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# Asynchronous switching control for fuzzy Markov jump systems with periodically varying delay and its application to electronic circuits 

Yinghong Zhao, Likui Wang, Xiangpeng Xie, Hak-Keung Lam, and Junhua Gu


#### Abstract

This article focuses on addressing the issue of asynchronous $H_{\infty}$ control for Takagi-Sugeno (T-S) fuzzy Markov jump systems with generally incomplete transition probabilities (TPs). The delay is assumed to vary periodically, resulting in one monotonically increasing interval and one monotonically decreasing interval during each period. Meanwhile, a new LyapunovKrasovskii functional (LKF) is devised, which depends on membership functions (MFs) and two looped functions formulated for the monotonic intervals. Since the modes and TPs of the original system are assumed to be unavailable, an asynchronous switching fuzzy controller on the basis of hidden Markov model is proposed to stabilize the fuzzy Markov jump systems (FMJSs) with generally incomplete TPs. Consequently, a stability criterion with improved practicality and reduced conservatism is derived, ensuring the stochastic stability and $H_{\infty}$ performance of the closed-loop system. Finally, this technique is employed to the tunnel diode circuit system, and a comparison example is given, which verifies the practicality and superiority of the method.


Note to Practitioners-As a category of stochastic hybrid nonlinear systems driven by continuous time and discrete events, FMJSs have significant applications in practical engineering such as aircraft control systems and large-scale manufacturing systems. However, the real-time acquisition of system mode information and TPs is difficult due to technological constraints and limited resources. Moreover, the presence of periodically varying delay is prevalent in many industrial processes, resulting in degradation of dynamic system performance and instability. Therefore, it is necessary to study the FMJSs with generally incomplete TPs and periodically varying delays in asynchronous framework. Accordingly, unlike previous work, the designed LKF relies on system modes, MFs, and two looped functions for monotonic intervals. This new LKF exhibits a high degree of flexibility, fully leveraging the information of MFs and periodically varying delays, which can significantly reduce conservatism.

Index Terms-Fuzzy Markov jump systems, membership functions, asynchronous control, generally incomplete transition probabilities.

## I. INTRODUCTION

[^0]IN the field of scientific and engineering, it is widely acknowledged that numerous physical systems exhibit nonlinear behavior, which presents considerable challenges in their control and design compared to linear systems [1]-[6]. To tackle these difficulties, the Takagi-Sugeno (T-S) fuzzy model emerged and proved to be an effective strategy for approximating and representing these nonlinear systems (NSs) [7]-[9]. Its practical applications encompass a wide range of fields, including robotics, process control, power systems, and autonomous vehicles. However, when dealing with NSs affected by unpredictable parameters and structural mutations, such as environmental shifts, component maintenance, or failures, the limitations of the T-S fuzzy model become evident. In this context, Markov jump systems (MJSs) present a practical and effective solution, becoming a hot topic with notable achievements in observer design [10], [11], stability analysis [12], sampled-data control [13], etc. Hence, it is essential to further research fuzzy Markov jump systems (FMJSs), and many relevant issues have been considered. For example, the asynchronous fault detection problem for nonhomogeneous higher-level FMJSs was considered in [14]. The problem of dynamic-memory asynchronous event-triggered control for singular FMJSs against multi-cyber attacks was discussed in [15].

In the majority of research on FMJSs, a notable assumption is that the transition probability (TP) matrices are completely known. Nevertheless, in practical applications, obtaining the entire transition probabilities (TPs) is a challenging task. This is mainly due to random information missing and time delays that occur in diverse operation cycles of various communication networks. Consequently, obtaining exhaustive samples of the TPs is not only costly but also time-consuming. In light of these challenges, further research on MJSs with generally incomplete TPs becomes significant and crucial. Recently, the importance of this research has been increasingly recognized by scholars, and many related results have been published in various aspects, such as observer design [16], suboptimal control [17], reachable set estimation [18], etc.

Furthermore, for MJSs, another prevalent assumption is that the controller must possess accurate and up-to-date information about the current mode of the system [10]-[13]. However, achieving precise measurements of these system modes in actual engineering operations is difficult due to a variety of technical and financial constraints. To tackle this issue, substantial research efforts have been dedicated to study the issue of mode asynchrony, and a plethora of achievements have been
achieved. For instance, the issue of asynchronous $L_{2}$ control for positive FMJSs was researched in [19]. The problem of finite time asynchronous control for nonhomogeneous FMJSs with multiple disturbances was investigated in [20]. Applying the asynchronous disturbance observer method, the antidisturbance control issue for MJSs with matched/mismatched disturbances was considered in [21]. However, the current research on the asynchronous control problem for FMJSs with generally incomplete TPs is limited, which motivates the present work.

On the other hand, time-varying delay is a pervasive factor leading to degraded performance and instability in dynamic systems like network control systems, chemical processes, and power systems [22]-[25]. Similarly, FMJSs are inevitably subjected to time-varying delays since the characteristics of the underlying Markov process. Therefore, the stability analysis and controller design for FMJSs with time-varying delay holds immense importance in both theoretical research and practical implementations. As commonly acknowledged, the LyapunovKrasovskii functional (LKF) method is recognized as one of the most efficient techniques for analyzing the stability of time-delay systems, and a variety of LKFs have been presented to reduce conservatism in recent years [26]-[28]. Lately, a novel LKF based on monotone delay intervals was introduced in [29] for the periodically varying delay, a common occurrence in mechanical systems. Studies indicate that this LKF can effectively reduce conservatism by accurately capturing the monotonic characteristics of delays. However, the study conducted in [29] solely focuses on the stability analysis of time-delay systems, without delving into the research on asynchronous controller design for FMJSs. This gap presents another motivation for this paper.

It is worth mentioning that majority of existing works on FMJSs are founded on the utilization of LKF that are independent of membership functions (MFs) [30]-[32]. However, as a unique characteristic of fuzzy systems, the consideration of MFs-independent LKF inevitably leads to conservatism. For this problem, the MFs-dependent LKF was discussed in [33][35]. Nevertheless, these approaches require the assumption that the time derivatives of the MFs are bounded, which is difficult to obtain in practice and imposes certain limitations. Recently, an alternative approach known as the switching method was proposed in [36]. This method effectively addresses the time derivatives of MFs, thus reducing conservatism. Consequently, designing an LKF that adequately considers the delay monotonicity and depends on MFs is a research topic of significance.

Inspired by the discussion mentioned above, we are motivated to investigate the issue of asynchronous switching control for FMJSs with periodically varying delays and generally incomplete TPs. The key contributions are highlighted as follows:

1) The switching method is employed to design a novel monotone-delay-interval-based membership functiondependent LKF (MDI-MF-LKF). Unlike the LKF proposed in [8], [26]-[28], the MDI-MF-LKF incorporates more information about MFs, allowing for flexible handling of delays across different intervals. This enhances
analysis accuracy and reduces conservatism, as demonstrated in Example 2.
2) Different from the existing works on synchronous control for FMJSs, the phenomenon of mode asynchronous is considered in this paper, utilizing hidden Markov model as its foundation. Furthermore, a more practical and universal asynchronous switching fuzzy controller is presented for the first time to stabilize the FMJSs with generally incomplete TPs.
Notation: In this paper, matrices are assumed to possess compatible dimensions unless explicitly specified otherwise. The main notations are shown in Table I.

TABLE I
Notations in this paper

| Symbol | Denotes |
| :---: | :---: |
| $\mathbb{R}^{\mathrm{n}}$ | $n$-dimensional Euclidean space |
| $\mathbb{R}^{\mathbf{n} \times \mathrm{m}}$ | $\mathrm{n} \times \mathrm{m}$ real matrices |
| $P>0(<0)$ | symmetric positive (negative) matrix |
| $\operatorname{sym}(G)$ | $G+G^{T}$ |
| $\operatorname{diag}\{\cdots\}$ | a block-diagonal matrix |
| $*$ | symmetric matrix |
| $\mathcal{L}$ | weak infinitesimal operator |
| $\operatorname{col}\left\{M_{1}, M_{2}, \ldots, M_{n}\right\}$ | $\left[M_{1}^{T}, M_{2}^{T}, \ldots, M_{n}^{T}\right]^{T}$ |
| $J^{\{\mathfrak{s}\}}, \mathfrak{s}=1,2,3$ | $\left[\begin{array}{ccc}J^{1} & J^{2} & J^{3}\end{array}\right]$ |
| $J^{\mathfrak{r}\{\mathfrak{s}\}}, \mathfrak{r}=1,2, \mathfrak{s}=1,2,3$ | $\left[\begin{array}{ccc}12 & J^{13} \\ J^{21} & J^{22} & J^{33}\end{array}\right]$ |

## II. Problem formulation and preliminaries

Consider the FMJSs as follows:
Rule $i$ : IF $\chi_{1}(t)$ is $M_{i 1}, \ldots, \chi_{g}(t)$ is $M_{i g}$, Then

$$
\left\{\begin{align*}
\dot{x}(t)= & A_{\wp(t) i} x(t)+A_{d \wp(t) i} x(t-d(t))+B_{1 \wp(t) i} u(t)  \tag{1}\\
& +D_{\wp(t) i} \omega(t) \\
z(t)= & C_{\wp(t) i} x(t)+C_{d_{\wp(t) i}} x(t-d(t))+B_{2 \wp(t) i} u(t) \\
& +E_{\wp(t) i} \omega(t) \\
x(t)= & \phi(t), t \in\left[-d_{2}, 0\right]
\end{align*}\right.
$$

where $x(t) \in \mathbb{R}^{\mathrm{n}}, z(t) \in \mathbb{R}^{\mathrm{w}}, u(t) \in \mathbb{R}^{\mathrm{s}}$, and $\omega(t) \in \mathbb{R}^{\mathrm{f}}$ represent the system state, control output and input, and disturbance, respectively. $\chi_{\aleph}(t)$ is the premise variable, $M_{i \aleph}(i=$ $1,2, \ldots, r, \aleph=1,2, \ldots, g)$ denotes the fuzzy set with $r$ rules. The matrices in system (1) are pre-established and real. The delay $d(t)$ is defined in the following form [29]:

$$
\begin{equation*}
d(t)=d_{0}+\bar{d} f(\Omega t) \tag{2}
\end{equation*}
$$

where $f: \mathbb{R} \rightarrow[-1,1]$ denotes a periodic function, characterized by a single monotonically increasing interval and a single monotonically decreasing interval within each period. The parameters $\bar{d}$ and $\Omega$ establish the amplitude and frequency of this variation, with $\bar{d} \Omega<1$. Obviously, $d(t)$ satisfies

$$
\begin{equation*}
0 \leq d_{1} \leq d(t) \leq d_{2},|\dot{d}(t)| \leq \kappa<1 \tag{3}
\end{equation*}
$$

in which $d_{1}=d_{0}-\bar{d}, d_{2}=d_{0}+\bar{d}, \kappa=\bar{d} \Omega$.

Additionally, $\{\wp(t), \forall t \geq 0\}$ represents the continuous Markov chain taking values in a finite set $\mathbb{N}=\{1,2, \ldots, N\}$, and the TP matrix $\Pi_{1}=\left[\pi_{l o}\right]$ is presented as:
$\operatorname{Pr}\left\{\wp_{t+\Delta t}=o \mid \wp_{t}=l\right\}=\left\{\begin{array}{l}\pi_{l o} \Delta t+o(\Delta t), \quad l \neq o, \\ 1+\pi_{l l} \Delta t+o(\Delta t), l=o,\end{array}\right.$
with $\Delta t>0$ and $\lim _{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t}=0, \pi_{l o}$ satisfies $\pi_{l o} \geq 0$ with $l \neq o$ and $\pi_{l l}=-\sum_{o=1, o \neq l}^{\Delta t \rightarrow 0} \pi_{l o}$.

Note that the TPs of the jumping process are considered to be generally incomplete in this article, where each TP is assumed to be either completely unknown or only partially estimated. As an illustration, the TP matrix $\Pi_{1}$ representing $N$ modes can be expressed as follows:
$\Pi_{1}=\left[\begin{array}{ccccc}? & ? & ? & \cdots & \hat{\pi}_{1 N}+\Delta \pi_{1 N} \\ \hat{\pi}_{21}+\Delta \pi_{21} & ? & ? & \cdots & ? \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & \hat{\pi}_{N 2}+\Delta \pi_{N 2} & ? & \cdots & ?\end{array}\right]$
where ? signifies the completely unknown $\mathrm{TPs}, \hat{\pi}_{l o}$ and $\Delta \pi_{l o} \in\left[-\mathfrak{f}_{l o}, \mathfrak{f}_{l o}\right]\left(\mathfrak{f}_{l o} \geq 0\right)$ represent the known estimate value and estimate error of $\pi_{l o}$, respectively. For the sake of brevity, $\forall l \in \mathbb{N}$, we denote $\mathbb{N}=\mathbb{N}_{k}^{l}+\mathbb{N}_{u k}^{l}$ with

$$
\mathbb{N}_{k}^{l}=\left\{o \mid \text { the estimate value of } \pi_{l o} \text { is known }\right\}
$$

$$
\mathbb{N}_{u k}^{l}=\left\{o \mid \text { the estimate value of } \pi_{l o} \text { is unknown }\right\}
$$

If $\mathbb{N}_{k}^{l} \neq \phi$, it is further represented as $\mathbb{N}_{k}^{l}=\left\{\ell_{1}^{l}, \ell_{2}^{l}, \ldots, \ell_{u}^{l}\right\}$, where $1 \leq u \leq N$ and $\ell_{u}^{l}$ denotes the $u$ th bound-known factor in the $l$ th row of matrix $\Pi_{1}$.

Remark 1: In [20], [21], the TP matrices are completely known. However, in practical applications, accurately estimating the TPs of certain jumping processes is challenging due to equipment limitations and the presence of uncertain factors. Consequently, the TP matrices discussed in [16], [17] are considered incomplete, and they can be represented as follows:

$$
\left[\begin{array}{ccccc}
? & ? & ? & \cdots & \pi_{1 N} \\
\pi_{21} & ? & ? & \cdots & ? \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
? & \pi_{N 2} & ? & \cdots & ?
\end{array}\right]
$$

Unlike the aforementioned studies, our research introduces a more general form of the TP matrix, denoted as $\Pi_{1}$. It includes not only elements that are completely known or unknown, but also elements that have known lower and upper bounds. Notably, if $\mathfrak{f}_{l o}=0$, the generally incomplete TP matrix $\Pi_{1}$ will reduce to an incomplete TP matrix; if $\mathbb{N}_{u k}^{l}=\phi$ and $\mathfrak{f}_{l o}=0$, $\Pi_{1}$ will simplify to a completely known TP matrix.

Utilizing the standard fuzzy inference, we obtain that the MF is $h_{i}(\chi(t))=\frac{\prod_{\aleph}^{g}=1}{\mu_{i \aleph}\left(\chi_{\aleph}(t)\right)}, \mu_{i \aleph}\left(\chi_{\aleph}(t)\right)$ is the grade of membership of $\chi_{\aleph}(t)$ in $\mu_{i \aleph}$. Evidently, $\sum_{i=1}^{r} h_{i}(\chi(t))=1$, $h_{i}(\chi(t)) \geq 0$.

For brevity of presentation, with $\wp(t)=l$, we employ the substitution of $u$ for $u(t)$. The single summations are represented as $G_{l h}=\sum_{i=1}^{r} h_{i} G_{l i}$ where $h_{i}=h_{i}(\chi(t))$, then
the system (1) is formulated as

$$
\left\{\begin{array}{l}
\dot{x}(t)=A_{l h} x(t)+A_{d l h} x(t-d(t))+B_{1 l h} u+D_{l h} \omega(t),  \tag{5}\\
z(t)=C_{l h} x(t)+C_{d l h} x(t-d(t))+B_{2 l h} u+E_{l h} \omega(t), \\
x(t)=\phi(t), t \in\left[-d_{2}, 0\right]
\end{array}\right.
$$

The asynchronous fuzzy controller with periodically varying delay is presented as follows:

$$
\begin{equation*}
u=K_{\vartheta(t) h} x(t)+K_{d \vartheta(t) h} x(t-d(t)), \tag{6}
\end{equation*}
$$

where $\vartheta(t)$ represents the controller mode and takes values from another finite set $\mathbb{F}=\{1,2, \ldots, F\}$. It is subject to the constraints imposed by the conditional probability ( CP ) matrix $\Pi_{2}=\left[\partial_{l \dagger}\right]$, which satisfies

$$
\begin{equation*}
\operatorname{Pr}\{\vartheta(t)=\dagger \mid \wp(t)=l\}=\partial_{l \dagger} \tag{7}
\end{equation*}
$$

with $\partial_{l \dagger} \in[0,1]$, and $\sum_{\dagger=1}^{F} \partial_{l \dagger}=1$.
For conciseness, $K_{\vartheta(t) h}, K_{d \vartheta(t) h}$ are denoted as $K_{\dagger h}, K_{d \dagger h}$, respectively. The two summations are represented as $G_{l \dagger h h}=$ $\sum_{i=1}^{r} \sum_{j=1}^{r} h_{i} h_{j} G_{l \dagger i j}$. Substituting (6) into (5), we derive the subsequent closed-loop system:
$\left\{\begin{array}{l}\dot{x}(t)=A_{l \dagger h h} x(t)+A_{d l \dagger h h} x(t-d(t))+D_{l h} \omega(t), \\ z(t)=C_{l \dagger h h} x(t)+C_{d l \dagger h h} x(t-d(t))+E_{l h} \omega(t), \\ x(t)=\phi(t), t \in\left[-d_{2}, 0\right],\end{array}\right.$
where $A_{l \dagger i j}=A_{l i}+B_{1 l i} K_{\dagger j}, A_{d l \dagger i j}=A_{d l i}+B_{1 l i} K_{d \dagger j}$, $C_{l \dagger i j}=C_{l i}+B_{2 l i} K_{\dagger j}, C_{d l \dagger i j}=C_{d l i}+B_{2 l i} K_{d \dagger j}$.

Considering an LKF $V(t)=x^{T}(t) \mathcal{U}_{h} x(t)$ that depends on the MFs, where $\mathcal{U}_{h}=\sum_{j=1}^{r} h_{j} \mathcal{U}_{j}$. Since $\sum_{j=1}^{r} \dot{h}_{j}=0$, we have

$$
\begin{equation*}
\dot{\mathcal{U}}_{h}=\sum_{j=1}^{r} \dot{h}_{j} \mathcal{U}_{j}=\sum_{k=1}^{r-1} \dot{h}_{k}\left(\mathcal{U}_{k}-\mathcal{U}_{r}\right), \tag{9}
\end{equation*}
$$

where $k=1,2, \ldots, r-1$. Similar to [36], a switching method is applied to ensure $\dot{\mathcal{U}}_{h} \leq 0$ :

$$
\left\{\begin{array}{l}
\text { if } \dot{h}_{k}<0, \text { then } \mathcal{U}_{k}-\mathcal{U}_{r}>0  \tag{10}\\
\text { if } \dot{h}_{k} \geq 0, \text { then } \mathcal{U}_{k}-\mathcal{U}_{r} \leq 0
\end{array}\right.
$$

Evidently, $2^{r-1}$ potential situations exist within equation (10). Define $\mathcal{H}_{\lambda}=\left\{\lambda:\right.$ The potential permutations of $\left.\dot{h}_{k}\right\}$, $\mathcal{C}_{\lambda}=\left\{\lambda:\right.$ The potential constraints of $\left.\mathcal{U}_{j}\right\}$ with $\lambda=$ $1,2, \ldots, 2^{r-1}$, then (10) can be presented as

$$
\begin{equation*}
\text { if } \mathcal{H}_{\lambda} \text {, then } \mathcal{C}_{\lambda} \tag{11}
\end{equation*}
$$

Furthermore, in the situation where the designed matrices within LKF are linked to Markov chain $\wp(t)(V(t)=$ $x^{T}(t) \mathcal{U}_{l h} x(t)$ ), each potential case described in (11) is transformed into corresponding $N$ modes, resulting in $2^{r-1}$ cases per mode. Therefore, (11) can be expressed as follows:

$$
\begin{equation*}
\text { if } \mathcal{H}_{\lambda} \text {, then } \mathcal{C}_{\lambda l} \tag{12}
\end{equation*}
$$

Lemma 1: [37] Let $\tilde{\xi} \in \mathbb{R}^{\mathfrak{s}}$, and $x:\left[\sigma_{1}, \sigma_{2}\right] \rightarrow \mathbb{R}^{\mathrm{n}}$ be a continuous differentiable function. For a positive definite matrix $Z \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, any matrix $\Re \in \mathbb{R}^{3 \mathrm{n} \times \mathfrak{s}}$, the following inequality holds:
$-\int_{\sigma_{1}}^{\sigma_{2}} \dot{x}^{T}(\alpha) Z \dot{x}(\alpha) d \alpha \leq 2 \xi^{T} \Pi^{T} \Re \tilde{\xi}+\left(\sigma_{2}-\sigma_{1}\right) \tilde{\xi}^{T} \Re^{T} \hat{Z}^{-1} \Re \tilde{\xi}$,
where

$$
\begin{aligned}
\xi= & \operatorname{col}\left\{x\left(\sigma_{1}\right), x\left(\sigma_{2}\right), \frac{1}{\sigma_{2}-\sigma_{1}} \int_{\sigma_{1}}^{\sigma_{2}} x(\alpha) d \alpha,\right. \\
& \left.\frac{1}{\left(\sigma_{2}-\sigma_{1}\right)^{2}} \int_{\sigma_{1}}^{\sigma_{2}} \int_{\beta}^{\sigma_{2}} x(\alpha) d \alpha d \beta\right\}, \\
\Pi= & \operatorname{col}\left\{\tilde{e}_{2}-\tilde{e}_{1}, \tilde{e}_{2}+\tilde{e}_{1}-2 \tilde{e}_{3}, \tilde{e}_{2}-\tilde{e}_{1}+6 \tilde{e}_{3}-12 \tilde{e}_{4}\right\}, \\
\hat{Z}= & \operatorname{diag}\{Z, 3 Z, 5 Z\}, \\
\tilde{e}_{\mathfrak{k}}= & {\left[0_{n \times(\mathfrak{k}-1) n} I_{n} 0_{n \times(4-\mathfrak{k}) n}\right], \mathfrak{k}=1,2,3,4 . }
\end{aligned}
$$

Lemma 2: [38] For a positive definite matrix $U \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, and a continuous differentiable function $x$ in $\left[\sigma_{1}, \sigma_{2}\right] \rightarrow \mathbb{R}^{\mathrm{n}}$, the following inequality holds:

$$
-\int_{\sigma_{1}}^{\sigma_{2}} \int_{\beta}^{\sigma_{2}} \dot{x}^{T}(\alpha) U \dot{x}(\alpha) d \alpha d \beta \leq-2 \mathfrak{T}_{1}^{T} U \mathfrak{T}_{1}-4 \mathfrak{T}_{2}^{T} U \mathfrak{T}_{2},
$$

where

$$
\begin{aligned}
\mathfrak{T}_{1}= & x\left(\sigma_{2}\right)-\frac{1}{\sigma_{2}-\sigma_{1}} \int_{\sigma_{1}}^{\sigma_{2}} x(\alpha) d \alpha \\
\mathfrak{T}_{2}= & x\left(\sigma_{2}\right)+\frac{2}{\sigma_{2}-\sigma_{1}} \int_{\sigma_{1}}^{\sigma_{2}} x(\alpha) d \alpha \\
& -\frac{6}{\left(\sigma_{2}-\sigma_{1}\right)^{2}} \int_{\sigma_{1}}^{\sigma_{2}} \int_{\beta}^{\sigma_{2}} x(\alpha) d \alpha d \beta
\end{aligned}
$$

## III. MAIN RESULTS

To facilitate convenient analysis, the notations are introduced as follows:

$$
\begin{aligned}
d_{1}(t)= & d(t)-d_{1}, d_{2}(t)=d_{2}-d(t), \dot{\Im}(t)=1-\dot{d}(t), \\
\rho_{1}(t)= & \operatorname{col}\left\{d_{1}(t)\left(e_{2}-e_{4}\right) \eta(t), d_{2}(t)\left(e_{3}-e_{2}\right) \eta(t),\right. \\
& \left.d_{1}(t) e_{10} \eta(t), d_{2}(t) e_{9} \eta(t)\right\}, \\
\rho_{2}(t)= & \operatorname{col}\left\{\eta_{1}(t), d_{1} e_{11} \eta(t), e_{9} \eta(t), e_{10} \eta(t)\right\}, \\
\rho_{3}(t)= & \operatorname{col}\left\{\left(e_{3}-e_{2}\right) \eta(t), e_{9} \eta(t)\right\}, \\
\rho_{4}(t)= & \operatorname{col}\left\{\left(e_{2}-e_{4}\right) \eta(t), e_{10} \eta(t)\right\}, \\
\rho_{5}(t)= & \operatorname{col}\left\{\eta_{1}(t), d_{1} e_{11} \eta(t), e_{9} \eta(t), e_{10} \eta(t), d_{1} e_{14} \eta(t),\right. \\
& \left.d_{1}(t) e_{15} \eta(t), d_{2}(t) e_{16} \eta(t)\right\}, \rho_{6}(t)=\operatorname{col}\{x(t), \dot{x}(t)\}, \\
\eta(t)= & \operatorname{col}\left\{\eta_{1}(t), \eta_{2}(t), \eta_{3}(t), \eta_{4}(t), \omega(t)\right\}, \\
\eta_{1}(t)= & \operatorname{col}\left\{x(t), x(t-d(t)), x\left(t-d_{1}\right), x\left(t-d_{2}\right)\right\}, \\
\eta_{2}(t)= & \operatorname{col}\left\{\dot{x}(t), \dot{x}(t-d(t)), \dot{x}\left(t-d_{1}\right), \dot{x}\left(t-d_{2}\right)\right\}, \\
\eta_{3}(t)= & \operatorname{col}\left\{\int_{t-d(t)}^{t-d_{1}} x(\alpha) d \alpha, \int_{t-d_{2}}^{t-d(t)} x(\alpha) d \alpha, \frac{1}{d_{1}} \int_{t-d_{1}}^{t} x(\alpha)\right. \\
& \left.\times d \alpha, \frac{1}{d_{1}(t)} \int_{t-d(t)}^{t-d_{1}} x(\alpha) d \alpha, \frac{1}{d_{2}(t)} \int_{t-d_{2}}^{t-d(t)} x(\alpha) d \alpha\right\}, \\
\eta_{4}(t)= & \operatorname{col}\left\{\frac{1}{d_{1}^{2}} \int_{t-d_{1}}^{t} \int_{\beta}^{t} x(\alpha) d \alpha d \beta, \frac{1}{d_{1}^{2}(t)} \int_{t-d(t)}^{t-d_{1}} \int_{\beta}^{t-d_{1}}\right. \\
& \left.\times x(\alpha) d \alpha d \beta, \frac{1}{d_{2}^{2}(t)} \int_{t-d_{2}}^{t-d(t)} \int_{\beta}^{t-d(t)} x(\alpha) d \alpha d \beta\right\}, \\
\bar{\Gamma}_{1}= & \operatorname{col}\left\{d_{1}(t)\left(e_{2}-e_{4}\right), d_{2}(t)\left(e_{3}-e_{2}\right), d_{1}(t) e_{10}, d_{2}(t) e_{9}\right\}, \\
\bar{\Gamma}_{2}= & \operatorname{col}\left\{e_{1}, e_{2}, e_{3}, e_{4}, d_{1} e_{11}, e_{9}, e_{10}\right\}, \\
\bar{\Gamma}_{3}= & \operatorname{col}\left\{e_{3}-e_{2}, e_{9}\right\}, \bar{\Gamma}_{4}=\operatorname{col}\left\{e_{2}-e_{4}, e_{10}\right\}, \\
\bar{\Gamma}_{5}= & \operatorname{col}\left\{\bar{\Gamma}_{2}, d_{1} e_{14}, d_{1}(t) e_{15}, d_{2}(t) e_{16}\right\}, \\
\bar{\Gamma}_{6 a}= & \operatorname{col}\left\{e_{1}, e_{5}\right\}, \bar{\Gamma}_{6 b}=\operatorname{col}\left\{e_{3}, e_{7}\right\}, \bar{\Gamma}_{6 c}=\operatorname{col\{ e_{2},e_{6}\} ,}
\end{aligned}
$$

$\bar{\Gamma}_{6 d}=\operatorname{col}\left\{e_{4}, e_{8}\right\}, \mathfrak{C}_{1}=d_{1}(t) e_{12}-e_{9}, \mathfrak{C}_{2}=d_{2}(t) e_{13}-e_{10}$,
$\mathfrak{D}_{1}=\operatorname{col}\left\{e_{3}-e_{2}, e_{3}+e_{2}-2 e_{12}, e_{3}-e_{2}+6 e_{12}-12 e_{15}\right\}$,
$\mathfrak{D}_{2}=\operatorname{col}\left\{e_{2}-e_{4}, e_{2}+e_{4}-2 e_{13}, e_{2}-e_{4}+6 e_{13}-12 e_{16}\right\}$,
$\mathfrak{D}_{3}=\operatorname{col}\left\{e_{1}-e_{11}, e_{1}+2 e_{11}-6 e_{14}\right\}$,
$\Gamma_{1}=\operatorname{col}\left\{\dot{d}(t)\left(e_{2}-e_{4}\right)+d_{1}(t)\left(\dot{\Im}(t) e_{6}-e_{8}\right),-\dot{d}(t)\right.$
$\times\left(e_{3}-e_{2}\right)+d_{2}(t)\left(e_{7}-\dot{\Im}(t) e_{6}\right), \dot{d}(t) e_{10}+d_{1}(t)$
$\left.\times\left(\dot{\Im}(t) e_{2}-e_{4}\right),-\dot{d}(t) e_{9}+d_{2}(t)\left(e_{3}-\dot{\Im}(t) e_{2}\right)\right\}$,
$\Gamma_{2}=\operatorname{col}\left\{e_{5}, \dot{\Im}(t) e_{6}, e_{7}, e_{8}, e_{1}-e_{3}, e_{3}-\dot{\Im}(t) e_{2}, \dot{\Im}(t) e_{2}\right.$
$\left.-e_{4}\right\}, \Gamma_{3}=\operatorname{col}\left\{e_{7}-\dot{\Im}(t) e_{6}, e_{3}-\dot{\Im}(t) e_{2}\right\}$,
$\Gamma_{4}=\operatorname{col}\left\{\dot{\Im}(t) e_{6}-e_{8}, \dot{\Im}(t) e_{2}-e_{4}\right\}, \Gamma_{5}=\operatorname{col}\left\{\Gamma_{2}, e_{1}\right.$
$\left.-e_{11}, e_{3}-\dot{\Im}(t) e_{12}-\dot{d}(t) e_{15}, \dot{\Im}(t) e_{2}-e_{13}+\dot{d}(t) e_{16}\right\}$,
$e_{\mathfrak{z}}=\left[\begin{array}{lll}0_{n \times(\mathfrak{z}-1) n} & I_{n} & 0_{n \times((16-\mathfrak{z}) n+f)}\end{array}\right]$,
$e_{17}=\left[\begin{array}{lll}0_{f \times 16 n} & I_{f}\end{array}\right], \mathfrak{z}=1,2, \ldots, 16$.
Theorem 1: For given scalars $\kappa \in[0,1), d_{\mathfrak{d}} \geq 0$, and $\gamma>0$, the system (8) with known TPs is stochastically stable with a prescribed $H_{\infty}$ performance level $\gamma$, if there exist positive definite matrices $S_{l j} \in \mathbb{R}^{10 \mathrm{n} \times 10 \mathrm{n}}, R_{1 l j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}$, $R_{2 j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, R_{3 j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, M_{j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, Z_{j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $U_{j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, and any matrices $P_{\Im \mathfrak{d}} \in \mathbb{R}^{4 \mathrm{n} \times 7 \mathrm{n}}, Q_{\Im \mathfrak{j} j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $\mathfrak{J}_{\mathfrak{d}}, \mathfrak{P}_{\mathfrak{d}} \in \mathbb{R}^{3 \mathrm{n} \times(16 \mathrm{n}+\mathrm{f})}, \mathfrak{G}_{\mathfrak{d}}, \mathfrak{F}_{\mathfrak{d}} \in \mathbb{R}^{\mathrm{n} \times(16 \mathrm{n}+\mathrm{f})}, \mathfrak{J}_{\mathfrak{d}}^{\mathfrak{r}\{\mathfrak{s}\}}, \mathfrak{P}_{\mathfrak{d}}^{\mathfrak{r}\{\mathfrak{s}\}} \in$ $\mathbb{R}^{3 \mathrm{n} \times 16 \mathrm{n}}, \mathfrak{J}_{\mathfrak{d}}^{\mathfrak{r}\{17\}}, \mathfrak{P}_{\mathfrak{d}}^{\mathfrak{r}\{17\}} \in \mathbb{R}^{3 \mathrm{n} \times \mathrm{f}}, \mathfrak{G}_{\mathfrak{d}}^{\{\mathfrak{s}\}}, \mathfrak{F}_{\mathfrak{d}}^{\{\mathfrak{j}\}} \in \mathbb{R}^{\mathrm{n} \times 16 \mathrm{n}}$, $\mathfrak{G}_{\mathfrak{d}}^{\{17\}}, \mathfrak{F}_{\mathfrak{d}}^{\{17\}} \in \mathbb{R}^{\mathrm{n} \times \mathrm{f}}, \mathfrak{W}_{1}, \mathfrak{W}_{2}, \mathfrak{W}_{3} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, such that (12) and the following inequalities hold for $l \in \mathbb{N}, \dagger \in \mathbb{F}$, $\Im \in\{I, D\}, \mathfrak{d}=1,2, \mathfrak{r}=1,2,3, \mathfrak{s}=1,2, \ldots, 16$

$$
\begin{equation*}
\sum_{o=1}^{N} \pi_{l o} R_{1 o j}-M_{j} \leq 0 \tag{13}
\end{equation*}
$$

$$
\begin{align*}
& \left\{\begin{array}{lr}
\Psi_{l \dagger i i}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, & \dot{d}(t) \\
\Psi_{l \dagger i j}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)+\Psi_{l \dagger j i}^{I} & \in[0, \kappa], \\
\left.d_{\mathfrak{d}}, \dot{d}(t)\right)<0, \dot{d}(t) & \in[0, \kappa],
\end{array}\right.  \tag{14}\\
& \left\{\begin{array}{lr}
\Psi_{l \dagger i i}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, & \dot{d}(t) \in[-\kappa, 0], \\
\Psi_{l \dagger i j}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)+\Psi_{l \dagger j i}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, \dot{d}(t) \in[-\kappa, 0],
\end{array}\right. \tag{15}
\end{align*}
$$

where

$$
\begin{aligned}
& \Psi_{l \dagger i j}^{I}\left(d_{1}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\Xi_{l \dagger i j}^{I}\left(d_{1}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \mathfrak{J}_{2}^{T} & \Upsilon_{l \dagger i j}^{T} \\
* & -\hat{Z}_{I 2 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \Psi_{l \dagger i j}^{I}\left(d_{2}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\Xi_{l \dagger i j}^{I}\left(d_{2}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \mathfrak{J}_{1}^{T} & \Upsilon_{l \dagger i j}^{T} \\
* & -\hat{Z}_{I 1 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \Psi_{l \dagger i j}^{D}\left(d_{1}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\Xi_{l \dagger i j}^{D}\left(d_{1}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \mathfrak{P}_{2}^{T} & \Upsilon_{l \dagger i j}^{T} \\
* & -\hat{Z}_{D 2 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \Psi_{l \dagger i j}^{D}\left(d_{2}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\Xi_{l \dagger i j}^{D}\left(d_{2}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \mathfrak{P}_{1}^{T} & \Upsilon_{l \dagger i j}^{T} \\
* & -\hat{Z}_{D 1 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
& \Xi_{l \dagger i j}^{I}(d(t), \dot{d}(t))=\bar{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))+\bar{\Xi}_{j}^{I}(d(t), \dot{d}(t)), \\
& \Xi_{l \dagger i j}^{D}(d(t), \dot{d}(t))=\bar{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))+\bar{\Xi}_{j}^{D}(d(t), \dot{d}(t)), \\
& \bar{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))=\operatorname{sym}\left(\bar{\Gamma}_{5}^{T} S_{l j} \Gamma_{5}+\mathfrak{W}_{0}^{T} \overline{\mathfrak{A}}_{l \dagger i j}\right)+\bar{\Gamma}_{6 a}^{T}\left(R_{1 l j}\right. \\
& \left.+d_{1} M_{j}\right) \bar{\Gamma}_{6 a}+\bar{\Gamma}_{6 b}^{T}\left(R_{2 j}-R_{1 l j}\right) \bar{\Gamma}_{6 b}+\dot{\Im}(t) \bar{\Gamma}_{6 c}^{T}\left(R_{3 j}\right. \\
& \left.-R_{2 j}\right) \bar{\Gamma}_{6 c}-\bar{\Gamma}_{6 d}^{T} R_{3 j} \bar{\Gamma}_{6 d}+e_{5}^{T}\left(\frac{1}{2} d_{1}^{2} U_{j}+\left(d_{2}-d_{1}\right) Z_{j}\right) \\
& \times e_{5}+\bar{\Gamma}_{5}^{T} \sum_{o=1}^{N} \pi_{l o} S_{o j} \bar{\Gamma}_{5}-\mathfrak{D}_{3}^{T} \hat{U}_{j} \mathfrak{D}_{3}-\gamma^{2} e_{17}^{T} e_{17}, \\
& \bar{\Xi}_{j}^{I}(d(t), \dot{d}(t))=\operatorname{sym}\left(\Gamma_{1}^{T} P_{I 1} \bar{\Gamma}_{2}+\bar{\Gamma}_{1}^{T} P_{I 1} \Gamma_{2}+\Gamma_{3}^{T} P_{I 2} \bar{\Gamma}_{4}\right. \\
& \left.+\bar{\Gamma}_{3}^{T} P_{I 2} \Gamma_{4}+\mathfrak{J}_{1}^{T} \mathfrak{D}_{1}+\mathfrak{J}_{2}^{T} \mathfrak{D}_{2}+\mathfrak{G}_{1}^{T} \mathfrak{C}_{1}+\mathfrak{G}_{2}^{T} \mathfrak{C}_{2}\right) \\
& +d_{2}(t)\left(e_{7}^{T} Q_{I 1 j} e_{7}-\dot{\Im}(t) e_{6}^{T} Q_{I 1 j} e_{6}\right) \\
& +d_{1}(t)\left(\dot{\Im}(t) e_{6}^{T} Q_{I 2 j} e_{6}-e_{8}^{T} Q_{I 2 j} e_{8}\right), \\
& \bar{\Xi}_{j}^{D}(d(t), \dot{d}(t))=\operatorname{sym}\left(\Gamma_{1}^{T} P_{D 1} \bar{\Gamma}_{2}+\bar{\Gamma}_{1}^{T} P_{D 1} \Gamma_{2}+\Gamma_{3}^{T} P_{D 2} \bar{\Gamma}_{4}\right. \\
& \left.+\bar{\Gamma}_{3}^{T} P_{D 2} \Gamma_{4}+\mathfrak{P}_{1}^{T} \mathfrak{D}_{1}+\mathfrak{P}_{2}^{T} \mathfrak{D}_{2}+\mathfrak{F}_{1}^{T} \mathfrak{C}_{1}+\mathfrak{F}_{2}^{T} \mathfrak{C}_{2}\right) \\
& +d_{2}(t)\left(e_{7}^{T} Q_{D 1 j} e_{7}-\dot{\Im}(t) e_{6}^{T} Q_{D 1 j} e_{6}\right) \\
& +d_{1}(t)\left(\dot{\Im}(t) e_{6}^{T} Q_{D 2 j} e_{6}-e_{8}^{T} Q_{D 2 j} e_{8}\right), \\
& \hat{Z}_{\Im \mathfrak{d} j}(\dot{d}(t))=\operatorname{diag}\left\{Z_{\Im \mathfrak{d} j}(\dot{d}(t)), 3 Z_{\Im \mathfrak{d} j}(\dot{d}(t)), 5 Z_{\Im \mathfrak{d} j}(\dot{d}(t))\right\}, \\
& Z_{\Im 1 j}(\dot{d}(t))=Z_{j}+\dot{d}(t) Q_{\Im 1 j}, Z_{\Im 2 j}(\dot{d}(t))=Z_{j}-\dot{d}(t) Q_{\Im 2 j}, \\
& \overline{\mathfrak{A}}_{l \dagger i j}=\sum_{\dagger=1}^{F} \partial_{l \dagger} A_{l \dagger i j} e_{1}+\sum_{\dagger=1}^{F} \partial_{l \dagger} A_{d l \dagger i j} e_{2}+D_{l i} e_{17}-e_{5}, \\
& A_{l \dagger i j}=A_{l i}+B_{1 l i} K_{\dagger j}, A_{d l \dagger i j}=A_{d l i}+B_{1 l i} K_{d \dagger j}, \\
& \Upsilon_{l \dagger i j}=\operatorname{col}\left\{\sqrt{\partial_{l 1}} z_{l 1 i j}, \sqrt{\partial_{l 2}} z_{l 2 i j}, \ldots, \sqrt{\partial_{l F}} z_{l F i j}\right\} \text {, } \\
& z_{l \dagger i j}=\left(C_{l i}+B_{2 l i} K_{\dagger j}\right) e_{1}+\left(C_{d l i}+B_{2 l i} K_{d \dagger j}\right) e_{2}+E_{l i} e_{17}, \\
& \mathfrak{W}_{0}=\mathfrak{W}_{1}^{T} e_{1}+\mathfrak{W}_{2}^{T} e_{2}+\mathfrak{W}_{3}^{T} e_{5}, \hat{U}_{j}=\operatorname{diag}\left\{2 U_{j}, 4 U_{j}\right\} \text {, } \\
& \mathfrak{J}_{\mathfrak{d}}=\left[\begin{array}{ll}
\mathfrak{J}_{\mathfrak{d}}^{\mathfrak{r}\{\mathfrak{s}\}} & \mathfrak{J}_{\mathfrak{d}}^{\mathfrak{r}\{17\}}
\end{array}\right], \mathfrak{P}_{\mathfrak{d}}=\left[\mathfrak{P}_{\mathfrak{d}}^{\mathfrak{r}\{\mathfrak{s}\}} \quad \mathfrak{P}_{\mathfrak{d}}^{\mathfrak{r}\{17\}}\right] \text {, } \\
& \mathfrak{G}_{\mathfrak{d}}=\left[\begin{array}{ll}
\mathfrak{G}_{\mathfrak{d}}^{\{\mathfrak{s}\}} & \mathfrak{G}_{\mathfrak{d}}^{\{17\}}
\end{array}\right], \mathfrak{F}_{\mathfrak{d}}=\left[\begin{array}{ll}
\mathfrak{F}_{\mathfrak{d}}^{\{\mathfrak{s}\}} & \mathfrak{F}_{\mathfrak{d}}^{\{17\}}
\end{array}\right] .
\end{aligned}
$$

Proof: Similar to the approach described in [29], it is assumed that for $t_{2 m-1}<t_{2 m}, m=\{1,2,3, \ldots\}$, the values $d\left(t_{2 m-1}\right)$ and $d\left(t_{2 m}\right)$ represent the extreme points of the delay function $d(t)$. Specifically, $d\left(t_{2 m-1}\right)=d_{1}$ corresponds to the minimum value, while $d\left(t_{2 m}\right)=d_{2}$ represents the maximum value. This assumption implies that $d(t)$ exhibits a monotonically increasing behavior within the intervals $t \in\left[t_{2 m-1}, t_{2 m}\right]$ and a monotonically decreasing behavior within the intervals $t \in\left[t_{2 m}, t_{2 m+1}\right]$. Consequently, it can be concluded that $\dot{d}(t) \in[0, \kappa]$ for $t \in\left[t_{2 m-1}, t_{2 m}\right]$ and $\dot{d}(t) \in[-\kappa, 0]$ for $t \in\left[t_{2 m}, t_{2 m+1}\right]$. Then, two separate looped functionals are constructed for these two categories of intervals. When $t \in\left[t_{2 m-1}, t_{2 m}\right)$, a looped functional is defined as:

$$
\begin{aligned}
V_{I}(t)= & 2 \rho_{1}^{T}(t) P_{I 1} \rho_{2}(t)+2 \rho_{3}^{T}(t) P_{I 2} \rho_{4}(t)+\left(d\left(t_{2 m}\right)\right. \\
& -d(t)) \int_{t-d(t)}^{t-d\left(t_{2 m-1}\right)} \dot{x}^{T}(\alpha) Q_{I 1 h} \dot{x}(\alpha) d \alpha+(d(t) \\
& \left.-d\left(t_{2 m-1}\right)\right) \int_{t-d\left(t_{2 m}\right)}^{t-d(t)} \dot{x}^{T}(\alpha) Q_{I 2 h} \dot{x}(\alpha) d \alpha .
\end{aligned}
$$

In the other case, when $t \in\left[t_{2 m}, t_{2 m+1}\right)$, a looped functional
is defined as:

$$
\begin{aligned}
V_{D}(t)= & 2 \rho_{1}^{T}(t) P_{D 1} \rho_{2}(t)+2 \rho_{3}^{T}(t) P_{D 2} \rho_{4}(t)+\left(d\left(t_{2 m}\right)\right. \\
& -d(t)) \int_{t-d(t)}^{t-d\left(t_{2 m+1}\right)} \dot{x}^{T}(\alpha) Q_{D 1 h} \dot{x}(\alpha) d \alpha+(d(t) \\
& \left.-d\left(t_{2 m+1}\right)\right) \int_{t-d\left(t_{2 m}\right)}^{t-d(t)} \dot{x}^{T}(\alpha) Q_{D 2 h} \dot{x}(\alpha) d \alpha .
\end{aligned}
$$

Utilizing the proposed looped functionals and accounting for the characteristics of the MFs, we construct the following MDI-MF-LKF:

$$
V(t)=\left\{\begin{array}{l}
V_{C}(t)+V_{I}(t), \quad t \in\left[t_{2 m-1}, t_{2 m}\right),  \tag{16}\\
V_{C}(t)+V_{D}(t), \quad t \in\left[t_{2 m}, t_{2 m+1}\right),
\end{array}\right.
$$

where

$$
\begin{aligned}
V_{C}(t)= & \rho_{5}^{T}(t) S_{l h} \rho_{5}(t)+\int_{t-d_{1}}^{t} \rho_{6}^{T}(\alpha) R_{1 l h} \rho_{6}(\alpha) d \alpha \\
& +\int_{t-d(t)}^{t-d_{1}} \rho_{6}^{T}(\alpha) R_{2 h} \rho_{6}(\alpha) d \alpha+\int_{t-d_{2}}^{t-d(t)} \rho_{6}^{T}(\alpha) \\
& \times R_{3 h} \rho_{6}(\alpha) d \alpha+\int_{-d_{1}}^{0} \int_{t+\beta}^{t} \rho_{6}^{T}(\alpha) M_{h} \rho_{6}(\alpha) d \alpha d \beta \\
& +\int_{-d_{2}}^{-d_{1}} \int_{t+\beta}^{t} \dot{x}^{T}(\alpha) Z_{h} \dot{x}(\alpha) d \alpha d \beta \\
& +\int_{t-d_{1}}^{t} \int_{\gamma}^{t} \int_{\beta}^{t} \dot{x}^{T}(\alpha) U_{h} \dot{x}(\alpha) d \alpha d \beta d \gamma
\end{aligned}
$$

It should be noted that $V_{C}(t) \geq 0, V_{I}\left(t_{2 m-1}\right)=V_{I}\left(t_{2 m}\right)=$ $V_{D}\left(t_{2 m}\right)=V_{D}\left(t_{2 m+1}\right)=0$. As a result, the MDI-MFLKF (16) is continuous in time and satisfies $V\left(t_{2 m-1}\right) \geq 0$, $V\left(t_{2 m}\right) \geq 0$.

To begin, we consider the case when $t \in\left[t_{2 m-1}, t_{2 m}\right)$, where $\dot{d}(t) \in[0, \kappa]$. Then, we have

$$
\begin{equation*}
\mathcal{L} V(t)=\sum_{\mathfrak{d}=1}^{2}\left(\mathcal{L} V_{I}^{\mathfrak{d}}(t)+\mathcal{L} V_{C}^{\mathfrak{d}}(t)\right), t \in\left[t_{2 m-1}, t_{2 m}\right) \tag{17}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathcal{L} V_{I}^{1}(t)= & \eta^{T}(t)\left\{2 \left(\Gamma_{1}^{T} P_{I 1} \bar{\Gamma}_{2}+\bar{\Gamma}_{1}^{T} P_{I 1} \Gamma_{2}+\Gamma_{3}^{T} P_{I 2} \bar{\Gamma}_{4}\right.\right. \\
& \left.+\bar{\Gamma}_{3}^{T} P_{I 2} \Gamma_{4}\right)+d_{2}(t)\left(e_{7}^{T} Q_{I 1 h} e_{7}-\dot{\Im}^{\prime}(t)\right. \\
& \left.\times e_{6}^{T} Q_{I 1 h} e_{6}\right)+d_{1}(t)\left(\dot{\Im}^{( }(t) e_{6}^{T} Q_{I 2 h} e_{6}\right. \\
& \left.\left.-e_{8}^{T} Q_{I 2 h} e_{8}\right)\right\} \eta(t)+\mathfrak{H}_{1}+\mathfrak{H}_{2}, \\
\mathcal{L} V_{I}^{2}(t)= & d_{2}(t) \int_{t-d(t)}^{t-d_{1}} \dot{x}^{T}(\alpha) \dot{Q}_{I 1 h} \dot{x}(\alpha) d \alpha \\
& +d_{1}(t) \int_{t-d_{2}}^{t-d(t)} \dot{x}^{T}(\alpha) \dot{Q}_{I 2 h} \dot{x}(\alpha) d \alpha, \\
\mathcal{L} V_{C}^{1}(t)= & \eta^{T}(t)\left\{2 \bar{\Gamma}_{5}^{T} S_{l h} \Gamma_{5}+\bar{\Gamma}_{6 a}^{T}\left(R_{1 l h}+d_{1} M_{h}\right) \bar{\Gamma}_{6 a}+\bar{\Gamma}_{6 b}^{T}\right. \\
& \times\left(R_{2 h}-R_{1 l h}\right) \bar{\Gamma}_{6 b}+\dot{\Im}^{\prime}(t) \bar{\Gamma}_{6 c}^{T}\left(R_{3 h}-R_{2 h}\right) \bar{\Gamma}_{6 c} \\
& -\bar{\Gamma}_{6 d}^{T} R_{3 h} \bar{\Gamma}_{6 d}+e_{5}^{T}\left(\frac{1}{2} d_{1}^{2} U_{h}+\left(d_{2}-d_{1}\right) Z_{h}\right) e_{5} \\
& \left.+\bar{\Gamma}_{5}^{T} \sum_{o=1}^{N} \pi_{l o} S_{o h} \bar{\Gamma}_{5}\right\} \eta(t)+\int_{t-d_{1}}^{t} \rho_{6}^{T}(\alpha)\left(\sum_{o=1}^{N} \pi_{l o}\right. \\
& \left.\times R_{1 o h}-M_{h}\right) \rho_{6}(\alpha) d \alpha+\mathfrak{H}_{3}+\mathfrak{H}_{4},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L} V_{C}^{2}(t)= & \sum_{j=1}^{r} \dot{h}_{j}\left(\rho_{5}^{T}(t) S_{l j} \rho_{5}(t)+\int_{t-d_{1}}^{t} \rho_{6}^{T}(\alpha) R_{1 l j} \rho_{6}(\alpha) d \alpha\right) \\
& +\int_{t-d(t)}^{t-d_{1}} \rho_{6}^{T}(\alpha) \dot{R}_{2 h} \rho_{6}(\alpha) d \alpha+\int_{t-d_{2}}^{t-d(t)} \rho_{6}^{T}(\alpha) \\
& \times \dot{R}_{3 h} \rho_{6}(\alpha) d \alpha+\int_{-d_{1}}^{0} \int_{t+\beta}^{t} \rho_{6}^{T}(\alpha) \dot{M}_{h} \rho_{6}(\alpha) d \alpha d \beta \\
& +\int_{-d_{2}}^{-d_{1}} \int_{t+\beta}^{t} \dot{x}^{T}(\alpha) \dot{Z}_{h} \dot{x}(\alpha) d \alpha d \beta \\
& +\int_{t-d_{1}}^{t} \int_{\gamma}^{t} \int_{\beta}^{t} \dot{x}^{T}(\alpha) \dot{U}_{h} \dot{x}(\alpha) d \alpha d \beta d \gamma,
\end{aligned}
$$

with

$$
\begin{aligned}
\mathfrak{H}_{1} & =-\dot{d}(t) \int_{t-d(t)}^{t-d_{1}} \dot{x}^{T}(\alpha) Q_{I 1 h} \dot{x}(\alpha) d \alpha, \\
\mathfrak{H}_{2} & =\dot{d}(t) \int_{t-d_{2}}^{t-d(t)} \dot{x}^{T}(\alpha) Q_{I 2 h} \dot{x}(\alpha) d \alpha, \\
\mathfrak{H}_{3} & =-\int_{t-d_{2}}^{t-d_{1}} \dot{x}^{T}(\alpha) Z_{h} \dot{x}(\alpha) d \alpha, \\
\mathfrak{H}_{4} & =-\int_{t-d_{1}}^{t} \int_{\beta}^{t} \dot{x}^{T}(\alpha) U_{h} \dot{x}(\alpha) d \alpha d \beta .
\end{aligned}
$$

According to Lemma 1 and Lemma 2, it follows that:

$$
\begin{aligned}
\sum_{\mathfrak{y}=1}^{3} \mathfrak{H}_{\mathfrak{y}} \leq & \eta^{T}(t)\left\{\operatorname{sym}\left(\mathfrak{J}_{1}^{T} \mathfrak{D}_{1}+\mathfrak{J}_{2}^{T} \mathfrak{D}_{2}\right)+d_{1}(t) \mathfrak{J}_{1}^{T}\right. \\
& \left.\times \hat{Z}_{I 1 h}^{-1}(\dot{d}(t)) \mathfrak{J}_{1}+d_{2}(t) \mathfrak{J}_{2}^{T} \hat{Z}_{I 2 h}^{-1}(\dot{d}(t)) \mathfrak{J}_{2}\right\} \eta(t), \\
\mathfrak{H}_{4} \leq & -\eta^{T}(t)\left(\mathfrak{D}_{3}^{T} \hat{U}_{h} \mathfrak{D}_{3}\right) \eta(t),
\end{aligned}
$$

with $\hat{Z}_{I \mathfrak{} h}(\dot{d}(t))=\operatorname{diag}\left\{Z_{I \mathfrak{o} h}(\dot{d}(t)), 3 Z_{I \mathfrak{} h}(\dot{d}(t)), 5 Z_{I \mathfrak{} h}(\dot{d}(t))\right\}$, $Z_{I 1 h}(\dot{d}(t))=Z_{h}+\dot{d}(t) Q_{I 1 h}>0, Z_{I 2 h}(\dot{d}(t))=Z_{h}-\dot{d}(t) Q_{I 2 h}>$ 0 . Additionally, for any invertible matrix $\mathfrak{W}_{\mathfrak{r}} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $\mathfrak{G}_{\mathfrak{d}} \in \mathbb{R}^{\mathrm{n} \times(16 \mathrm{n}+\mathrm{f})}, \mathfrak{d}=1,2, \mathfrak{r}=1,2,3$, based on system (8) and $\left(d_{1}(t) e_{12}-e_{9}\right) \eta(t)=0,\left(d_{2}(t) e_{13}-e_{10}\right) \eta(t)=0$, it follows

$$
\begin{gather*}
0=\eta^{T}(t) \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i} h_{j} \operatorname{sym}\left(\mathfrak{W}_{0}^{T} \overline{\mathfrak{A}}_{l \dagger i j}\right) \eta(t),  \tag{18}\\
0=2 \eta^{T}(t)\left\{\mathfrak{G}_{1}^{T} \mathfrak{C}_{1}+\mathfrak{G}_{2}^{T} \mathfrak{C}_{2}\right\} \eta(t) . \tag{19}
\end{gather*}
$$

Combining (12), (13), (17)-(19), we can obtain

$$
\begin{align*}
\mathcal{L} V(t) \leq & \eta^{T}(t) \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i} h_{j}\left(\Xi_{l \neq i j}^{I}(d(t), \dot{d}(t))\right. \\
& +\gamma^{2} e_{17}^{T} e_{17}+d_{1}(t) \mathfrak{J}_{1}^{T} \hat{Z}_{I 1 j}^{-1}(\dot{d}(t)) \mathfrak{J}_{1}  \tag{20}\\
& \left.+d_{2}(t) \mathfrak{J}_{2}^{T} \hat{Z}_{I 2 j}^{-1}(\dot{d}(t)) \mathfrak{J}_{2}\right) \eta(t) .
\end{align*}
$$

The following $H_{\infty}$ performance function $\mathcal{J}$ is considered:

$$
\begin{align*}
\mathcal{J} \leq & \int_{0}^{\infty} \eta^{T}(t) \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i} h_{j}\left(\mathfrak{Z}_{l \dagger i j}-\gamma^{2} e_{17}^{T} e_{17}\right) \eta(t)  \tag{21}\\
& +\mathcal{L} V(t) d t
\end{align*}
$$

with $\mathfrak{Z}_{l \dagger i j}=\sum_{\dagger=1}^{F} \partial_{l \dagger} z_{l \dagger i j}^{T} z_{l \dagger i j}$.

By employing the Schur complement, it can be deduced from (14) that $\mathcal{J}<0$. Further, when $\omega(t)=0$, it becomes evident that $\mathcal{L} V(t) \leq-\varepsilon_{1} x^{T}(t) x(t)$, with $\varepsilon_{1}>0$. Similarly, for the case where $t \in\left[t_{2 m}, t_{2 m+1}\right)$ and $\dot{d}(t) \in[-\kappa, 0]$, utilizing a comparable procedure, we can get

$$
\begin{align*}
\mathcal{L} V(t) \leq & \eta^{T}(t) \sum_{i=1}^{r} \sum_{j=1}^{r} h_{i} h_{j}\left(\Xi_{l \dagger i j}^{D}(d(t), \dot{d}(t))\right. \\
& +\gamma^{2} e_{17}^{T} e_{17}+d_{1}(t) \mathfrak{P}_{1}^{T} \hat{Z}_{D 1 j}^{-1}(\dot{d}(t)) \mathfrak{P}_{1}  \tag{22}\\
& \left.+d_{2}(t) \mathfrak{P}_{2}^{T} \hat{Z}_{D 2 j}^{-1}(\dot{d}(t)) \mathfrak{P}_{2}\right) \eta(t) .
\end{align*}
$$

Similarly, it can be deduced from (15) that $\mathcal{J}<0$. Moreover, when $\omega(t)=0$, we can easily obtain $\mathcal{L} V(t) \leq-\varepsilon_{2} x^{T}(t) x(t)$, with $\varepsilon_{2}>0$. Therefore, it is concluded that $\mathcal{L} V(t) \leq$ $-\varepsilon_{\mathfrak{v}} x^{T}(t) x(t)$ for $t \in\left[t_{0}, \infty\right)$, where $\varepsilon_{\mathfrak{v}}=\min \left\{\varepsilon_{1}, \varepsilon_{2}\right\}$. Considering that $V\left(t_{2 m-1}\right) \geq V(t) \geq V\left(t_{2 m+1}\right) \geq 0$ for $t \in\left[t_{2 m-1}, t_{2 m+1}\right)$, we can conclude that $V(t)$ is bounded and satisfies $V(t) \geq 0$ for $t \in\left[t_{0}, \infty\right)$. This completes the proof.

Remark 2: The MDI-MF-LKF (16) constructed in this paper introduces two distinct looped functionals to handle delays on monotonically increasing intervals and monotonically decreasing intervals separately. For $t \in\left[t_{2 m-1}, t_{2 m}\right)$, based on $V\left(t_{2 m-1}\right) \geq 0, V\left(t_{2 m}\right) \geq 0$, and $\mathcal{L} V(t) \leq-\varepsilon_{\mathfrak{v}} x^{T}(t) x(t)$, it implies that $V\left(t_{2 m-1}\right) \geq V(t) \geq V\left(t_{2 m}\right) \geq 0$. Similarly, for $t \in\left[t_{2 m}, t_{2 m+1}\right)$, the inequality $V\left(t_{2 m}\right) \geq V(t) \geq$ $V\left(t_{2 m+1}\right) \geq 0$ holds. Consequently, the condition $V(t) \geq 0$ can be ensured without the requirement of $V_{I}(t) \geq 0$ and $V_{D}(t) \geq 0$, resulting in less conservatism compared to the traditional LKF [8], [26]-[28]. In addition, the introduction of these two separate looped functionals allows for a more flexible treatment of delay on different intervals, thus enhancing the accuracy of the analysis.

Remark 3: Unlike the LKF proposed in [29], the MDI-MFLKF constructed in this article depends on MFs, providing a comprehensive framework for analyzing FMJSs. And the switching technique based on switching rule (12) is employed to ensure $\mathcal{L} V_{I}^{2}(t) \leq 0, \mathcal{L} V_{D}^{2}(t) \leq 0$, and $\mathcal{L} V_{C}^{2}(t) \leq 0$.

Theorem 2: For given scalars $\kappa \in[0,1), d_{\mathfrak{d}} \geq 0$, and $\gamma>0$, the system (8) with known TPs is stochastically stable with a prescribed $H_{\infty}$ performance level $\gamma_{2}$ if there exist positive definite matrices $\tilde{S}_{l j} \in \mathbb{R}_{\tilde{R}}^{10 \mathrm{n} \times 10 \mathrm{n}}, \tilde{R}_{1 l j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}$, $\tilde{R}_{2 j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \tilde{R}_{3 j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \tilde{M}_{j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \tilde{Z}_{j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $\tilde{U}_{j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, and any matrices $\tilde{P}_{\Im \mathfrak{d}} \in \mathbb{R}^{4 \mathrm{n} \times 7 \mathrm{n}}, \tilde{Q}_{\Im \mathfrak{d} j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $\tilde{\mathfrak{J}}_{\mathfrak{d}}, \tilde{\mathfrak{P}}_{\mathfrak{d}} \in \mathbb{R}^{3 \mathrm{n} \times(16 \mathrm{n}+\mathrm{f})}, \tilde{\mathfrak{G}}_{\mathfrak{d}}, \tilde{\mathfrak{F}}_{\mathfrak{d}} \in \mathbb{R}^{\mathrm{n} \times(16 \mathrm{n}+\mathrm{f})}, \mathfrak{W} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $\mathfrak{Y}_{\dagger j}, \mathfrak{Y}_{d \dagger j} \in \mathbb{R}^{\mathrm{s} \times \mathrm{n}}$, such that (12) and the following inequalities hold for $l \in \mathbb{N}, \dagger \in \mathbb{F}, \Im \in\{I, D\}, \mathfrak{d}=1,2, \mathfrak{r}=1,2,3$, $\mathfrak{s}=1,2, \ldots, 16$

$$
\begin{equation*}
\sum_{o=1}^{N} \pi_{l o} \tilde{R}_{1 o j}-\tilde{M}_{j} \leq 0 \tag{23}
\end{equation*}
$$

$$
\begin{align*}
& \left\{\begin{array}{lr}
\tilde{\Psi}_{l \dagger i i}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, & \dot{d}(t) \in[0, \kappa], \\
\tilde{\Psi}_{l \dagger i j}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)+\tilde{\Psi}_{l \dagger j i}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, \dot{d}(t) \in[0, \kappa],
\end{array}\right.  \tag{24}\\
& \left\{\begin{array}{lr}
\tilde{\Psi}_{l+i i}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, & \dot{d}(t) \in[-\kappa, 0], \\
\tilde{\Psi}_{l \dagger i j}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)+\tilde{\Psi}_{l \dagger j i}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, \dot{d}(t) \in[-\kappa, 0],
\end{array}\right. \tag{25}
\end{align*}
$$

where

$$
\begin{aligned}
& \tilde{\Psi}_{l \dagger i j}^{I}\left(d_{1}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\tilde{\Xi}_{l \dagger i j}^{I}\left(d_{1}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{J}}_{2}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{I 2 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \tilde{\Psi}_{l \dagger i j}^{I}\left(d_{2}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\tilde{\Xi}_{l \dagger i j}^{I}\left(d_{2}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{J}}_{1}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{I 1 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \tilde{\Psi}_{l \dagger i j}^{D}\left(d_{1}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\tilde{\Xi}_{l \dagger i j}^{D}\left(d_{1}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{P}}_{2}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{D 2 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \tilde{\Psi}_{l \dagger i j}^{D}\left(d_{2}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\tilde{\Xi}_{l \dagger i j}^{D}\left(d_{2}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{P}}_{1}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{D 1 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \tilde{\Xi}_{l \dagger i j}^{I}(d(t), \dot{d}(t))=\hat{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))+\hat{\Xi}_{j}^{I}(d(t), \dot{d}(t)), \\
& \tilde{\Xi}_{l \dagger i j}^{D}(d(t), \dot{d}(t))=\hat{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))+\hat{\Xi}_{j}^{D}(d(t), \dot{d}(t)), \\
& \hat{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))=\operatorname{sym}\left(\bar{\Gamma}_{5}^{T} \tilde{S}_{l j} \Gamma_{5}+\tilde{\mathfrak{W}}_{0}^{T} \tilde{\mathfrak{A}}_{l \dagger i j}\right)+\bar{\Gamma}_{6 a}^{T}\left(\tilde{R}_{1 l j}\right. \\
& \left.+d_{1} \tilde{M}_{j}\right) \bar{\Gamma}_{6 a}+\bar{\Gamma}_{6 b}^{T}\left(\tilde{R}_{2 j}-\tilde{R}_{1 l j}\right) \bar{\Gamma}_{6 b}+\dot{\Im}(t) \bar{\Gamma}_{6 c}^{T}\left(\tilde{R}_{3 j}\right. \\
& \left.-\tilde{R}_{2 j}\right) \bar{\Gamma}_{6 c}-\bar{\Gamma}_{6 d}^{T} \tilde{R}_{3 j} \bar{\Gamma}_{6 d}+e_{5}^{T}\left(\frac{1}{2} d_{1}^{2} \tilde{U}_{j}+\left(d_{2}-d_{1}\right) \tilde{Z}_{j}\right) \\
& \times e_{5}+\bar{\Gamma}_{5}^{T} \sum_{o=1}^{N} \pi_{l o} \tilde{S}_{o j} \bar{\Gamma}_{5}-\mathfrak{D}_{3}^{T} \bar{U}_{j} \mathfrak{D}_{3}-\gamma^{2} e_{17}^{T} e_{17}, \\
& \hat{\Xi}_{j}^{I}(d(t), \dot{d}(t))=\operatorname{sym}\left(\Gamma_{1}^{T} \tilde{P}_{I 1} \bar{\Gamma}_{2}+\bar{\Gamma}_{1}^{T} \tilde{P}_{I 1} \Gamma_{2}+\Gamma_{3}^{T} \tilde{P}_{I 2} \bar{\Gamma}_{4}\right. \\
& \left.+\bar{\Gamma}_{3}^{T} \tilde{P}_{I 2} \Gamma_{4}+\tilde{\mathfrak{J}}_{1}^{T} \mathfrak{D}_{1}+\tilde{\mathfrak{J}}_{2}^{T} \mathfrak{D}_{2}+\tilde{\mathfrak{G}}_{1}^{T} \mathfrak{C}_{1}+\tilde{\mathfrak{G}}_{2}^{T} \mathfrak{C}_{2}\right) \\
& +d_{2}(t)\left(e_{7}^{T} \tilde{Q}_{I 1 j} e_{7}-\dot{\Im}(t) e_{6}^{T} \tilde{Q}_{I 1 j} e_{6}\right) \\
& +d_{1}(t)\left(\dot{\Im}(t) e_{6}^{T} \tilde{Q}_{I 2 j} e_{6}-e_{8}^{T} \tilde{Q}_{I 2 j} e_{8}\right), \\
& \hat{\Xi}_{j}^{D}(d(t), \dot{d}(t))=\operatorname{sym}\left(\Gamma_{1}^{T} \tilde{P}_{D 1} \bar{\Gamma}_{2}+\bar{\Gamma}_{1}^{T} \tilde{P}_{D 1} \Gamma_{2}+\Gamma_{3}^{T} \tilde{P}_{D 2} \bar{\Gamma}_{4}\right. \\
& \left.+\bar{\Gamma}_{3}^{T} \tilde{P}_{D 2} \Gamma_{4}+\tilde{\mathfrak{P}}_{1}^{T} \mathfrak{D}_{1}+\tilde{\mathfrak{P}}_{2}^{T} \mathfrak{D}_{2}+\tilde{\mathfrak{F}}_{1}^{T} \mathfrak{C}_{1}+\tilde{\mathfrak{F}}_{2}^{T} \mathfrak{C}_{2}\right) \\
& +d_{2}(t)\left(e_{7}^{T} \tilde{Q}_{D 1 j} e_{7}-\dot{\Im}(t) e_{6}^{T} \tilde{Q}_{D 1 j} e_{6}\right) \\
& +d_{1}(t)\left(\dot{\Im}(t) e_{6}^{T} \tilde{Q}_{D 2 j} e_{6}-e_{8}^{T} \tilde{Q}_{D 2 j} e_{8}\right), \\
& \bar{Z}_{\Im \mathfrak{d} j}(\dot{d}(t))=\operatorname{diag}\left\{\tilde{Z}_{\Im \mathfrak{d} j}(\dot{d}(t)), 3 \tilde{Z}_{\Im \mathfrak{d} j}(\dot{d}(t)), 5 \tilde{Z}_{\Im d j}(\dot{d}(t))\right\}, \\
& \tilde{Z}_{\Im 1 j}(\dot{d}(t))=\tilde{Z}_{j}+\dot{d}(t) \tilde{Q}_{\Im 1 j}, \tilde{Z}_{\Im 2 j}(\dot{d}(t))=\tilde{Z}_{j}-\dot{d}(t) \tilde{Q}_{\Im 2 j}, \\
& \tilde{\Upsilon}_{l+i j}=\operatorname{col}\left\{\sqrt{\partial_{l 1}} \tilde{z}_{l i i j}, \sqrt{\partial_{l 2}} \tilde{z}_{l 2 i j}, \ldots, \sqrt{\partial_{l F}} \tilde{z}_{l F i j}\right\} \text {, } \\
& \tilde{z}_{l \dagger i j}=C_{l \dagger i j}^{\mathfrak{Y}} e_{1}+C_{l \dagger i j}^{d \mathfrak{Y}} e_{2}+E_{l i} e_{17}, \tilde{\mathfrak{W}}_{0}=e_{1}+\lambda_{1} e_{2}+\lambda_{2} e_{5}, \\
& C_{l \dagger i j}^{\mathfrak{Y}}=C_{l i} \mathfrak{W}^{T}+B_{2 l i} \mathfrak{Y}_{\dagger j}, C_{l \dagger i j}^{d \mathfrak{Y}}=C_{d l i} \mathfrak{W}^{T}+B_{2 l i} \mathfrak{Y}_{d \dagger j}, \\
& \tilde{\mathfrak{A}}_{l \dagger i j}=A_{l \dagger i j}^{\mathfrak{Y}} e_{1}+A_{l \dagger i j}^{d \mathfrak{Y}} e_{2}+D_{l i} e_{17}-\mathfrak{W}^{T} e_{5}, \\
& A_{l \dagger i j}^{\mathfrak{Y}}=A_{l i} \mathfrak{W}^{T}+\sum_{\dagger=1}^{F} \partial_{l \dagger} B_{1 l i} \mathfrak{Y}_{\dagger j}, \bar{U}_{j}=\operatorname{diag}\left\{2 \tilde{U}_{j}, 4 \tilde{U}_{j}\right\}, \\
& A_{l \dagger i j}^{d \mathfrak{Y}}=A_{d l i} \mathfrak{W}^{T}+\sum_{\dagger=1}^{F} \partial_{l \dagger} B_{1 l i} \mathfrak{Y}_{d \dagger j}, \bar{I}=\operatorname{diag} \overbrace{\{I, I, \ldots, I\}}^{F},
\end{aligned}
$$

and the controller gains are

$$
\begin{equation*}
K_{\dagger j}=\mathfrak{Y}_{\dagger j} \mathfrak{W}^{-T}, K_{d \dagger j}=\mathfrak{Y}_{d \dagger j} \mathfrak{W}^{-T} \tag{26}
\end{equation*}
$$

Proof: To reduce the number of parameters, let $\mathfrak{W}_{2}=$
$\lambda_{1} \mathfrak{W}_{1}, \mathfrak{W}_{3}=\lambda_{2} \mathfrak{W}_{1}$, and $\mathfrak{W}_{1}^{-1}=\mathfrak{W}$. Furthermore, define

$$
\left\{\begin{array}{l}
\mathfrak{U}_{1}=\operatorname{diag}\{\mathfrak{W}, \mathfrak{W}\}, \mathfrak{U}_{2}=\operatorname{diag}\{\mathfrak{W}, \mathfrak{W}, \mathfrak{W}\}, \\
\mathfrak{U}_{3}=\operatorname{diag}\{\mathfrak{W}, \mathfrak{W}, \mathfrak{W}, \mathfrak{W}\}, \mathfrak{U}_{4}=\operatorname{diag}\left\{\mathfrak{U}_{2}, \mathfrak{U}_{3}\right\}, \\
\mathfrak{U}_{5}=\operatorname{diag}\left\{\mathfrak{U}_{2}, \mathfrak{U}_{4}\right\}, \mathfrak{U}_{6}=\operatorname{diag}\left\{\mathfrak{U}_{1}, \mathfrak{U}_{3}, \mathfrak{U}_{5}\right\}, \\
\mathfrak{U}_{7}=\operatorname{diag}\left\{\mathfrak{U}_{6}, I, \mathfrak{U}_{2}, \bar{I}\right\} .
\end{array}\right.
$$

Based on Schur complement, pre- and post-multiplying (13)(15) by $\mathfrak{U}_{1}, \mathfrak{U}_{7}, \mathfrak{U}_{7}$ and their transpositions respectively and defining
we have (23)-(25).
Remark 4: To enhance expression clarity, it is assumed that each period of the delay consists of a single monotone increasing interval and a single monotone decreasing interval. It is worth noting that the derived condition from the proof process only requires the delay to exhibit repetitive variations between extreme values, which can occur multiple times within a period. Thus, the proposed approach can be extended to accommodate cases where there are multiple monotonically increasing intervals and monotonically decreasing intervals in each period.
Remark 5: In this paper, an asynchronous switching controller is proposed based on (12), which can be expressed as follows:

$$
\begin{equation*}
u^{\lambda}=K_{\dagger h}^{\lambda} x(t)+K_{d \dagger h}^{\lambda} x(t-d(t)), \tag{27}
\end{equation*}
$$

where $K_{\dagger h}^{\lambda}=\sum_{j=1}^{r} h_{j} K_{\dagger j}^{\lambda}, K_{d \dagger h}^{\chi}=\sum_{j=1}^{r} h_{j} K_{d \dagger j}^{\chi}, \lambda=$ $1,2, \ldots, 2^{r-1}$. Unlike the synchronous controller proposed in existing works [30]-[32], the controller (27) presented in this paper is general and can be transformed into some special cases under specific conditions. On the other hand, according to switching rule (12), the proposed controller can be dynamically switched based on different $\mathcal{H}_{\lambda}$. If the matrices in LKF are unrelated to the MFs (i.e., $Q_{I 1}, Q_{I 2}, Q_{D 1}, Q_{D 2}$, $\left.S_{l}, R_{1 l}, R_{2}, R_{3}, M, Z, U\right)$, the controller would simplify into the non-switching version proposed in the existing works.

Based on Theorem 2, the subsequent corollary presents a method for designing an asynchronous controller with generally incomplete TPs.

Corollary 1: For given scalars $\kappa \in[0,1), d_{\mathfrak{d}} \geq 0, \mathfrak{f}_{l o} \geq 0$, and $\gamma>0$, the system (8) with generally incomplete TPs is stochastically stable with a prescribed $H_{\infty}$ performance level $\gamma$, if there exist positive definite matrices $\tilde{S}_{l j} \in \mathbb{R}_{\tilde{\sim}}^{10 \mathrm{n} \times 10 \mathrm{n}}, \tilde{R}_{1 l j} \in$ $\mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \tilde{R}_{2 j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \tilde{R}_{3 j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \tilde{M}_{j} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}$, $\tilde{Z}_{j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}, \tilde{U}_{j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}, \mathcal{F}_{l o} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \mathcal{W}_{l o} \in \mathbb{R}^{10 \mathrm{n} \times 10 \mathrm{n}}$, and any matrices $\tilde{P}_{I \mathfrak{d}}, \tilde{P}_{D \mathfrak{d}} \in \mathbb{R}^{4 \mathrm{n} \times 7 \mathrm{n}}, \tilde{Q}_{I \mathfrak{d} j}, \tilde{Q}_{D \mathfrak{d} j} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $\tilde{\mathfrak{J}}_{\mathfrak{d}}, \tilde{\mathfrak{P}}_{\mathfrak{d}} \in \mathbb{R}^{3 \mathrm{n} \times(16 \mathrm{n}+\mathrm{f})}, \tilde{\mathfrak{G}}_{\mathfrak{d}}, \tilde{\mathfrak{F}}_{\mathfrak{d}} \in \mathbb{R}^{\mathrm{n} \times(16 \mathrm{n}+\mathrm{f})}, \mathfrak{W} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$, $\mathfrak{Y}_{\dagger j}, \mathfrak{Y}_{d \dagger j} \in \mathbb{R}^{\mathrm{s} \times \mathrm{n}}, \mathfrak{L}_{l} \in \mathbb{R}^{2 \mathrm{n} \times 2 \mathrm{n}}, \mathfrak{X}_{l} \in \mathbb{R}^{10 \mathrm{n} \times 10 \mathrm{n}}$, such that
(12) and the following inequalities hold for $l \in \mathbb{N}, \dagger \in \mathbb{F}$, $\Im \in\{I, D\}, \mathfrak{d}=1,2$,

$$
\begin{align*}
& {\left[\begin{array}{cccc}
\Theta_{l o j} & \tilde{R}_{11 j}-\mathfrak{L}_{l} & \cdots & \tilde{R}_{1 u j}-\mathfrak{L}_{l} \\
* & -\mathcal{F}_{l 1} & \cdots & 0 \\
* & * & \ddots & \vdots \\
* & * & * & -\mathcal{F}_{l u}
\end{array}\right]<0,}  \tag{28}\\
& \left\{\begin{array}{lr}
\breve{\Psi}_{l \dagger i i}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, & \dot{d}(t) \in[0, \kappa], \\
\breve{\Psi}_{l \dagger i j}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)+\breve{\Psi}_{l \dagger j i}^{I}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, \dot{d}(t) \in[0, \kappa],
\end{array}\right.  \tag{29}\\
& \left\{\begin{array}{lr}
\breve{\Psi}_{l+i i}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, & \dot{d}(t) \in[-\kappa, 0], \\
\breve{\Psi}_{l+i j}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)+\breve{\Psi}_{l \dagger j i}^{D}\left(d_{\mathfrak{d}}, \dot{d}(t)\right)<0, \dot{d}(t) \in[-\kappa, 0],
\end{array}\right.  \tag{30}\\
& \tilde{S}_{o j}-\mathfrak{X}_{l}<0, \tilde{R}_{1 o j}-\mathfrak{L}_{l}<0, \forall o \in \mathbb{N}_{u k}^{l}, l \in \mathbb{N}_{k}^{l},  \tag{31}\\
& \tilde{S}_{o j}-\mathfrak{X}_{l}>0, \tilde{R}_{1 o j}-\mathfrak{L}_{l}>0, \forall o \in \mathbb{N}_{u k}^{l}, l \in \mathbb{N}_{u k}^{l}, l=o, \\
& \tilde{S}_{o j}-\mathfrak{X}_{l}<0, \tilde{R}_{1 o j}-\mathfrak{L}_{l}<0, \forall o \in \mathbb{N}_{u k}^{l}, l \in \mathbb{N}_{u k}^{l}, l \neq o, \tag{32}
\end{align*}
$$

where

$$
\begin{aligned}
& \breve{\Psi}_{l \dagger i j}^{\Im}(d(t), \dot{d}(t))=\left[\begin{array}{cccc}
\vec{\Psi}_{l \dagger i j}^{\Im}(d(t), \dot{d}(t)) & \overline{\mathfrak{X}}_{l 1 j} & \cdots & \overline{\mathfrak{X}}_{l u j} \\
* & -\mathcal{W}_{l 1} & \cdots & 0 \\
* & * & \ddots & \vdots \\
* & * & * & -\mathcal{W}_{l u}
\end{array}\right], \\
& \vec{\Psi}_{l \dagger i j}^{I}\left(d_{1}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\vec{\Xi}_{l \dagger i j}^{I}\left(d_{1}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{J}}_{2}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{I 2 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \vec{\Psi}_{l \dagger i j}^{I}\left(d_{2}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\vec{\Xi}_{l \dagger i j}^{I}\left(d_{2}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{J}}_{1}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{I 1 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \vec{\Psi}_{l \dagger i j}^{D}\left(d_{1}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\vec{\Xi}_{l \dagger i j}^{D}\left(d_{1}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{P}}_{2}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{D 2 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \vec{\Psi}_{l \dagger i j}^{D}\left(d_{2}, \dot{d}(t)\right)=\left[\begin{array}{ccc}
\vec{\Xi}_{l \dagger i j}^{D}\left(d_{2}, \dot{d}(t)\right) & \sqrt{d_{2}-d_{1}} \tilde{\mathfrak{P}}_{1}^{T} & \tilde{\Upsilon}_{l \dagger i j}^{T} \\
* & -\bar{Z}_{D 1 j}(\dot{d}(t)) & 0 \\
* & * & -I
\end{array}\right], \\
& \vec{\Xi}_{l \dagger i j}^{I}(d(t), \dot{d}(t))=\vec{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))+\hat{\Xi}_{j}^{I}(d(t), \dot{d}(t)), \\
& \vec{\Xi}_{l \dagger i j}^{D}(d(t), \dot{d}(t))=\vec{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))+\hat{\Xi}_{j}^{D}(d(t), \dot{d}(t)) \text {, } \\
& \vec{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))=\operatorname{sym}\left(\bar{\Gamma}_{5}^{T} \tilde{S}_{l j} \Gamma_{5}+\tilde{\mathfrak{W}}_{0}^{T} \tilde{\mathfrak{A}}_{l \dagger i j}\right)+\bar{\Gamma}_{6 a}^{T}\left(\tilde{R}_{1 l j}\right. \\
& \left.+d_{1} \tilde{M}_{j}\right) \bar{\Gamma}_{6 a}+\bar{\Gamma}_{6 b}^{T}\left(\tilde{R}_{2 j}-\tilde{R}_{1 l j}\right) \bar{\Gamma}_{6 b}+\dot{\Im}(t) \bar{\Gamma}_{6 c}^{T}\left(\tilde{R}_{3 j}\right. \\
& \left.-\tilde{R}_{2 j}\right) \bar{\Gamma}_{6 c}-\bar{\Gamma}_{6 d}^{T} \tilde{R}_{3 j} \bar{\Gamma}_{6 d}+e_{5}^{T}\left(\frac{1}{2} d_{1}^{2} \tilde{U}_{j}+\left(d_{2}-d_{1}\right) \tilde{Z}_{j}\right) \\
& \times e_{5}+\bar{\Gamma}_{5}^{T} \sum_{o \in \mathbb{N}_{k}^{l}}\left[\hat{\pi}_{l o}\left(\tilde{S}_{o j}-\mathfrak{X}_{l}\right)+\frac{1}{4}\left(\mathfrak{f}_{l o}\right)^{2} \mathcal{W}_{l o}\right] \bar{\Gamma}_{5} \\
& -\mathfrak{D}_{3}^{T} \bar{U}_{j} \mathfrak{D}_{3}-\gamma^{2} e_{17}^{T} e_{17}, \\
& \overline{\mathfrak{X}}_{l 1 j}=\left[\left(\bar{\Gamma}_{5}^{T}\left(\tilde{S}_{1 j}-\mathfrak{X}_{l}\right)\right)^{T} \quad 0 \quad 0 \quad\right]^{T}, \\
& \overline{\mathfrak{X}}_{l u j}=\left[\left(\bar{\Gamma}_{5}^{T}\left(\tilde{S}_{u j}-\mathfrak{X}_{l}\right)\right)^{T} \quad 0 \quad 0 \quad\right]^{T}, \\
& \Theta_{l o j}=\sum_{o \in \mathbb{N}_{k}^{l}}\left[\hat{\pi}_{l o}\left(\tilde{R}_{1 o j}-\mathfrak{L}_{l}\right)+\frac{1}{4}\left(\mathfrak{f}_{l o}\right)^{2} \mathcal{F}_{l o}\right]-\tilde{M}_{j} .
\end{aligned}
$$

Proof: Rewrite $\sum_{o=1}^{N} \pi_{l o} \tilde{S}_{o j}$ in $\hat{\Xi}_{l \dagger i j}^{C}(d(t), \dot{d}(t))$ by taking into account the condition $\sum_{o=1}^{N} \pi_{l o}=0$, which yields:

$$
\sum_{o=1}^{N} \pi_{l o} \tilde{S}_{o j}=\sum_{o \in \mathbb{N}_{k}^{l}} \pi_{l o}\left(\tilde{S}_{o j}-\mathfrak{X}_{l}\right)+\sum_{o \in \mathbb{N}_{u k}^{l}} \pi_{l o}\left(\tilde{S}_{o j}-\mathfrak{X}_{l}\right)
$$

For $\pi_{l o}=\hat{\pi}_{l o}+\Delta \pi_{l o}$, applying Lemma 2 in [39], it follows

$$
\begin{align*}
\sum_{o=1}^{N} \pi_{l o} \tilde{S}_{o j} \leq & \sum_{o \in \mathbb{N}_{u k}^{l}} \pi_{l o}\left(\tilde{S}_{o j}-\mathfrak{X}_{l}\right)+\sum_{o \in \mathbb{N}_{k}^{l}}\left[\hat{\pi}_{l o}\right. \\
& \times\left(\tilde{S}_{o j}-\mathfrak{X}_{l}\right)+\frac{1}{4}\left(\mathfrak{f}_{l o}\right)^{2} \mathcal{W}_{l o}  \tag{33}\\
& \left.+\left(\tilde{S}_{o j}-\mathfrak{X}_{l}\right) \mathcal{W}_{l o}^{-1}\left(\tilde{S}_{o j}-\mathfrak{X}_{l}\right)^{T}\right]
\end{align*}
$$

Case 1: $l \in \mathbb{N}_{k}^{l}$
It is evident that $\sum_{o \in \mathbb{N}_{k}^{l}} \pi_{l o} \leq 0$. By using Schur complement and based on (33), one can obtain (29)-(31).

Case 2: $l \in \mathbb{N}_{u k}^{l}$
In this case, we have $\pi_{l l}<0, \pi_{l o}>0\left(\forall o \in \mathbb{N}_{u k}^{l}, o \neq l\right)$. By using Schur complement and based on (33), we can get (29), (30), and (32).

By employing the same method to address (23), we can deduce (28), (31), and (32).

## IV. Numerical example

Example 1: In this example, a tunnel diode circuit system (illustrated in Fig. 1) is considered [9], whose dynamic equation is given by

$$
\left\{\begin{aligned}
C \dot{v}_{C}(t) & =-0.01 v_{C}(t)-\alpha_{i} v_{C}^{3}(t)+i_{L}(t) \\
L \dot{i}_{L}(t) & =-v_{C}(t)-R i_{L}(t)+u(t)+0.1 \omega(t)
\end{aligned}\right.
$$

where $\alpha_{i}$ is the mode-dependent characteristic parameter, with $\alpha_{1}=0.01$ and $\alpha_{2}=0.02$. Furthermore, the TP matrix representing the relationship between the two modes is presented as follows:

$$
\Pi_{1}=\left[\begin{array}{cc}
-0.4+\Delta \pi_{11} & ? \\
0.6+\Delta \pi_{21} & ?
\end{array}\right]
$$

where $\Delta \pi_{11} \in[-0.06,0.06], \Delta \pi_{21} \in[-0.08,0.08]$. Consider the following controlled output:

$$
z(t)=J\left[\begin{array}{c}
v_{C}(t) \\
i_{L}(t)
\end{array}\right]+0.1 \omega(t)
$$

Let $x_{1}(t)=v_{C}(t), x_{2}(t)=i_{L}(t)$. Assuming that $C=$ $100 \mathrm{mF}, L=1 \mathrm{H}, R=10 \Omega, J=\left[\begin{array}{cc}1 & 0\end{array}\right]$, and $\left|x_{1}(t)\right| \leq 3$, the circuit system can be described by the FMJSs with the following parameters:

$$
\begin{aligned}
A_{11} & =\left[\begin{array}{cc}
-0.1 & 10 \\
-1 & -10
\end{array}\right], A_{12}=\left[\begin{array}{cc}
-1 & 10 \\
-1 & -10
\end{array}\right] \\
A_{21} & =\left[\begin{array}{cc}
-0.1 & 10 \\
-1 & -10
\end{array}\right], A_{22}=\left[\begin{array}{cc}
-1.9 & 10 \\
-1 & -10
\end{array}\right] \\
B_{1 l i} & =\left[\begin{array}{c}
0 \\
1
\end{array}\right], D_{l i}=\left[\begin{array}{c}
0 \\
0.1
\end{array}\right], C_{l i}=\left[\begin{array}{cc}
1 & 0
\end{array}\right], E_{l i}=0.1 \\
h_{1} & =1-\frac{x_{1}^{2}(t)}{9}, h_{2}=1-h_{1}
\end{aligned}
$$



Fig. 1. Tunnel diode circuit (Example 1).

TABLE II
CONDITIONAL PROBABILITY $\Pi_{2}$ (EXAMPLE 1)

|  | case I | case II | case III |
| :---: | :---: | :---: | :---: |
| $\Pi_{2}$ | $\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$ | $\left[\begin{array}{cc}1 & 0 \\ 0.5 & 0.5\end{array}\right]$ | $\left[\begin{array}{cc}0.3 & 0.7 \\ 0.5 & 0.5\end{array}\right]$ |

TABLE III
OPTIMAL $\gamma$ FOR DIFFERENT $d_{2}$ AND CONDITIONAL PROBABILITIES (EXAMPLE 1)

| $d_{2}$ | case I | case II | case III |
| :---: | :---: | :---: | :---: |
| 0.3 | 0.1000 | 0.1002 | 0.1003 |
| 0.4 | 0.1000 | 0.1004 | 0.1010 |
| 0.5 | 0.1000 | 0.1008 | 0.1015 |
| 0.6 | 0.1001 | 0.1008 | 0.1020 |

To address the problem of asynchronous control in FMJSs (1), the delay function is defined as $d(t)=\frac{d_{2}+d_{1}}{2}+$ $\frac{d_{2}-d_{1}}{2} \cos \frac{2 \kappa}{d_{2}-d_{1}} t$. And the following additional matrices are introduced:

$$
\begin{aligned}
& A_{d 11}=\left[\begin{array}{cc}
-0.1 & 2 \\
1 & -0.2
\end{array}\right], A_{d 12}=\left[\begin{array}{cc}
2 & 0.1 \\
0.1 & -3
\end{array}\right], \\
& A_{d 21}=\left[\begin{array}{cc}
1 & 0.1 \\
0.1 & -2
\end{array}\right], A_{d 22}=\left[\begin{array}{cc}
0.1 & 0.1 \\
0.2 & 0
\end{array}\right], \\
& C_{d 11}=\left[\begin{array}{ll}
0.4 & 0.1
\end{array}\right], C_{d 12}=\left[\begin{array}{cc}
0.2 & 0.1
\end{array}\right], \\
& C_{d 21}=\left[\begin{array}{ll}
0.2 & 0.1
\end{array}\right], C_{d 22}=\left[\begin{array}{cc}
0 & 0.4
\end{array}\right], \\
& B_{211}=B_{212}=B_{221}=B_{222}=0.1 .
\end{aligned}
$$

The associated constraints are: if $\dot{h}_{1}<0$, it has

$$
\mathcal{T}_{1}=\left\{\begin{array}{l}
Q_{I \mathfrak{D} 1}>Q_{I \mathfrak{} 2}, Q_{D \mathfrak{D} 1}>Q_{D \mathfrak{} 2}, S_{\mathfrak{O} 1}>S_{\mathfrak{0} 2}, R_{1 \mathfrak{1} 1}>R_{102}, \\
R_{21}>R_{22}, R_{31}>R_{32}, M_{1}>M_{2}, Z_{1}>Z_{2}, U_{1}>U_{2},
\end{array}\right.
$$

if $\dot{h}_{1} \geq 0$, it has

$$
\mathcal{T}_{2}=\left\{\begin{array}{l}
Q_{I \mathrm{D} 1} \leq Q_{I \mathrm{~d} 2}, Q_{D \mathfrak{D} 1} \leq Q_{D \mathrm{D} 2}, S_{\mathfrak{0} 1} \leq S_{\mathrm{o} 2}, R_{101} \leq R_{102}, \\
R_{21} \leq R_{22}, R_{31} \leq R_{32}, M_{1} \leq M_{2}, Z_{1} \leq Z_{2}, U_{1} \leq U_{2},
\end{array}\right.
$$

with $\mathfrak{d}=1,2$.
In order to analyze the relationship between the optimal $H_{\infty}$ performance, maximum delay bound, and different asynchronous degrees, three distinct cases, including synchronous, partially asynchronous, and completely asynchronous, are illustrated in Table II. Letting $d_{1}=0.2, \kappa=0.2, \lambda_{1}=\lambda_{2}=1$,


Fig. 2. Trajectories of system state and periodically varying delay $d(t)$ (Example 1).
and keeping other parameters unchanged, the optimal index $\gamma$ for different values of $d_{2}$ and CP matrices is presented in Table III. For example, in case III, $d_{2}=0.6$, the optimal index is $\gamma_{1}=0.1020$ under $\mathcal{T}_{1}$ and $\gamma_{2}=0.1019$ under $\mathcal{T}_{2}$. Thus, the final optimal $H_{\infty}$ performance index is $\gamma_{\min }=$ $\max \left\{\gamma_{1}, \gamma_{2}\right\}=0.1020$. From Table III, we can easily find out that the optimal $H_{\infty}$ performance index $\gamma$ exhibits an upward trend with increasing values of $d_{2}$ or intensifying asynchrony. This indirectly indicates that it is necessary and reasonable to consider the asynchronous phenomenon.
In particular, in case III, with $d_{2}=0.6, \gamma=1$, the corresponding controller matrices are
$K_{11}^{1}=\left[\begin{array}{ll}-3.843 & -10.950\end{array}\right], K_{12}^{1}=\left[\begin{array}{ll}-3.397 & -10.859\end{array}\right]$,
$K_{21}^{1}=\left[\begin{array}{ll}-4.021 & -9.795\end{array}\right], K_{22}^{1}=\left[\begin{array}{ll}-3.694 & -9.134\end{array}\right]$,
$K_{d 11}^{1}=\left[\begin{array}{ll}2.163 & 8.552\end{array}\right], K_{d 12}^{1}=\left[\begin{array}{ll}0.934 & -6.133\end{array}\right]$,
$K_{d 21}^{1}=\left[\begin{array}{ll}-1.318 & -2.271\end{array}\right], K_{d 22}^{1}=\left[\begin{array}{ll}-0.092 & 8.699\end{array}\right]$,
$K_{11}^{2}=\left[\begin{array}{ll}-3.701 & -6.506\end{array}\right], K_{12}^{2}=\left[\begin{array}{ll}-2.765 & -7.153\end{array}\right]$,
$K_{21}^{2}=\left[\begin{array}{ll}-2.760 & -6.133\end{array}\right], K_{22}^{2}=\left[\begin{array}{ll}-3.462 & -6.186\end{array}\right]$,
$K_{d 11}^{2}=\left[\begin{array}{ll}2.207 & 8.718\end{array}\right], K_{d 12}^{2}=\left[\begin{array}{ll}1.133 & -6.342\end{array}\right]$,
$K_{d 21}^{2}=\left[\begin{array}{ll}-1.336 & -2.282\end{array}\right], K_{d 22}^{2}=\left[\begin{array}{ll}-0.199 & 8.561\end{array}\right]$.
Given $d(t)=0.4+0.2 \cos (t), \omega(t)=e^{-0.3 t} \cos (t)$ with bounded energy $\left(\int_{0}^{t} \omega^{T}(t) \omega(t) d t \leq \varpi=\frac{5}{3}\right)$, under initial conditions $x(0)=\left[\begin{array}{cc}-2.8 & 1.1\end{array}\right]^{T}$, the evolution of system state and periodically varying delay $d(t)$ is depicted in Fig. 2. Besides, the trajectories of the system mode $\wp(t)$ and the controller mode $\vartheta(t)$ are shown in Fig. 3. These figures collectively illustrate the effectiveness of the proposed asynchronous controller design approach. Additionally, Fig. 4 depicts the progression of the control input $u$ along with $d h_{1} / d t$, where $S_{1}(t=0.600)$ and $S_{2}(t=0.777)$ represent the switching points. At these switching points, the values of $\dot{h}_{1}(0.600)=-0.089$ and $\dot{h}_{1}(0.777)=3.056 e-03$. Notably, it can be observed that the controller is $u_{2}$ within the interval $\left[0, S_{1}\right]$, subsequently switches to $u_{1}$ within the interval $\left[S_{1}, S_{2}\right.$ ], and finally switches back to $u_{2}$ within the interval $\left[S_{2},+\infty\right)$.

Example 2: When $\mathbb{N}=\{1\}$, consider the T-S fuzzy system under $u=0$ with the parameters as follows, which is borrowed


Fig. 3. The trajectories of $\wp(t)$ and $\vartheta(t)$ (Example 1).


Fig. 4. The evolution of $d h_{1} / d t$ and control input $u$ (Example 1).
TABLE IV
ALLOWABLE UPPER BOUNDS $d_{2}$ FOR DIFFERENT $\kappa$ (EXAMPLE 2)

| Methods | $\kappa=0.03$ | $\kappa=0.1$ | $\kappa=0.5$ | $\kappa=0.9$ |
| :---: | :---: | :---: | :---: | :---: |
| Th.1 $(\alpha=0.5),[8]$ | 0.9192 | 0.7985 | 0.7630 | 0.7541 |
| Th.1, $[26]$ | 1.9137 | 1.4354 | 1.3123 | 1.2063 |
| Th.1 $(\sigma=5),[27]$ | 2.2782 | 1.6065 | 1.4819 | 1.3686 |
| Th.1, $[28]$ | 2.8231 | 2.1807 | 1.9914 | 1.8705 |
| Th.1 with (i) | 3.2092 | 2.8256 | 2.4466 | 2.3088 |
| Th.1 with (ii) | 3.0603 | 2.7164 | 2.3926 | 2.2735 |
| Th.1 with (iii) | 2.6085 | 2.3435 | 2.2201 | 2.0902 |

from [28]:

$$
\begin{aligned}
A_{1} & =\left[\begin{array}{cc}
-3.2 & 0.6 \\
0 & -2.1
\end{array}\right], A_{2}=\left[\begin{array}{cc}
-1 & 0 \\
1 & -3
\end{array}\right] \\
A_{d 1} & =\left[\begin{array}{cc}
1 & 0.9 \\
0 & 2
\end{array}\right], A_{d 2}=\left[\begin{array}{cc}
0.9 & 0 \\
1 & 1.6
\end{array}\right]
\end{aligned}
$$

In order to make a comprehensive comparison, setting $d_{1}=$ 0 , we consider various cases:
(i) MFs-dependent LKF $V(t) . P_{I 1}, P_{I 2}, P_{D 1}, P_{D 2}, Q_{I 1 h}$, $Q_{I 2 h}, Q_{D 1 h}, Q_{D 2 h}, S_{l h}, R_{1 l h}, R_{2 h}, R_{3 h}, M_{h}, Z_{h}, U_{h}$.
(ii) MFs-independent LKF $V(t) . P_{I 1}, P_{I 2}, P_{D 1}, P_{D 2}, Q_{I 1}$, $Q_{I 2}, Q_{D 1}, Q_{D 2}, S_{l}, R_{1 l}, R_{2}, R_{3}, M, Z, U$.
(iii) MFs-dependent LKF $V_{C}(t) . P_{I 1}=0, P_{I 2}=0, P_{D 1}=0$, $P_{D 2}=0, Q_{I 1 h}=0, Q_{I 2 h}=0, Q_{D 1 h}=0, Q_{D 2 h}=0$, $S_{l h}, R_{1 l h}, R_{2 h}, R_{3 h}, M_{h}, Z_{h}, U_{h}$.
In this example, the common brute-force algorithm [36] is employed to search for the values of $\lambda_{1}$ and $\lambda_{2}$. For instance,
in case (i), $\kappa=0.1$, a search for $\lambda_{1}=1, \lambda_{2}=1.2$ yields a maximum delay bound $d_{2}$ of $d_{2}^{1}=3.2545$ under $\mathcal{T}_{1}$. Similarly, for $\lambda_{1}=0.8, \lambda_{2}=1.4$, the maximum delay bound $d_{2}$ is determined as $d_{2}^{2}=2.8256$ under $\mathcal{T}_{2}$. Hence, the final maximum delay bound for $d_{2}$ is determined as $d_{2}^{\max }=$ $\min \left\{d_{2}^{1}, d_{2}^{2}\right\}=2.8256$. Meanwhile, Table IV illustrates the allowable upper bounds $d_{2}$ obtained by different methods under identical parameters. For example, when $\kappa=0.03$, the maximum delay bound $d_{2}$ is 3.2092 for case (i) and 3.0603 for case (ii). Therefore, the validity of incorporating MFs-dependent terms is substantiated through a comparison of results between case (i) and case (ii). At the same time, the maximum delay bound $d_{2}$ obtained from case (iii) is 2.6085 . And the effectiveness of the monotone-delay-interval-based LKF is confirmed by contrasting the results of case (i) with those of case (iii). Moreover, it can be found that the value of the maximum delay bound $d_{2}$ obtained from case (i) is larger than that of [8], [26]-[28], which proves that the presented method in this article exhibits less conservatism compared to the approaches proposed in [8], [26]-[28].

## V. Conclusion

This paper has investigated the asynchronous switching control issue for FMJSs. By incorporating the specific characteristics of periodically varying delays and T-S fuzzy systems, an MDI-MF-LKF has been formulated based on monotonedelay intervals and MFs, leading to less conservative stability conditions. In addition, an asynchronous switching fuzzy controller has been developed, characterized by non-synchronous modes and MFs time derivative information. Ultimately, two numerical examples have been proposed to demonstrate the effectiveness and practicality of the presented approach. In future research, it is essential to explore new methods to reduce conservatism, such as matrices dependent on polynomial MFs.

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