$. Then for $\sigma\in W$, $C\in\mathcal{Q}$ there are translations $\delta_{\sigma,C}p$ in pX uniquely defined by the condition $\sigma(C)+\delta_{\sigma,C}p \subset X_p^+$. The maps $\delta_\sigma : \mathcal{Q}\to\mathcal{Q}$, defined by $\delta_\sigma : C \longrightarrow \sigma(C)+\delta_{\sigma,C}p$ are permutations of $\mathcal{Q}$. Let $C\in\mathcal{Q}$ denote the p-alcove

$\{\lambda\in H_{\mathbb{R}}^* : n_\alpha p < (\lambda,\alpha^\vee) < (n_\alpha+1)p\ ,\ \alpha\in\Phi^+,\ n_\alpha\in Z_{\geq 0}\}$ and

$\hat{C} = \{\lambda\in H_{\mathbb{R}}^* : n_\alpha p < (\lambda,\alpha^\vee) \leq (n_\alpha+1)p\ ,\ \alpha\in\Phi^+\ ,\ n_\alpha\in Z_{\geq 0}\}$ its upper closure.

Then we have a partition $X_p^+ + \rho = \dot{\bigcup}_{C\in\mathcal{Q}} \hat{C}$.

(2.2a) The Iteration Procedure

For $\lambda = \lambda_{(i)} + \lambda^{(i)}p^i \in X^+$, $\lambda_{(i)} \in X_p^+ i$, define $\chi_i(\lambda) = \chi(\lambda_{(i)})\,\chi^{Fr^i}(\lambda^{(i)})$. Then if $\lambda\in X_p^+2$, (2.1.1) and (2.1.3) imply that $\chi(\lambda)$ may be written in the form,

(2.2.1) $\quad \chi(\lambda) = \sum_{w\in W_p} \Delta(\lambda,w)\,\chi_1(w.\lambda) \qquad \Delta(\lambda,w)\in Z.$

We call this the generating equation for D'. Let $\lambda = \lambda_0 + \lambda_1 p$, $\lambda_0,\lambda_1 \in X_p^+$ and $\lambda_0+\rho \in C$. If $w = tp\circ\sigma \in W_p$ with $t\in X$, $\sigma\in W$ then

$$\chi_1(w.\lambda) = \chi(\delta_\sigma(\lambda_0+\rho)-\rho)\,\chi^{Fr}(\sigma(\lambda_1)+t-\delta_{\sigma,C}) \qquad (1)$$

Now the essential point about the generating equation is that it may be considered as an identity in $Z[X]$ with λ_0, λ_1 as indeterminates. Furthermore p may be treated as an integer variable. Hence if $\lambda_0+\rho, \lambda_0'+\rho \in C$, (1) implies that $\Delta(\lambda,w) = \Delta(\lambda',w)$. Extend the action of δ_σ to $\hat{C}$, $\delta_\sigma(\lambda+\rho) = \sigma(\lambda+\rho)+\delta_{\sigma,C}p$, $\lambda+\rho\in\hat{C}$. Clearly we have $-1 \leqslant (\delta_\sigma(\lambda+\rho)-\rho,\check{\alpha}) < p$, $\sigma\in W, \alpha\in\Delta$, $\lambda\in X_p^+$. But if $(\delta_\sigma(\lambda+\rho)-\rho,\check{\alpha}) = -1$ for some α, then $\chi(\delta_\sigma(\lambda+\rho)-\rho) = 0$ since the element $s_{\alpha,0}$ of W will fix $\delta_\sigma(\lambda+\rho)$. Hence we may write $\Delta(\lambda,w) = \Delta(\lambda',w)$ if $\lambda_0+\rho, \lambda_0'+\rho\in\hat{C}$. Thus there are $|\mathcal{Q}| = \frac{|W|}{[X:X']}$ essentially different types of generating equation. If $\lambda_0+\rho\in\hat{C}$ we write $\Delta(\lambda_0,w)$, or $\Delta(C,w)$, for $\Delta(\lambda,w)$.

Let $\sigma_i : H^*_{\mathbb{R}} \longrightarrow H^*_{\mathbb{R}}$ denote the linear map $\sigma_i x = p^i x$ and set ${}_i\lambda = \sigma_{i-1}^{-1}(\lambda_{(i)})$. Then if $\lambda\in X_p^+{}^{i+1}$ the change of variables $p\longrightarrow p^i$, $\lambda_0\longrightarrow\lambda_{(i)}$, $\lambda_1\longrightarrow\lambda^{(i)}$ in (2.2.1) yields,

$$(2.2.2) \qquad \chi(\lambda) = \sum_{w\in W_p} \Delta({}_i\lambda,w)\,\chi_i(\sigma_{i-1}w\sigma_{i-1}^{-1}.\lambda)$$

<u>Remarks</u> 1. $w^{\sigma_{i-1}} \in W_{p^i}$.

2. ${}_i\lambda$ might not, of course, be a weight but clearly $\Delta({}_i\lambda,w) = \Delta(C,w)$ if ${}_i\lambda+\rho\in\hat{C}$.

Suppose now that $\lambda\in X_p^+{}^n$ and define $\{w_{(i)}\}_{i=1,\dots,n-1}$, $w\in W_p$, by $w_{(i)}.\lambda = (w^{\sigma_{i-1}}.\lambda)_{(i)}$. Let $f_{i,\lambda} = f_{i,\lambda}(w_i,\dots,w_{n-1}) = w_i^{\sigma_{i-1}}.(w_{i+1})_{(i+1)}\dots(w_{n-1})_{(n-1)}.\lambda$, $w_j\in W_p$, $i\leqslant j\leqslant n-1$, $0\leqslant i\leqslant n-1$, $w_0 = 1$.

Then iterating (2.2.2) for $i = n-1, n-2, \dots, 1$ successively yields,

$$(2.2.3) \quad \chi(\lambda) = \sum_{w_1}\cdots\sum_{w_{n-1}} \prod_{i=1}^{n-1} \Delta({}_i(f_{i+1,\lambda})_{(i+1)}, w_i) \prod_{i=0}^{n-1} \chi^{Fr^i}(f^{(i)}_{i,\lambda})$$

$$(f_{n,\lambda} = \lambda,\ f^{(0)}_{0,\lambda} = f_{0,\lambda})$$

(2.2b) Conversion into Pseudo - Characters

The aim of this section is to put (2.2.3) into the form of (2.1.1). To this end define $\rho_i^{(w_i)} \in W_p i$, $1 \leq i \leq n-1$, by $\rho_i^{(w_i)} . \lambda = w_i^{\sigma_{i-1}} . \lambda_{(i+1)} + \lambda^{(i+1)} p^{i+1}$. Then $\sum_{i=0}^{n-1} f_{i,\lambda}^{(i)} p^i = \rho_1^{(w_1)} \cdots \rho_{n-1}^{(w_{n-1})} . \lambda$.

Hence if $f_{i,\lambda}^{(i)} \in X_p^+$, $1 \leq i \leq n-1$, then

(2.2.4) $$\prod_{i=0}^{n-1} \chi^{Fr^i}(f_{i,\lambda}^{(i)}) = \psi(\rho_1^{(w_1)} \cdots \rho_{n-1}^{(w_{n-1})} . \lambda)$$

Now suppose that $f_{i,\lambda}^{(i)} \notin X_p^+$ for some i. Then the following method will enable $\prod_{i=0}^{n-1} \chi^{Fr^i}(f_{i,\lambda}^{(i)})$ to be expressed as a linear combination of pseudo-characters. Firstly we may assume that $f_{i,\lambda}^{(i)} \in X^+$; for if not then use of the formula $\chi(\nu) = \det\sigma\, \chi(\sigma,\nu)$, $\sigma \in W$, will rectify the situation. Let t be the least i such that $f_{i,\lambda}^{(i)} \notin X_p^+$. Then the generating equation (2.2.1) will express $\chi(f_{t,\lambda}^{(t)})$ in the form,

$$\chi(f_{t,\lambda}^{(t)}) = \sum_{\kappa} \pm \chi(\kappa_0)\, \chi^{Fr}(\kappa_1) , \quad \kappa = \kappa_0 + \kappa_1 p, \quad \kappa_0 \in X_p^+ .$$

Utilising the product formula $\chi(\kappa_1)\, \chi(f_{t+1,\lambda}^{(t+1)}) = \sum_{\alpha} \chi(\alpha)$,

$\prod_{i=0}^{n-1} \chi^{Fr^i}(f_{i,\lambda}^{(i)})$ can be expressed as a linear combination of certain

$\prod_{i=0}^{n-1} \chi^{Fr^i}(\tau_i)$ with $t+1$ as the least i such that $\tau_i \notin X_p^+$.

Continuing this process gives the desired expression in terms of pseudo-characters.

Finally we note that (2.2b) is required in addition to part (a) if and only if there is an i such that $\rho_i^{(w_i)} \cdots \rho_{n-1}^{(w_{n-1})} . \lambda$ and $\rho_{i+1}^{(w_{i+1})} \cdots \rho_{n-1}^{(w_{n-1})} . \lambda$ lie in different translates of $X_p^+ i+1$ via $p^{i+1} X$, where $\prod_{j=i}^{n-1} \Delta\left({}_j(f_{j+1,\lambda}(w_{j+1},\ldots,w_{n-1}))_{(j+1)}, w_j\right) \neq 0$.

(2.2c) Determination of the Generating Equation

Let $\lambda \in X_p^+ 2$, and $\chi(\lambda) = \sum_{\mu \leq \lambda} \Delta(\lambda,\mu)\, \chi_1(\mu)$ as in (2.2.1).

Then writing $\chi_1(\mu) = \chi(\mu_0)\,\chi^{Fr}(\mu_1)$ and substituting the Weyl formula (1.9.2),

$$\sum_{S,T \in W} \det ST\,[S(\lambda+\rho) + T\rho p] = \sum_{\mu \leq \lambda} \Delta(\lambda,\mu) \sum_{S,T \in W} \det ST\,[S(\mu_0+\rho) + T(\mu_1+\rho)p]$$

(2.2.5) Hence $(\sum_{S \in W} S)Y = (\sum_{S \in W} S)Z$ where $Y = \sum_{T \in W} \det T\,[\lambda+\rho+ T\rho p]$

$$Z = \sum_{\mu} \Delta(\lambda,\mu) \sum_{T \in W} \det T\,[T(\mu_0+\rho) + (\mu_1+\rho)p]$$

As remarked in (2.2a), (2.2.5) is an identity in λ_0, λ_1 where $\lambda = \lambda_0 + \lambda_1 p$.

Let $\bar{\Omega}(\lambda) = \{\mu : \Delta(\lambda,\mu) \neq 0\}$.

(2.2.6) <u>Conjecture</u> The set $\{T(\mu_0+\rho) + (\mu_1+\rho)p\}_{\mu \in \bar{\Omega}(\lambda), T \in W}$.

is contained in the convex linear subspace of X with vertices $\{\lambda+\rho+ T\rho p\}_{T \in W}$.

Henceforth we assume (2.2.6), which implies that the equation $Y = Z$ holds. Moreover if $Y = Z$, then the coefficients $(\Delta(\lambda,\mu))$ appearing in it will be precisely those of (2.2.1). Since $\Delta(\lambda,\lambda) = 1$ we have,

$$\sum_{T \in W} \det T\,[\lambda+\rho+ T\rho p] - \sum_{T \in W} \det T\,[T(\lambda_0+\rho) + (\lambda_1+\rho)p]$$
$$= \sum_{\mu \in \bar{\Omega}(\lambda)} \Delta(\lambda,\mu) \sum_{T \in W} \det T\,[T(\mu_0+\rho) + (\mu_1+\rho)p].$$

Choose the highest weight, together with its sign, which appears in the left hand side after cancellation. Since $T(\mu_0+\rho) < \mu_0+\rho$ when $T \neq 1$, it must equal $\Delta(\lambda,\mu)\,[\mu+\rho+\rho p]$, some $\mu \in \bar{\Omega}(\lambda)$. Now subtract $\sum_{T \in W} \Delta(\lambda,\mu)\,[T(\mu_0+\rho) + (\mu_1+\rho)p]$ from each side and repeat the operation. The procedure is continued until the left hand side vanishes. The coefficients $(\Delta(\lambda,\mu))$ thus extracted are those required.

This algorithmic procedure admits a graphical interpretation for types of small rank which will be illustrated in (2.3).

2.3 <u>Application to types A_1, A_2, B_2</u>

(2.3.1) <u>Theorem</u> Let Φ be of type A_1, A_2, B_2. For $\lambda \in X^+_p 2$ define $\{\mu_T = \mu_{0,T} + \mu_{1,T}\,p : T \in W\}$ by (i) $\mu_{0,T} \in X^+_p$ and

(ii) $\lambda+\rho+T\rho p = T(\mu_{0,T}+\rho)+(\mu_{1,T}+\rho)p$. Then

$\chi(\lambda) = \sum_{T\in W}\chi_1(\mu_T) + \chi_1'$, where $\chi_1' = 0$, $\Phi = A_1, A_2$

$\chi_1' = \sum_{i=1}^{4}\chi_1(\nu_{i,\lambda})$, $\nu_{i,\lambda}$ distinct, for $\Phi = B_2$.

Furthermore if $\{\lambda+\rho+T\rho p : T\in W\}\subset X_p^+2$, then all weights above are dominant.

Proof The equation $Y = Z$ of (2.2.5) has a solution, and from it the theorem follows. Conjecture (2.2.6) also holds. This will be demonstrated later in the A_1, A_2 cases. The 4 extra terms appearing in the B_2 case can be deduced from (2.3.2) and (2.3.3).

(2.3.2) Theorem (Braden [3])

Let Φ be of type A_1, A_2, B_2 and $\lambda\in X_p^+$. Then $\overline{V(\lambda)} = M(\lambda) + M(s_{\alpha,np}\cdot\lambda)$ if there exist $\alpha\in\Phi^+$,$n\in\mathbb{N}$ such that $s_{\alpha,np}\cdot\lambda < \lambda$ and $s_{\alpha,np}\cdot\lambda \in X_p^+$. ($\alpha$,n will then be unique.) Otherwise $\overline{V(\lambda)} = M(\lambda)$.

Let C_0^2 be the p^2-alcove in X^+ whose closure contains 0.

(2.3.3) Corollary Let Φ be of type A_1, A_2, B_2 and define, for $\lambda\in X_p^+2$, the set $D(\lambda) = \{\mu = \mu_0+\mu_1 p : \mu_0\in X_p^+ , \mu_0\leq\mu_{0,T} , \mu_0+\rho \equiv \mu_{0,T}+\rho$ mod W_p , $\mu_1 = \mu_{1,T}$, $T\in W\}$.

Then if λ satisfies (i) $\lambda+\rho\in C_0^2$ (ii) $\{\lambda+\rho+T\rho p : T\in W\}\subset X^+$. (iii) $(\lambda_0+\rho,\alpha^\vee)\neq p$, $\alpha\in\Phi^+$ (iv) $(\lambda_1,\alpha^\vee)\neq p-1$, $\alpha\in\Phi^+$ we have

$d_{\lambda\mu} = 1$ if $\mu\in D(\lambda)$

$= 0$ otherwise.

The corollary follows from (2.3.1) and (2.3.2). The reader should now consult FIG 1, configurations I and II, and FIG 4, which exhibit the set $D(\lambda)$ in the A_2 and B_2 cases. Note that each μ_T occurs in a different translate X_T of X_p^+ and μ_T is the highest weight of $D(\lambda)$ occuring in X_T . All other elements of $D(\lambda)$ are obtained by 'filling up' the translates X_T in the obvious sense. We now tabulate some numerical information.

	$\lvert\mathcal{A}\rvert$	$\lvert\Omega(\lambda)\rvert$	$\lvert D(\lambda)\rvert$
A_1	1	2	2
A_2	2	6	9
B_2	4	12	20

(i) <u>Φ of type A_1</u>

From (2.3.2), $\overline{V(\lambda)}$ remains irreducible for $\lambda \in X_p^+$. Hence modular and pseudo-characters are the same. Identify weights with integers. We first prove (2.3.1) following the method of (2.2c). Suppose

$\lambda = \lambda_0 + \lambda_1 p$, $\lambda_1 \neq 0$.

$$\sum_{T \in W} \det T\,[\lambda + 1 + Tp] = [\lambda + 1 + p] - [\lambda + 1 - p] \qquad (1)$$

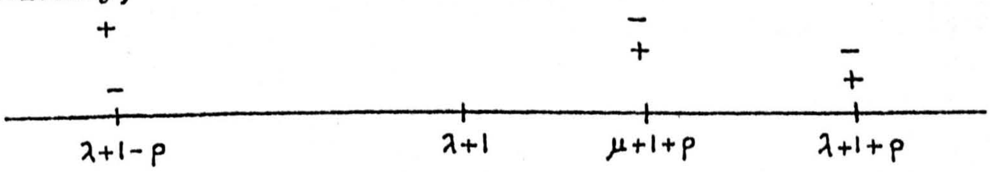
$[\lambda + 1 + p]$ is the highest weight, hence we subtract

$[(\lambda_0 + 1) + (\lambda_1 + 1)p] - [-(\lambda_0 + 1) + (\lambda_1 + 1)p]$ from (1) giving remainder

$$[-(\lambda_0 + 1) + (\lambda_1 + 1)p] - [\lambda + 1 - p] \qquad (2)$$

If $\lambda_0 + 1 = p$, we are finished. If not then $[\mu + 1 + p] = [-(\lambda_0 + 1) + (\lambda_1 + 1)p]$ is the highest weight in (2) and subtracting $[-(\lambda_0 + 1) + (\lambda_1 + 1)p] - [-(p - \lambda_0 - 1) + \lambda_1 p]$ from (2) gives zero.

Diagramatically,



Hence $\chi(\lambda_0 + \lambda_1 p) = \chi(\lambda_0)\,\chi^{Fr}(\lambda_1) + \chi(p - 2 - \lambda_0)\,\chi^{Fr}(\lambda_1 - 1)$

and $\Omega(\lambda) = \{\lambda, \mu\}$ where μ occurs if and only if $\lambda_0 + 1 \neq p$ and $\lambda_1 \neq 0$ and $\mu = w.\lambda$, $w = 2\lambda_1 p \circ (-1) \in W_p$. (see (2.2.1)).

In particular $\chi(\lambda) = \varphi(\lambda) + \varphi(\mu)$.

This proves (2.3.1) and (2.3.3).

We now perform the iteration of (2.2a). By definition $\rho_i^{(1)}$ is the identity, and $\rho_i^{(w)} = \rho_i \in W_p{}^i$ is given by

$\rho_i . \lambda = w^{\sigma^{i-1}} . \lambda_{(i+1)} + \lambda^{(i+1)} p^{i+1} = \lambda - 2(\lambda_{(i)} + 1)$, where $\rho_i . \lambda$ is defined if and only if $\lambda_{(i)} + 1 \neq p^i$, $\lambda_i \neq 0$.

In other words $\rho_i(\lambda + 1)$ is the reflection of $\lambda + 1$ in the highest multiple of p^i less than $\lambda + 1$, provided $\lambda + 1$ is not a multiple of p^i and $\rho_i \neq \rho_{i+1}$. If $\lambda = \lambda_{(i+1)}$, then (2.2.2) reads,

$$\chi(\lambda) = \chi_i(\lambda) + \chi_i(\rho_i\cdot\lambda)\ , \quad \rho_i\cdot\lambda = (p^i - 2 - \lambda_{(i)}) + (\lambda_i - 1)p^i.$$

Iterating as in (2.2a) and using the fact that $\chi(-1) = 0$ gives,

(2.3.4) <u>Theorem</u> If $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$, $\lambda_i \in X_p^+$ then

(i) $d_{\lambda\mu} = 1$ when $\mu = \rho_{i_1} \cdots\cdots \rho_{i_t} \cdot \lambda$, $1 \leqslant i_1 < \cdots\cdots < i_t \leqslant n-1$.

$$\rho_{i_s} \neq \rho_{i_s+1} \qquad 1 \leqslant s \leqslant t.$$

$= 0$ otherwise.

(ii) $$\varphi(\lambda) = \sum_{i_1=0}^{\lambda_1} \cdots\cdots \sum_{i_{n-1}=0}^{\lambda_{n-1}} (-1)^{i_1+\cdots\cdots+i_{n-1}} \chi(\rho_{n-1}^{i_{n-1}} \cdots\cdots \rho_1^{i_1} \cdot\lambda).$$

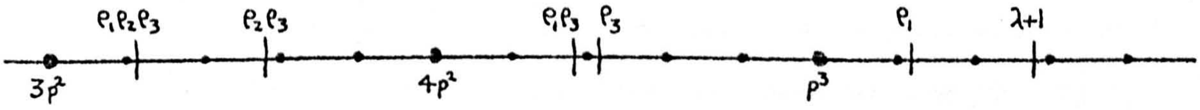
(ρ_i^0 = identity).

Part (ii) is an easy consequence of (i). In particular we see that there are at most 2^{n-1} composition factors of $\overline{V(\lambda)}$, each occuring with multiplicity 1.

<u>Remarks</u> 1. (2.3.4(i)) was originally proved, though in a rather different form, in[18] following a method of Srinivasan[14].

2. For different descriptions of the numbers $d_{\lambda\mu}$ see (3.2.5).

<u>Examole</u> $\lambda = 138$, $p = 5$.



$\rho_2(\lambda + 1) = \rho_3(\lambda + 1)$, hence ρ_2, $\rho_1\rho_2$ do not occur.

$$\overline{V(138)} = M(138) + M(130) + M(110) + M(108) + M(88) + M(80)$$

$$\varphi(\lambda) = \chi(\lambda) - \chi(\rho_1\cdot\lambda) - \chi(\rho_3\cdot\lambda) + \chi(\rho_3\rho_1\cdot\lambda) + \chi(\rho_1^2\cdot\lambda) - \chi(\rho_3\rho_1^2\cdot\lambda)$$

i.e. $\varphi(138) = \chi(138) - \chi(130) - \chi(110) + \chi(108) + \chi(128) - \chi(120)$.

(ii) Φ of type A_2

$\Delta = \{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ are the fundamental dominant weights defined by $(\beta_i, \alpha_j^\vee) = \delta_{ij}$, $i,j = 1,2$.

Each $\lambda = r\beta_1 + s\beta_2 \in X$ is represented by the point (r,s).

$\Phi^+ = \{\alpha_1, \alpha_2, \rho = \alpha_1 + \alpha_2\}$ and therefore Γ_p consists of a regular hexagonal 'lattice'. Every p-alcove has an equilateral triangle of side p as boundary and X_p^+ contains 2 p-alcoves C_0, C_1. Let $0 \in \overline{C}_0$.

We first determine the matrix T of (2.1.1). Define

$$\rho_i, \overline{\rho}_i, \; i \geq 0, \text{ by } \rho_i \cdot \lambda = s_{\rho, m_i} \cdot \lambda, \quad m_i = ((\lambda + \rho)^{(i)}, \rho^\vee)$$
$$\overline{\rho}_i \cdot \lambda = s_{\rho, m_i + p^i} \cdot \lambda, \quad (\rho_0 = \text{identity}).$$

(2.3.5) <u>Proposition</u>. Let $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$, $\lambda_i \in X_p^+$.

Then $\psi(\lambda) = \sum_S \varphi(\lambda(S))$, $\lambda(S) = \prod_{i \in S} \rho_i \overline{\rho}_{i+1} \cdot \lambda$, where the sum is over all subsets S of $I = \{i : \lambda_i + \rho \in C_1, 0 \leq i \leq n-1\}$.

<u>Proof</u>. From theorem (2.3.2),

$$\chi(\lambda) = \varphi(\lambda) + \varphi(s_{\rho,p} \cdot \lambda) \quad \text{if } \lambda \in C_1' = C_1 \sim \rho$$
$$= \varphi(\lambda) \quad \text{if } \lambda \in X_p^+ \sim C_1'.$$

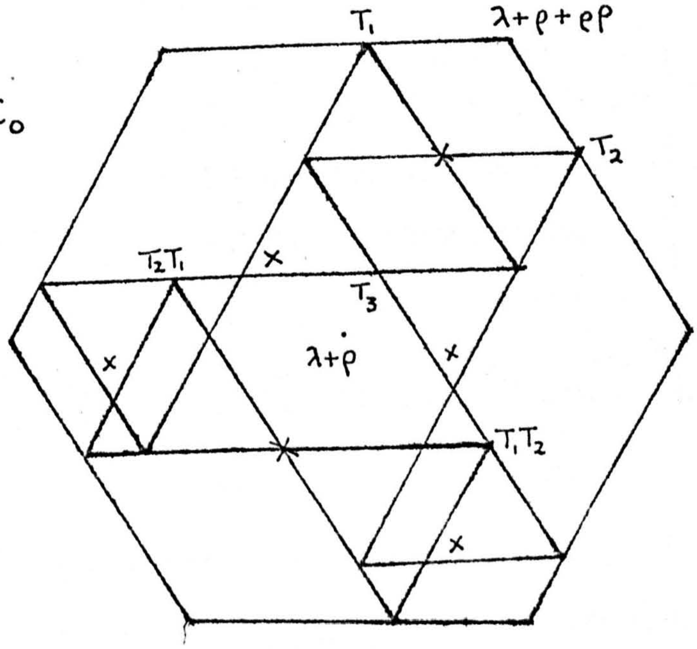
Hence $\psi(\lambda) = \prod_{i=0}^{n-1} \chi^{Fr^i}(\lambda_i) = \prod_{i \in I} (\varphi(\lambda_i) + \varphi(s_{\rho,p} \cdot \lambda_i))^{Fr^i} \prod_{i \notin I} \varphi(\lambda_i)^{Fr^i}$.

Therefore $\psi(\lambda) = \sum_{S \subseteq I} \varphi(\lambda(S))$, $\lambda(S) = \sum_{i \in S} (s_{\rho,p} \cdot \lambda_i) p^i + \sum_{i \notin S} \lambda_i p^i$.

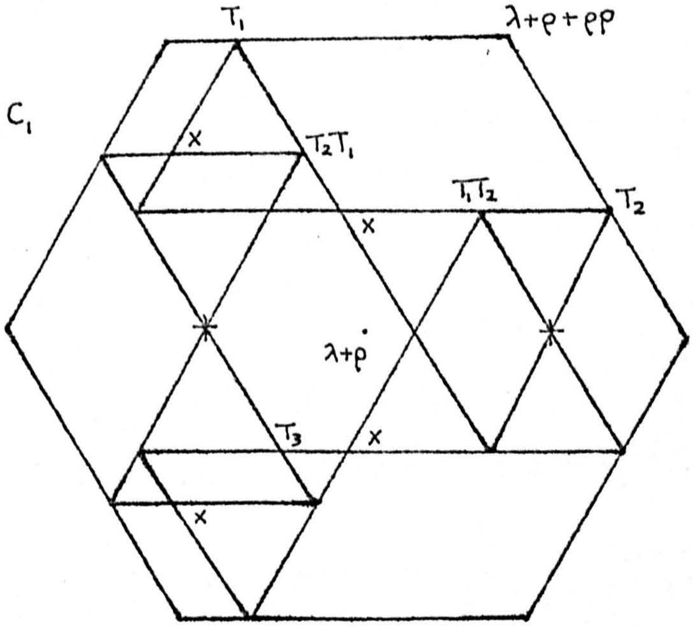
It is easy to check that $\lambda(S)$ is as given in the proposition.

As far as the generating equation for D' is concerned, the method of (2.2c) may be represented by the diagrams on the next page. Each vertex labelled $T \in W$ ($\cong S_3$) in the figures represents the weight $\mu_T + \rho + \rho p$ and forms the 'dominant vertex' of a hexagon with centre $(\mu_{1,T} + \rho)p$ and radius $\mu_{0,T} + \rho$. The number of positive and negative signs at each vertex of every such hexagon are equal. These have been deleted for clarity. In terms of co-ordinates the weights μ_T are as below.

$\lambda_0 + \rho \in C_0$



$\lambda_0 + \rho \in C_1$



Let $\lambda = \lambda_0 + \lambda_1 p$, $\lambda_0 = (r_0, s_0) \in X_p^+$, $\lambda_1 = (r_1, s_1) \in X_p^+$.

	$\lambda_0+\rho \in \hat{C}_0$		$\lambda_0+\rho \in \hat{C}_1$	
T	$\mu_{0,T}$	$\mu_{1,T}$	$\mu_{0,T}$	$\mu_{1,T}$
1	(r_0, s_0)	(r_1, s_1)	(r_0, s_0)	(r_1, s_1)
T_1	$(p-2-r_0, r_0+s_0+1)$	(r_1-1, s_1)	$(p-2-r_0, r_0+s_0+1-p)$	(r_1-1, s_1+1)
T_2	$(r_0+s_0+1, p-2-s_0)$	(r_1, s_1-1)	$(r_0+s_0+1-p, p-2-s_0)$	(r_1+1, s_1-1)
T_3	$(p-2-s_0, p-2-r_0)$	(r_1-1, s_1-1)	$(p-2-s_0, p-2-r_0)$	(r_1-1, s_1-1)
T_1T_2	$(p-2-(r_0+s_0+1), r_0)$	(r_1, s_1-2)	$(p-2-(r_0+s_0+1-p), r_0)$	(r_1, s_1-1)
T_2T_1	$(s_0, p-2-(r_0+s_0+1))$	(r_1-2, s_1)	$(s_0, p-2-(r_0+s_0+1-p))$	(r_1-1, s_1)

$T_1 = s_{\alpha_1,0}$; $T_2 = s_{\alpha_2,0}$; $T_3 = s_{\rho,0} = T_1T_2T_1 = T_2T_1T_2$.

If $\lambda_0+\rho \in \Gamma_p$, it is evident from the above table that for some $T \in W$, $\alpha \in \Delta$, $(\mu_{0,T}, \check{\alpha}) = -1$, giving $\chi(\mu_{0,T}) = 0$. We list the cases for which this occurs.

(a) $\lambda_0+\rho \in \hat{C}_0 \setminus C_0 \subset H_{\rho,p}$ i.e. $r_0+s_0+2=p$; $T = T_1T_2, T_2T_1$.

(b) $\lambda_0+\rho \in \hat{C}_1 \setminus C_1$.

(i) $\lambda_0+\rho \in H_{\alpha_1,p}$ i.e. $r_0+1 = p$; $T = T_1, T_3$

(ii) $\lambda_0+\rho \in H_{\alpha_2,p}$ i.e. $s_0+1 = p$; $T = T_2, T_3$.

If $\mu_{1,T} \notin X_p^+$, then either $\chi(\mu_{1,T}) = 0$ or

(2.3.6) (i) $\lambda_0+\rho \in \hat{C}_0$; $T = T_1T_2$, $s_1 = 0$; $T = T_2T_1$, $r_1 = 0$.

(ii) $\lambda_0+\rho \in \hat{C}_1$; $T = T_1$, $s_1+1 = p$; $T = T_2$, $r_1+1 = p$.

In (i), $\chi(r_1,-2) = -\chi(r_1-1,0)$; $\chi(-2,s_1) = -\chi(0,s_1-1)$.

using $\chi(\lambda) = \det S\, \chi(S.\lambda)$ $S \in W$ (Υ)

In (ii), $\chi(r_1-1,p) = \chi(r_1-1,0)\, \chi^{Fr}(0,1) + \chi(r_1,p-2) - \chi(0,p-r_1-2)$

$\chi(p,s_1-1) = \chi(0,s_1-1)\, \chi^{Fr}(1,0) + \chi(p-2,s_1) - \chi(p-2-s_1,0)$

using the generating equation and (Υ).

$\chi(\lambda)$, $\lambda \in X_{p^2}^+$, can now be expressed as a linear combination of pseudo-characters.

Define $\rho_i^{T_j} \in W_p i$, $i > 0$, $j = 1,2,3$ by

$\rho_i^{T_j} \cdot = s_{\alpha_j, n_{ij}} \cdot$, $n_{ij} = ((\lambda+\rho)^{(i)}, \alpha_j^\vee)$, where $\rho_i^{T_j}$ is defined only if

$\rho_i^{T_j} \neq \rho_{i+1}^{T_j}$ and n_{ij} is not a multiple of p^i, $j = 1,2$. Note that

$\rho_i^{T_3} = \rho_i$ (see (2.3.5)). Let C_0^i, C_1^i denote the p^i-alcoves in $X_p^+ i$ with

$0 \in \overline{C_0^i}$. Let $\underline{C}_l^i$ denote the lower closure $C_l^i \cup (\overline{C_l^i} \setminus \hat{C}_l^i)$ of C_l^i, $l = 0,1$.

Then define $\rho_i^{T_1T_2}$, $\rho_i^{T_2T_1}$ by

$$\rho_i^{T_1T_2} \cdot \lambda = \rho_i^{T_2} \rho_i^{T_3} \cdot \lambda , \quad \rho_i^{T_2T_1} \cdot \lambda = \rho_i \rho_i^{T_3} \cdot \lambda , \qquad (\lambda+\rho)_{(i)} \in \underline{C}_0^i$$

$$\rho_i^{T_1T_2} \cdot \lambda = \rho_i^{T_1} \rho_i^{T_2} \cdot \lambda , \quad \rho_i^{T_2T_1} \cdot \lambda = \rho_i^{T_2} \rho_i^{T_1} \cdot \quad , \qquad (\lambda+\rho)_{(i)} \in \underline{C}_1^i$$

Restating information already obtained in terms of these operators, we have,

(2.3.7) <u>Proposition</u> Let $\lambda \in X_p^+ 2$. Then $\chi(\lambda) = \sum_{T \in W} \chi_1(\rho_1^T \cdot \lambda)$.

If $\rho_1^T \cdot \lambda \in X_p^+ 2$ then $\chi_1(\rho_1^T \cdot \lambda) = \psi(\rho_1^T \cdot \lambda)$.

If $\rho_1^T \cdot \lambda \notin X_p^+ 2$ then either $\chi_1(\rho_1^T \cdot \lambda) = 0$ or

(1) (2.3.6(i)) holds, in which case

$$\chi_1(\rho_1^{T_1T_2} \cdot \lambda) = -\psi(\bar{\rho}_1 \rho_1^{T_1} \cdot \lambda) , \quad \lambda_1 = (r_1 \neq 0, 0).$$

$$\chi_1(\rho_1^{T_2T_1} \cdot \lambda) = -\psi(\bar{\rho}_1 \rho_1^{T_2} \cdot \lambda) , \quad \lambda_1 = (0, s_1 \neq 0), \quad \text{or}$$

(2) (2.3.6(ii)) holds, in which case

$$\chi_1(\rho_1^{T_1} \cdot \lambda) = \psi(\rho_1^{T_1} \cdot \lambda) + \psi(\bar{\rho}_1 \rho_1^{T_1T_2} \cdot \lambda) - \psi(\bar{\rho}_1 \rho_1^{T_2} \bar{\rho}_2 \cdot \lambda)$$

$\lambda_1 = (r_1, p-1)$ $r_1 = 0, p-1$.

$$\chi_1(\rho_1^{T_2} \cdot \lambda) = \psi(\rho_1^{T_2} \cdot \lambda) + \psi(\bar{\rho}_1 \rho_1^{T_2T_1} \cdot \lambda) - \psi(\bar{\rho}_1 \rho_1^{T_1} \bar{\rho}_2 \cdot \lambda)$$

$\lambda_1 = (p-1, s_1)$ $s_1 = 0, p-1$.

If $\lambda_1 = (p-1, p-1)$ then the terms $\psi(\bar{\rho}_1 \rho_1^{T_2} \bar{\rho}_2 \cdot \lambda)$ and $\psi(\bar{\rho}_1 \rho_1^{T_1} \bar{\rho}_2 \cdot \lambda)$ do not appear in the above.

(2.3.8) <u>Corollary</u> If $\lambda_0 + \rho \in C_1$, $\lambda \in X_p^+ 2$, then

$d_{\lambda\mu} = 2$ if (i) $\mu = \bar{\rho}_1 \rho_1^{T_1T_2} \cdot \lambda$, $\lambda_1 = (r_1 \neq 0, p-1)$.

or (ii) $\mu = \bar{\rho}_1 \rho_1^{T_2T_1} \cdot \lambda$, $\lambda_1 = (p-1, s_1 \neq 0)$.

The reader should now consult FIG 1. in which 16 basic configurations are represented. Note that, for $\lambda \in X_p^+ 2$, $d_{\lambda\mu} \leq 2$. Performing the iteration of (2.2a) and combining it with (2.3.5) we have,

(2.3.9) <u>Theorem</u> If $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$, $\lambda_i \in X_p^+$, $(\lambda_i, \alpha^\vee) \neq 0, p-1$ all $\alpha \in \Phi^+$ $1 \leq i \leq n-1$, then

$$d_{\lambda\mu} = 1 \quad \text{when} \quad \mu = \bar{\rho}_1 \rho_1^{\tau_1} \bar{\rho}_2 \rho_2^{\tau_2} \cdots \bar{\rho}_{n-1} \rho_{n-1}^{\tau_{n-1}} \bar{\rho}_n \cdot \lambda$$

where $\tau_i \in W$ and $\bar{\rho}_i$ may or may not occur.

$= 0$ othewise.

<u>Remark</u> The condition in (2.3.9) is slightly stronger than necessary. See FIG 2 for an example of (2.3.9). Most weights decompose in such a regular pattern with the configurations I and II of FIG 1. providing the basic motif.

If the method of (2.2b) is required then, in addition to the above, the formulae

$$\chi(1,0)\,\chi(\lambda_1,\lambda_2) = \chi(\lambda_1+1,\lambda_2) + \chi(\lambda_1-1,\lambda_2+1) + \chi(\lambda_1,\lambda_2-1)$$
$$\chi(0,1)\,\chi(\lambda_1,\lambda_2) = \chi(\lambda_1,\lambda_2+1) + \chi(\lambda_1+1,\lambda_2-1) + \chi(\lambda_1-1,\lambda_2)$$

suffice to derive the $d_{\lambda\mu}$. See FIG 3 for an example of an irregular decomposition. In that example $(\lambda_1, \alpha_2^\vee) = p-1$, $\lambda_0 + \rho \in C_1$, so $\rho_1^{T_1} \cdot \lambda, \lambda$ lie in different translates of $X_p^+ 2$ by $p^2 X$ and (2.2b) is needed. Finally we give 2 corollaries which follow from the analysis of the $\lambda \in X_p^+ 2$ case and the iteration of (2.2a).

(2.3.10) <u>Corollary</u> If $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$, then $d_{\lambda\mu} \leq 2^{n-1}$.

With $\rho_i, \bar{\rho}_i$ as in (2.3.5), then $d_{\lambda\mu} = 2^{n-1}$ if $(\lambda_i, \rho^\vee) = p-1$, $1 \leq i \leq n-1$, and $\mu = \rho_1 \bar{\rho}_2 \rho_2 \cdots \bar{\rho}_{n-1} \rho_{n-1} \bar{\rho}_n \cdot \lambda$

(2.3.11) <u>Corollary</u> $\overline{V(\lambda)}$ remains irreducible if and only if $\lambda + \rho \in \sigma_i(P)$ for some $i \geq 0$, where $P = \{\lambda + \rho : \lambda \in X_p^+ , \overline{V(\lambda)} = M(\lambda)\}$ $= (\hat{C}_0 \cup \hat{C}_1 \smallsetminus C_1) \cap X^+$.

(iii) <u>Φ of type B_2</u>

This case lends itself to treatment similar to that carried out for Φ of type A_2. We omit the details but remark that, as in (2.3.9), $d_{\lambda\mu}$ is 1 or 0 for most weights .

<u>Remark</u> Recently Jantzen [11] has given a form of the generating equation with the coefficients in $Z[X]^{Fr}$, for general Lie type.

A_2 : Some configurations for $(d_{\lambda\mu})_{\lambda \in X_p^+ 2, \mu \in X^+}$

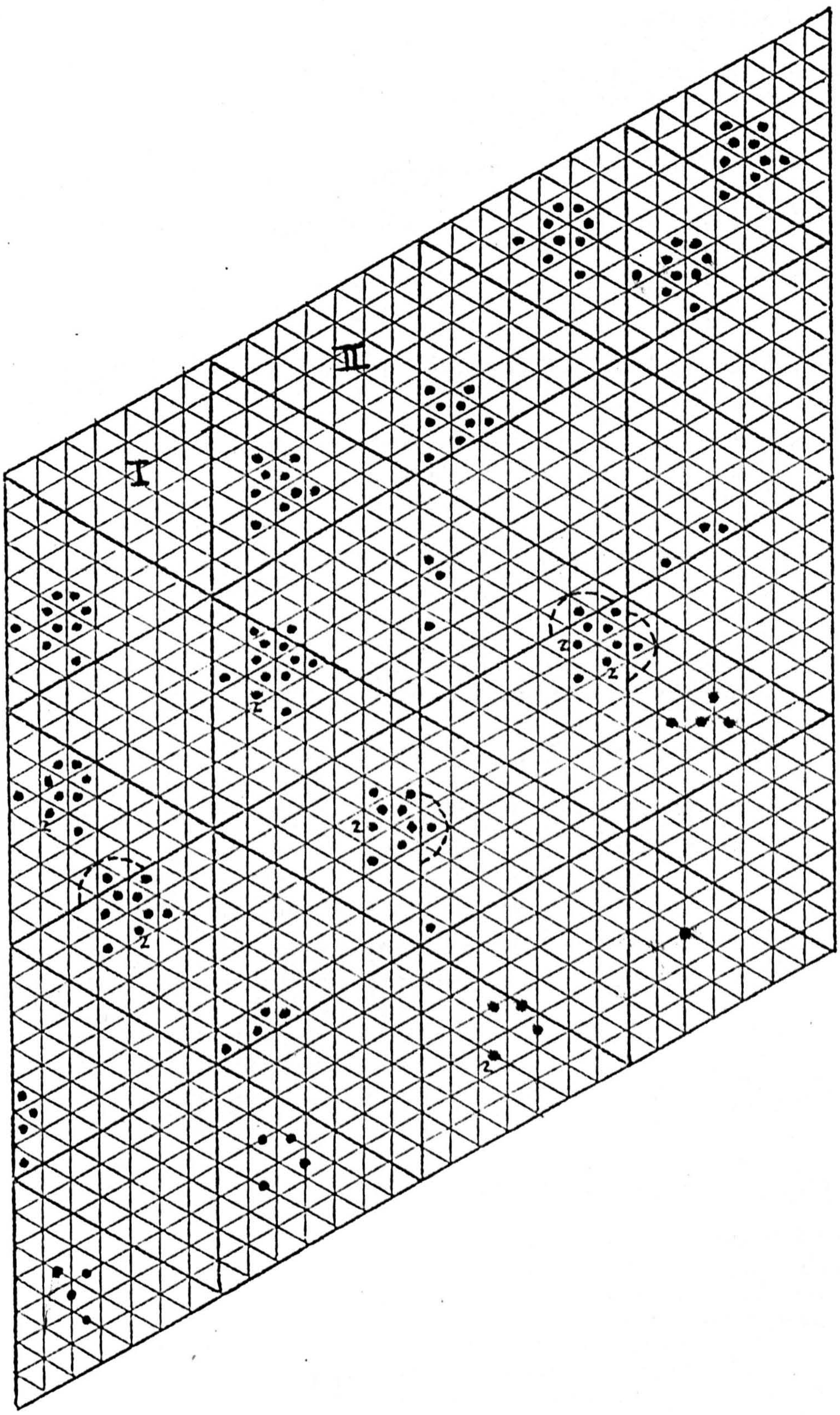


FIG 1.

A_2 : $\lambda = \lambda_0 + \lambda_1 p + \lambda_2 p^2$, $p = 5$, $\lambda_0 + \rho \in C_1$, $\lambda_1 = (1,1)$, $\lambda_2 = (3,3)$.

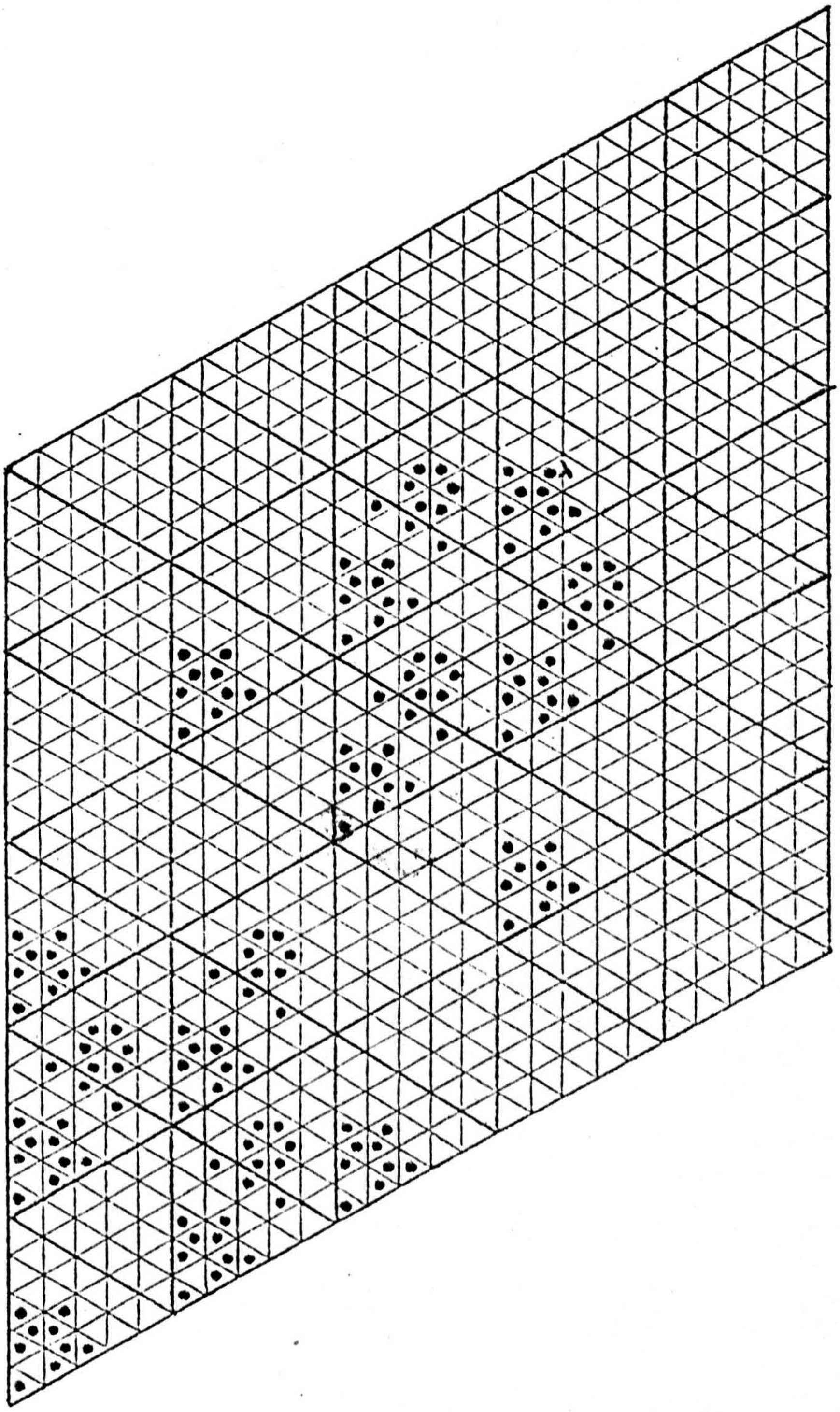


FIG 2.

A_2 : $\lambda = \lambda_0 + \lambda_1 p + \lambda_2 p^2$, $p = 5$, $\lambda_0 + \rho \in C_1$, $\lambda_1 = (3,4)$, $\lambda_2 = (3,3)$.

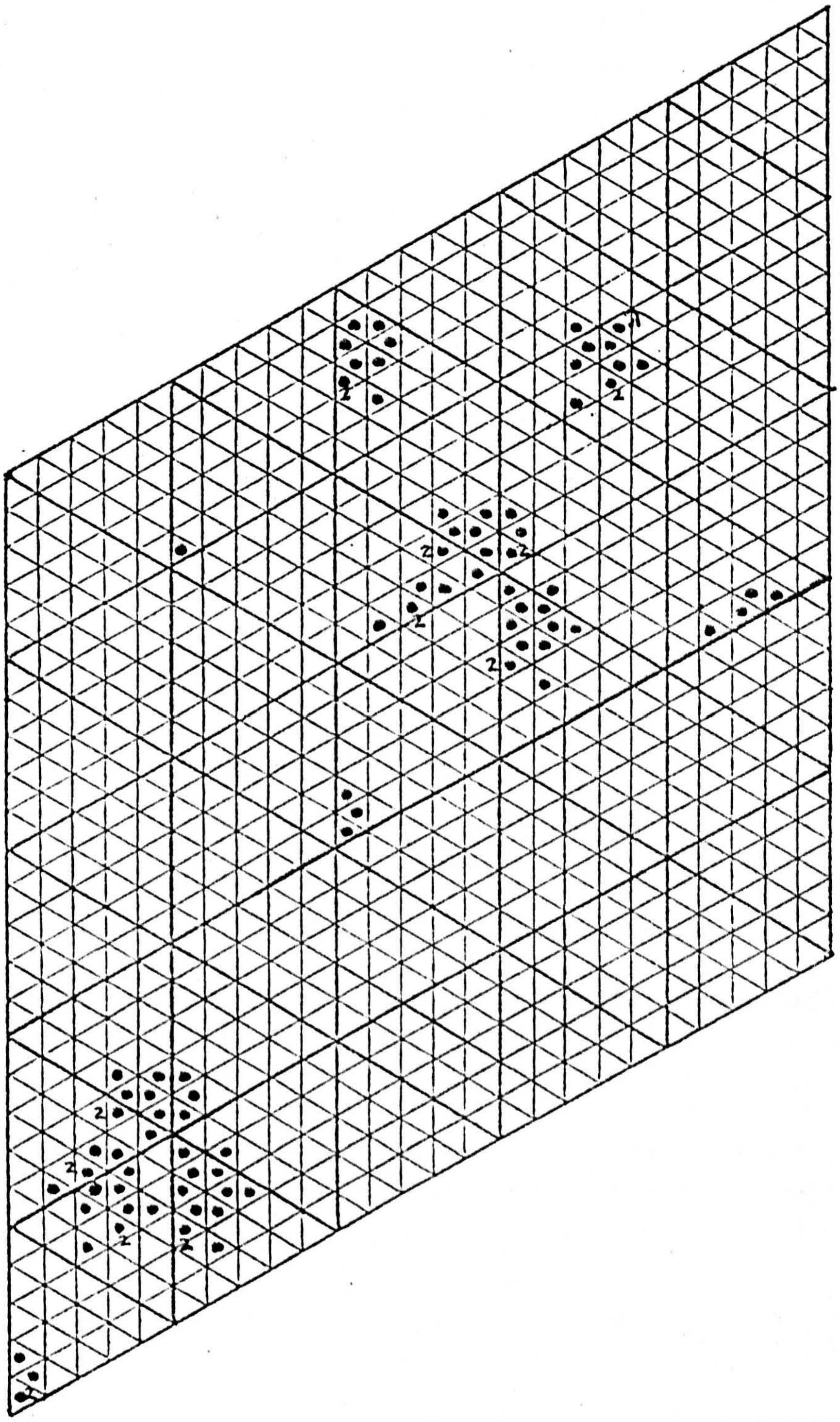
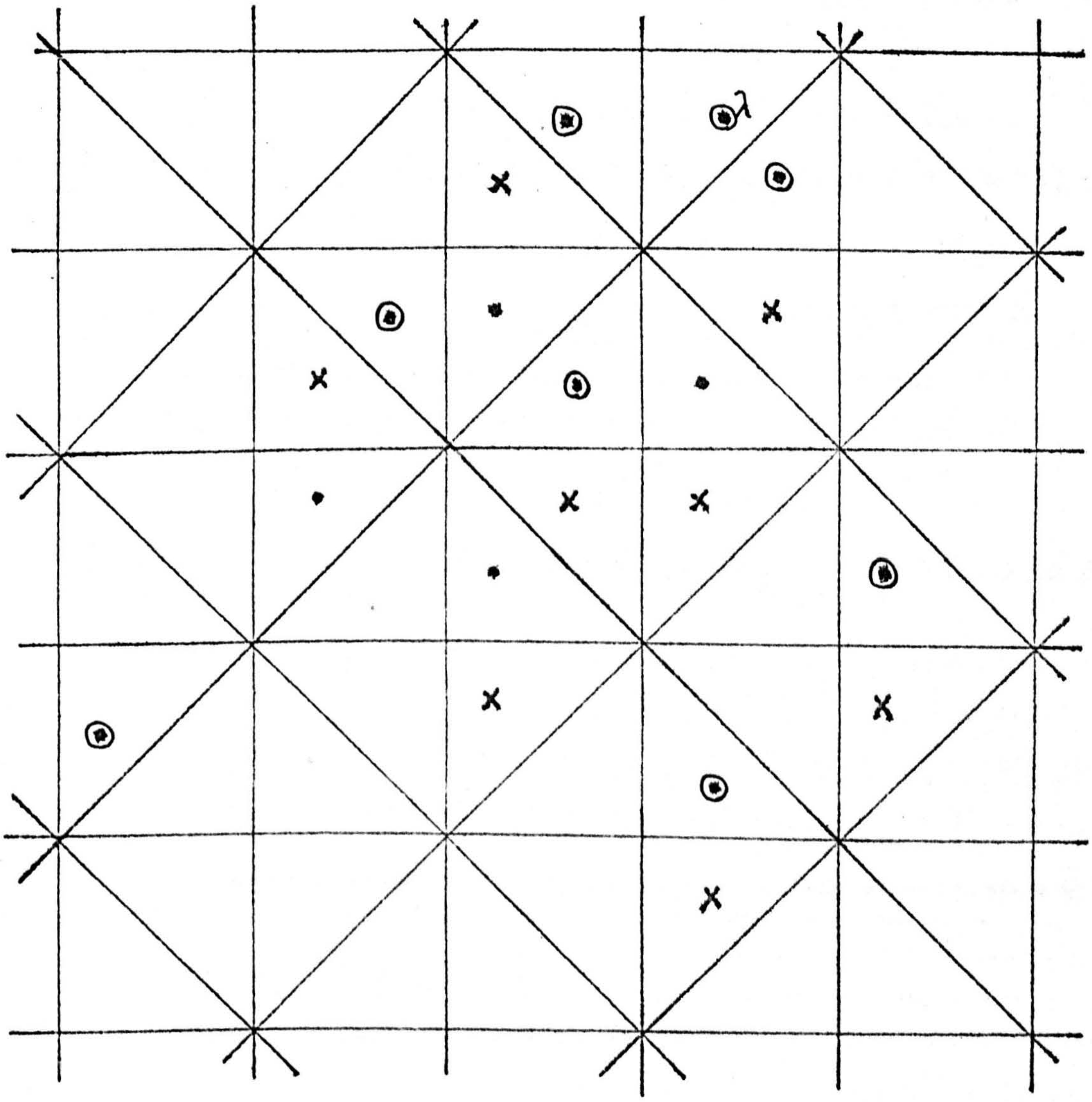


FIG 3.

B_2 : $\lambda_0 + \rho \in C_0$, $0 \leq (\lambda_1, \alpha^\vee) < p^2$, $\alpha \in \Phi^+$



The dots ⊙ represent the elements μ_T, $T \in W$.

FIG 4.

§3 THE MODULAR REPRESENTATION THEORY OF SL(2,K)

Throughout, K is an algebraically closed field of characteristic $p \neq 0$, G is the special linear group SL(2,K) and $A = K[G]$, the affine ring of G.

(3.1) The Group SL(2,K).

G is the universal Chevalley group of type A_1 over K. (see 1.2). The Lie algebra $\underline{g}$ used in the construction of G is 3-dimensional, consisting of the 2×2 matrces over C with trace 0. The root system $\Phi = \{\alpha, -\alpha\}$ and the fundamental system $\Delta = \Phi^+ = \{\alpha\}$. X, the full lattice of weights, consists of integral multiples of the fundamental dominant weight $\beta = \frac{1}{2}\alpha$, and is thus identified with Z.

As an affine algebraic group G has underlying set $\{(x_1, x_2, x_3, x_4) : x_1 x_4 - x_2 x_3 = 1\} \subset K^4$. The unipotent subgroups U, U^- and the maximal torus H of G have the form,

$U = \langle u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in K \rangle \cong K^+$, $H = \langle h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in K \rangle \cong K^*$,

$U^- = \langle u^-(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in K \rangle \cong K^+$. Gis generated by the groups U, U^-.

X(H), the group of rational characters of H, is isomorphic to Z and so to X. The pair (B,N), where B is the Borel subgroup HU and $N = N_G(H)$, is a B.N pair with Weyl group $W = \{I, s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$ and root system Φ.

As such G has a Bruhat decomposition,

(3.1.1) $G = B \dot{\cup} BsB$ into double cosets of B.

(3.2) The structure of $\overline{V(\lambda)}_{max}$ (or $\overline{V(\lambda)}_{min}$.)

We begin with a description of the modules in the title. Let x,y be a basis of C^2. The natural action of $\underline{g} = \underline{sl}_2$ on $Cx + Cy$ can be extended to the polynomial algeba $C[x,y]$ by derivations. The irreducible

$\underline{g}$-module V_λ of highest weight λ can then be realised as the subspace of $C[x,y]$ consisting of the homogeneous polynomials of degree λ in x and y. The vector $x^{\lambda-i}y^i$ has weight $\lambda-2i$, and $v_o = x^\lambda$ is a highest weight vector. Let $\{e,f,h\}$ be the standard Chevalley basis of $\underline{g}$ and $\mathcal{U}_Z$ be the Z-module $< \frac{e^r}{r!}, \frac{f^r}{r!} : r \in Z^+ >$. Then the minimal admissible ($\mathcal{U}_Z$-invariant) Z-form of V is $V^{min}_{\lambda,Z} = \mathcal{U}_Z v_o = \mathcal{U}^-_Z v_o$ where $\mathcal{U}^-_Z = < \frac{f^r}{r!} : r \in Z^+ >$. Set $v_i = \frac{f^i}{i!} v_o$. Then $v_i = \binom{\lambda}{i} . x^{\lambda-i}y^i$ and $(v_i)_{i=o,\ldots,\lambda}$ is a Z-basis of $V^{min}_{\lambda,Z}$.

Now $\sum_{i=0}^{\lambda} Zx^{\lambda-i}y^i$ is also an admissible Z-form. It is clearly the maximal one and is denoted by $V^{max}_{\lambda,Z}$. These extremal Z-forms are related by the fact that $V^{max}_{\lambda,Z}$ is isomorphic to the dual of $V^{min}_{\lambda,Z}$. To see this, define the basis $(w_i)_{i=0,\ldots,\lambda}$ of V^*_λ by $w_i(v_{\lambda-j}) = (-1)^i \delta_{ij}$. Then with the following action on V^*_λ :-

$(lf)(v) = -f(lv)$, $l \in \underline{g}$, $v \in V_\lambda$, $f \in V^*_\lambda$, we find that w_o is a highest weight vector and $\sum_{i=0}^{\lambda} Zw_i$ an admissible Z-form, the maximal one. The $\underline{g}$-isomorphism identifying $V^{max}_{\lambda,Z}$ in V is then given by the map $w_i \mapsto x^{\lambda-i}y^i$. (NB. admissible Z-forms are detd.up to dilatation.)

Let $V_{\lambda,Z}$ be any admissible Z-form of V_λ. Then it is easy to show that $V^{min}_{\lambda,Z} \subset V_{\lambda,Z} \subset V^{max}_{\lambda,Z}$. $\overline{V(\lambda)} = V_{\lambda,Z} \otimes K$ can be made into a $K[G]$-module under the action :

$$u(t)(v \otimes 1_K) = \sum_{r\geq 0} t^r \left(\frac{e^r}{r!}v \otimes 1_K\right)$$

$$u^-(t)(v \otimes 1_K) = \sum_{r\geq 0} t^r \left(\frac{f^r}{r!}v \otimes 1_K\right)$$

Now the $K[G]$-modules $\overline{V(\lambda)}_{min}$, $\overline{V(\lambda)}_{max}$ are defined to be $V^{min}_{\lambda,Z} \otimes K$, $V^{max}_{\lambda,Z} \otimes K$ resp. $\overline{V(\lambda)}_{min}$ has K-basis $\{v_{i,K} = v_i \otimes 1_K\}$ and $\overline{V(\lambda)}_{max}$

has K-basis $\{w_{i,K} = w_i \otimes 1_K : 0 \leqslant i \leqslant \lambda\}$. The G-action on each module is as follows.

(3.2.1) $$u(t)v_{i,K} = \sum_{j=0}^{i} \binom{\lambda-j}{\lambda-i} t^{i-j} v_{j,K} \;;\quad u^-(t)v_{i,K} = \sum_{j=i}^{\lambda} \binom{j}{i} t^{j-i} v_{j,K} .$$

(3.2.2) $$u(t)w_{i,K} = \sum_{j=0}^{i} \binom{i}{j} t^{i-j} w_{j,K} \;;\quad u^-(t)w_{i,K} = \sum_{j=i}^{\lambda} \binom{\lambda-i}{\lambda-j} t^{j-i} w_{j,K} .$$

<u>Remarks</u> 1. $\overline{V(\lambda)}_{min}$ and $\overline{V(\lambda)}_{max}$ are indecomposable.

2. $\overline{V(\lambda)}_{max} \cong (\overline{V(\lambda)}_{min})^*$

3. $\overline{V(\lambda)}_{min}$ has a unique maximal proper $K[G]$-submodule (contained in the sum of the weight spaces except the highest one), the quotient by which is $M(\lambda)$. Hence $\overline{V(\lambda)}_{max}$ has $M(\lambda)$ as its unique irreducible submodule.

4. The injection $i_{\lambda,Z} : V^{min}_{\lambda,Z} \longrightarrow V^{max}_{\lambda,Z}$ gives a homomorphism $i_\lambda : \overline{V(\lambda)}_{min} \longrightarrow \overline{V(\lambda)}_{max}$, with $\ker i_\lambda$ as the maximal proper submodule of $\overline{V(\lambda)}_{min}$ and $\operatorname{im} i_\lambda = M(\lambda)$.

We are now in a position to set about determining the structure of $\overline{V(\lambda)}_{max}$ (and hence by 2. above, $\overline{V(\lambda)}_{min}$).

Let $\{m_i = w_{i,K} : 0 \leqslant i \leqslant \lambda\}$ be the K-basis of $\overline{V(\lambda)}_{max}$ described above. Then thanks to (3.1.1), the G-action is described by the equations,

(3.2.3) (i) $$u(t)m_i = \sum_{j=0}^{i} \binom{i}{j} t^{i-j} m_j .$$

(ii) $$h(t)m_i = t^{\lambda-2i} m_i .$$

(iii) $$sm_i = (-1)^{\lambda-i} m_{\lambda-i} .$$

We seek the G-invariant subspaces of $\sum_{i \in V} Km_i$ where $V = \{0,1,\ldots,\lambda\}$.

Since the weight spaces are 1-dimensional, such subspaces will be of the form $S_I = \sum_{i \in I} Km_i$, $I \subset V$.

Call I symmetric if $i \in I$ implies $\lambda - i \in I$.

I complete if $i \in I$ and $\binom{i}{j} \neq 0$ implies $j \in I$.

Equations (3.2.3) imply that $_I$ is a complete symmetric set. For future reference we state the following well known lemma without proof.

(3.2.4) Lemma Let $i = \sum_{k=0}^{n-1} i_k p^k \quad 0 \leq i_k < p, \quad j = \sum_{k=0}^{n-1} j_k p^k \quad 0 \leq j_k < p$.

Then $\binom{i}{j} \neq 0 \Leftrightarrow \binom{i_k}{j_k} \neq 0 \Leftrightarrow j_k \leq i_k, \quad 0 \leq k \leq n-1$.

If $\binom{i}{j} \neq 0$, write $j \preccurlyeq i$.

Recall from (2.3.4) that if $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i, \quad 0 \leq \lambda_i < p$, then

$$d_{\lambda\mu} = 1 \quad \text{when } \mu = \rho_{i_1} \cdots\cdots \rho_{i_k} \cdot \lambda, \quad 1 \leq i_1 < \cdots\cdots < i_k \leq n-1.$$
$$= 0 \quad \text{otherwise.}$$

Direct calculation gives the following,

(3.2.5) Proposition Let $D_\lambda = \{\mu : d_{\lambda\mu} \neq 0\}$. Then $\mu \in D_\lambda$ if and only if there exists a t such that

$$\mu = \rho_{i_1} \cdots\cdots \rho_{i_{2t}} \cdot \lambda = \lambda - 2 \sum_{j=1}^{t} \left(\sum_{h=i_{2j-1}}^{i_{2j}-1} \lambda_h p^h + p^{i_{2j-1}} \right)$$

where $0 \leq i_1 < \cdots\cdots < i_{2t} \leq n-1$ and $\lambda_{i_{2j}} \neq 0$, $\lambda_{i_{2j-1}} \neq p-1$ for $1 \leq j \leq t$.

(Note that ρ_0 is the identity).

Remark For another description of the elements of D_λ see the proof of (3.5.3).

Let $\theta : \{\lambda - 2i\}_{i=0}^{\lambda} \longrightarrow V$ be defined by $\theta(k) = \frac{\lambda - k}{2}$. Then (3.2.3(ii)) shows that θ assigns to a weight of $\overline{V(\lambda)}_{max}$ the index of the corresponding basis vector. Let $\mu = \sum_{i=0}^{n-1} \mu_i p^i, \quad 0 \leq \mu_i < p, \quad 0 \leq i \leq n-1$, and let $T_\mu =$

$\Pi_{M(\mu)}$, the set of weights of $M(\mu)$. Then $T_\mu = \{\mu - 2k : k \preccurlyeq \mu\}$.
Let $\theta_{(\mu)}$ denote the image of T_μ under θ. Then we see that $\theta_{(\mu)}$ is a symmetric set but not necessarily complete. In particular, $\theta_{(\lambda)} = \{ i : \binom{\lambda}{i} \neq 0 \}$ and $S_{\theta_{(\lambda)}} \cong M(\lambda)$. ($= KU^- m_0$ by (1.3.3).)
Now suppose that S_I is a submodule. Then clearly each composition factor of S_I is a composition factor of S_V. Hence $I = \bigcup_{\mu \in \Lambda} \theta_{(\mu)}$ for some subset Λ of D_λ. Hence I must be symmetric and only completeness need be considered. Now let $\hat{I}(\mu)$, $\mu \in D_\lambda$, be the smallest set in V containing $\frac{1}{2}(\lambda - \mu)$ and admitting a submodule. Then $S_{\hat{I}(\mu)} = KGm_\mu$.
Trivially, $S_{I \cup J} = S_I + S_J$, $S_{I \cap J} = S_I \cap S_J$ and $I \subset J \Leftrightarrow S_I \subset S_J$.

We see that the submodule lattice is generated by taking all unions of elements of the set $\{\hat{I}(\mu)\}_{\mu \in D_\lambda}$, which will now be determined.

Let $\mu = \rho_{i_1} \cdots \rho_{i_{2t}} \cdot \lambda \in D_\lambda$ then from (3.2.5),

$$(3.2.6) \qquad \frac{\lambda - \mu}{2} = \sum_{j=1}^{t} \Big(\sum_{h=i_{2j-1}}^{i_{2j}-1} \lambda_h p^h + p^{i_{2j-1}} \Big)$$

Using the fact that $\rho_k \cdot \lambda$ has p-adic expansion represented by the vector $(p-2-\lambda_0, p-1-\lambda_1, \ldots, p-1-\lambda_{k-1}, \lambda_k - 1, \lambda_{k+1}, \ldots, \lambda_{n-1})$ we easily find that the μ_i in the expansion $\mu = \sum \mu_i p^i$ are given by,

(3.2.7) $\mu_{i_{2j-1}} = p-2-\lambda_{i_{2j-1}}$, $\mu_{i_{2j}} = \lambda_{i_{2j}} - 1$, $\mu_h = p-1-\lambda_h$,

$i_{2j-1} < h \leqslant i_{2j} - 1$, $1 \leqslant j \leqslant t$.

$\mu_i = \lambda_i$ otherwise.

Put $C(\mu) = \bigcup_{1 \leqslant l \leqslant t} \{ i_{2l-1}, i_{2l-1}+1, \ldots, i_{2l}-1 \}$. (3.2.6), (3.2.7), give

(3.2.8) <u>Lemma</u> $\theta_{(\mu)}$ consits of all $\nu = \sum \nu_i p^i$ such that

$\lambda_h \leqslant \nu_h < p$ if $h \in C(\mu)$,

$0 \leqslant \nu_h \leqslant \lambda_h$ otherwise, but $\nu_h \neq \lambda_h$ for $h = i_{2l-1}, i_{2l}$.

Recall that if $\theta_{(\mu)}$ is contained in a complete set I, then I must contain all integers $i \preccurlyeq \nu$ for each $\nu \in \theta_{(\mu)}$. Hence by (3.2.8) μ is complete if and only if $C(\mu) = \phi$ i.e. $\mu = \lambda$. Since every subset of $C(\mu)$ may be realised as some $C(\nu)$, it is clear that $\hat{I}(\mu) = \bigcup_{\nu} \theta_{(\nu)}$ where the union is over all $\nu \in D_{\lambda}$ such that $C(\nu) \subset C(\mu)$.

The above considerations yield immediately the following result.

(3.2.9) <u>Theorem</u> Let $\mu = \rho_{i_1} \cdots \rho_{i_{2t}} \cdot \lambda \in D_{\lambda}$. Then

$\hat{I}(\mu) = \bigcup_{\nu \in I_{\mu}} \theta_{(\nu)}$ where $\theta_{(\nu)} = \left\{ \frac{\lambda - \nu}{2} + k : k \preccurlyeq \nu \right\}$ and I_{μ} is the set of all weights $\nu = \left(\prod_{l=1}^{t} \rho_{j_{1},l} \cdots \rho_{j_{n_l},l} \right) \cdot \lambda$

$i_{2l-1} \leqslant j_{1,l} < \cdots < j_{n_l,l} \leqslant i_{2l}$, $1 \leqslant l \leqslant t$, and n_l is always even, except when $i_1 = 0$ in which case n_1 may be even or odd.

(Notation : the product of reflections should be read with ρ_i occurring before ρ_j if $i < j$. Also theset $\{ j_{1,l}, \ldots, j_{n_l,l} \}$ may be empty).

We have then that every submodule of $\overline{V(\lambda)}_{max}$ is of the form $S_I = \sum_{i \in I} Km_i$, where I is a union of sets $\hat{I}(\mu)$ given above, and every such S_I is a submodule.

<u>The Loewy Series</u>

We now seek a series of submodules $V_0 \subset V_1 \subset \cdots \subset V_k = \overline{V(\lambda)}_{max}$, such that $V_0 = 0$ and $V_j / V_{j-1} = \sigma(V_k / V_{j-1})$.

This is called the <u>Loewy series</u> of $\overline{V(\lambda)}_{max}$.

Putting $S_{T_j} = V_j$, we get $T_0 = \phi$ and $T_k = V$.

Let $T_j = \bigcup_{\nu \in \Lambda_j} \theta_{(\nu)}$. Then clearly $V_1 = \sigma(S_V) = M(\lambda)$ i.e. $\Lambda_1 = \{\lambda\}$.

If $S_I / M(\lambda)$ is an irruducible submodule of $S_V / M(\lambda)$ isomorphic to $M(\nu)$, then $I = \theta_{(\lambda)} \cup \theta_{(\nu)}$ must be complete. Hence $C(\nu)$ must have no non-trivial subsets i.e. $|C(\nu)| = 1$. Since $C(\rho_i \rho_{i+1} \cdot \lambda) = \{i\}$ we have

$$\Lambda_2 = \{ \lambda, \rho_i \rho_{i+1} \cdot \lambda : 0 \leqslant i \leqslant n-2 \} \text{ and } V_2 = M(\lambda) + \sum_{i=0}^{n-2} M(\rho_i \rho_{i+1} \cdot \lambda).$$

In fact we have,

(3.2.10) Theorem If $\lambda = \sum_{i=0}^{n-1} \lambda_i p^i$, $0 \leqslant \lambda_i < p$, then the Loewy series of $\overline{V(\lambda)}_{max}$ is $0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \overline{V(\lambda)}_{max}$, where

$$V_{s+1} = M(\lambda) + \sum M\left(\prod_{l=1}^{s} \rho_{i_l} \cdot \rho_{i_l+1} \cdot \lambda\right)$$ and the sum is over all sequences $0 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n-2$, $1 \leqslant k \leqslant s$.

In addition, if $i_l + 1 = i_{l+1}$ then $\rho_{i_l+1} \rho_{i_{l+1}}$ is deleted in the product and if then $\rho_i = \rho_{i+1}$ some i , the whole product is deleted.

Proof By induction on s. The theorem is true for $s = 0,1$ by the above. Suppose it is true for all $0 \leqslant i \leqslant s$. Then it is easy to see that $\bigcup_{\nu \in \Lambda_s} C(\nu)$ is pecisely the union of all subsets of $\{0, 1, \ldots, n-2\}$ of order less than s. Now if $\nu \in \Lambda_{s+1} \setminus \Lambda_s$ then I claim that $|C(\nu)| = s$. For if not then there will exist subsets of $C(\nu)$ of order greater than or equal to s, and hence completeness of $T_s \cup \theta_{(\nu)}$ will not be satisfied. But $|C(\nu)| = s$ if and only if $\nu = \left(\prod_{i=1}^{s} \rho_{i_l} \cdot \rho_{i_l+1}\right) \cdot \lambda$, $0 \leqslant i_1 < \ldots < i_s \leqslant n-2$. This completes the proof. //

(3.2.11) Corollary The Loewy factors of $\overline{V(\lambda)}_{max}$ are given by ,

$$V_j / V_{j-1} = \sum M\left(\prod_{l=1}^{j-1} \rho_{i_l} \cdot \rho_{i_l+1} \cdot \lambda\right)$$ where the sum is over all $j-1$ element sequences $0 \leqslant i_1 < \ldots < i_{j-1} \leqslant n-2$.

In particular, $V_n / V_{n-1} = M(\rho_{n-1} \cdot \lambda)$ and V_{n-1} is the unique maximal submodule of $\overline{V(\lambda)}_{max}$.

Remarks 1. Since $\overline{V(\lambda)}_{min} = \overline{V(\lambda)}^*_{max}$ the structure of $\overline{V(\lambda)}_{min}$ can easily be found from the above theorems. For if $\overline{V(\lambda)}_{max}$ has a composition series $0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \overline{V(\lambda)}_{max}$, then $\overline{V(\lambda)}_{min}$ has composition series $0 \subset W_1 \subset \ldots \subset W_n = \overline{V(\lambda)}_{min}$, where $W_{n-k} = \left(V_n / V_k\right)^*$. further if $V_k = S_{J_k}$ then

$W_{n-k} = S'_{V-J_k} = \sum_{i \in V-J_k} K v_i$ where v'_i is of weight $\lambda - 2i$ and

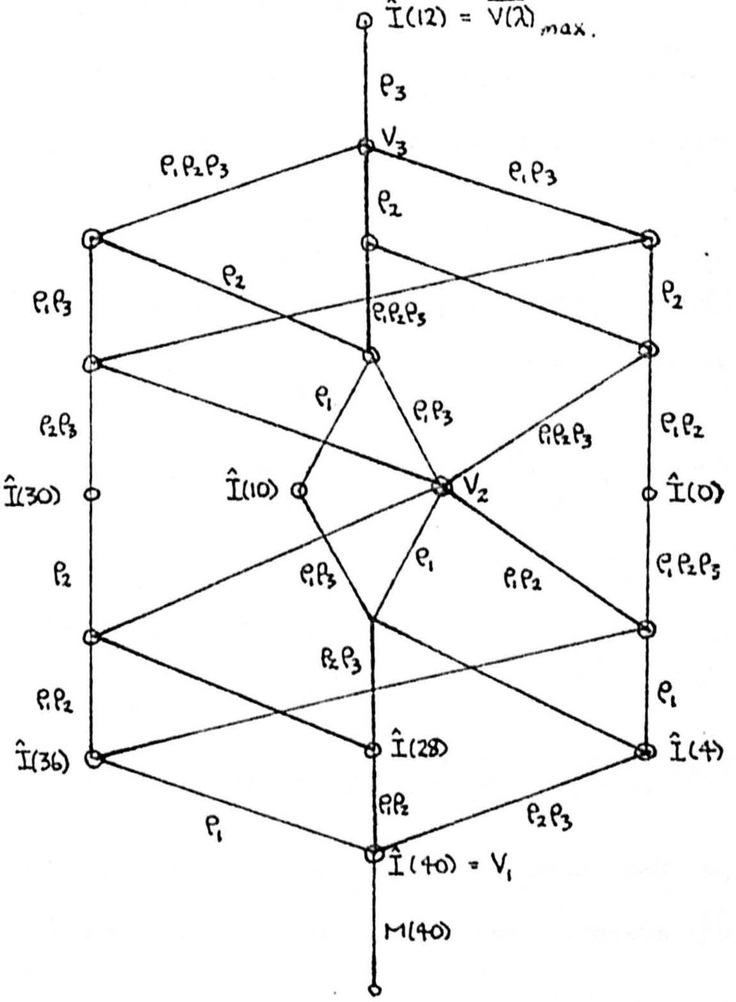
$V = \{0, \ldots, \lambda\}$.

2. For a natural embedding of the modules $S_{\hat{I}(\mu)}$ in $K[G]$ see (3.4.6).

An Example $\lambda = 40$, $p = 3$.

$\rho_1 \cdot \lambda = 36$, $\rho_2 \cdot \lambda = 30$, $\rho_3 \cdot \lambda = 12$, $\rho_1 \rho_2 \cdot \lambda = 28$, $\rho_2 \rho_3 \cdot \lambda = 4$, $\rho_1 \rho_3 \cdot \lambda = 10$, $\rho_1 \rho_2 \rho_3 \cdot \lambda = 0$.

From (3.2.9), $I_{(36)} = \{\lambda, \rho_1 \cdot \lambda\}$, $I_{(30)} = \{\lambda, \rho_1 \cdot \lambda, \rho_2 \cdot \lambda, \rho_1 \rho_2 \cdot \lambda\}$, $I_{(12)} = D_\lambda$, $I_{(28)} = \{\lambda, \rho_1 \rho_2 \cdot \lambda\}$, $I_{(4)} = \{\lambda, \rho_2 \rho_3 \cdot \lambda\}$, $I_{(10)} = \{\lambda, \rho_1 \rho_2 \cdot \lambda, \rho_2 \rho_3 \cdot \lambda, \rho_1 \rho_3 \cdot \lambda\}$, $I_{(0)} = \{\lambda, \rho_1 \cdot \lambda, \rho_2 \rho_3 \cdot \lambda, \rho_1 \rho_2 \rho_3 \cdot \lambda\}$.



The Loewy series $0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = \overline{V(\lambda)}_{max}$ is marked on the lattice diagram, and is in accordance with theorem (3.2.10).

3.3 The Affine Ring ; Submodules and decompositions.

Let a, b, c, d denote the coefficient functions on G given by $g = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$, so that $ad - bc = \underline{1}$.

Then from (1.1a), the affine ring $A = K[G] = K[a, b, c, d]$.

Hence we may express any $f \in A$ as a polynomial

$$f = \sum \lambda_{ijkl}\, a^i b^j c^k d^l \,, \quad \lambda_{ijkl} \in K .$$

A may be regarded as a 2-sided A-module, with the action of $g \in G$ on $f \in A$ given by,

$$g.f = R_g f \quad (: x \mapsto f(xg))$$

$$f.g = L_g f \quad (: x \mapsto f(gx)).$$

Both actions are K-linear and multiplicative, and hence the actions of U, H, U^- on a, b, c, d, tabulated below for future reference, determine the action of G on A,

(3.3.1)

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} . \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ta & t^{-1}b \\ tc & t^{-1}d \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} ta & tb \\ t^{-1}c & t^{-1}d \end{pmatrix}$$

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & ta+b \\ c & tc+d \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} . \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tb & b \\ c+td & d \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ ta+c & tb+d \end{pmatrix}$$

(3.3.2)

Recalling equations (3.2.2) we see that there is a commutative diagram

$$\begin{array}{ccc} \overline{V(r)}_{max} & \xrightarrow{\ \theta\ } & A \\ {\scriptstyle inc}\uparrow & \nearrow_{\varphi} & \\ M(r) & & \end{array}$$

with θ, φ (left) A-monomorphisms and $\theta(w_{i,K}) = a^{r-i}b^i$. Also $\varphi(M(r)) = KGa^r$ has K-basis $\left\{ a^{r-i}b^i : \binom{r}{i} \neq 0 \right\}$. Note that a,b could equally well be replaced by c,d respectively. The K-basis $\{a, b\}$ of $\overline{V(1)}_{max}$ affords the natural representation of G,

$$\sigma_1 : g \longmapsto g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad \text{with invariant matrix}$$

$\sigma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The K-basis $\{a^{r-i}b^i\}$ of $\overline{V(r)}_{max}$ affords σ_r, the r^{th}

symmetric power of σ_1 ; $\sigma_r(g)$ is sometimes called the r^{th} induced matrix of g [13].

e.g.

$$\sigma_2 = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

The entries of the invariant matrix σ_r will, of course, be homogeneous polynomials of degree r in a, b, c, d. with integral coefficients.

Let $\sigma_{(r)}$ be the invariant matrix of the representation of G afforded by $M(r)$.

Then if $r = \sum_{i=0}^{n-1} r_i p^i$, Steinberg's theorem gives

$$\sigma_{(r)} = \sigma_{r_0} \times \sigma_{r_1}^{Fr} \times \ldots\ldots \times \sigma_{r_{n-1}}^{Fr^{n-1}} \quad \text{(Kronecker product).}$$

(3.3.3) <u>Some notation</u> Write λ_r for the character $\lambda_r : \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \longmapsto t^r$ of H. Then the group of rational characters $X(H) = \{\lambda_r : r \in Z\} \cong Z$.

Let $\Pi_r = \{r, r-2, \ldots, -r\}$ be the set of weights of $\overline{V(r)}$ and $\Pi(n) = \{r \in Z : \frac{r+n}{2}, \frac{r-n}{2} \in Z_{\geqslant 0}\}$. Hence $r \in \Pi(n) \Leftrightarrow n \in \Pi_r$.

We now introduce a grading on A as follows. Consider A as a right A-module and let

$$A(n) = \{f \in A : L_h f = \lambda_n(h) f, \text{ all } h \in H\}.$$

Then $A(n)$ is a left A-submodule of A, and

(3.3.4) $$A = \bigoplus_{n \in Z} A(n).$$

Since A is injective (1.1.12), it follows that each $A(n)$ is an injective left A-module. Using equations (3.3.1) we find that $A(n) = \sum K a^i b^j c^k d^l$, where the sum is over all i, j, k, $l \geqslant 0$ such that $(i + j) - (k + l) = n$.

Trivially $A(r).A(s) = A(r + s)$, and so A is graded.

(3.3.5) <u>Lemma</u> For any fixed $\lambda_1, \lambda_2 \geqslant 0$, $S(\lambda_1, \lambda_2) = \{a^i b^j c^k d^l : i + j = \lambda_1, k + l = \lambda_2\}$ is a linearly independent set.

<u>Proof</u> Since $a^{\lambda_1 - i} b^i c^{\lambda_2 - l} d^l$ has weight $(\lambda_1 + \lambda_2) - 2(i + l)$ and elements of different weight cannot be linearly dependent, it suffices to prove that $\sum_{i+l=s} \lambda_{i,l}\, a^{\lambda_1 - i} b^i c^{\lambda_2 - l} d^l = 0$, $\lambda_{i,l} \in K^*$, implies $\lambda_{i,l} = 0$.

Assume $s \leqslant \lambda_1, \lambda_2$. Then $a^{\lambda_1 - s} c^{\lambda_2 - s} \sum_{l=0}^{s} \lambda_l\, a^l b^{s-l} c^{s-l} d^l = 0$

where $\lambda_l = \lambda_{s-l,l}$.

But A is an integral domain. Hence we need only consider

$$\sum \lambda_l\, a^l b^{s-l} c^{s-l} d^l = 0 .$$

Applying this last equation to the group element $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ we find that

$\lambda_o = 0$. Factoring by d and repeating the operation a sufficient number of times gives all $\lambda_l = 0$.

The proof is similar for the cases $\lambda_1 \leqslant s \leqslant \lambda_2$, $\lambda_2 \leqslant s \leqslant \lambda_1$ and $s \geqslant \lambda_1, \lambda_2$. //

(3.3.6) The action of G on a monomial is given by,

$$u(t).a^i b^j c^k d^l = \sum_{\alpha=0}^{j} \sum_{\beta=0}^{l} t^{\alpha+\beta} \binom{j}{\alpha} \binom{l}{\beta} a^{i+\alpha} b^{j-\alpha} c^{k+\beta} d^{l-\beta} .$$

$$u^-(t).a^i b^j c^k d^l = \sum_{\alpha=0}^{j} \sum_{\beta=0}^{k} t^{\alpha+\beta} \binom{i}{\alpha} \binom{k}{\beta} a^{i-\alpha} b^{j+\alpha} c^{k-\beta} d^{l+\beta} .$$

We observe that $i+j$ and $k+l$ are kept fixed.

Let $\lambda_1 = \frac{r+n}{2}$, $\lambda_2 = \frac{r-n}{2}$, where $r \in \Pi(n)$. Then (3.3.5) and (3.3.6) show that $N(r,n) = \sum_{f \in S(\lambda_1, \lambda_2)} Kf$ is an A-submodule of A with K-basis $S(\lambda_1, \lambda_2)$.

By definition, $A(n) = \sum_{r \in \Pi(n)} N(r,n)$. In fact $\{N(r,n)\}_{r \in \Pi(n)}$ is a filtration of $A(n)$. Since for every $s \geqslant 0$ there is an inclusion $\iota_s : N(r,n) \longrightarrow N(r + 2s, n)$ given by $\iota_s(f) = (ad - bc)^s f$. Hence

(3.3.7) $$A(n) = \bigcup_{r \in \Pi(n)} N(r,n) .$$

Define $H(r)$ to be the K-span of all monomials in A of degree r.

Then $H(r) = \bigoplus_{\alpha \in \Pi_r} N(r,\alpha)$ is a 2-sided A-module and $A = \sum_{r \geqslant 0} H(r)$.

As above we have inclusions $\iota_s : H(r) \longrightarrow H(r+2s)$, and hence

(3.3.8) $$A = \bigcup_{i \geq 0} H(2i) \oplus \bigcup_{i \geq 0} H(2i+1) .$$

The following result will be needed in (3.6).

(3.3.9) <u>Proposition</u>

Suppose $r \in \Pi(n)$, $r = \sum_{i=0}^{k} r_i p^i$, $n = \sum_{i=0}^{k} n_i p^i$ and set

$$\lambda_1 = \frac{r+n}{2} , \ \lambda_2 = \frac{r-n}{2} , \ \lambda_{1i} = \frac{r_i + n_i}{2} , \ \lambda_{2i} = \frac{r_i - n_i}{2} , \ 0 \leq i \leq k .$$

Then there is a diagram ,

$$\begin{array}{ccc} \overline{V(\lambda_1)}_{max} \otimes \overline{V(\lambda_2)}_{max} & \xrightarrow{\ \psi\ } & N(r,n) \\ \uparrow \iota & & \uparrow \varphi \\ M(\lambda_1) \otimes M(\lambda_2) & \xrightarrow[\ \theta\ ]{} & \bigotimes_{i=0}^{k} N(r_i,n_i)^{Fr^i} \end{array}$$

where ι is inclusion, such that :

(i) ψ is an A-isomorphism.

(ii) If $0 \leq \lambda_{1,i}$, $\lambda_{2,i} < p$, $0 \leq i \leq k$, then θ is an A-isomorphism, φ an A-monomorphism , and the diagram commutes .

<u>Proof</u> Let (v_m) , (w_l) be bases of $\overline{V(\lambda_1)}_{max}$, $\overline{V(\lambda_2)}_{max}$ as in (3.2.2) . Then comparison of the equations in (3.2.2) and (3.3.6) show that

$\psi : v_m \otimes w_l \longmapsto a^{\lambda_1 - m} b^m c^{\lambda_2 - l} d^l$ is an A-map, and by definition of $N(r,n)$ an A-isomorphism .

(ii) The condition $0 \leq \lambda_{1,i}$, $\lambda_{2,i} < p$ implies that $N(r_i,n_i) \cong M(\lambda_{1,i}) \otimes M(\lambda_{2,i})$ by (i). Steinberg's theorem (1.3.7) then establishes an A-isomorphism θ which may be regarded as a composite of isomorphisms ,

$$v_m \otimes w_l \longmapsto a^{\lambda_1 - m} b^m c^{\lambda_2 - l} d^l \longmapsto a^{\lambda_{10} - m_0} b^{m_0} \otimes \ldots \otimes a^{(\lambda_{1k} - m_k)p^k} b^{m_k p^k}$$

$$\otimes\, c^{\lambda_{20} - l_0} d^{l_0} \otimes \ldots \otimes c^{(\lambda_{2k} - l_k)p^k} d^{l_k p^k} \longmapsto a^{\lambda_{10} - m_0} b^{m_0} c^{\lambda_{20} - l_0} d^{l_0} \otimes \ldots$$

$$\otimes\, (a^{\lambda_{1k} - m_k} b^{m_k} c^{\lambda_{2k} - l_k} d^{l_k})^{p^k} , \text{ where } m \leq \lambda_1,\ l \leq \lambda_2 .$$

(See (3.3.2) et. seq.).

Now it is easy to check that if $f_i \in N(r_i,n_i)$ then the map φ defined

by $\varphi : f_1 \otimes f_2^p \otimes \ldots \otimes f_k^{p^k} \longmapsto f_1 . f_2^p \ldots . f_k^{p^k}$ and extended

K-linearly is an A-isomorphism, with image $\sum_{\substack{m \leq \lambda_1 \\ l \leq \lambda_2}} K a^{\lambda_1 - m} b^m c^{\lambda_2 - l} d^l$,

which makes the diagram commute. //

Attention will now be focussed on the decomposition of A into injective indecomposables. Define a set $T(n)$ by the rule,

$r \in T(n) \Leftrightarrow n \in T_r$, the set of weights of $M(r)$.

(3.3.10) <u>Lemma</u> $A(n)$ contains a copy of $M(r)$ if and only if $r \in T(n)$.

Moreover this copy will be unique and equal to $KGc_{r,n}$, where

$c_{r,n} = a^{\frac{r+n}{2}} c^{\frac{r-n}{2}}$.

<u>Proof</u> From (1.3.8(i)), $cf(M(r)) = KGa^r KG$. Since $U^- B$ is dense in G,

$a^r KG = \sum_{i \leq r} Ka^{r-i} c^i$, giving $cf(M(r)) = \bigoplus_{i \leq r} KGa^{r-i} c^i$. This is Burnside's

decomposition (1.1.6(ii)), because $KGa^{r-i} c^i$ is isomorphic to $M(r)$ by (3.4.7(ii)). The lemma follows from the fact that $cf(M(r))$ contains all copies of $M(r)$ in A (1.1.6(ii)). //

Let $M(r,n) = KGc_{r,n}$ be the unique copy of $M(r)$ in $A(n)$

for $r \in T(n)$, and $I(r,n)$ its injective cover. Recalling (1.3.8) and (1.6.1) we have,
$$\sigma(A) = \sum_{r \geq 0}^{\oplus} \Big(\bigoplus_{n \in T_r} M(r,n) \Big).$$

This decomposition extends to A,

(3.3.11)
$$A = \sum_{r \geq 0}^{\oplus} \Big(\bigoplus_{n \in T_r} I(r,n) \Big).$$

We note that $\{ I(r) \cong I(r,n) \}_{r \geq 0}$ is a full set of injective indecomposables by (1.1.11), and that

(3.3.12)
$$A(n) = \bigoplus_{r \in T(n)} I(r,n).$$

Our picture of A is now quite extensive, but there remains the problem of finding the A-submodules $I(r,n)$ of A. Much of the remainder of this chapter will be devoted to its solution.

We end this chapter by saying more about the precise nature of $T(n)$.

Clearly $T(n) = T(-n)$, so we may assume without loss of generality that

$$n = \sum_{i=0}^{k-1} n_i p^i , \quad 0 \leqslant n_i < p .$$

(3.3.13) <u>Proposition</u> $T(n) = \dot{\bigcup_{\alpha}} S(\alpha)$ where

$S(\alpha) = \{ r : \alpha_i \leqslant r_i < p , \ r_i \in \Pi(\alpha_i) \}$, and α varies over all sequences

$\alpha = (\alpha_i)_{i=0,1,\ldots}$ given by

$\alpha_0 = n_0, p-n_0$; $\alpha_i = n_i, n_i+1$ implies $\alpha_{i+1} = n_{i+1}, p-n_{i+1}$;

$\alpha_i = p-n_i, p-n_i-1$ implies $\alpha_{i+1} = n_{i+1}+1, p-n_{i+1}-1$ $\quad 0 \leqslant i \leqslant k-1$.

If $\alpha_{k-1} = n_{k-1}, n_{k-1}+1$, then $\alpha_l = 0$, $l \geqslant k$.

If $\alpha_{k-1} = p-n_{k-1}, p-n_{k-1}-1$, then $\alpha_k = \ldots = \alpha_{m-1} = p-1$, $\alpha_m = 1$ and

$\alpha_l = 0$ for any m such that $l > m \geqslant k$.

(If $\alpha_i = p$, delete the sequence in which it occurs).

<u>Proof</u> Since $0 \leqslant n < p^k$, we seek all representations of n in the form ,

$$n = \sum \epsilon(i) \alpha_i p^i + \delta p^k , \quad \epsilon(i) = \pm 1 , \quad 0 \leqslant \alpha_i < p.$$

where $\delta = 0$ if $\sum \epsilon(i) \alpha_i p^i \geqslant 0$ (implying $\alpha_{k-1} \geqslant 0$)

$\delta = 1$ otherwise.

Putting the right hand side in p-adic form and comparing coefficients with those of n, it is easily seen that the sequences (α) in the proposition are the only ones that can occur. Clearly n can be expressed as above in at most 2^k different ways in 1-1 correspondence with the sequences $(\epsilon(i))$. Further the union over α is disjoint since the sequnces $(\alpha_i \bmod 2)$ are all distinct. //

The sequences α such that $S(\alpha) \subset T(n)$ may be described iteratively as follows. Consider n as having infinitely many p-adic coefficients all but finitely many being zero. For reasons which will become apparent later let $S_m = \sum_{(\alpha)} \prod_{i=0}^{m} (p-\alpha_i)$ represent the first $m+1$ terms in all sequences α with $S(\alpha) \subset T(n)$ and $\alpha_m = n_m, n_m+1$.

It is not hard to see that , using (3.3.13),

(3.3.14) $$S_m = S_{m-1}(p-n_m) + S'_{m-1}(p-n_m-1)$$ where S'_i is given by

$S'_i = S_{i-1}.n_i + S'_{i-1}.(n_i+1)$, $1 \leqslant i \leqslant m+1$, with initial values $S_0 = p-n_0$, $S'_0 = n_0$.

Notice that the exclusion condition $\alpha_i = p$ is accounted for, because the corresponding product in S_m will become zero. Hence by making m arbitrarily large, we may recover all sequences α with $S(\alpha) \subset T(n)$. Finally there is an interesting identity which will be encountered again in (3.5.4).

<u>Corollary</u> (3.3.15) Let $n_{(m)} = \sum_{i=0}^{m} n_i p^i$, then

$$S_m = p^{m+1} - n_{(m)} .$$

<u>Proof</u> By induction on m. The claim is true for $m=0$ clearly. Suppose the result holds for all $i < m$. From (3.3.14),

$$S_m = (p-n_m).S_{m-1} + (p-n_m-1).S'_{m-1} \qquad \text{(i)}$$

$$S'_{m-1} = n_{m-1}.S_{m-2} + (n_{m-1}+1).S'_{m-2} \qquad \text{(ii)}$$

Eliminating S'_{m-2} ,

$$S'_{m-1} = n_{m-1}.S_{m-2} + \frac{(n_{m-1}+1)}{(p-n_{m-1}-1)} . (S_{m-1} - (p-n_{m-1}).S_{m-2}).$$

By hypothesis, $S_{m-1} - (p-n_{m-1}).S_{m-2} = p^m - n_{(m-1)} - (p-n_{m-1}).(p^{m-1} - n_{(m-2)})$

$$= (p-n_{m-1}-1).n_{(m-2)}$$

Hence $S'_{m-1} = n_{m-1}(p^{m-1} - n_{(m-2)}) + (n_{m-1}+1).n_{(m-2)} = n_{(m-1)}$.

Substituting in (i) yields the result. //

3.4 <u>Restriction to the Borel subgroup</u> .

In this section we find the socle of $I(r,n)$ (and hence $A(n)$) as a $K[B]$-module. Though not essential in the sequel it is instructive and relevant to (3.5).

The simple $K[B]$-modules are given as follows. For each $r \in Z$, let $\lambda_r \in X(H)$ be extended to $\lambda_{r,B} : B \longrightarrow K^*$ thus :

$$\lambda_{r,B}(b) = \lambda_{r,B}(hu) = \lambda_r(h) \quad \text{all } b \in B,$$

where $b = hu$ is the Jordan decomposition. Then $\{\lambda_{r,B} : r \in Z\}$ is the set of all rational representations of B. Say a left $K[B]$-module V is of <u>type $\lambda_{r,B}$</u> , if $V = Kv$ where $v \neq 0$ and

(3.4.1) $$bv = \lambda_{r,B}(b)v \qquad \text{all } b \in B$$

Call such a v a B-vector (of weight r) and let $\mathcal{B}_r$ denote the space of such. Then $\mathcal{B}_r$ is a right A-submodule of A.

Let $A^U = \{ f \in A : R_u f = f \text{ all } u \in U \}$, $\mathcal{H}_r = \{ f \in A : R_h f = \lambda_r(h) f, \ h \in H \}$

Then these spaces are right A-submodules of A and $\mathcal{B}_r = A^U \cap \mathcal{H}_r$.

Using (3.3.1) we find that $A^U = K[a,c]$ and $\mathcal{H}_r = \sum K\, a^i b^j c^k d^l$,

where the sum is over all $i,j,k,l \geqslant 0$ such that $(i+k) - (j+l) = r$.

Hence $\mathcal{B}_r = \sum_{i+k=r} K\, a^i c^k$, and we have proved,

(3.4.2) <u>Proposition</u> The space of all B-vectors of type r in $A(n)$ has dimension 1 if $r \in \Pi(n)$, being spanned by $c_{r,n} = a^{\frac{r+n}{2}} c^{\frac{r-n}{2}}$, dimension 0 otherwise.

(3.4.3) <u>Corollary</u> $\sigma_B\, A(n) = \sum_{r \in \Pi(n)} K c_{r,n}$, $\sigma_B\, N(r,n) = \sum_{\substack{s \in \Pi(n) \\ s \leqslant r}} K c_{s,n}$

In particular it follows that $\sigma_G\, A(n)$ is multiplicity free, a fact which has already been observed in (3.3.12). Combining (3.3.12) with (3.4.3) we see that there must exist a partition,

$$\Pi(n) = \dot{\bigcup_{r \in T(n)}} \Pi_r(n) \quad \text{with} \quad \sigma_B\, I(r,n) = \sum_{s \in \Pi_r(n)} K c_{s,n}.$$

The following lemma is immediate from the above.

(3.4.4) <u>Lemma</u> (i) $r \in T(n) \Leftrightarrow r \in \Pi(n)$ and $KG.c_{r,n}$ is simple.

(ii) Given $r \in T(n)$, $s \in \Pi_r(n) \Leftrightarrow c_{r,n} \in KGc_{s,n}$.

Part (i) of course was given by (3.3.10). This now prompts investigation into the structure of $KGc_{r,n}$.

Let $c_{r,n} = a^{\lambda_1} c^{\lambda_2}$ where $r = \lambda_1 + \lambda_2$, $n = \lambda_1 - \lambda_2$, $\lambda_1, \lambda_2 \geqslant 0$. Since $U^- B$ is dense in G, $KGc_{r,n} = KU^- c_{r,n}$.

Hence we consider

$$\begin{aligned} u^-(t).c_{r,n} &= (a+tb)^{\lambda_1} (c+td)^{\lambda_2} \\ &= \sum_{l=0}^{\lambda_1} \sum_{m=0}^{\lambda_2} t^{l+m} \binom{\lambda_1}{l} a^{\lambda_1 - l} b^l \binom{\lambda_2}{m} c^{\lambda_2 - m} d^m \\ &= \sum_{l,m} t^{l+m} y_{l,m} \end{aligned}$$

$$y_{l,m} = \binom{\lambda_1}{l}\binom{\lambda_2}{m} a^{\lambda_1 - l} b^l c^{\lambda_2 - m} d^m .$$

Put $y_j = \sum_{l+m=j} y_{l,m}$, then $u^-(t).c_{r,n} = \sum_{j=0}^{r} t^j y_j$.

Hence the set $\{y_j \neq 0\}$ forms a basis of $KGc_{r,n}$. ($y_j \in KGc_{r,n}$ since the matrix (t^j) is invertible.)

Since $\{y_{l,m} \neq 0\}$ is a linearly independent set (3.3.5), we have

$y_j \neq 0 \Leftrightarrow \binom{\lambda_1}{l}\cdot\binom{\lambda_2}{m} \neq 0$ for some l,m such that $l \leqslant \lambda_1$, $m \leqslant \lambda_2$, $l + m = j$.

Therefore the set $\{j : y_j \neq 0\}$ is precisely the subset S_{λ_1,λ_2} of distinct elements in $\{l + m : l \leqslant \lambda_1, m \leqslant \lambda_2\}$.

<u>Remark</u> Since $\binom{\lambda_1+\lambda_2}{j} = \sum_{l+m=j} \binom{\lambda_1}{l}\cdot\binom{\lambda_2}{m}$, we have $\binom{r}{j} \neq 0$ implies $y_j \neq 0$.

Also $y_j \neq 0 \Leftrightarrow y_{r-j} \neq 0$ and y_0, y_{λ_1}, y_{λ_2}, y_r are always non-zero.

Now it can easily be checked that (y_j) satisfies the same equations as the standard basis of $\overline{V(r)}_{min}$ (see (3.2.1)). Hence $KGc_{r,n}$ is a factor module of $\overline{V(r)}_{min}$ (or equivalently the dual of a submodule of $\overline{V(r)}_{max}$).

Let $r = \sum_{\alpha=0}^{m-1} r_\alpha p^\alpha$, $\lambda_1 = \sum_{\alpha=0}^{m-1} \lambda_{1\alpha} p^\alpha$, $\lambda_2 = \sum_{\alpha=0}^{m-1} \lambda_{2\alpha} p^\alpha$ $0 \leqslant r_\alpha, \lambda_{1\alpha}, \lambda_{2\alpha} < p$,

where $r = \sum \nu_\alpha p^\alpha$, $\nu_\alpha = \lambda_{1\alpha} + \lambda_{2\alpha}$.

(3.4.5) <u>Proposition</u> $(KGc_{r,n})^* \cong S_{\hat{I}(\mu_{\lambda_1,\lambda_2})}$ (see (3.2))

where $\mu_{\lambda_1,\lambda_2} = (\prod_{l=1}^{t} \rho_{i_l}\cdot\rho_{i_l+1}).r$ and $\{i_l : 1 \leqslant l \leqslant t, 0 \leqslant i_1 < \ldots < i_l \leqslant m-2\}$ $\{\alpha : \nu_\alpha \geqslant p\}$.

(Remarks : 1. If we define $\{\alpha_l : 1 \leqslant l \leqslant t\}$ by $\nu_\alpha = p-1$ $i_l < \alpha \leqslant i_l + \alpha_l$ $\neq p-1$ $\alpha = i_l + 1 + \alpha_l$ then $\rho_{i_l}\cdot\rho_{i_l+1} = \rho_{i_l}\rho_{i_l+1+\alpha_l}$.

2. If $\nu_\alpha = p-1$, $0 \leqslant \alpha < i_1$, then $\rho_{i_1}\cdot\rho_{i_1+1} = \rho_{i_1+1}$

3. If $i_1+1+\alpha_l = i_{l+1}$ then $\rho_{i_1+1+\alpha_l}\rho_{i_{l+1}}$ is deleted from $\mu_{\lambda_1,\lambda_2}$.)

<u>Proof</u> We require to show that $S_{\lambda_1,\lambda_2} = \hat{I}(\mu_{\lambda_1,\lambda_2})$ for then

$$S^*_{\hat{I}(\mu_{\lambda_1,\lambda_2})} \cong \overline{V(r)}_{min} / S'_{V-\hat{I}(\mu_{\lambda_1,\lambda_2})} \quad \text{(see remark following (3.2.11))},$$

$$\cong KGc_{r,n} .$$

The proof is suggested by considering the case t=1.

Suppose $\nu_{i_1} \geqslant p$, $\nu_\alpha = p-1 \quad i_1 < \alpha \leqslant i_1 + \alpha_1$

$\nu_\alpha < p-1$ otherwise.

Then $r_{i_1} = \nu_{i_1} - p$, $r_\alpha = 0 \quad i_1 < \alpha \leqslant i_1 + \alpha_1$

$r_{i_1+\alpha_1+1} = \nu_{i_1+\alpha_1+1} + 1$, $r_\alpha = \nu_\alpha$ otherwise.

Therefore $S_{\lambda_1,\lambda_2} = \{ a = \sum a_i p^i : a_i \leqslant \nu_i \} = \{ a : a_{i_1} < p, a_i \leqslant \nu_i \text{ otherwise} \}$

$\cup \{ a : a_{i_1} \geqslant p, a_i \leqslant \nu_i \text{ otherwise} \}$.

If $p \leqslant a_{i_1} \leqslant \nu_{i_1}$ then $a_{i_1} = p + k$, $0 \leqslant k \leqslant \nu_{i_1} - p = r_{i_1}$.

Hence $S_{\lambda_1,\lambda_2} = \{ a : a_\alpha < p, \; i_1 \leqslant \alpha \leqslant i_1 + \alpha_1 \; , \; a_{i_1+\alpha_1+1} \leqslant r_{i_1+\alpha_1+1} - 1$

$a_\alpha \leqslant r_\alpha \text{ otherwise} \} \cup \{ a : a_\alpha \leqslant r_\alpha \}$

$= \hat{I}(\rho_{i_1} \rho_{i_1+\alpha_1+1} . r)$ by (3.2.8) and (3.2.9). //

We note that the subset of distinct elements of $\{ (KGc_{r,n})^* : n \in \Pi_r \}$ is precisely the set $\{ S_{\hat{I}(\mu)} : \mu \in D_r \}$. This follows from the fact that every $\mu \in D_r$ is of the form $\mu = \mu_{\lambda_1,\lambda_2}$, $\lambda_1 + \lambda_2 = r$.

Let $\mathcal{V} = \{ (r,n) : r \in \Pi(n), n \in Z$ and $KGc_{r,n} \cong \overline{V(r)}_{min} \}$

$\mathcal{M} = \{ (r,n) : r \in \Pi(n), n \in Z$ and $KGc_{r,n}$ irreducible$\}$.

and $\varphi : \bigcup_{n \in Z} \Pi(n) \times \{n\} \longrightarrow Z_{\geqslant 0} \times Z_{\geqslant 0}$ denote the bijection

$\varphi : (r,n) \longmapsto (\frac{r+n}{2}, \frac{r-n}{2})$

(3.4.6) <u>Corollary</u> (i) $\varphi\mathcal{V} = \{ (\lambda_1,\lambda_2) : \lambda_{1\alpha} + \lambda_{2\alpha} \geqslant p-1, \; 0 \leqslant \alpha \leqslant m-2 \}$

(ii) $\varphi\mathcal{M} = \{ (\lambda_1,\lambda_2) : \lambda_{1\alpha} + \lambda_{2\alpha} \leqslant p-1 \}$.

(3.4.7) <u>Theorem</u> $\sigma_B I(r,n) = \sum_{s \in B(r)} Kc_{s,n}$. where $B(r) = \{ s : d_{sr} \neq 0 \}$.

<u>Proof</u> We require to prove that $\Pi_r(n) = B(r)$.

By (3.3.10),(3.4.4), $s \in \Pi_r(n) \Leftrightarrow M(r) = KGc_{r,n}$ is a submodule of $KGc_{s,n}$.

But by (3.4.5), this is true if and only if $M(r)$ is a top composition factor of $S_{\hat{I}(\mu_{\lambda_1,\lambda_2})}$, $\lambda_1 + \lambda_2 = s$. This in turn is true if and only if $r = \mu_{\lambda_1,\lambda_2}$ by (3.2). Hence $\Pi_r(n) \subseteq B(r)$.

Now in terms of weights the decomposition $\overline{V(m)} = \sum d_{mr} M(r)$ yields the partition $\Pi_m = \dot{\bigcup}_{r \in D_m} T_r$. This partition has a 'dual', $\Pi(n) = \dot{\bigcup}_{r \in T(n)} B(r)$.

Hence $\Pi_r(n) = B(r)$, since we already know that $\Pi(n) = \dot{\bigcup}_{r \in T(n)} \Pi_r(n)$. //

<u>Summary</u> The decompositions in (3.3) and (3.4) will now be summarised.

Let $I_\Pi = \{(r,n) : r \in \Pi(n),\ n \in Z\}$ denote the tableau,

.						.	
.					4		.
.				3		.	
$\Pi(2)$			2		4		.
$\Pi(1)$		1		3		.	
$\Pi(0)$	0		2		4		.
$\Pi(-1)$		1		3		.	
$\Pi(-2)$			2		4		.
.				3		.	
.					4		.
.						.	

with n^{th} row $\Pi(n)$ and 'r^{th} column' Π_r, and $I_T \subset I_\Pi$ the subtableau $\{(r,n) : r \in T(n),\ n \in Z\}$. Then

$$A = \bigoplus_{n \in Z} A(n) \qquad (3.3.4)$$

$$= \bigoplus_n \left(\bigcup_r N(r,n) \right), \quad (r,n) \in I_\Pi \qquad (3.3.7)$$

$$= \bigoplus I(r,n), \quad (r,n) \in I_T \qquad (3.3.11)$$

$$\sigma_G A = \bigoplus KGc_{r,n}, \quad (r,n) \in I_T \qquad (3.3.10)$$

$$\sigma_B A = \bigoplus Kc_{r,n}, \quad (r,n) \in I_\Pi \qquad (3.4.2)$$

The n^{th} row of I_T, I_Π gives the corresponding decompositions for $A(n)$. Finally by (3.4.7) there is a partition,

$$\Pi(n) = \dot{\bigcup}_{r \in T(n)} B(r) \qquad \text{and for } r \in T(n),$$

$$\sigma_B I(r,n) = \sum Kc_{s,n}, \qquad s \in B(r).$$

3.5 <u>The Injective Indecomposables.</u>

We now construct the injective indecomposable modules $I(r,n)$ of (3.3.11). This is done by using certain finite dimensional indecomposable A-modules utilised by Jeyakumar [12] in fiding the principal indecomposables for $SL(2,q)$.

Let $r \in \Pi(n)$, $0 \leq r < p$, $\lambda_1 = \frac{r' + n}{2}$, $\lambda_2 = \frac{r' - n}{2}$ where $r' = 2(p - 1) - r$.

Consider the A-module $V_{r,n} = \overline{V(\lambda_1)} \otimes \overline{V(\lambda_2)}$ $(= M(\lambda_1) \otimes M(\lambda_2))$. Following an argument of Humphreys [8], $V_{r,n}$ may be identified with $\overline{V(\lambda_1) \otimes V(\lambda_2)}$. Since the weights of $V_{r,n}$ and $V(\lambda_1) \otimes V(\lambda_2)$ are the same modulo conanical identifications, the composition factors of $V_{r,n}$ are those of the $\overline{V(s)}$ for which $V(s)$ occurs as a constituent of $V(\lambda_1) \otimes V(\lambda_2)$. Now consider the indecomposable direct summand $J(r,n)$ of $V_{r,n}$ in which the highest weight $\lambda_1 + \lambda_2 = r'$ occurs. Clearly $M(r')$ is a composition factor of $J(r,n)$. All other composition factors have highest weights in the same orbit of W_p as r'. Hence the Clebsch-Gordan expansion of $V(\lambda_1) \otimes V(\lambda_2)$ shows that $J(r,n)$ has composition factors $M(r')$, $M(r)$, $M(r)$ when $r \neq p - 1$ and $M(p - 1)$ when $r = p - 1$. The modules $J(r,n)$, $0 \leq r < p$, are the aforementioned modules of Jeyakumar. (Actually he considered only the case $n = -r$, but the modules $J(r,n)$ are all isomorphic for fixed r.)

Clearly $V_{r,n}$ contains a unique copy of $\overline{V(r')}_{min}$, which must be generated by a vector of weight r'. With trivial modifications, the construction in [12] shows that $\overline{V(r')}_{min}$ has an essentially unique extension by $M(r)$ in $V_{r,n}$. Hence we have an s.e.s.,

(3.5.1) $$0 \longrightarrow \overline{V(r')}_{min} \longrightarrow J(r,n) \longrightarrow M(r) \longrightarrow 0.$$

Let $\psi_{r,n} : V_{r,n} \longrightarrow N(r',n)$ be the A-isomorphism given by (3.3.9) and identify $J(r,n)$ with its image under $\psi_{r,n}$.

The following proposition is an easy consequence of the above considerations together with (3.3.10) and (3.4.2).

(3.5.2) <u>Proposition</u> (i) $\chi_{J(r,n)} = \chi(r) + \chi(r'), \quad r \neq p - 1.$
$= \chi(r), \quad r = p - 1.$

(ii) $J(r,n)$ has a unique maximal submodule $KGc_{r',n}$ isomorphic to $\overline{V(r')}_{min}$.

(iii) $\sigma_B J(r,n) = Kc_{r,n} + Kc_{r',n}$.

(iv) $\sigma_G J(r,n) = KGc_{r,n}$.

(Note also that $J(r,n)$ will be isomorphic to its dual.)

Let $r = \sum_{i=0}^{m-1} r_i p^i$ where $0 \leqslant r_i < p$, $0 \leqslant i \leqslant m-1$.

Then for $r \in T(n)$ define $J_m(r,n) = \bigotimes_{i=0}^{m-1} J(r_i,\alpha_i)^{Fr^i}$, where $n = \sum_{i=0}^{m-1} \alpha_i p^i$

with $r_i \in \Pi(\alpha_i)$. Since $J(r_i,\alpha_i)$ is contained in $N(r'_i,\alpha_i)$,

(3.3.9) shows that there is an embedding,

$$\psi^{(m)}_{r,n} : J_m(r,n) \longrightarrow N(\rho_m^{-1}\cdot r,n).$$

(Recall $\rho_m^{-1}\cdot r = 2(p^m - 1) - r$.). Identify $J_m(r,n)$ with its image under $\psi^{(m)}_{r,n}$.

(3.5.3) <u>Proposition</u> (i) $\chi_{J_m(r,n)} = \sum_{s\in B(r)_m} \chi(s)$, where

$B(r)_m = \{ s \in B(r) : s \leqslant \rho_m^{-1}\cdot r \}$.

(ii) $\sigma_B\, J_m(r,n) = \sum_{s\in B(r)_m} Kc_{s,n}$.

(iii) $\sigma_G\, J_m(r,n) = KGc_{r,n}$.

[We assume $r_i \neq p-1$ in the proofs of (i) and (ii), which may easily be adapted to the case $r_i = p-1$ some i.]

<u>Proof</u> Recall from (2.3.4) the following identity in $Z[X]$,

$$(1) \quad \chi(s + tp) = \chi(s)(\chi(t))^{Fr} + \chi(p - 2 - s)(\chi(t - 1))^{Fr}$$

where $s,t \in Z$ and the domain of definition of χ is extended from X^+ to X.

Let $r = \sum_{i=0}^{m-1} r_i p^i$, $0 \leqslant r_i < p$, and let I be any subset of $\{0,1,\ldots,m-1\}$.

Then (2) $\prod_{i\in I} (\rho_i\cdot\rho_{i+1})^{-1}\cdot r = \sum_{i\in I} r'_i p^i + \sum_{i\notin I} r_i p^i$.

We now prove (i) by induction on m. First note that

$$\chi_{J_m(r,n)} = \prod_{i=0}^{m-1} (\chi(r_i) + \chi(r'_i))^{Fr^i} \quad \text{by (3.5.2(i))}.$$

Hence (i) is true for $m = 1$. Suppose it is true for all k, $1 \leqslant k < m$.

From (1) we obtain,

$$(3) \quad \sum_s \chi(s + r_m p^m) + \chi(s + r'_m p^m) = \sum_s \chi(s)(\chi(r_m) + \chi(r'_m))^{Fr^m}$$
$$+ \sum_s \chi(p^m - 2 - s)(\chi(r_m - 1) + \chi(r'_m - 1))^{Fr^m}$$

where the sum is over all $s \in B(r)_m$.

Now I claim that $\sum_{s\in B(r)_m} \chi(p^m - 2 - s) = 0$.

Let $s = \sum_{i\in I} r'_i p^i + \sum_{i\notin I} r_i p^i = S_I$, say, and let I' be the complement of I in $\{0,1,\dots,m-1\} = I_m$. Then clearly,

$$S_I + S_{I'} = 2\sum_{i\in I}(p-1)p^i + 2\sum_{i\in I'}(p-1)p^i = 2(p^m - 1).$$

Hence $\chi(p^m - 2 - S_I) = \chi(-(p^m - S_{I'})) = -\chi(p^m - 2 - S_{I'})$.

This proves the claim, for (2) implies that $B(r)_m = \{S_I : I\subset I_m\}$. (with the conventions of (3.2.10)).

Invoking the induction hypothesis, equation (3) now proves (i) for $k = m$, thus completing the induction.

(ii) This is an immediate consequence of (3.5.2(iii)),(3.3.9(ii)) and the fact that $B(r)_m = \{S_I : I\subset I_m\}$ as in (i).

(iii) Using (3.5.1) it is clear that $J_m(r,n)$ has a unique maximal submodule, the quotient by which is isomorphic to $M(r)$. But by its construction, $J_m(r,n)$ must be isomorphic to its dual, and so have a unique minimal submodule $M(r)$. Applying (3.3.10) gives the result.

Alternatively, (iii) follows from (ii) and (3.3.10), since if $s\in T(n)\cap B(r)$, with $r\in T(n)$, then $s = r$. //

<u>Remarks</u> 1. The B-vectors of type $s = S_I$ in $J_m(r,r)$ are of a very simple form. Let $S_I = a^{\lambda_1}c^{\lambda_2}$. Then $\lambda_{1i} = p-1$, $\lambda_{2i} = p-1-r_i$ for $i\in I$; $\lambda_{1i} = r_i$, $\lambda_{2i} = 0$ otherwise.

We now decompose $N(\rho_m^{-1}.n,n)$ into a direct sum of indecomposable A-modules, where $n<p^m$.

(3.5.4) <u>Theorem</u> For $n<p^m$, $N(\rho_m^{-1}.n,n) = \bigoplus_{r\in T(n)_m} J_m(r,n)$,

where $T(n)_m = \{r\in T(n) : r<p^m\}$.

<u>Proof</u> Since $J_m(r,n)\subset N(\rho_m^{-1}.r,n)$, we have inclusions

$$\iota_{\frac{r-n}{2}} : J_m(r,n)\longrightarrow N(\rho_m^{-1}.n,n) \qquad \text{(see (3.3.7))}.$$

$N(\rho_m^{-1}.n,n)$ is isomorphic to $\overline{V(p^m-1)}\otimes\overline{V(p^m-1-n)}_{max}$

and so has character $\chi(p^m - 1)\,\chi(p^m - 1 - n)$.

$$\chi(p^m - 1)\,\chi(p^m - 1 - n) = \sum_{s \in \Pi(n)_m} \chi(s), \qquad \Pi(n)_m = \{s \in \Pi(n) : s \leqslant \rho_m^{-1}.n\}$$

$$= \sum_{r \in T(n)_m} \sum_{s \in B(r)_m} \chi(s) \quad , \text{ since } \Pi(n) = \dot{\bigcup}_{r \in T(n)} B(r),$$

$$= \sum_{r \in T(n)_m} \chi_{J_m(r,n)} \qquad \text{by (3.5.3(i)).}$$

Since $\sigma_G\, J_m(r,n)$ is irreducible (3.5.3(iii)), the sum must be direct and so we have the theorem. //

Remarks 1. Comparing dimensions in (3.5.4) and dividing by p^m, gives the equation $p^m - n = S_{m-1}$ of (3.3.15).

2. $J_m(r,n)$ may be identified with the indecomposable direct summand of $\overline{V(p^m - 1 - (\frac{r+n}{2}))}_{max} \otimes \overline{V(p^m - 1 - (\frac{r-n}{2}))}_{max}$ containing the highest weight $2(p^m - 1) - r$.

We now come to the main result.

(3.5.5) Theorem If $n < p^t$ then $I(r,n) = \bigcup_{m \geqslant t} J_m(r,n)$.

Proof First observe that the union makes sense. By (3.5.2(iv)), $K\underline{1}$ is a submodule of $J(0,0)$. ($\underline{1}$ is the identity of A). Hence for $m \geqslant t$ (3.3.9) shows that $J_m(r,n)$ is a submodule of $J_{m+1}(r,n)$ in $A(n)$. Let $J_{r,n} = \bigcup_{m \geqslant t} J_m(r,n)$. Then I claim that $\sum_{r \in T(n)} J_{r,n}$ is a direct sum. For suppose $f_1 + f_2 + \dots + f_s = 0$, where $f_i \in J_{m_i}(r^{(i)},n)$. Now if $k = \max(m_i)$ then $f_i \in J_k(r^{(i)},n)$, $1 \leqslant i \leqslant s$. Hence each $f_i = 0$, since $\sum_{r \in T(n)_k} J_k(r,n)$ is direct by (3.5.4). Clearly $\bigoplus_{r \in T(n)} J_{r,n} \subset A(n)$. On the other hand (3.3.7) gives $A(n) = \bigcup_{m \geqslant 0} N(\rho_m^{-1}.n,n)$, and (3.5.4) gives $N(\rho_m^{-1}.n,n) \subset \bigoplus_{r \in T(n)} J_{r,n}$. Hence $A(n) = \bigoplus_{r \in T(n)} J_{r,n}$.

But $\sigma_G\, J_{r,n} = KGc_{r,n}$ and so we must have $J_{r,n} = I(r,n)$. //

Remark In [12] it is shown that the restrictions of the A-modules $J_m(r,-r)$ to the group $SL(2,p^m)$ are projective indecomposable

The Injective Indecomposables (see Th. (3·5·5))

$$N_m(n) = N(2(p^m-1)-n,\, n) \;, \qquad T(n)_m = \{ r_1, r_2, \dots, r_{i_m} \} \qquad \text{where} \quad n < p^t \,,\; m \geq t$$

$$
\begin{array}{ccccccccccccc}
A(n) & = & I(r_1,n) & \oplus & I(r_2,n) & \oplus \cdots \oplus & I(r_{i_t},n) & \oplus \cdots \oplus & I(r_{i_{t+1}},n) & \oplus \cdots \oplus & I(r_{i_m},n) & \oplus \cdots\cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
N_m(n) & = & J_m(r_1,n) & \oplus & J_m(r_2,n) & \oplus \cdots \oplus & J_m(r_{i_t},n) & \oplus \cdots \oplus & J_m(r_{i_{t+1}},n) & \oplus \cdots \oplus & J_m(r_{i_m},n) & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\
N_{t+1}(n) & = & J_{t+1}(r_1,n) & \oplus & J_{t+1}(r_2,n) & \oplus \cdots \oplus & J_{t+1}(r_{i_t},n) & \oplus \cdots \oplus & J_{t+1}(r_{i_{t+1}},n) & & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & & & & \\
N_t(n) & = & J_t(r_1,n) & \oplus & J_t(r_2,n) & \oplus \cdots \oplus & J_t(r_{i_t},n) & & & & &
\end{array}
$$

for $r \neq 0$. When $r = 0$ however, a factor module must be taken. A full set of P.I.M.s for $SL(2,p^m)$ is thus obtained.

Theorem (3.5.5) is illustrated on the page following it.

3.6 Cartan Invariants and Blocks.

In this section we dtermine the Cartan invariants $c_{\lambda\mu}$ and the blocks, as defined in 1.6 and 1.7.

Suppose that for $\lambda, \mu \in X^+$, $c_{\lambda\mu} \neq 0$. Then (1.7.2) says that there exists a $\nu \in X^+$ such that $d_{\nu\lambda} \neq 0$, $d_{\nu\mu} \neq 0$. Hence there are sequences $0 \leqslant i_1 < \dots < i_t$, $0 \leqslant j_1 < \dots < j_s$ such that $\lambda = \rho_{i_1} \cdots \rho_{i_t} \cdot \nu$, $\mu = \rho_{j_1} \cdots \rho_{j_s} \cdot \nu$. Let $k = \max(i_t, j_s)$. Then given any sequence $k < k_1 < \dots < k_l$, $\rho_{k_l}^{-1} \cdots \rho_{k_1}^{-1} \cdot \nu$ occurs in $B(\lambda) \cap B(\mu)$. This proves,

(3.6.1) <u>Proposition</u> If $c_{\lambda\mu} \neq 0$ then $c_{\lambda\mu}$ must be infinite.

Moreover $c_{\lambda\mu} \neq 0$ if and only if there exist sequences of integers $0 \leqslant i_1 < \dots < i_t$, $0 \leqslant j_1 < \dots < j_s$ such that $\lambda = \rho_{j_1} \cdots \rho_{j_s} \rho_{i_t}^{-1} \cdots \rho_{i_1}^{-1} \cdot \mu$.

The next result gives the block partition of X^+. (see (1.8.2)).

(3.6.2) <u>Theorem</u> $X^+ = \dot{\bigcup}_{\substack{i \geqslant 0 \\ 0 < n < p}} B_{i,n}$ is the partition of X^+ into blocks $B_{i,n} = \{\rho_{i+1}^{-j} \cdot (np^i - 1) : j \geqslant 0\}$, where ρ_{i+1}^{-j} denotes the j^{th}-fold iteration of ρ_{i+1}^{-1}. In particular the blocks $B_{i,n}$ are infinite in number.

<u>Proof</u> Let $\lambda \in X^+$, then since (3.6.2) is clearly a partition, $\lambda \in B_{i,n}$ for some $i \geqslant 0$, $0 < n < p$. The p-adic coefficients of λ then satisfy $\lambda_0 = \dots = \lambda_i = p - 1$, $\lambda_{i+1} \neq p - 1$. Hence if $d_{\lambda\mu} \neq 0$, then by (3.2.5) there must exist a $j \geqslant 0$ such that $\mu = \rho_{i+1}^{j} \cdot \lambda$.

This shows that if $d_{\lambda\mu} \neq 0$, then λ and μ must belong to the same set $B_{i,n}$. Now $\lambda, \mu \in X^+$ are adjacent if and only if $c_{\lambda\mu} \neq 0$ (1.8.1). But $c_{\lambda\mu} \neq 0$ if and only if $d_{\nu\lambda} \neq 0$ and $d_{\nu\mu} \neq 0$ for some $\nu \in X^+$ which

implies, by the above, that λ and μ belong to the same $B_{i,n}$.
Hence $B_{i,n}$ is a union of blocks.

On the other hand suppose that λ, μ belong to the same set $B_{i,n}$. Then assuming $\lambda \geqslant \mu$, we have $\mu = \rho_{i+1}^{j} \cdot \lambda$ for some $j \geqslant 0$.
Hence there exists a sequence $\mu_0 = \lambda$, $\mu_1 = \rho_{i+1} \cdot \lambda, \ldots, \mu_{j+1} = \rho_{i+1}^{j} \cdot \lambda = \mu$ such that $d_{\mu_t \mu_{t+1}} \neq 0$, $0 \leqslant t \leqslant j$. (see (3.2.5)).
But $d_{\mu_t \mu_{t+1}} \neq 0$ implies that $c_{\mu_t \mu_{t+1}} \neq 0$. Hence by (1.8.1), λ, μ belong to the same block. Thus $B_{i,n}$ is a block and we are finished. //

The matrices C and D can now be decomposed in accordance with (1.8.3).

Finally a proposition concerning block components.

(3.6.3) <u>Proposition</u> Let H_1 (H_2) be the sum of the block components containing the even (odd) weights. Then,

$$A = H_1 \oplus H_2 \ , \quad \text{where } H_1 = \bigcup_{i \geqslant 0} H(2i) \ , \quad H_2 = \bigcup_{i \geqslant 0} H(2i+1).$$

<u>Proof</u> Clear from (1.8.5(ii)) and (3.3.8). //

Conjectures .

Let $S = \{\lambda \in X^+ : \overline{V(\lambda)}$ irreducible$\}$. Define S_0 to be the intersection of S with X_p^+ but excluding the Steinberg weight $(p-1)\rho$.

Define S_i , $i \geqslant 0$, by $S_i + \rho = p^i(S_0 + \rho)$.

Conjecture 1. $\quad S = \dot{\bigcup}_{i \geqslant 0} S_i$

Conjecture 2. (The Block Conjecture).

The set S is an index set for the blocks, and

$$X^+ = \dot{\bigcup}_{\substack{i \geqslant 0 \\ \lambda \in S_i}} (W_{p^{i+1}} . \lambda \cap X^+) \quad \text{is the block partition.}$$

Conjecture 3. The Cartan invariants $c_{\lambda\mu}$ are either infinite or zero.

All of these conjectures have been proved for Φ of type A_1. Using the material of (2.3) it is not hard to see that they are all true for Φ of type A_2.

—— · ——