

From Bounded Checking to Verification of Equivalence via Symbolic Up-to Techniques ^{*}

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Abstract. We present a bounded equivalence verification technique for higher-order programs with local state. This technique combines fully abstract *symbolic environmental bisimulations* similar to symbolic game semantics, novel *up-to techniques*, and lightweight *state invariant annotations*. This yields an equivalence verification technique with no false positives or negatives. The technique is bounded-complete, in that all inequivalences are automatically detected given large enough bounds. Moreover, several hard equivalences are proved automatically or after being annotated with state invariants. We realise the technique in a tool prototype called HOBbit and benchmark it with an extensive set of new and existing examples. HOBbit can prove many classical equivalences including all Meyer and Sieber examples.

Keywords: Contextual equivalence · bounded model checking · symbolic bisimulation · up-to techniques · operational game semantics.

1 Introduction

Contextual equivalence is a relation over program expressions which guarantees that related expressions are interchangeable in any program context. It encompasses verification properties like safety and termination. It has attracted considerable attention from the semantics community (cf. the 2017 Alonzo Church Award), and has found its main applications in the verification of cryptographic protocols [4], compiler correctness [25] and regression verification [10,11,9,17].

In its full generality, contextual equivalence is hard as it requires reasoning about the behaviour of all program contexts, and becomes even more difficult in languages with higher-order features (e.g. callbacks) and local state. Advances in bisimulations [16,28,3], logical relations [1,13,15] and game semantics [18,24,8,20] have offered powerful theoretical techniques for hand-written proofs of contextual equivalence in higher-order languages with state. However, these advancements have yet to be fully integrated in verification tools for contextual equivalence in

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programming languages, especially in the case of bisimulation techniques. Existing tools [12,23,14] only tackle carefully delineated language fragments.

In this paper we aim to push the frontier further by proposing a bounded model checking technique for contextual equivalence for the entirety of a higher-order language with local state (Sec. 3). This technique, realised in a tool called HOBBIT,³ automatically detects inequivalent program expressions given sufficient bounds, and proves hard equivalences automatically or semi-automatically.

Our technique uses a labelled transition system (LTS) for open expressions in order to express equivalence as a bisimulation. The LTS is symbolic both for higher-order arguments (Sec. 4), similarly to symbolic game models [8,20] and derived proof techniques [3,15], and first-order ones (Sec. 6), following established techniques (e.g. [6]). This enables the definition of a fully abstract *symbolic environmental bisimulation*, the bounded exploration of which is the task of the HOBBIT tool. Full abstraction guarantees that our tool finds all inequivalences given sufficient bounds, and only reports true inequivalences. As is corroborated by our experiments, this makes HOBBIT a practical inequivalence detector, similar to traditional bounded model checking [2] which has been proved an effective bug detection technique in industrial-scale C code [6,7,29].

However, while proficient in bug finding, bounded model checking can rarely prove the absence of errors, and in our setting prove an equivalence: a bound is usually reached before all—potentially infinite—program runs are explored. Inspired by hand-written equivalence proofs, we address this challenge by proposing two key technologies: new *bisimulation up-to techniques*, and lightweight user guidance in the form of *state invariant annotations*. Hence we increase significantly the number of equivalences proven by HOBBIT, including for example all classical equivalences due to Meyer and Sieber [21].

Up-to techniques [27] are specific to bisimulation and concern the reduction of the size of bisimulation relations, oftentimes turning infinite transition systems into finite ones by focusing on a core part of the relation. Although extensively studied in the theory of bisimulation, up-to techniques have not been used in practice in an equivalence checker. We specifically propose three novel up-to techniques: *up to separation* and *up to re-entry* (Sec. 5), dealing with infinity in the LTS due to the higher-order nature of the language, and *up to state invariants* (Sec. 7), dealing with infinity due to state updates. Up to separation allows us to reduce the knowledge of the context the examined program expressions are running in, similar to a frame rule in separation logic. Up to re-entry removes the need of exploring unbounded nestings of higher-order function calls under specific conditions. Up to state invariants allows us to abstract parts of the state and make finite the number of explored configurations by introducing state invariant predicates in configurations.

State invariants are common in equivalence proofs of stateful programs, both in handwritten (e.g. [16]) and tool-based proofs. In the latter they are expressed manually in annotations (e.g. [9]) or automatically inferred (e.g. [14]). In HOB-

³ Higher Order Bounded Bisimulation Tool (HOBBIT), <https://github.com/Lai fsV1/Hobbit>.

BIT we follow the manual approach, leaving heuristics for automatic invariant inference for future work. An important feature of our annotations is the ability to express relations between the states of the two compared terms, enabled by the up to state invariants technique. This leads to finite bisimulation transition systems in examples where concrete value semantics are infinite state.

The above technology, combined with standard up-to techniques, transform HOBbit from a bounded checker into an equivalence prover able to reason about infinite behaviour in a finite manner in a range of examples, including classical example equivalences (e.g. all in [21]) and some that previous work on up-to techniques would not algorithmically decide [3] (cf. Ex. 23). We have benchmarked HOBbit on examples from the literature and newly designed ones (Sec. 8). Due to the undecidable nature of contextual equivalence, up-to techniques are not exhaustive: no set of up-to techniques is guaranteed to finitise all examples. Indeed there are a number of examples where the bisimulation transition system is still infinite and HOBbit reaches the exploration bound. For instance, HOBbit is not able to prove examples with inner recursion and well-bracketing properties, which we leave to future work. Nevertheless, our approach provides a contextual equivalence tool for a higher-order language with state that can prove many equivalences and inequivalences which previous work could not handle due to syntactic restrictions and other limitations (Sec. 9).

Related work Our paper marries techniques from environmental bisimulations up-to [16,28,27,3] with the work on fully abstract game models for higher-order languages with state [18,8,20]. The closest to our technique is that of Biernacki et al. [3], which introduces up-to techniques for a similar symbolic LTS to ours, albeit with symbolic values restricted to higher-order types, resulting in infinite LTSs in examples such as Ex. 22, and with inequivalence decided outside the bisimulation by (non-)termination, precluding the use up-to techniques in examples such as Ex. 23. Close in spirit is the line of research on logical relations [1,13,15] which provides a powerful tool for hand-written proofs of contextual equivalence. Also related are the tools HECTOR [12] and CONEQT [23], and SYTECI [14], based on game semantics and step-indexed logical relations respectively (cf. Sec. 9).

2 High-Level Intuitions

Contextual equivalence requires that two program expressions lead to the same observable result *in any program context* these may be fed in. This quantification is hard to work with e.g. due to redundancy in program contexts. Alternatively, we can translate programs into a semantic model that is *fully abstract*, i.e. it assigns to program expressions the same denotation just if these are contextually equivalent. Thus doing, contextual equivalence is reduced to semantic equality.

The semantic model we use is that of Game Semantics [18]. We model programs as formal interactions between two *players*: a *Proponent* (corresponding to the program) and an *Opponent* (standing for any program context). Concretely, these interactions are sets of traces produced from a Labelled Transition System (LTS), the nodes and labels of which are called *configurations* and *moves* respec-

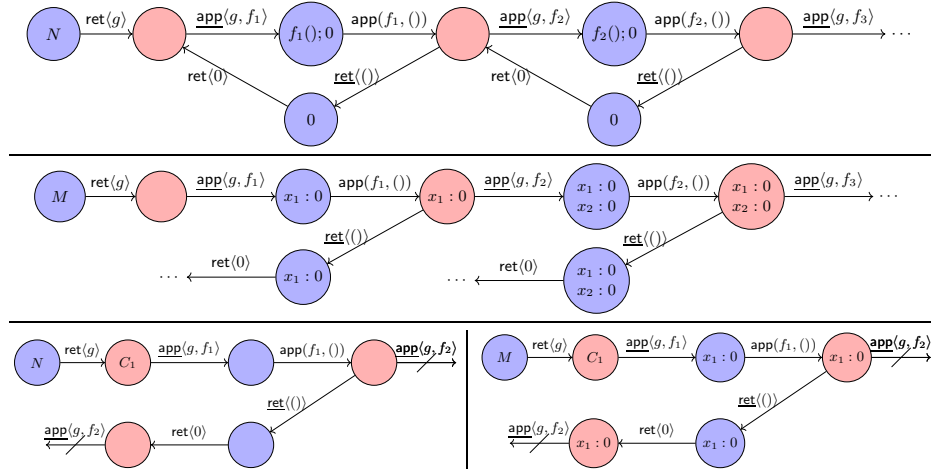


Fig. 1. Sample LTS's modelling expressions in Section 2. Note that in these diagrams we include in each configuration only part of their components, for better exposition.

tively. The LTS captures the interaction of the program with its environment, which is realised via function applications and returns: moves can be *questions* (i.e. function applications) or *answers* (returns), and belong to proponent or opponent. E.g. a program calling an external function will issue a proponent question, while the return of the external function will be an opponent answer. In the examples that follow, moves that correspond to the opponent shall be underlined.

Example 1. Consider the following expression of type $(\text{unit} \rightarrow \text{unit}) \rightarrow \text{int}$.

$$N = \mathbf{fun} \ f \ \rightarrow \ f \ (); \ \mathbf{0}$$

Evaluating N leads to a function, call it g , being returned (i.e. g is $\lambda f.f();0$). When g is called with some input f_1 , it will always return 0 but in the process it may call the external function f_1 . The call to f_1 may immediately return or it may call g again (i.e. reenter), and so on. The LTS for N is as in Fig. 1 (top).

Given two expressions M, N , checking their equivalence will amount to checking bisimulation equivalence of their (generally infinite) LTS's. Our checking routine performs a bounded analysis that aims to either find a finite counterexample and thus prove inequivalence, or build a bisimulation relation that shows the equivalence of the expressions. The former case is easier as it is relatively rapid to explore a bisimulation graph up to a given depth. The latter one is harder, as the target bisimulation can be infinite. To tackle part of this infinity, we use three novel *up-to techniques* for environmental bisimulation.

Up-to techniques roughly assert that if a core set of configurations in the bisimulation graph explored can be proven to be part of a relation satisfying a definition that is more permissive than standard bisimulation, then a superset of configurations forms a proper bisimulation relation. This has the implication

that a bounded analysis can be used to explore a finite part of the bisimulation graph to verify potentially infinitely many configurations. As there can be no complete set of up-to techniques, the pertaining question is how useful they are in practice. In the remainder of this section we present the first of our up-to techniques, called *up to separation*, via an example equivalence. The intuition behind this technique comes from Separation Logic and amounts to saying that functions that access separate regions of the state can be explored independently. As a corollary, a function that manipulates only its own local references may be explored independently of itself, i.e. it suffices to call it once.

Example 2. Consider the following pair of expressions, where N is from Ex. 1.

$$M = \mathbf{fun} \ f \ \rightarrow \ \mathbf{ref} \ x = \mathbf{0} \ \mathbf{in} \ f \ (); \ !x \qquad N = \mathbf{fun} \ f \ \rightarrow \ f \ (); \ \mathbf{0}$$

The LTS corresponding to M and N are shown in Fig. 1 (middle and top). Regarding M , we can see that opponent is always allowed to reenter the proponent function g , which creates a new reference x_n each time. This makes each configuration unique, which prevents us from finding cycles and thus finitise the bisimulation graph. Moreover, both the LTS for M and N are infinite because of the stack discipline they need to adhere to when \mathbf{O} issues reentrant calls.

With separation, however, we could prune the two LTS's as in Fig. 1 (bottom). We denote the configurations after the first opponent call as C_1 . Any opponent call after C_1 leads to a configuration which differs from C_1 either by a state component that is not accessible anymore and can thus be separated, or by a stack component that can be similarly separated. Hence, the LTS's that we need to consider are finite and thus the expressions are proven equivalent.

3 Language and Semantics

We develop our technique for the language λ^{imp} , a simply typed lambda calculus with local state whose syntax and reduction semantics are shown in Fig. 2. Expressions (**Exp**) include the standard lambda expressions with recursive functions ($\mathbf{fix} f(x).e$), together with location creation ($\mathbf{ref} \ l = v \ \mathbf{in} \ e$), dereferencing ($!l$), and assignment ($l := e$), as well as standard base type constants (c) and operations ($op(\vec{e})$). Locations are mapped to values, including function values, in a store (**St**). We write \cdot for the empty store and let $\text{fl}(\chi)$ denote the set of free locations in χ .

The language λ^{imp} is simply-typed with typing judgements of the form $\Delta; \Sigma \vdash e : T$, where Δ is a type environment (omitted when empty), Σ a store typing and T a value type (**Type**); Σ_s is the typing of store s . The rules of the type system are standard and omitted here (Appendix A). Values consist of boolean, integer, and unit constants, functions and arbitrary length tuples of values. To keep the presentation of our technique simple we do not include reference types as value types, effectively keeping all locations local. Exchange of locations between expressions can be encoded using get and set functions. In Ex. 23 we show the encoding of a classic equivalence with location exchange between expressions and their context. Future work extensions to our technique to handle location types can be informed from previous work [18,14].

Loc:	l, k	Var: x, y, z	Const: c
Type:	$T ::= \text{bool} \mid \text{int} \mid \text{unit} \mid T \rightarrow T \mid T_1 * \dots * T_n$		
Exp:	$e, M, N ::= v \mid (\vec{e}) \mid \text{op}(\vec{e}) \mid ee \mid \text{if } e \text{ then } e \text{ else } e \mid \text{ref } l = v \text{ in } e \mid !l \mid l := e \mid \text{let } (\vec{x}) = e \text{ in } e$		
Val:	$u, v ::= c \mid x \mid \text{fix } f(x).e \mid (\vec{v})$		
ECxt:	$E ::= [\cdot]_T \mid (\vec{v}, E, \vec{e}) \mid \text{op}(\vec{v}, E, \vec{e}) \mid Ee \mid vE \mid l := E \mid \text{if } E \text{ then } e \text{ else } e \mid \text{let } (\vec{x}) = E \text{ in } e$		
Cxt:	$D ::= [\cdot]_{i,T} \mid e \mid (\vec{D}) \mid \text{op}(\vec{D}) \mid DD \mid l := D \mid \text{if } D \text{ then } D \text{ else } D \mid \text{fix } f(x).D$ $\mid \text{ref } l = D \text{ in } D \mid \text{let } (\vec{x}) = D \text{ in } D$		
St:	$s, t \in \text{Loc} \xrightarrow{\text{fin}} \text{Val}$		
	$\langle s; \text{op}(\vec{c}) \rangle$	$\hookrightarrow \langle s; w \rangle$	if $\text{op}^{\text{arith}}(\vec{c}) = w$
	$\langle s; \text{fix } f(x).e \ v \rangle$	$\hookrightarrow \langle s; e[v/x][\text{fix } f(x).e/f] \rangle$	
	$\langle s; \text{let } (\vec{x}) = (\vec{v}) \text{ in } e \rangle$	$\hookrightarrow \langle s; e[\vec{v}/\vec{x}] \rangle$	
	$\langle s; \text{ref } l = v \text{ in } e \rangle$	$\hookrightarrow \langle s[l \mapsto v]; e \rangle$	if $l \notin \text{dom}(s)$
	$\langle s; !l \rangle$	$\hookrightarrow \langle s; v \rangle$	if $s(l) = v$
	$\langle s; l := v \rangle$	$\hookrightarrow \langle s[l \mapsto v]; () \rangle$	
	$\langle s; \text{if } c \text{ then } e_1 \text{ else } e_2 \rangle$	$\hookrightarrow \langle s; e_i \rangle$	if $(c, i) \in \{(\text{tt}, 1), (\text{ff}, 2)\}$
	$\langle s; E[e] \rangle$	$\rightarrow \langle s'; E[e'] \rangle$	if $\langle s; e \rangle \hookrightarrow \langle s'; e' \rangle$

Fig. 2. Syntax and reduction semantics of the language λ^{imp} .

The reduction semantics is by small-step transitions between configurations containing a store and an expression, $\langle s; e \rangle \rightarrow \langle s'; e' \rangle$, defined using single-hole evaluation contexts (ECxt) over a base relation \hookrightarrow . Holes $[\cdot]_T$ are annotated with the type T of closed values they accept, which we may omit to lighten notation. Beta substitution of x with v in e is written as $e[v/x]$. We write $\langle s; e \rangle \Downarrow$ to denote $\langle s; e \rangle \rightarrow^* \langle t; v \rangle$ for some t, v . We write \vec{x} to mean a syntactic sequence, and assume standard syntactic sugar from the lambda calculus. In our examples we assume an ML-like syntax and implementation of the type system, which is also the concrete syntax of HOBBIT.

We consider environments $\Gamma \in \mathbb{N} \xrightarrow{\text{fin}} \text{Val}$ which map natural numbers to closed values. The concatenation of two such environments Γ_1 and Γ_2 , written Γ_1, Γ_2 is defined when $\text{dom}(\Gamma_1) \cap \text{dom}(\Gamma_2) = \emptyset$. We write $(^{i_1}v_1, \dots, ^{i_n}v_n)$ for a concrete environment mapping i_1, \dots, i_n to v_1, \dots, v_n , respectively. When indices are unimportant we omit them and treat Γ environments as lists.

General contexts D contain multiple, non-uniquely indexed holes $[\cdot]_{i,T}$, where T is the type of value that can replace the hole. Notation $D[\Gamma]$ denotes the context D with each hole $[\cdot]_{i,T}$ replaced with $\Gamma(i)$, provided that $i \in \text{dom}(\Gamma)$ and $\Sigma \vdash \Gamma(i) : T$, for some Σ . We omit hole types where possible and indices when all holes in D are annotated with the same i . In the latter case we write $D[v]$ instead of $D[({}^i v)]$ and allow to replace all holes of D with a closed expression e , written $D[e]$. We assume the Barendregt convention for locations, thus replacing context holes avoids location capture. Standard contextual equivalence [22] follows.

Definition 3 (Contextual Equivalence). *Expressions $\vdash e_1 : T$ and $\vdash e_2 : T$ are contextually equivalent, written as $e_1 \equiv e_2$, when for all contexts D such that $\vdash D[e_1] : \text{unit}$ and $\vdash D[e_2] : \text{unit}$ we have $\langle \cdot; D[e_1] \rangle \Downarrow$ iff $\langle \cdot; D[e_2] \rangle \Downarrow$.*

$$\begin{array}{l}
\text{PROPAPP} : \langle A; \Gamma; K; s; E[\alpha v] \rangle \xrightarrow{\text{app}(\alpha, D)} \langle A; \Gamma; \Gamma'; E[\cdot], K; s; \cdot \rangle \text{ if } (D, \Gamma') \in \text{ulpatt}(v) \\
\text{PROPRET} : \langle A; \Gamma; K; s; v \rangle \xrightarrow{\text{ret}(D)} \langle A; \Gamma; \Gamma'; K; s; \cdot \rangle \text{ if } (D, \Gamma') \in \text{ulpatt}(v) \\
\text{OPAPP} : \langle A; \Gamma; K; s; \cdot \rangle \xrightarrow{\text{app}(i, D[\vec{\alpha}])} \langle A \uplus \vec{\alpha}; \Gamma; K; s; e \rangle \text{ if } \Sigma_s \vdash \Gamma(i) : T \rightarrow T' \\
\text{OPRET} : \text{ if } (D, \vec{\alpha}) \in \text{ulpatt}(T) \\
\text{ if } \Gamma(i) D[\vec{\alpha}] \succ e \\
\langle A; \Gamma; E[\cdot]_T, K; s; \cdot \rangle \xrightarrow{\text{ret}(D[\vec{\alpha}])} \langle A \uplus \vec{\alpha}; \Gamma; K; s'; E[D[\vec{\alpha}]] \rangle \text{ if } (D, \vec{\alpha}) \in \text{ulpatt}(T) \\
\text{TAU} : \langle A; \Gamma; K; s; e \rangle \xrightarrow{\tau} \langle A; \Gamma; K; s; e' \rangle \text{ if } \langle s; e \rangle \rightarrow \langle s'; e' \rangle \\
\text{RESPONSE} : C \xrightarrow{\eta} \langle \perp \rangle \text{ if } \eta \neq \downarrow \\
\text{TERM} : \langle A; \Gamma; \cdot; s; \cdot \rangle \xrightarrow{\downarrow} \langle \perp \rangle
\end{array}$$

Fig. 3. The Labelled Transition System.

4 LTS with Symbolic Higher-Order Transitions

Our Labelled Transition System (LTS) has symbolic transitions for both higher-order and first-order transitions. For simplicity we first present our LTS with symbolic higher-order and concrete first-order transitions. We develop our theory and most up-to techniques on this simpler LTS. We then show its extension with symbolic first-order transitions and develop up to state invariants which relies on this extension. We extend the syntax with abstract function names α :

$$\text{Val: } u, v, w ::= c \mid \text{fix } f(x).e \mid (\vec{v}) \mid \alpha$$

We assume that α 's are annotated by the type of function they represent, written $\alpha_{T \rightarrow T'}$, and omitted where possible; $\text{an}(\chi)$ is the set of abstract names in χ .

We define our LTS (shown in Fig. 3) by opponent and proponent call and return transitions, based on Game Semantics [18]. Proponent transitions are the moves of an expression interacting with its context. Opponent transitions are the moves of the context surrounding this expression. These transitions are over proponent and opponent configurations $\langle A; \Gamma; K; s; e \rangle$ and $\langle A; \Gamma; K; s; \cdot \rangle$, respectively. In these configurations:

- A is a set of abstract function names been used so far in the interaction;
- Γ is an environment indexing proponent functions known to opponent;⁴
- K is a stack of proponent continuations, created by nested proponent calls;
- s is the store containing proponent locations;
- e is the expression reduced in proponent configurations; \hat{e} denotes e or \cdot .

In addition, we introduce a special configuration $\langle \perp \rangle$ which is used in order to represent expressions that cannot perform given transitions (cf. Remark 6). We let a *trace* be a sequence of app and ret moves (i.e. labels), as defined in Fig. 3.

For the LTS to provide a fully abstract model of the language, it is necessary that functions which are passed as arguments or return values from proponent

⁴ thus, Γ is encoding the environment of Environmental Bisimulations (e.g. [16])

to opponent be abstracted away, as the actual syntax of functions is not directly observable in λ^{imp} . This is achieved by deconstructing such values v to:

- an *ultimate pattern* D (cf. [19]), which is a context obtained from v by replacing each function in v with a distinct numbered hole; together with
- an environment Γ whose domain is the indices of these holes, and $D[\Gamma] = v$.

We let $\text{ulpatt}(v)$ contain all such pairs (D, Γ) for v ; e.g.: $\text{ulpatt}((\lambda x.e_1, 5)) = \{([\cdot]_i, 5), [^i\lambda x.e_1] \mid \text{for any } i\}$.

Ultimate pattern matching is extended to types through the use of symbolic function names: $\text{ulpatt}(T)$ is the largest set of pairs $(D, \vec{\alpha})$ such that $\vdash D[\vec{\alpha}] : T$, where $\vec{\alpha}$ is an environment with indices omitted, and D does not contain functions.

In Fig. 3, proponent application and return transitions (PROPAPP , PROPRET) use ultimate pattern matching for values and accumulate the functions generated by the proponent in the Γ environment of the configuration, leaving only their indices on the label of the transition itself. Opponent application and return transitions (OPAPP , OPRET) use ultimate pattern matching for types to generate opponent-generated values which can only contain abstract functions. This eliminates the need for quantifying over all functions in opponent transitions but still includes infinite quantification over all base values. Symbolic first-order values in Sec. 6 will obviate the latter.

At opponent application the following preorder performs a beta reduction when opponent applies a concrete function. This technicality is needed for soundness.

Definition 4 (\succ). *For application vu we write $vu \succ e$ to mean $e = \alpha u$, when $v = \alpha$; and $e = e'[u/x][\text{fix}f(x).e/f]$, when $v = \text{fix}f(x).e'$.*

In our LTS, C ranges over configurations and η over transition labels; $\xrightarrow{\eta}$ means $\xrightarrow{\tau}^*$, when $\eta = \tau$, and $\xrightarrow{\tau} \xrightarrow{\eta} \xrightarrow{\tau}$ otherwise. Standard weak (bi-)simulation follows.

Definition 5 (Weak Bisimulation). *Binary relation \mathcal{R} is a weak simulation when for all $C_1 \mathcal{R} C_2$ and $C_1 \xrightarrow{\eta} C'_1$, there exists C'_2 such that $C_2 \xrightarrow{\eta} C'_2$ and $C'_1 \mathcal{R} C'_2$. If $\mathcal{R}, \mathcal{R}^{-1}$ are weak simulations then \mathcal{R} is a weak bisimulation. Similarity ($\overset{\sim}{\approx}$) and bisimilarity (\approx) are the largest weak simulation and bisimulation, respectively.*

Remark 6. Any proponent configuration that cannot match a standard bisimulation transition challenge can trivially respond to the challenge by transitioning into $\langle \perp \rangle$ by the RESPONSE rule in Fig. 3. By the same rule, this configuration can trivially perform all transitions except a special termination transition, labelled with \downarrow . However, regular configurations that have no pending proponent calls ($K = \cdot$), can perform the special termination transition (TERM rule), signalling the end of a *complete trace*, i.e. a completed computation. This mechanism allows us to encode complete trace equivalence, which coincides with contextual

equivalence [18], as bisimulation equivalence. In a bisimulation proof, if a proponent configuration is unable to match a bisimulation transition with a regular transition, it can still transition to $\langle \perp \rangle$ where it can simulate every transition of the other expression, apart from \downarrow leading to a complete trace.

Our mechanism for treating unmatched transitions has the benefit of enabling us to use the standard definition of bisimulation over our LTS. This is in contrast to previous work [3,15], where termination/non-termination needed to be proven independently or baked in the simulation conditions. More importantly, our approach allows us to use bisimulation up-to techniques even when one of the related configurations diverges, which is not possible in previous symbolic LTSs [18,15,3], and is necessary in examples such as Ex. 23.

Definition 7 (Bisimilar Expressions). *Expressions $\vdash e_1 : T$ and $\vdash e_2 : T$ are bisimilar, written $e_1 \approx e_2$, when $\langle \cdot ; \cdot ; \cdot ; \cdot ; e_1 \rangle \approx \langle \cdot ; \cdot ; \cdot ; \cdot ; e_2 \rangle$.*

Theorem 8 (Soundness and Completeness). *$e_1 \approx e_2$ iff $e_1 \equiv e_2$ (see proof in Appendices B, E and F).*

As a final remark, the LTS presented in this section is finite state only for a small number of trivial equivalence examples, such as the following one.

Example 9. The following two implementations of conjunction (`bool * bool → bool`) have a finite transition system according to the rules of Fig. 3.

```
M = fun xy -> let (x,y) = xy in if x then y else false
N = fun xy -> let (x,y) = xy in x && y
```

However, even simple modifications to this example, such as allocation of a fresh location within one of the functions, leads to infinite state transition systems. The following section addresses multiple sources of infinity in the transition systems through bisimulation up-to techniques.

5 Up-to Techniques

We start by the definition of a sound up-to technique.

Definition 10 (Weak Bisimulation up to f). *\mathcal{R} is a weak simulation up to f when for all $C_1 \mathcal{R} C_2$ and $C_1 \xrightarrow{\eta} C'_1$, there is C'_2 with $C_2 \xrightarrow{\eta} C'_2$ and $C'_1 f(\mathcal{R}) C'_2$. If $\mathcal{R}, \mathcal{R}^{-1}$ are weak simulations up to f then \mathcal{R} is a weak bisimulation up to f .*

Definition 11 (Sound up-to technique). *A function f is a sound up-to technique when for any \mathcal{R} which is a simulation up to f we have $R \subseteq \langle \underline{\mathcal{R}} \rangle$.*

HOBbit employs the standard techniques: up to identity, up to garbage collection, up to beta reductions and up to name permutations (see Appendix D). Here we present two novel up-to techniques: up to separation and up to reentry.

Up to Separation Our experience with HOBbit has shown that one of the most effective up-to techniques for finitising bisimulation transition systems is the novel *up to separation* which we propose here. The intuition of this technique is that if different functions operate on disjoint parts of the store, they can be explored in disjoint parts of the bisimulation transition system. Taken to the extreme, a function that does not contain free locations can be applied only once in a bisimulation test as two copies of the function will not interfere with each other, even if they allocate new locations after application. To define up to separation we need to define a separating conjunction for configurations.

Definition 12 (Stack Interleaving). *Let K_1, K_2 be lists of evaluation contexts from ECxt (Fig. 2); we define the interleaving operation $K_1 \#_{\vec{k}} K_2$ inductively, and write $K_1 \# K_2$ to mean $K_1 \#_{\vec{k}} K_2$ for unspecified \vec{k} . We let $\cdot \# \cdot = \cdot$ and:*

$$E_1, K_1 \#_{(1, \vec{k})} K_2 = E_1, (K_1 \#_{\vec{k}} K_2) \quad K_1 \#_{(2, \vec{k})} E_2, K_2 = E_2, (K_1 \#_{\vec{k}} K_2).$$

Definition 13 (Separating Conjunction). *Let $C_1 = \langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle$ and $C_2 = \langle A_2; \Gamma_2; K_2; s_2; \hat{e}_2 \rangle$ be well-formed configurations. We define:*

- $C_1 \oplus_{\vec{k}}^1 C_2 \stackrel{\text{def}}{=} \langle A_1 \cup A_2; \Gamma_1, \Gamma_2; K_1 \#_{\vec{k}} K_2; s_1, s_2; \hat{e}_1 \rangle$ when $\hat{e}_2 = \cdot$
- $C_1 \oplus_{\vec{k}}^2 C_2 \stackrel{\text{def}}{=} \langle A_1 \cup A_2; \Gamma_1, \Gamma_2; K_1 \#_{\vec{k}} K_2; s_1, s_2; \hat{e}_2 \rangle$ when $\hat{e}_1 = \cdot$

provided $\text{dom}(s_1) \cap \text{dom}(s_2) = \emptyset$. We let $C_1 \oplus C_2$ denote $\exists i, \vec{k}. C_1 \oplus_{\vec{k}}^i C_2$.

The function `sep` provides the up to separation technique; it is defined as:

$$\frac{\text{UPTo}\oplus \quad C_1 \mathcal{R} C_2 \quad C_3 \mathcal{R} C_4}{C_1 \oplus_{\vec{k}}^i C_3 \text{sep}(\mathcal{R}) C_2 \oplus_{\vec{k}}^i C_4} \quad \frac{\text{UPTo}\oplus\perp_L \quad C_1 \mathcal{R} \langle \perp \rangle \quad C_3 \mathcal{R} C_4}{C_1 \oplus C_3 \text{sep}(\mathcal{R}) \langle \perp \rangle} \quad \frac{\text{UPTo}\oplus\perp_R \quad C_1 \mathcal{R} C_2 \quad C_3 \mathcal{R} \langle \perp \rangle}{C_1 \oplus C_3 \text{sep}(\mathcal{R}) \langle \perp \rangle}$$

Its soundness follows by extending [27,26] with a more powerful proof obligation (see Appendix D.1).

Lemma 14. *Function `sep` is a sound up-to technique.*

Many example equivalences have a finite transition system when using up to separation in conjunction with the simple techniques of the preceding section.

Example 15. The following is a classic example equivalence from Meyer and Sieber [21]. The following expressions are equivalent at type $(\text{unit} \rightarrow \text{unit}) \rightarrow \text{unit}$.

$$M = \mathbf{fun} \ f \ -> \ \mathbf{ref} \ x = \mathbf{0} \ \mathbf{in} \ f \ () \quad N = \mathbf{fun} \ f \ -> \ f \ ()$$

After initial application of the function by the opponent, the proponent calls `f`, growing the stack K in the two configurations. At that point the opponent can apply the same functions again. The LTS of this example is thus infinite because K can grow indefinitely. It is additionally infinite because the opponent can keep applying the initial function applications even after these return. However,

$$\begin{array}{c}
\text{UPTOREENTRY} \\
C_1 = \langle A; \Gamma_1; K_1; s_1; \cdot \rangle \mathcal{R} \langle A; \Gamma_2; K_2; s_2; \cdot \rangle = C_2 \\
\forall \bar{\eta}, C, A', \Gamma'_1, \Gamma'_2, s'_1, s'_2. [(\mathbf{app}(i, _)) \notin \{\bar{\eta}\}] \text{ and} \\
\langle A; \Gamma_1; \cdot; s_1; \cdot \rangle \xrightarrow{\mathbf{app}(i, C)} \bar{\eta} \succ \langle A'; \Gamma'_1; \cdot; s'_1; \cdot \rangle \text{ and} \\
\langle A; \Gamma_2; \cdot; s_2; \cdot \rangle \xrightarrow{\mathbf{app}(i, C)} \bar{\eta} \succ \langle A'; \Gamma'_2; \cdot; s'_2; \cdot \rangle \\
\text{implies } \Gamma'_1 = \Gamma_1 \text{ and } \Gamma'_2 = \Gamma_2 \text{ and } s_1 = s'_1 \text{ and } s_2 = s'_2] \\
\frac{C_1 \xrightarrow{\mathbf{app}(i, C)} \bar{\eta}' \xrightarrow{\mathbf{app}(i, C')} \langle A'; \Gamma_1; K'_1, K_1; s_1; e'_1 \rangle}{C_2 \xrightarrow{\mathbf{app}(i, C)} \bar{\eta}' \xrightarrow{\mathbf{app}(i, C')} \langle A'; \Gamma_2; K'_2, K_2; s_2; e'_2 \rangle} \\
\hline
\langle A'; \Gamma_1; K'_1, K_1; s_1; e'_1 \rangle \text{ reent}(\mathcal{R}) \langle A'; \Gamma_2; K'_2, K_2; s_2; e'_2 \rangle
\end{array}$$

Fig. 4. Up to Proponent Function Re-entry (omitting rules for \perp -configurations).

if we apply the up-to separation technique immediately after the first opponent application, the Γ environments become empty, and thus no second application of the same functions can happen. The LTS thus becomes trivially small. Note that no other up to technique is needed here.

Example 16. This example is due to Bohr and Birkedal [5] which includes a non-synchronised divergence.

```

M = fun f ->
  ref l1 = false in ref l2 = false in
  f (fun () -> if !l1 then _bot_ else l2 := true);
  if !l2 then _bot_ else l1 := true

N = fun f -> f (fun () -> _bot_)

```

Note that `_bot_` is a diverging computation. This is a hard example to prove using environmental bisimulation even with up to techniques; requiring quantification over contexts within the proof. However, with up-to separation after the opponent applies the two functions, the Γ environments are emptied, thus leaving only one application of M and N that needs to be explored by the bisimulation. Applications of the inner function provided as argument to `f` only leads to a small number of reachable configurations. HOBBIT can indeed prove this equivalence.

Up to Proponent Function Re-entry The higher-order nature of λ^{imp} and its LTS allows infinite nesting of opponent and proponent calls. Although up to separation avoids those in a number of examples, here we present a second novel up-to technique, which we call *up to proponent function re-entry* (or simply, up to re-entry). This technique has connections to the induction hypothesis in the definition of environmental bisimulations in [16]. However up to re-entry is specifically aimed at avoiding nested calls to proponent functions, and it is designed to work with our symbolic LTS. In combination with other techniques this eliminates the need to consider configurations with unbounded stacks K in many classical equivalences, including those in [21].

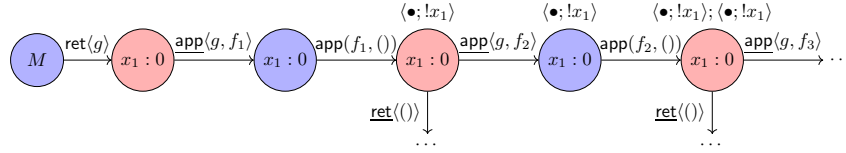
Up to re-entry is realised by function `reent` in Fig. 4. The intuition of this up-to technique is that if the application of related functions at i in the Γ environments has no potential to change the local stores (up to garbage collection) or increase the Γ environments, then there are no additional observations to be made by nested calls to the i -functions. Soundness follows similarly to up-to separation.

In HOBBIT we require the user to flag the functions to be considered for the up to re-entry technique. This annotation is later combined with state invariant annotations, as they are often used together. Below is an example where the state invariant needed is trivial and up to separation together with up to re-entry are sufficient to finitise the LTS and thus prove the equivalence.

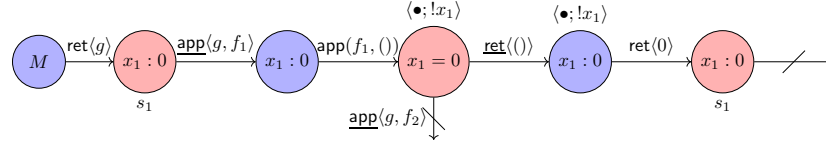
Example 17.

$$M = \mathbf{ref} \ x = \mathbf{0} \ \mathbf{in} \ \mathbf{fun} \ f \ \rightarrow f \ (); \ !x \qquad N = \mathbf{fun} \ f \ \rightarrow f \ (); \ \mathbf{0}$$

This is like Ex. 2 except the reference in M is created outside of the function body. The LTS for this is as follows. Labels $\langle \bullet; !x_1 \rangle$ are continuations.



Again, the opponent is allowed to reenter g as before. With up-to reentry, however, the opponent skips nested calls to g as these do not modify the state.



6 Symbolic First-Order Transitions

We extend λ^{imp} constants (`Const`) with a countable set of symbolic constants ranged over by κ . We define symbolic environments $\sigma ::= \cdot \mid (\kappa \frown e), \sigma$, where \frown is either $=$ or \neq , and e is an arithmetic expression over constants, and interpret them as conjunctions of (in-)equalities, with the empty set interpreted as \top .

Definition 18 (Satisfiability). *Symbolic environment σ is satisfiable if there exists an assignment δ , mapping the symbolic constants of σ to actual constants, such that $\delta\sigma$ is a tautology; we then write $\delta \models \sigma$.*

We extend reduction configurations with a symbolic environment σ , written as $\sigma \vdash \langle s; e \rangle$. These constants are implicitly annotated with their type. We modify the reduction semantics from Fig. 2 to consider symbolic constants:

$$\begin{aligned} \sigma \vdash \langle s; \text{op}(\vec{c}) \rangle &\hookrightarrow \sigma \wedge (\kappa = \text{op}(\vec{c})) \vdash \langle s; \vec{w} \rangle \text{ if } \kappa \text{ fresh} \\ \sigma \vdash \langle s; \text{if } \vec{c} \text{ then } e_1 \text{ else } e_2 \rangle &\hookrightarrow \sigma \wedge (\vec{c} = \mathbf{tt}) \vdash \langle s; e_1 \rangle \quad \text{if } \sigma \wedge (\vec{c} = \mathbf{tt}) \text{ is sat.} \\ \sigma \vdash \langle s; \text{if } \vec{c} \text{ then } e_1 \text{ else } e_2 \rangle &\hookrightarrow \sigma \wedge (\vec{c} = \mathbf{ff}) \vdash \langle s; e_2 \rangle \quad \text{if } \sigma \wedge (\vec{c} = \mathbf{ff}) \text{ is sat.} \end{aligned}$$

All other reduction semantics rules carry the σ . The LTS from Sec. 4 is modified to operate over configurations of the form $\sigma \vdash C$ or $\cdot \vdash \langle \perp \rangle$. We let \tilde{C} range over both forms of configurations. All LTS rules for proponent transitions simply carry the σ ; rule **TAU** may increase σ due to the inner reduction. Opponent transitions generate fresh symbolic constants, instead of actual constants: labels $\underline{\text{app}}(i, D[\vec{\alpha}])$ and $\underline{\text{ret}}(D[\vec{\alpha}])$ in rules **OPAPP** and **OPRET** of Fig. 3, respectively, contain D with symbolic, instead of concrete constants. We adapt (bi-)simulation as follows.

Definition 19. *Binary relation \mathcal{R} on symbolic configurations is a weak simulation when for all $\tilde{C}_1 \mathcal{R} \tilde{C}_2$ and $\tilde{C}_1 \xrightarrow{\eta_1} \tilde{C}'_1$, there exists \tilde{C}'_2 such that $\tilde{C}_2 \xrightarrow{\eta_2} \tilde{C}'_2$ and $\tilde{C}'_1 \mathcal{R} \tilde{C}'_2$ and $(\tilde{C}'_1.\sigma, \tilde{C}'_2.\sigma)$ is sat. and for all $\delta \models (\tilde{C}'_1.\sigma, \tilde{C}'_2.\sigma)$ it is $\delta\eta_1 = \delta\eta_2$.*

Lemma 20. $(\sigma_1 \vdash C_1) \sqsubseteq (\sigma_2 \vdash C_2)$ iff for all $\delta \models \sigma_1, \sigma_2$ we have $\delta C_1 \sqsubseteq \delta C_2$.

Corollary 21 (Soundness, Completeness). $(\cdot \vdash C_1) \sqsubseteq (\cdot \vdash C_2)$ iff $C_1 \sqsubseteq C_2$.

The up-to techniques we have developed in previous sections apply unmodified to the extended LTS as the techniques do not involve symbolic constants, with the exception of up to beta which requires adapting the definition of a beta move to consider all possible δ . The introduction of symbolic first-order transitions allows us to prove many interesting first-order examples, such as the equivalence of bubble sort and insertion sort, an example borrowed from **HECTOR** [12] (omitted here, see the **HOBBIT** distribution). Below is a simpler example showing the equivalence of two integer swap functions which **HOBBIT** is able to prove.

Example 22.

$M = \text{let swap } xy =$ $\quad \text{let } (x, y) = xy$ $\quad \text{in } (y, x)$ in swap	$N = \text{fun } xy \text{ -> let } (x, y) = xy \text{ in}$ $\quad \text{ref } x = x \text{ in ref } y = y \text{ in}$ $\quad x := !x - !y; y := !x + !y;$ $\quad x := !y - !x; (!x, !y)$
--	---

7 Up to State Invariants

The addition of symbolic constants into λ^{imp} and the LTS not only allows us to consider all possible opponent-generated constants simultaneously in a symbolic execution of proponent expressions, but also allows us to define an additional powerful up-to technique: *up to state invariants*. We define this technique in two parts: *up to abstraction* and *up to tautology* realised by **abs** and **taut**.⁵

UPToabs $\frac{(\sigma_1 \vdash C_1) \mathcal{R} (\sigma_2 \vdash C_2)}{(\sigma_1 \vdash C_1)[\vec{c}/\vec{\kappa}] \text{abs}(\mathcal{R}) (\sigma_2 \vdash C_2)[\vec{c}/\vec{\kappa}]}$	UPTotaut $\frac{(\sigma_1, \sigma'_1 \vdash C_1) \mathcal{R} (\sigma_2, \sigma'_2 \vdash C_2)$ $\quad \sigma_1, \sigma_2, \sigma'_1, \sigma'_2 \text{ is sat.}$ $\quad \sigma_1, \sigma_2 \wedge \neg(\sigma'_1, \sigma'_2) \text{ is not sat.}}{(\sigma_1 \vdash C_1) \text{taut}(\mathcal{R}) (\sigma_2 \vdash C_2)}$
--	--

The first function **abs** allows us to derive the equivalence of configurations by abstracting constants with fresh symbolic constants (of the same type) and instead prove equivalent the more abstract configurations. The second function

⁵ **HOBBIT** also implements an *up to σ -normalisation and garbage collection* technique.

`taut` allows us to introduce tautologies into the symbolic environments. These are predicates which are valid; i.e., they hold for all instantiations of the abstract variables. Combining the two functions we can introduce a tautology $I(\vec{c})$ into the symbolic environments, and then abstract constants \vec{c} from the predicate but also from the configurations with symbolic ones, obtaining $I(\vec{\kappa})$, which encodes an invariant that always holds.

Currently in HOBBIT, up to abstraction and tautology are combined and applied in a principled way. Functions can be annotated with the following syntax:

$$F = \mathbf{fun} \ x \ \{ \vec{\kappa} \mid l_1 \ \mathbf{as} \ C_1[\vec{\kappa}], \ \dots, \ l_n \ \mathbf{as} \ C_n[\vec{\kappa}] \mid \phi \} \rightarrow e$$

The annotation instructs HOBBIT to use the two techniques when opponent applies related functions where at least one of them has such an annotation. If both functions contain annotations, then they are combined and the same $\vec{\kappa}$ are used in both annotations. The techniques are used again when proponent returns from the functions, and proponent calls opponent from within the functions.⁶ As discussed in Sec. 5, the same annotation enables up to reentry in HOBBIT.

When HOBBIT uses the above two up-to techniques it 1) pattern-matches the values currently in each location l_i with the value context C_i where fresh symbolic constants $\vec{\kappa}$ are in its holes, obtaining a substitution $[\vec{c}/\vec{\kappa}]$; 2) the up to tautology technique is applied for the formula $\phi[\vec{c}/\vec{\kappa}]$; and 3) the up to abstraction technique is applied by replacing $\phi[\vec{c}/\vec{\kappa}]$ in the symbolic environment with ϕ , and the contents of locations l_i with $C_i[\vec{\kappa}]$.

Example 23. Following is an example by Meyer and Sieber [21] featuring location passing, adapted to λ^{imp} where locations are local. Full example in Appendix G.3.

```
M = let loc_eq loc1loc2 = [...] in
      fun q -> ref x = 0 in
          let locx = (fun () -> !x) , (fun v -> x := v) in
              let almostadd_2 locz {w | x as w | w mod 2 == 0} =
                  if loc_eq (locx, locz) then x := 1 else x := !x + 2
              in q almostadd_2; if !x mod 2 = 0 then _bot_ else ()

N = fun q -> _bot_
```

In this example we simulate general references as a pair of read-write functions. Function `loc_eq` implements a standard location equality test (see Appendix G.3). The two higher-order expressions are equivalent because the opponent can only increase the contents of `x` through the function `almostadd_2`. As the number of times the opponent can call this function is unbounded, the LTS is infinite. However, the annotation of function `almostadd_2` applies the up to state invariants technique when the function is called (and, less crucially, when it returns), replacing the concrete value of `x` with a symbolic integer constant w satisfying the invariant $w \bmod 2 == 0$. This makes the LTS finite, up to permutations of symbolic constants. Moreover, up to separation removes the outer functions from the Γ environments, thus preventing re-entrant calls to these

⁶ Finer-grain control of application of these up-to techniques is left to future work.

functions. Note the up-to techniques are applied even though one of the configurations is diverging (`_bot_`). This would not be possible with the LTS and bisimulation of [3].

8 Implementation and Evaluation

We implemented the LTS and up-to techniques for λ^{imp} in a tool prototype called HOBBIT, which we ran on a test-suite of 105 equivalences and 68 inequivalences—3338 and 2263 lines of code for equivalences and inequivalences respectively.

HOBBIT is bounded in the total number of function calls it explores per path. We ran HOBBIT with a default bound of 6 calls except where a larger bound was found to prove or disprove equivalence—46 examples required a larger bound, and the largest bound used was 348. To illustrate the impact of up-to techniques, we checked all files (pairs of expressions to be checked for equivalence) in five configurations: default (all up-to techniques on), up to separation off, annotations (up to state invariants and re-entry) off, up to re-entry off, and everything off. The tool stops at the first trace that disproves equivalence, after enumerating all traces up to the bound, or after timing out at 150 seconds. Time taken and exit status (equivalent, inequivalent, inconclusive) were recorded for each file; an overview of the experiment can be seen in the following table. All experiments ran on an Ubuntu 18.04 machine with 32GB RAM, Intel Core i7 1.90GHz CPU, with intermediate calls to Z3 4.8.10 to prune invalid internal symbolic branching and decide symbolic bisimulation conditions. All constraints passed to Z3 are of propositional satisfiability in conjunctive normal form (CNF).

	default	sep. off	annot. off	ree. off	all off
eq.	72 0 [5.6s]	32 0 [1622.9s]	47 0 [178.3s]	57 0 [177.6s]	3 0 [2098.5s]
ineq.	0 68 [20.0s]	0 66 [312.8s]	0 68 [19.6s]	0 68 [20.1s]	0 65 [515.7s]
<i>a</i> <i>b</i> [<i>c</i>] for <i>a</i> (out of 105) equivalences and <i>b</i> (out of 68) inequivalences reported taking <i>c</i> seconds in total.					

We can observe that HOBBIT was sound and bounded-complete for our examples; no false reports and all inequivalences were identified. Up-to techniques also had a significant impact on proving equivalence. With all techniques on, it proved 68.6% of our equivalences; a dramatic improvement over 2.9% proven with none on. The most significant technique was up-to separation—necessary for 55.6% of equivalences proven and reducing time taken by 99.99%—which was useful when functions could be independently explored by the context. Following was annotations—necessary for 34.7% of equivalences and decreasing time by 96.9%—and up-to re-entry—20.8% of files and decreased time by 96.8%. Although the latter two required manual annotation, they enabled equivalences where our language was able to capture the proof conditions. Note that, since turning off invariant annotations also turns off re-entry, only 10 files needed up-to re-entry on top of invariant annotations. In contrast, inequivalences did not benefit as much. This was expected as without up-to techniques HOBBIT is still based on bounded model checking, which is theoretically sound and complete for inequivalences. Nonetheless, three files timed out with techniques turned off,

which suggests that the reduction in state space is still relevant when searching for counterexamples.

9 Comparison with Existing Tools

There are two main classes of tools for contextual equivalence checking. The first one includes semantics-driven tools that tackle higher-order languages with state like ours. In this class belong game-based tools HECTOR [12] and CONEQCT [23], which can only address carefully crafted fragments of the language, delineated by type restrictions and bounded data types. The most advanced tool in this class is SYTECI [14], which is based on logical relations and removes a good part of the language restrictions needed in the previous tools. The second class concerns tools that focus on first-order languages, typically variants of C, with main tools including RÊVE [9], SYMDIFF [17] and RVT [11]. These are highly optimised for handling *internal loops*, a problem orthogonal to handling the interactions between higher-order functions and their environment, addressed by HOBBIT and related tools. We believe the techniques used in these tools may be useful when adapted to HOBBIT, which we leave for future work.

In the higher-order contextual equivalence setting, the most relevant tool to compare with HOBBIT is SYTECI. This is because SYTECI supersedes previous tools by proving examples with fewer syntactical limitations. We ran the tools on examples from both SYTECI’s and our own benchmarks—7 and 15 equivalences, and 2 and 7 inequivalences from SYTECI and HOBBIT respectively—with a timeout of 150s and using Z3. Unfortunately, due to differences in parsing and SYTECI’s syntactical restrictions, the input languages were not entirely compatible and only few manually translated programs were chosen.

	SyTeCi	Hobbit
SyTeCi eq. examples	3 0 4 (0.03s)	1 0 6 (<0.01s)
Hobbit eq. examples	8 0 7 (0.4s)	15 0 0 (<0.01s)
SyTeCi ineq. examples	0 2 0 (0.06s)	0 2 0 (0.02s)
Hobbit ineq. examples	2 3 2 (0.52s)	0 7 0 (0.45s)
<i>a</i> <i>b</i> <i>c</i> (<i>d</i>) for <i>a</i> eq’s, <i>b</i> ineq’s and <i>c</i> inconclusive’s reported taking <i>d</i> sec in total		

We were unable to translate many of our examples because of restrictions in the input syntax supported by SYTECI. Some of these restrictions were inessential (e.g. absence of tuples) while others were substantial: the tool does not support programs where references are allocated both inside and outside functions (e.g. Ex. 16), or with non-synchronisable recursive calls. Moreover, SYTECI relies on Constrained Horn Clause satisfiability which is undecidable. In our testing SYTECI sometimes timed out on examples; in private correspondence with its creator this was attributed to Z3’s ability to solve Constrained Horn Clauses. Finally, SYTECI was sound for equivalences, but not always for inequivalences as can be seen in the table above; the reason is unclear and may be due to bugs. On the other hand, SYTECI was able to solve equivalences we are not able to handle; e.g. synchronisable recursive calls and examples like the well-bracketed state problem:


```

M = ref x = 0 in fun f -> x:=0; f(); x:=1; f(); !x
N = fun f -> f(); f(); 1

```

10 Conclusion

Our experience with HOBbit suggests that our technique provides a significant contribution to verification of contextual equivalence. In the higher-order case, HOBbit does not impose language restrictions as present in other tools. Our tool is able to solve several examples that can not be solved by SYTECI, which is the most advanced tool in this family. In the first-order case, the problem of contextual equivalence differs significantly as the interactions that a first-order expression can have with its context are limited; e.g. equivalence analyses do not need to consider callbacks or re-entrant calls. Moreover, the distinction between global and local state is only meaningful in higher-order languages where a program phrase can invoke different calls of the same function, each with its own state. Therefore, tools for first-order languages focus on what in our setting are internal transitions and the complexities arising from e.g. unbounded datatypes and recursion, whereas we focus on external interactions with the context.

As for limitations, HOBbit does not handle synchronised internal recursion and well-bracketed state, which SYTECI can often solve. More generally, HOBbit is not optimised for internal recursion as first-order tools are. In this work we have also disallowed reference types in λ^{imp} to simplify the technical development; location exchange is encoded via function exchange (cf. Ex. 23). We intend to address these limitations in future work.

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This appendix is provided for the benefit of the reviewers, and will not appear in a final version of this paper.

A Typing rules of λ^{imp}

$$\begin{array}{c}
 \frac{c \text{ cons. of type } T}{\Delta; \Sigma \vdash c : T} \quad \frac{(x : T) \in \Delta}{\Delta; \Sigma \vdash x : T} \quad \frac{\Delta; \Sigma \vdash e_1 : T_1 \quad \dots \quad \Delta; \Sigma \vdash e_n : T_n}{\Delta; \Sigma \vdash (e_1, \dots, e_n) : T_1 * \dots * T_n} \\
 \\
 \frac{op : \vec{T} \rightarrow T \quad \Delta; \Sigma \vdash (\vec{e}) : \vec{T}}{\Delta; \Sigma \vdash op(\vec{e}) : T} \quad \frac{\Delta; \Sigma \vdash e : \text{bool} \quad \Delta; \Sigma \vdash (e_1, e_2) : T * T}{\Delta; \Sigma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : T} \\
 \\
 \frac{\Delta; \Sigma \vdash v : T \quad \Delta; \Sigma, l : T \vdash e : T'}{\Delta; \Sigma \vdash \text{ref } l = v \text{ in } e : T'} \quad \frac{(l : T) \in \Sigma}{\Delta; \Sigma \vdash !l : T} \quad \frac{(l : T) \in \Sigma \quad \Delta; \Sigma \vdash e : T}{\Delta; \Sigma \vdash l := e : \text{unit}} \\
 \\
 \frac{\Delta; \Sigma \vdash e : T \rightarrow T' \quad \Delta; \Sigma \vdash e' : T}{\Delta; \Sigma \vdash ee' : T'} \quad \frac{\Delta, f : T \rightarrow T', x : T; \Sigma \vdash e : T'}{\Delta; \Sigma \vdash \text{fix } f(x).e : T \rightarrow T'} \\
 \\
 \frac{\Delta, x_1 : T_1, \dots, x_n : T_n; \Sigma \vdash e : T \quad \Delta; \Sigma \vdash e' : \vec{T}}{\Delta; \Sigma \vdash \text{let } (\vec{x}) = e' \text{ in } e : T}
 \end{array}$$

B Proof of Thm. 8

We let a *trace* be a sequence of app and ret moves (i.e. labels), as defined in Fig. 3. A trace is *complete* if it starts with a fully bracketed segment, followed by a proponent return and is afterwards again fully bracketed, i.e. it adheres to the grammar:

$$\begin{aligned}
 CT &::= Y \text{ ret}(D) X \\
 X &::= \varepsilon \mid \underline{\text{app}}(\alpha, D) Y \text{ ret}(D) X \\
 Y &::= \varepsilon \mid \text{app}(\alpha, D) X \underline{\text{ret}}(D) Y
 \end{aligned}$$

Theorem 24 ([18]). *Expressions $\vdash e_1 : T$ and $\vdash e_2 : T$ are contextually equivalent iff the configurations $\langle \cdot; \cdot; \cdot; \cdot; e_1 \rangle, \langle \cdot; \cdot; \cdot; \cdot; e_2 \rangle$ produce the same complete traces.*

Proof of Thm. 8 We first note that our LTS is deterministic modulo the selection of fresh locations in tau transitions. Moreover, by Thm. 24, it suffices to show that e_1, e_2 are bisimilar iff they have the same complete traces.

If the two expressions have the same complete traces then each of them can match any challenge posed by the other, so long as such a challenge can lead to a complete trace. If a challenge is doomed to not complete, then it can be matched by a transition to $\langle \perp \rangle$. Conversely, if e_1, e_2 are bisimilar then any transition sequence yielding a complete trace of e_1 can be simulated by e_2 , and viceversa, so the two expressions have the same complete traces. \square

Remark 25. Soundness is also proved via the up to techniques in Appendix E. Completeness is also proved directly in Appendix F.

C Theory of Enhancements

We develop our up-to techniques using the theory of bisimulation enhancements from [27,26]. Here we summarise main definitions, starting with the notions of progressions and compatible functions [27]. The main result of this section is a set of proof obligations with which we can proof an up-to technique sound, shown in Lem. 37. We start by defining basic operations on monotone functions.

Definition 26. Consider monotone functions $f, g : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ on some set X . We write $f \circ g$ for the composition of f and g , and $f \sqcup g$ for the function $\mathcal{S} \mapsto f(\mathcal{S}) \sqcup g(\mathcal{S})$. For any set F of functions, we write $\bigsqcup F$ for the function $\mathcal{S} \mapsto \bigcup_{f \in F} f(\mathcal{S})$. We also write c_X to be the constant function with range $\{X\}$. We let $f^0 \stackrel{\text{def}}{=} \text{id}$ and $f^{n+1} \stackrel{\text{def}}{=} f \circ f^n$. Moreover, we write f^ω to mean $\bigsqcup_{k < \omega} f^k$. We write $f \sqsubseteq g$ when, for all $\mathcal{S} \in \mathcal{P}(X)$, $f(\mathcal{S}) \subseteq g(\mathcal{S})$.

The theory of enhancements we use here is based on the notion of *weak progression*. Weak progression is first defined as a monotone function on configuration relations ($\mathbf{wp}(\mathcal{R})$), and then used for a pre-fixpoint predicate on configuration relations ($\mathcal{R} \overset{\mathbf{wp}}{\rightsquigarrow} \mathcal{S}$) and one on monotone functions over said relations ($f \overset{\mathbf{wp}}{\rightsquigarrow} g$). The latter functions are meant to encode up-to techniques.

Definition 27 (Progressions (\rightsquigarrow)).

- $\mathbf{wp}(\mathcal{R}) = \{(C_1, C_2) \mid \forall C'_1, \eta. C_1 \xrightarrow{\eta} C'_1 \text{ implies } \exists C'_2, C_2 \xrightarrow{\eta} C'_2 \text{ and } C'_1 \mathcal{R} C'_2\}$.
- \mathcal{R} weakly progresses to \mathcal{S} , and we write $\mathcal{R} \overset{\mathbf{wp}}{\rightsquigarrow} \mathcal{S}$ when $\mathcal{R} \subseteq \mathbf{wp}(\mathcal{S})$.
- For monotone functions f, g we write $f \overset{\mathbf{wp}}{\rightsquigarrow} g$ when $f \circ \mathbf{wp} \sqsubseteq \mathbf{wp} \circ g$.

Lemma 28. \mathcal{R} is a weak simulation when $\mathcal{R} \overset{\mathbf{wp}}{\rightsquigarrow} \mathcal{R}$. Also, $(\overline{\sqsubseteq}) = (\mathbf{gfp}(\mathbf{wp}))$. \square

The following gives the definition of an up-to technique, what it means to be sound, and the stronger notion of compatibility.

Definition 29.

- Bisimulation up-to: \mathcal{R} is a weak simulation up to f when $\mathcal{R} \overset{\mathbf{wp}}{\rightsquigarrow} f(\mathcal{R})$.
- Sound up-to technique: Function f is \mathbf{wp} -sound when $\mathbf{gfp}(\mathbf{wp} \circ f) \subseteq \mathbf{gfp}(\mathbf{wp})$.
- Compatibility: Monotone function f is \mathbf{wp} -compatible when $f \overset{\mathbf{wp}}{\rightsquigarrow} f$.

Lemma 30 ([27], Lem. 6.3.12). $f \overset{\mathbf{wp}}{\rightsquigarrow} f$ if and only if for all $\mathcal{R} \overset{\mathbf{wp}}{\rightsquigarrow} \mathcal{S}$ we have $f \circ \mathbf{wp}(\mathcal{R}) \subseteq \mathbf{wp} \circ g(\mathcal{S})$. \square

Lemma 31 ([27], Thm. 6.3.9). If f is \mathbf{wp} -compatible then it is \mathbf{wp} -sound. \square

Lemma 32 ([27], Prop. 6.3.11 and 6.3.12). The following functions are \mathbf{wp} -compatible:

- the reflexive c_{refl} and identity id functions;
- $f \circ g$, for any \mathbf{wp} -compatible monotone functions f, g ;

– $\sqcup F$, for any set F of **wp**-compatible monotone functions. \square

Pous [26] extends the theory of enhancements with the notion of companion of **wp**, the largest **wp**-compatible function.

Definition 33 (Companion). $\mathbf{t}_{\mathbf{wp}} \stackrel{\text{def}}{=} \sqcup \{f : \mathcal{P}(\text{Conf}^2) \rightarrow \mathcal{P}(\text{Conf}^2) \mid f \overset{\mathbf{wp}}{\rightsquigarrow} f\}$.

Lemma 34 ([26]).

1. $\mathbf{t}_{\mathbf{wp}}$ is **wp**-compatible: $\mathbf{t}_{\mathbf{wp}} \rightsquigarrow \mathbf{t}_{\mathbf{wp}}$;
2. **wp** is **wp**-compatible: $\mathbf{wp} \sqsubseteq \mathbf{t}_{\mathbf{wp}}$;
3. $\mathbf{t}_{\mathbf{wp}}$ is idempotent: $\text{id} \sqsubseteq \mathbf{t}_{\mathbf{wp}}$ and $\mathbf{t}_{\mathbf{wp}} \circ \mathbf{t}_{\mathbf{wp}} \sqsubseteq \mathbf{t}_{\mathbf{wp}}$;
4. $\mathbf{t}_{\mathbf{wp}}$ is **wp**-sound: $\text{gfp}(\mathbf{wp} \circ \mathbf{t}_{\mathbf{wp}}) \sqsubseteq \text{gfp}(\mathbf{wp})$. \square

This gives rise a proof technique for proving up-to techniques sound.

Lemma 35. *Let $f \sqsubseteq \mathbf{t}_{\mathbf{wp}}$. Then f is **wp**-sound.*

Proof. By showing that $f \cup \mathbf{t}_{\mathbf{wp}} \overset{\mathbf{wp}}{\rightsquigarrow} f \cup \mathbf{t}_{\mathbf{wp}}$ and using Lem. 31.

Lemma 36 (Function Composition Laws). *Consider monotone functions $f, g, h : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ and set $\mathcal{S} \in \mathcal{P}(X)$. We have*

1. $\mathbf{c}_{\mathcal{S}} \circ f = \mathbf{c}_{\mathcal{S}}$
2. $(f \sqcup g) \circ h = (f \circ h) \sqcup (g \circ h)$
3. $h \circ (f \sqcup g) = (h \circ f) \sqcup (h \circ g)$
4. $(f \sqcup g) \sqsubseteq (f \sqcup h)$ and $(f \circ g) \sqsubseteq (f \circ h)$ and $(g \circ f) \sqsubseteq (h \circ f)$, when $g \sqsubseteq h$.
5. $f \sqsubseteq f^\omega$ and $f \circ f^\omega = f^\omega \circ f \sqsubseteq f^\omega \circ f^\omega \sqsubseteq f^\omega$.
6. $f^\omega \circ g = \sqcup_{i < \omega} (f^i \circ g)$. \square

We distil this up-to technique to the following three proof obligations, each sufficient for proving the soundness of up-to techniques.

Lemma 37 (POs for Up-To Soundness). *Let f be a monotone function and \mathcal{R} be a weak simulation; f is **wp**-sound when one of the following holds:*

1. $f \overset{\mathbf{wp}}{\rightsquigarrow} f$; or
2. $f \overset{\mathbf{wp}}{\rightsquigarrow} (f \circ g)$, for some $g \sqsubseteq \mathbf{t}_{\mathbf{wp}}$; or
3. $f = \sqcup_{f_i \in F} f_i \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})}$, where F is a set of monotone functions and, for all $f_i \in F$, there exists $g_i \sqsubseteq \mathbf{t}_{\mathbf{wp}}$ such that $f_i \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \overset{\mathbf{wp}}{\rightsquigarrow} (f \sqcup g_i)^\omega \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})}$.

Proof.

1. By Lem. 31.
2. By Lem. 35, it suffices to show $f \sqsubseteq \mathbf{t}_{\mathbf{wp}}$. Because $f \sqsubseteq f \circ (\text{id} \sqcup \mathbf{t}_{\mathbf{wp}}) \sqsubseteq f \circ \mathbf{t}_{\mathbf{wp}}$, it suffices to show $f \circ \mathbf{t}_{\mathbf{wp}} \overset{\mathbf{wp}}{\rightsquigarrow} f \circ \mathbf{t}_{\mathbf{wp}}$ by unfolding definitions and the premise:

$$f \circ \mathbf{t}_{\mathbf{wp}} \circ \mathbf{wp} \sqsubseteq f \circ \mathbf{wp} \circ \mathbf{t}_{\mathbf{wp}} \sqsubseteq \mathbf{wp} \circ f \circ g \circ \mathbf{t}_{\mathbf{wp}} \sqsubseteq \mathbf{wp} \circ f \circ \mathbf{t}_{\mathbf{wp}} \circ \mathbf{t}_{\mathbf{wp}} \sqsubseteq \mathbf{wp} \circ f \circ \mathbf{t}_{\mathbf{wp}}.$$

3. Let $g = \sqcup_{f_i \in F} g_i$. By Lem. 35, it suffices to show $f \sqsubseteq \mathbf{t}_{\mathbf{wp}}$. Because

$$\begin{aligned} f_i \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} &= f_i \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \sqsubseteq f \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \sqsubseteq (f \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}) \sqcup (\mathbf{t}_{\mathbf{wp}} \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}) \\ &= (f \sqcup \mathbf{t}_{\mathbf{wp}}) \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \sqsubseteq (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \end{aligned}$$

it suffices to show that $(f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \overset{\mathbf{wp}}{\rightsquigarrow} (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}$. This is proven by showing that for all k ,

$$(f \sqcup \mathbf{t}_{\mathbf{wp}})^k \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \overset{\mathbf{wp}}{\rightsquigarrow} (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}. \quad (P(k))$$

We proceed by induction on k . The base case is straightforward:

$$\text{id} \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \circ \mathbf{wp} = \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} = \mathbf{wp} \circ \text{id} \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \sqsubseteq \mathbf{wp} \circ (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}$$

In the inductive case we assume $P(k)$ and prove $P(k+1)$ as follows:

$$\begin{aligned} &(f \sqcup \mathbf{t}_{\mathbf{wp}})^{k+1} \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \circ \mathbf{wp} \\ &= (f \sqcup \mathbf{t}_{\mathbf{wp}}) \circ (f \sqcup \mathbf{t}_{\mathbf{wp}})^k \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \circ \mathbf{wp} \\ &\sqsubseteq (f \sqcup \mathbf{t}_{\mathbf{wp}}) \circ \mathbf{wp} \circ h \quad (P(k), h = (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}) \\ &\sqsubseteq (f \circ \mathbf{wp} \circ h) \sqcup (\mathbf{t}_{\mathbf{wp}} \circ \mathbf{wp} \circ h) \quad (\text{Lem. 36}) \\ &= \left(\bigsqcup_{f_i \in F} (f_i \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \circ \mathbf{wp} \circ h) \right) \sqcup (\mathbf{t}_{\mathbf{wp}} \circ \mathbf{wp} \circ h) \quad (\text{definition of } f \text{ and Lem. 36}) \\ &\sqsubseteq \left(\bigsqcup_{f_i \in F} (\mathbf{wp} \circ (f \sqcup g_i)^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \circ h) \right) \sqcup (\mathbf{t}_{\mathbf{wp}} \circ \mathbf{wp} \circ h) \quad (\text{premise}) \\ &\sqsubseteq \left(\bigcup_{f_i \in F} (\mathbf{wp} \circ (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}) \right) \sqcup (\mathbf{t}_{\mathbf{wp}} \circ \mathbf{wp} \circ h) \quad (\text{Lem. 36 and premise on } g_i) \\ &\sqsubseteq (\mathbf{wp} \circ (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}) \sqcup (\mathbf{wp} \circ \mathbf{t}_{\mathbf{wp}} \circ h) \quad (\text{Lem. 34 (1)}) \\ &\sqsubseteq (\mathbf{wp} \circ \text{id} \circ h) \sqcup (\mathbf{wp} \circ \mathbf{t}_{\mathbf{wp}} \circ h) \quad (\text{definition of } h) \\ &= \mathbf{wp} \circ (\text{id} \sqcup \mathbf{t}_{\mathbf{wp}}) \circ h \quad (\text{Lem. 36}) \\ &= \mathbf{wp} \circ \mathbf{t}_{\mathbf{wp}} \circ h \quad (\text{Lem. 34 (3)}) \\ &\sqsubseteq \mathbf{wp} \circ (f \sqcup \mathbf{t}_{\mathbf{wp}}) \circ h \quad (\text{Lem. 36}) \\ &\sqsubseteq \mathbf{wp} \circ (f \sqcup \mathbf{t}_{\mathbf{wp}})^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \quad (\text{Lem. 36 and definition of } h) \end{aligned}$$

As we are only interested in weak progression, in the following we drop the \mathbf{wp} annotation from progressions, compatibility and companion.

D Simple Up-To Techniques

We develop our up-to techniques using the theory of bisimulation enhancements from [27,26] (see Appendix C). We start by presenting three straightforward up-to techniques which nevertheless are needed to reduce the configurations considered by bisimulation, achieving finite LTSs in many examples. These techniques

are up to permutations, beta reductions, garbage collection, and weakening of knowledge environments. To present these techniques we first need the following definitions.

Definition 38 (Permutations). *We consider permutations of store locations, π_l , abstract names, π_α and environment indices, π_i , respectively. When applying a permutation π_l to a store s , the former acts on both the domain and range of the latter. When applying a permutation π_i to an environment Γ , it only acts on its domain; other types of permutations only act on the codomain of Γ .*

Definition 39 (β -move). *A τ -transition $C \xrightarrow{\tau} C'$ is called a β -move, and we write $C \xrightarrow{\tau}_\beta C'$, when for all transitions $C \xrightarrow{\eta} C''$, one of the following holds:*

- $\eta = \tau$ and $C' = C''$; or
- there exists C''' such that $C' \xrightarrow{\eta} C'''$ and $C'' \xrightarrow{\tau}_\beta C'''$ or $C'' = C'''$.

Definition 40 (Garbage Collection). *We let (\asymp) be the largest equivalence relation between well-formed configurations with the axioms:*

- $\langle A; \Gamma; K; s; \hat{e} \rangle \asymp \langle A \uplus A'; \Gamma; K; s; \hat{e} \rangle$
- $\langle A; \Gamma; K; s; \hat{e} \rangle \asymp \langle A; \Gamma; K; s, s_g; \hat{e} \rangle$

for any $A, A', K, s, s_g, \hat{e}$ with $A' \cap \text{an}(\Gamma, K, s, \hat{e}) = \emptyset$, $\text{dom}(s_g) \cap \text{fl}(\Gamma, K, s, \hat{e}) = \emptyset$.

Lemma 41. *Let π_l, π_α , and π_i be permutations on locations, abstract names, and indices, respectively, and $\pi = \pi_l \pi_\alpha \pi_i$. If $C \xrightarrow{\eta} C'$ then $C\pi \xrightarrow{\eta \pi_\alpha \pi_i} C'\pi$.*

Proof. By nominal sets reasoning (all transition rules are closed under permutations).

Lemma 42. *Let $C = \langle A; \Gamma; K; s; e \rangle \xrightarrow{\eta} \langle A'; \Gamma'; K'; s'; e' \rangle = C'$; then for all finite L_0, A_0, I_0 there exist π_l, π_α, π_i such that*

$$C \xrightarrow{\eta \pi} C'\pi \text{ and } (A' \setminus A) \cap A_0 = (\text{dom}(s'\pi) \setminus \text{dom}(s)) \cap L_0 = (\text{dom}(\Gamma'\pi) \setminus \text{dom}(\Gamma)) \cap I_0 = \emptyset$$

where $\pi = \pi_l \pi_\alpha \pi_i$.

Proof. By Lem. 41, picking permutations π that rename new names in C' to fresh ones, and therefore such that $C\pi = C$.

Corollary 43. *Let $C = \langle A; \Gamma; K; s; e \rangle \xrightarrow{\eta} \langle A'; \Gamma'; K'; s'; e' \rangle = C'$; then for all finite L_0, A_0, I_0 there exist π_l, π_α, π_i such that*

$$C \xrightarrow{\eta \pi} C'\pi \text{ and } (A' \setminus A) \cap A_0 = (\text{dom}(s'\pi) \setminus \text{dom}(s)) \cap L_0 = (\text{dom}(\Gamma'\pi) \setminus \text{dom}(\Gamma)) \cap I_0 = \emptyset$$

where $\pi = \pi_l \pi_\alpha \pi_i$.

Proof. By induction on the length of the transition from C_1 , using Lem. 42.

$$\begin{array}{c}
\text{UPToBETA} \\
\frac{C_1' \mathcal{R} C_2'}{C_1 \xrightarrow{\tau}_{\beta}^* C_1' \quad C_2 \xrightarrow{\tau}_{\beta}^* C_2'} \\
\frac{}{C_1 \text{ beta}(\mathcal{R}) C_2}
\end{array}
\quad
\begin{array}{c}
\text{UPToPERM} \\
\frac{}{C_1 \mathcal{R} C_2} \\
\frac{}{C_1 \pi_{l_1} \pi_{\alpha} \pi_i \text{ perm}(\mathcal{R}) C_2 \pi_{l_2} \pi_{\alpha} \pi_i}
\end{array}
\quad
\begin{array}{c}
\text{UPToGC} \\
\frac{}{C_1 \simeq_{\mathcal{R}} C_2} \\
\frac{}{C_1 \text{ gc}(\mathcal{R}) C_2}
\end{array}
\quad
\begin{array}{c}
\text{UPToGC}\perp \\
\frac{}{C_1 \simeq_{\mathcal{R}} \langle \perp \rangle} \\
\frac{}{C_1 \text{ gc}(\mathcal{R}) \langle \perp \rangle}
\end{array}$$

$$\begin{array}{c}
\text{UPToWEAKENING} \\
\frac{\langle A_1; \Gamma_1, {}^i v_1; K_1; s_1; \hat{e}_1 \rangle \mathcal{R} \langle A_2; \Gamma_2, {}^i v_2; K_2; s_2; \hat{e}_2 \rangle}{\langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle \text{ weak}(\mathcal{R}) \langle A_2; \Gamma_2; K_2; s_2; \hat{e}_2 \rangle}
\end{array}
\quad
\begin{array}{c}
\text{UPToWEAKENING}\perp \\
\frac{\langle A_1; \Gamma_1, {}^i v_1; K_1; s_1; \hat{e}_1 \rangle \mathcal{R} \langle \perp \rangle}{\langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle \text{ weak}(\mathcal{R}) \langle \perp \rangle}
\end{array}$$

Fig. 5. Simple Up-to techniques.

The monotone functions on relations `perm`, `beta`, `gc`, and `weak`, as shown on Fig. 5, define the sound enhancement techniques: up to permutations, up to beta reductions, up to garbage collection, and up to weakening, respectively.

Lemma 44. *Functions `perm`, `beta`, `gc`, and `weak` are sound up-to techniques.*

Soundness follows by the bisimulation enhancement technique [27,26] (Lem. 37), showing `perm` \rightsquigarrow `perm`, `beta` \rightsquigarrow `beta`, `gc` \rightsquigarrow `gc` \circ `perm`, and `weak` \rightsquigarrow `weak` \circ `perm`.

Lemma 45. *Function `perm` is a sound up-to technique.*

Proof. From Lem. 37 (1), it suffices to show that `perm` is compatible; i.e., `perm(wp(R))` \sqsubseteq `wp(perm(R))`, for any configuration relation \mathcal{R} .

Let $C_1 \text{ wp}(\mathcal{R}) C_2$ and $C_1 \pi_1 \text{ perm}(\mathcal{R}) C_2 \pi_2$, where $\pi_1 = \pi_{\alpha} \pi_{l_1}$ and $\pi_1 = \pi_{\alpha} \pi_{l_1}$. Moreover, let $C_1 \pi_1 \xrightarrow{\eta} C_1'$. Because of $\pi_1 \pi_1 = \text{id}$ and Lem. 41 we get $C_1 \xrightarrow{\eta \pi_{\alpha}} C_1' \pi_1$. By definition of `wp(R)`, there exists C_2' such that $C_2 \xrightarrow{\eta \pi_{\alpha}} C_2'$ and $C_1' \pi_1 \mathcal{R} C_2'$. By Lem. 41 $C_2 \pi_2 \xrightarrow{\eta} C_2' \pi_2$, and by definition of `perm(R)`: $C_1' \text{ perm}(\mathcal{R}) C_2' \pi_2$.

Lemma 46. *Function `beta` is a sound up-to technique.*

Proof. From Lem. 37 (1), it suffices to show that `beta` is compatible; i.e., `beta(wp(R))` \sqsubseteq `wp(beta(R))`, for any configuration relation \mathcal{R} . Let $C_1 \text{ beta}(\text{wp}(\mathcal{R})) C_2$ and $C_1' \text{ wp}(\mathcal{R}) C_2'$ and $C_1 \xrightarrow{\tau}_{\beta}^* C_1'$ and $C_2 \xrightarrow{\tau}_{\beta}^* C_2'$. We need to show that for all C_1'' such that $C_1 \xrightarrow{\eta} C_1''$ there exists C_2'' such that $C_2 \xrightarrow{\eta} C_2''$ and $C_1'' \text{ beta}(\mathcal{R}) C_2''$.

Let $C_1 \xrightarrow{\eta} C_1''$. By definition of a β -move (definition 39), $C_1'' = C_1'$ and $\eta = \tau$ or there exists C_3 such that $C_1' \xrightarrow{\eta} C_3$ and $C_1'' \xrightarrow{\tau}_{\beta}^* C_3$. In the former case the proof is trivial. In the latter case, by definition of `wp(R)`, there exists C_4 such that $C_2' \xrightarrow{\eta} C_4$ and $C_3 \mathcal{R} C_4$. Moreover, $C_2 \xrightarrow{\tau} C_2' \xrightarrow{\eta} C_4$, and $C_1'' \xrightarrow{\tau}_{\beta}^* C_3 \mathcal{R} C_4$ which implies $C_1'' \text{ beta}(\mathcal{R}) C_4$, concluding the proof.

Lemma 47. *Function `gc` is a sound up-to technique.*

Proof. From Lem. 37 (2) and Lem. 45, it suffices to show that $\mathbf{gc} \rightsquigarrow \mathbf{gc} \circ \mathbf{perm}$.

Let $C_1 \mathbf{gc}(\mathbf{wp}(\mathcal{R})) C_2$. By case analysis on this derivation we have two cases:

UPToGC: $C_1 \asymp C_3 \mathbf{wp}(\mathcal{R}) C_4 \asymp C_2$. Consider $C_1 \xrightarrow{\eta} C'_1$.

By Lem. 42, there exists $\pi = \pi_\alpha \pi_{l1}$ such that $C_1 \xrightarrow{\eta \pi_\alpha} C'_1 \pi$ and $(\mathbf{an}(C'_1 \pi) \setminus \mathbf{an}(C_1)) \cap \mathbf{an}(C_3, C_2) = \emptyset$ and $(\mathbf{fl}(C'_1 \pi) \setminus \mathbf{fl}(C_1)) \cap \mathbf{fl}(C_3) = \emptyset$.

By Lem. 56, there exists C'_3 such that $C_3 \xrightarrow{\eta \pi_\alpha} C'_3$ and $C'_1 \asymp C'_3 \pi$.

By definition of \mathbf{wp} , there exists C'_4 such that $C_4 \xrightarrow{\eta \pi_\alpha} C'_4$ and $C'_3 \mathcal{R} C'_4$.

By Cor. 43 there exists π_{l2} such that $C_4 \xrightarrow{\eta \pi_\alpha} C'_4 \pi_{l2}$ and $(\mathbf{fl}(C'_4 \pi_{l2}) \setminus \mathbf{fl}(C_4)) \cap \mathbf{fl}(C_2) = \emptyset$.

By Lem. 54, $(\mathbf{an}(C'_4 \pi_{l2}) \setminus \mathbf{an}(C_4)) \cap \mathbf{an}(C_2) = \emptyset$.

By Cor. 57, there exists C'_2 such that $C_2 \xrightarrow{\eta \pi_\alpha} C'_2$ and $C'_2 \asymp C'_4 \pi_{l1}$.

By Lem. 41 $C_2 \xrightarrow{\eta} C'_2 \pi_\alpha$, and moreover, by Lem. 55, $C'_2 \pi_\alpha \asymp C'_4 \pi_{l1} \pi_\alpha$.

Therefore, $C'_1 \asymp C_3 \pi_\alpha \pi_{l1} \mathcal{R} C'_4 \pi_\alpha \pi_{l2} \asymp C'_2$, from which we derive $C'_1 \mathbf{gc}(\mathbf{perm}(\mathcal{R})) C'_2$ as required.

UPToGC \perp : $C_3 \mathcal{R} \langle \perp \rangle$ and $C_1 \asymp C_3$ and $C_2 = \langle \perp \rangle$. We proceed with the same reasoning as in the above case, with the exception that $C'_4 = \langle \perp \rangle$.

Lemma 48. *Function weak is a sound up-to technique.*

Proof. Similar to the preceding proof, using Lem. 59.

The up to beta technique is useful in reducing the configurations considered in bisimulation, focusing only on the configurations before the observable transitions of Fig. 3; τ -transitions are all beta transitions and can be considered all at once.

Many simple example equivalences have infinite transition systems without these up-to techniques, even when combined with the more sophisticated up-to techniques in the following sections. A simple example is the following.

Example 49. Consider the equivalent functions

$$M = (\mathbf{fun} () \rightarrow \mathbf{ref} \ \mathfrak{l} = \mathbf{0} \ \mathbf{in} \ 5) \qquad M = (\mathbf{fun} () \rightarrow 5)$$

Due to the allocation of \mathfrak{l} the LTS of this equivalence is finite only using the up to \mathbf{gc} technique.

D.1 Proof of Lem. 14

Proof. By Lem. 37 (3) it suffices to show $\mathbf{sep} \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \rightsquigarrow (\mathbf{sep} \cup g)^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}$, where $g = \mathbf{id} \sqcup \mathbf{perm}$. We need to show $\mathbf{sep} \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})} \circ \mathbf{wp}(\mathcal{R}) \subseteq \mathbf{wp} \circ (\mathbf{sep} \cup g)^\omega \circ \mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}(\mathcal{R})$. Because $\mathbf{c}_{\mathbf{gfp}(\mathbf{wp})}$ is the constant function mapping its argument to $(\underline{\mathfrak{k}})$, we need to show $\mathbf{sep}(\underline{\mathfrak{k}}) \subseteq \mathbf{wp} \circ (\mathbf{sep} \cup g)^\omega(\underline{\mathfrak{k}})$. We proceed by induction on the derivation of $C_1 \mathbf{sep}(\underline{\mathfrak{k}}) C_2$.

Let $C_1 \mathbf{sep}(\underline{\mathfrak{k}}) C_2$, and $C_1 \xrightarrow{\eta} C'_1$. If $C_1 \xrightarrow{\eta} C'_1$ is produced by rule RESPONSE then the proof is trivial as C_2 can perform the same transition and $C'_1 =$

$\langle \perp \rangle \text{id}(\overline{\cong}) \langle \perp \rangle = C'_2$. We thus consider only non-RESPONSE transitions from C_1 . We proceed by case analysis on the derivation $C_1 \text{sep}(\overline{\cong}) C_2$. There are three cases: $\text{UPTo}\oplus$, $\text{UPTo}\oplus\perp_L$ and $\text{UPTo}\oplus\perp_R$. We only show the proof for the first case; the last two cases are handled in a similar but simpler fashion.

W.l.o.g. we let $i = 1$ and we have

$$\begin{aligned} C_1 &= C_3 \oplus_{\vec{k}}^i C_5 = \langle A_3 \cup A_5; \Gamma_3, \Gamma_5; K_1 \#_{\vec{k}} K_3; s_1, s_3; \hat{e}_3 \rangle \\ C_2 &= C_4 \oplus_{\vec{k}}^i C_6 = \langle A_3 \cup A_5; \Gamma_4, \Gamma_6; K_2 \#_{\vec{k}} K_4; s_2, s_4; \hat{e}_4 \rangle \\ C_3 &= \langle A_3; \Gamma_3; K_3; s_3; \hat{e}_3 \rangle \overline{\cong} \langle A_3; \Gamma_4; K_4; s_4; \hat{e}_4 \rangle = C_4 & \text{dom}(\Gamma_3) &= \text{dom}(\Gamma_4) \\ C_5 &= \langle A_5; \Gamma_5; K_5; s_5; \hat{e}_5 \rangle \overline{\cong} \langle A_5; \Gamma_6; K_6; s_6; \hat{e}_6 \rangle = C_6 & \text{dom}(\Gamma_5) &= \text{dom}(\Gamma_6) \end{aligned}$$

We proceed by cases on the transition $C_1 \xrightarrow{\eta} C'_1$.

?, PROPAPP, ?, PROPRET, TAU: In all proponent transitions the proof is similar. We show the case PROPAPP. In this case $e_3 = E[\alpha v]$ and

$$\begin{aligned} C_1 &= \langle A_3 \cup A_5; \Gamma_3, \Gamma_5; K_3 \#_{\vec{k}} K_5; s_3, s_5; E[\alpha v] \rangle \xrightarrow{\text{app}(\alpha, i)} \langle A_3 \cup A_5; \Gamma_3, \Gamma_5, {}^i v; K_3 \#_{\vec{k}} K_5; s_3, s_5; \cdot \rangle = C'_1 \\ & \text{and } i \notin \text{dom}(\Gamma_3, \Gamma_5) \\ C_3 &= \langle A_3; \Gamma_3; K_3; s_3; E[\alpha v] \rangle \xrightarrow{\text{app}(\alpha, i)} \langle A_3; \Gamma_3, {}^i v; K_3; s_3; \cdot \rangle = C'_3 \end{aligned}$$

Moreover, $C'_1 = C'_3 \oplus_{\vec{k}}^1 C_5$. By $C_3 \overline{\cong} C_4$ there are two possibilities:

- $C_4 \xrightarrow{\text{app}(\alpha, i)} \langle \perp \rangle$ and $C'_3 \overline{\cong} \langle \perp \rangle$. The proof in this case is completed by $C'_2 \xrightarrow{\text{app}(\alpha, i)} \langle \perp \rangle$ and $C_1 \text{sep}(\overline{\cong}) \langle \perp \rangle$ by either rule $\text{UPTo}\oplus\perp_L$ or $\text{UPTo}\oplus\perp_R$.
- $C_4 = \langle A_3; \Gamma_4; K_4; s_4; \hat{e}_4 \rangle \xrightarrow{\tau} \text{app}(\alpha, i) \langle A_3; \Gamma_4, {}^i v_4; K_4; s'_4; \cdot \rangle = C'_4$ and $C'_3 \overline{\cong} C'_4$. By $\text{dom}(\Gamma_3, \Gamma_5) = \text{dom}(\Gamma_4, \Gamma_6)$ and Lem. 61 we derive:

$$C_2 = \langle A_3 \cup A_5; \Gamma_4, \Gamma_6; K_4 \#_{\vec{k}} K_6; s_4, s_6; \hat{e}_4 \rangle \xrightarrow{\text{app}(\alpha, i)} \langle A_3 \cup A_5; \Gamma_4, \Gamma_6, {}^i v_4; K_4 \#_{\vec{k}} K_6; s'_4, s_6; \cdot \rangle = C'_2$$

Moreover, $C'_2 = C'_4 \oplus_{\vec{k}}^1 C_6$ and $C'_1 \text{sep}(\overline{\cong}) C'_2$.

?, OPAPP: We show the case for OPAPP. In this case $\eta = \text{app}(i, \alpha)$, and $\hat{e}_3 = \cdot$, and by Lem. 62, $\hat{e}_4 = \cdot$. Moreover $i \in \text{dom}(\Gamma_3)$ or $i \in \text{dom}(\Gamma_5)$. If it is the former, then $C_3 \xrightarrow{\text{app}(i, \alpha)} C'_3$ and $C'_1 = C'_3 \oplus_{\vec{k}}^1 C_5$, otherwise $C_5 \xrightarrow{\text{app}(i, \alpha)} C'_5$ and $C'_1 = C_3 \oplus_{\vec{k}}^2 C'_5$. Moreover by the simulation we can show that $C_4 \xrightarrow{\text{app}(i, \alpha)} C'_4$ or $C_6 \xrightarrow{\text{app}(i, \alpha)} C'_6$, respectively. In both cases $C_2 \xrightarrow{\text{app}(i, \alpha)} C'_2$ and $C'_2 = C'_4 \oplus_{\vec{k}}^1 C_6$ or $C'_2 = C_4 \oplus_{\vec{k}}^2 C'_6$, respectively. Therefore $C'_1 \text{sep}(\overline{\cong}) C'_2$.

?, OPRET: We show the cases for OPRET. Here \vec{k} is either $(1, \vec{k}')$ or $(2, \vec{k}')$. We consider the former case, the latter is symmetric. In this case $K_3 = E, K'_3$ and

$K_3 \#_{\vec{k}} K_5 = E, (K'_3 \#_{\vec{k}'} K_5)$. Therefore we have:

$$C_1 = \langle A_3 \cup A_5; \Gamma_3, \Gamma_5; E, (K'_3 \#_{\vec{k}'} K_5); s_3, s_5; \cdot \rangle \xrightarrow{\text{ret}(\alpha)} \langle A_3 \cup A_5 \uplus \alpha; \Gamma_3, \Gamma_5; K'_3 \#_{\vec{k}'} K_5; s_3, s_5; E[\alpha] \rangle = C'_1$$

and $\alpha \notin \text{dom}(A_3, A_5)$

$$C_3 = \langle A_3; \Gamma_3; K_3; s_3; \cdot \rangle \xrightarrow{\text{ret}(\alpha)} \langle A_3 \uplus \alpha; \Gamma_3; K_3; s_3; \cdot \rangle = C'_3$$

By $C_3 \sqsubseteq C_4$ there are two possibilities:

- $C_4 \xrightarrow{\text{ret}(\alpha)} \langle \perp \rangle$ and $C'_3 \sqsubseteq \langle \perp \rangle$. The proof in this case is completed by $C'_2 \xrightarrow{\text{ret}(\alpha)} \langle \perp \rangle$ and $C_1 \text{ sep}(\sqsubseteq) \langle \perp \rangle$ by either rule $\text{UPTo} \oplus \perp_L$ or $\text{UPTo} \oplus \perp_R$.
- $C_4 = \langle A_3; \Gamma_4; K_4; s_4; \cdot \rangle \xrightarrow{\tau} \xrightarrow{\text{ret}(\alpha)} \langle A_3 \uplus \alpha; \Gamma_4, {}^i v_4; K'_4; s'_4; E_4[\alpha] \rangle = C'_4$ and $C'_3 \sqsubseteq C'_4$ and $K_4 = E_4, K'_4$. Thus $K_4 \#_{\vec{k}} K_6 = K_4 \#_{\vec{k}'} K_6 = E_4, (K'_4 \#_{\vec{k}'} K_6)$. By Lem. 61 we derive:

$$C_2 = \langle A_3 \cup A_5; \Gamma_4, \Gamma_6; E_4, (K'_4 \#_{\vec{k}'} K_6); s_4, s_6; \cdot \rangle$$

$$\xrightarrow{\text{app}(\alpha, i)} \langle A_3 \cup A_5 \uplus \alpha; \Gamma_4, \Gamma_6; K'_4 \#_{\vec{k}'} K_6; s'_4, s_6; E_4[\alpha] \rangle = C'_2$$

Moreover, $C'_2 = C'_4 \oplus_{\vec{k}} C_6$ and $C'_1 \text{ sep}(\sqsubseteq) C'_2$.

TAU: In this case we have

$$C_1 = C_3 \oplus_{\vec{k}} C_5 \xrightarrow{\tau} C'_3 \oplus_{\vec{k}} C_5 = C'_1$$

$$C_3 \xrightarrow{\tau} C'_3$$

By $C_3 \sqsubseteq C_4$ we have $C_4 \xrightarrow{\tau} C'_4 = \langle \perp \rangle$ or $C_4 \xrightarrow{\tau} C'_4 \neq \langle \perp \rangle$ and $C'_3 \sqsubseteq C'_4$. In the former case the proof is completed by $C_2 \xrightarrow{\tau} \langle \perp \rangle$ and $C'_1 \text{ sep}(\sqsubseteq) \langle \perp \rangle$ by rule $\text{UPTo} \oplus \perp_L$. In the latter case, by Cor. 43, there exists π_{l_4} such that $C_4 \xrightarrow{\tau} C'_4 \pi_{l_4}$ and $(\text{fl}(C'_4 \pi_{l_2}) \setminus \text{fl}(C_4)) \cap \text{fl}(C_6) = \emptyset$. We have $C'_3 \text{ perm}(\sqsubseteq) C'_4 \pi_{l_4}$. Moreover we derive $C_2 = C_4 \oplus_{\vec{k}} C_6 \xrightarrow{\tau} C'_4 \pi_{l_4} \oplus_{\vec{k}} C_6 = C'_2$ and $C'_1 \text{ sep}(\text{perm}(\sqsubseteq)) C'_2$.

TERM: In this case we have $K_3 = K_5 = \cdot$ and $\hat{e}_3 = \hat{e}_5 = \cdot$. Therefore $C_3 \xrightarrow{\downarrow} \langle \perp \rangle$ and $C_5 \xrightarrow{\downarrow} \langle \perp \rangle$. Therefore by $C_3 \sqsubseteq C_4$ and $C_5 \sqsubseteq C_6$, we have $C_4 \xrightarrow{\downarrow} \langle \perp \rangle$ and $C_6 \xrightarrow{\downarrow} \langle \perp \rangle$. Therefore $K_4 = K_6 = \cdot$ and $\hat{e}_4 = \hat{e}_6 = \cdot$, and thus $C_2 \xrightarrow{\downarrow} \langle \perp \rangle$. The resulting $\langle \perp \rangle$ configurations are related by $\text{id}(\sqsubseteq)$.

E Soundness of (\approx)

E.1 Language Lemmas

The following lemmas hold for λ^{imp} extended with abstract names.

Lemma 50 (Unique Decomposition). *Let $e = E[e]$ and $\langle s; e \rangle \hookrightarrow$. Then for any E' and e' such that $e = E'[e']$ and $\langle s; e' \rangle \hookrightarrow$, we have $E = E'$ and $e = e'$.*

Proof. By induction on E .

Lemma 51. *Let π_l, π_α, π_i be permutations on locations, abstract names and indices respectively, and $\pi = \pi_l \pi_\alpha \pi_i$. If $\langle s; e \rangle \hookrightarrow \langle s'; e' \rangle$ then $\langle s\pi; e\pi \rangle \hookrightarrow \langle s'\pi; e'\pi \rangle$. Moreover, if $\langle s; e \rangle \rightarrow \langle s'; e' \rangle$ then $\langle s\pi; e\pi \rangle \rightarrow \langle s'\pi; e'\pi \rangle$.*

Proof. By nominal sets reasoning (all reduction rules are closed under permutations).

Lemma 52. *Let $\sigma = \{\vec{v}/\vec{\alpha}\}$, where \vec{v} are closed λ -abstractions with $\text{fl}(\vec{v}) \subseteq \text{dom}(s)$ and $\langle s\sigma; e\sigma \rangle \hookrightarrow \langle s'; e' \rangle$. Then one of the following holds:*

1. $e = (c\ c')$ and $e' = w\sigma$ and $s' = s\sigma$ and $c^{\text{arith}}(c') = w$;
2. $e = (\text{ref } l = u \text{ in } e')$ and $s' = (s[l \mapsto u])\sigma$ and $l \notin \text{dom}(s)$;
3. $e = !l$ and $e' = u\sigma$ and $s' = s\sigma$ and $\sigma(l) = u$;
4. $e = l := u$ and $e' = \text{tt}\sigma$ and $s' = (s[l \mapsto u])\sigma$;
5. $e = ((\text{fix } f(x).e'')\ u)$ and $e' = (e''[u/x])\sigma$ and $s' = s$;
6. $e = (\alpha\ u)$ and $e' = (e''[u/x])\sigma$ and $s' = s$ and $\sigma(\alpha) = (\text{fix } f(x).e'')$.

Proof. By case analysis on the transition.

Lemma 53. *Let $\sigma = \{\vec{v}/\vec{\alpha}\}$, where \vec{v} are closed λ -abstractions with $\text{an}(\vec{v}) \cap \{\vec{\alpha}\} = \emptyset$ and $\text{fl}(\vec{v}) \subseteq \text{dom}(s)$ and $\langle s\sigma; e\sigma \rangle \rightarrow \langle s'; e' \rangle$. Then there exists s'' such that $s' = s''\sigma$ and one of the following holds:*

1. there exists e'' such that and $e' = e''\sigma$ and $\langle s; e \rangle \rightarrow \langle s'; e' \rangle$; or
2. there exist E, u and $\alpha \in \vec{\alpha}$ such that $e = E[\alpha\ u]$ and $e' = (E[e''[u/x]])\sigma$ and $\sigma(\alpha) = \text{fix } f(x).e''$.

Proof. By definition of the transition, using Lem. 52.

E.2 LTS Lemmas

Lemma 54. *Let $\langle A; \Gamma; K; s; e \rangle \xrightarrow{\eta} \langle A'; \Gamma'; K'; s'; e' \rangle$; then*

1. if $\eta \notin \{\underline{\text{app}}(i, \alpha), \underline{\text{ret}}(\alpha) \mid \text{any } i, \alpha\}$ then $A = A'$;
2. if $\eta = \underline{\text{app}}(i, \alpha)$ or $\eta = \underline{\text{ret}}(\alpha)$ then $A \uplus \alpha = A'$;
3. if $\eta \notin \{\underline{\text{app}}(\alpha, i), \underline{\text{ret}}(i) \mid \text{any } i, \alpha\}$ then $\text{dom}(\Gamma) = \text{dom}(\Gamma')$;
4. if $\eta = \underline{\text{app}}(\alpha, i)$ or $\eta = \underline{\text{ret}}(i)$ then $\text{dom}(\Gamma) \uplus i = \text{dom}(\Gamma')$;
5. $\text{dom}(s) \subseteq \text{dom}(s')$.

Proof. By cases on the transition.

Lemma 55. *If $C \asymp C'$ then $C\pi \asymp C'\pi$. □*

Lemma 56. *Let $C_1 \asymp C_2$ and $C_1 \xrightarrow{\eta} C'_1$; then $C_2 \xrightarrow{\eta} C'_2$ and $C'_1 \asymp C'_2$, provided that $(\text{an}(C'_1) \setminus \text{an}(C_1)) \cap \text{an}(C_2) = \emptyset$ and $(\text{fl}(C'_1) \setminus \text{fl}(C_1)) \cap \text{fl}(C_2) = \emptyset$.*

Proof. By induction on the derivation of $C_1 \asymp C_2$ and case analysis on the transition from C_1 .

Corollary 57. *Let $C_1 \asymp C_2$ and $C_1 \xrightarrow{\eta} C'_1$; then $C_2 \xrightarrow{\eta} C'_2$ and $C'_1 \asymp C'_2$, provided that $(\text{an}(C'_1) \setminus \text{an}(C_1)) \cap \text{an}(C_2) = \emptyset$ and $(\text{fl}(C'_1) \setminus \text{fl}(C_1)) \cap \text{fl}(C_2) = \emptyset$.*

Proof. By induction on the length of the transition from C_1 and Lem.(s) 54 and 56.

Lemma 58. *Any transition $C \xrightarrow{\tau} C'$ is a β -move, provided $C' = \langle \perp \rangle$ implies $C = \langle \perp \rangle$.*

Proof. Case analysis on the transition relation gives us two cases: the transition is derived either by the RESPONSE or TAU rule. The former case is trivial because $C = C'$. The latter is also trivial because unique decomposition (Lem. 50) implies that transitions derived by the TAU rule can only perform that transition and the transition derived by RESPONSE. The TAU-transition satisfies the first condition of definition 39 and the RESPONSE-transition satisfies the second condition of the same definition.

Lemma 59. *Let $C_1 = \langle A; \Gamma; K; s; \hat{e} \rangle$ and $C_2 = \langle A; \Gamma; {}^i v; K; s; \hat{e} \rangle$ be well formed configurations. Then the following hold:*

1. *If $C_1 \xrightarrow{\eta} C'_1 = \langle A'; \Gamma'; K'; s'; \hat{e}' \rangle$, where $\eta \notin \{\text{app}(\alpha, i), \text{ret}(i) \mid \text{any } \alpha\}$, then*

$$C_2 \xrightarrow{\eta} \langle A'; \Gamma'; {}^i v; K'; s'; \hat{e}' \rangle.$$

2. *If $C_1 \xrightarrow{\eta} C'_1 = \langle A'; \Gamma; {}^i u; K'; s'; \hat{e}' \rangle$ where $\eta = \text{app}(\alpha, i)$ or $\eta = \text{ret}(i)$, then*

$$C_2 \xrightarrow{\eta'} \langle A'; \Gamma; {}^j u; K'; s'; \hat{e}' \rangle$$

where $\eta' = \text{app}(\alpha, j)$ or $\eta' = \text{ret}(j)$, respectively, and $j \neq i$.

3. *If $C_2 \xrightarrow{\eta} \langle A'; \Gamma'; {}^i v; K'; s'; \hat{e}' \rangle$, where $\eta \notin \{\text{app}(i, \alpha) \mid \text{any } \alpha\}$, then*

$$C_1 \xrightarrow{\eta} \langle A'; \Gamma; {}^i u; K'; s'; \hat{e}' \rangle$$

Proof. By case analysis on the transitions.

Lemma 60. *Let $C_1 = \langle A; \Gamma; K; s; \hat{e} \rangle$ and $C_1\{v/\alpha\}$ be well-formed configurations; then the following hold:*

1. *If $C_1 \xrightarrow{\tau} C'_1$ then $C_1\{v/\alpha\} \xrightarrow{\tau} C'_1\{v/\alpha\}$.*
2. *If $C_1\{v/\alpha\} \xrightarrow{\tau} C'_2$ and $\hat{e} \neq E[\alpha u]$ (for any E, u) then there exists C'_1 such that $C'_2 = C'_1\{v/\alpha\}$ and $C_1 \xrightarrow{\tau} C'_1$.*

Proof. By case analysis on the transitions.

Lemma 61. *Let $C_1 = \langle A_1; \Gamma_1; K_1; s; e \rangle \xrightarrow{\tau} \langle A'_1; \Gamma'_2; K'_1; s'; e' \rangle$. For any A_2 , Γ_2 , K_2 , and E such that the configurations $C_2 = \langle A_2; \Gamma_2; K_2; s; e \rangle$ and $C'_2 = \langle A_2; \Gamma_2; K_2; s; E[e] \rangle$ are well-formed:*

$$C_2 \xrightarrow{\tau} \langle A_2; \Gamma_2; K_2; s'; e' \rangle \quad \text{and} \quad C'_2 \xrightarrow{\tau} \langle A_2; \Gamma_2; K_2; s'; E[e'] \rangle$$

Proof. By case analysis on the transitions.

$$\begin{array}{c}
\frac{\langle A_1; \Gamma_1; E_1, K_1; s_1; e_1 \rangle \mathcal{R} \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle}{\langle A_1; \Gamma_1; K_1; s_1; E_1[e_1] \rangle \text{fde}(\mathcal{R}) \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle} \text{UTFE1} \\
\frac{\langle A_1; \Gamma_1; K_{11}, E_1, E'_1, K_{12}; s_1; \hat{e}_1 \rangle \mathcal{R} \langle A_2; \Gamma_2; K_{21}, E_2, E'_2, K_{22}; s_2; \hat{e}_2 \rangle}{|K_{11}| = |K_{21}|} \text{UTFE2} \\
\frac{\langle A_1; \Gamma_1; K_{11}, E'_1[E_1], K_{12}; s_1; \hat{e}_1 \rangle \text{fde}(\mathcal{R}) \langle A_2; \Gamma_2; K_{21}, E'_2[E_2], K_{22}; s_2; \hat{e}_2 \rangle}{\langle A_1; \Gamma_1; E_1, K_1; s_1; e_1 \rangle \mathcal{R} \langle \perp \rangle} \text{UTFD1} \quad \frac{\langle A_1; \Gamma_1; K_{11}, E_1, E'_1, K_{12}; s_1; \hat{e}_1 \rangle \mathcal{R} \langle \perp \rangle}{\langle A_1; \Gamma_1; K_{11}, E'_1[E_1], K_{12}; s_1; \hat{e}_1 \rangle \text{fde}(\mathcal{R}) \langle \perp \rangle} \text{UTFD2} \\
\hline
\frac{\langle A_1 \uplus \alpha; \Gamma_1, {}^i v_1; K_1; s_1; \hat{e}_1 \rangle \mathcal{R} \langle A_2 \uplus \alpha; \Gamma_2, {}^i v_2; K_2; s_2; \hat{e}_2 \rangle}{\kappa_1 = \{v_1/\alpha\} \quad \kappa_2 = \{v_2/\alpha\} \quad \alpha \notin \text{an}(v_1, v_2)} \text{UTFV} \\
\frac{\langle A_1 \uplus \alpha; \Gamma_1, {}^i v_1; K_1; s_1; \hat{e}_1 \rangle \mathcal{R} \langle \perp \rangle \quad \kappa_1 = \{v_1/\alpha\} \quad \alpha \notin \text{an}(v_1)}{\langle A_1; \Gamma_1 \kappa_1; K_1 \kappa_1; s_1 \kappa_1; \hat{e}_1 \kappa_1 \rangle \text{fldv}(\mathcal{R}) \langle \perp \rangle} \text{UTFD3} \\
\hline
\end{array}$$

Fig. 6. Up-to fold.

E.3 Simple Simulation Results

Lemma 62 (Equivalent Knowledge Environments). Consider $C_1 = \langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle$ and $C_2 = \langle A_2; \Gamma_2; K_2; s_2; \hat{e}_2 \rangle$ with $C_1 \cong C_2$.

1. $\hat{e}_1 = \cdot$ if and only if $\hat{e}_2 = \cdot$.
2. If there exists a trace $C_1 \xrightarrow{t\downarrow} C_2$ then $|K_1| = |K_2|$.
3. $\text{dom}(\Gamma_1) \subseteq \text{dom}(\Gamma_2)$
4. If $C_1 \xrightarrow{\eta} C'_1$ and $\eta \in \{\text{app}(\alpha, i), \text{ret}(i) \mid \text{some } i, \alpha\}$ then $\text{dom}(\Gamma_1) = \text{dom}(\Gamma_2)$.
5. If $\Gamma_1 \neq \cdot$ or $K_1 \neq \cdot$ then $A_1 \supseteq A_2$. \square

E.4 Up to Fold

Definition 63 (Concretisation). A concretisation $\{v/\alpha\}$ is defined when $\alpha \notin \text{an}(v)$, and we write $e\{v/\alpha\}$ for the expression obtained after substituting α for v in e . We let $\text{dom}(\{v/\alpha\}) = \{\alpha\}$ and $\text{rng}(\{v/\alpha\}) = \{v\}$, and let κ range over concretisations. We lift concretisation to contexts, environments and stores pointwise; we also lift it to configurations, writing $C\kappa$ to mean $\langle A \setminus \text{dom}(\kappa); \Gamma\kappa; K\kappa; s\kappa; e\kappa \rangle$, when $C = \langle A; \Gamma; K; s; e \rangle$.

Lemma 64. Let $f = \text{fde} \sqcup \text{fldv}$; then $\text{fde} \circ \text{c}_{\text{gfp}(\text{wp})} \rightsquigarrow f^\omega \circ \text{c}_{\text{gfp}(\text{wp})}$.

Proof. We need to show $\text{fde} \circ \text{c}_{\text{gfp}(\text{wp})} \circ \text{wp}(\mathcal{R}) \sqsubseteq \text{wp} \circ f^\omega \circ \text{c}_{\text{gfp}(\text{wp})}(\mathcal{R})$. Because $\text{c}_{\text{gfp}(\text{wp})}$ is the constant function mapping its argument to (\cong) , we need to show $\text{fde}(\cong) \sqsubseteq \text{wp} \circ f^\omega(\cong)$.

Let $C_1 \text{ flde}(\overline{\cong}) C_2$, and $C_1 \xrightarrow{\eta} C'_1$. If $C_1 \xrightarrow{\eta} C'_1$ is produced by rule **RESPONSE** then the proof is trivial as C_2 can perform the same transition and $C'_1 = \langle \perp \rangle f^0(\overline{\cong}) \langle \perp \rangle = C'_2$. We thus consider only non-**RESPONSE** transitions from C_1 . We proceed by case analysis on the derivation $C_1 \text{ flde}(\overline{\cong}) C_2$.

UTFE1: $C_1 = \langle A_1; \Gamma_1; K_1; s_1; E_1[e_1] \rangle$ and $C_2 = \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle$ and $C_3 = \langle A_1; \Gamma_1; E_1, K_1; s_1; e_1 \rangle$ and $C_4 = \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle$ and $C_3 \overline{\cong} C_4$. In this case C_1 is a proponent configuration thus the transition can only be produced by rules **?**, **PROPAPP**, **?**, **PROPRET**, and **TAU**.

– **?** and **PROPAPP:** Here we show only the case for the latter rule. We have $\eta = \text{app}(\alpha, i)$ and $E_1[e_1] = E[\alpha v]$ and $C'_1 = \langle A_1; \Gamma_1, {}^i v_1; E'_1; K_1; s_1; \cdot \rangle$. We have the following three cases:

- $e_1 = E'_1[\alpha v_1]$, $E = E_1[E'_1[\cdot]]$: In this case configuration C_3 can perform the same transition: $C_3 \xrightarrow{\eta} C'_3 = \langle A_1; \Gamma_1, {}^i v_1; E'_1, E_1, K; s_1; \cdot \rangle$. Because $C_3 \overline{\cong} C_4$, $C_4 \xrightarrow{\eta} C'_4 = \langle A_2; \Gamma'_2; E'_2, E_2, K; s'_2; \cdot \rangle$ or $C_4 \xrightarrow{\eta} C'_4 = \langle \perp \rangle$, and $C'_3 \overline{\cong} C'_4$. In the case where C_4 goes to $\langle \perp \rangle$, $C_2 \xrightarrow{\eta} \langle \perp \rangle$ and $C'_1 \text{ flde}(\overline{\cong}) \langle \perp \rangle$ by rule **UTFD2**. In the other case we derive $C_2 \xrightarrow{\eta} C'_2 = \langle A_2; \Gamma'_2; E_2[E'_2]; s'_2; \cdot \rangle$ and $C'_1 \text{ flde}(\mathcal{S}) C'_2$ by rule **UTFE2**.
- $e_1 = \alpha$, $E_1 = E'_1[[\cdot] v_1]$ and $C'_1 = \langle A_1; \Gamma_1, {}^i v_1; E'_1; K_1; s_1; \cdot \rangle$:
In this case configuration C_3 can perform the transitions:

$$\begin{aligned}
 C_3 &\xrightarrow{\text{ret}(j)} \langle A_1; \Gamma_1, {}^j \alpha; E'_1[[\cdot] v_1], K_1; s_1; \cdot \rangle \\
 &\xrightarrow{\text{ret}(\alpha')} \langle A_1 \uplus \alpha'; \Gamma_1, {}^j \alpha; K_1; s_1; E'_1[\alpha' v_1] \rangle \\
 &\xrightarrow{\text{app}(\alpha', j')} \langle A_1 \uplus \alpha'; \Gamma_1, {}^j \alpha, {}^{j'} v_1; E'_1, K_1; s_1; \cdot \rangle && (\alpha' \notin \text{an}(\Gamma_1, K_1, s_1, E'_1, v_1)) \\
 &\xrightarrow{\text{app}(j, \alpha'')} \langle A_1 \uplus \alpha', \alpha''; \Gamma_1, {}^j \alpha, {}^{j'} v_1; E'_1, K_1; s_1; \alpha \alpha'' \rangle && (\alpha'' \notin \text{an}(\Gamma_1, K_1, s_1, E'_1, v_1)) \\
 &\xrightarrow{\text{app}(\alpha, i)} \langle A_1 \uplus \alpha', \alpha''; \Gamma_1, {}^j \alpha, {}^{j'} v_1, {}^i \alpha''; [\cdot], E'_1, K_1; s_1; \cdot \rangle = C'_3
 \end{aligned}$$

Because $C_3 \overline{\cong} C_4$, there exists C'_4 such that

$$C_4 \xrightarrow{\text{ret}(j)} \xrightarrow{\text{ret}(\alpha')} \xrightarrow{\text{app}(\alpha', j')} \xrightarrow{\text{app}(j, \alpha'')} \xrightarrow{\text{app}(\alpha, i)} C'_4$$

and $C'_3 \overline{\cong} C'_4$. By analysis of the transitions we have two possibilities:

* $C'_4 = \langle \perp \rangle$ and in this case we have $C_2 \xrightarrow{\eta} \langle \perp \rangle$. Moreover:

$$\begin{aligned}
 C'_3 &= \langle A_1 \uplus \alpha', \alpha''; \Gamma_1, {}^j \alpha, {}^{j'} v_1, {}^i \alpha''; [\cdot], E'_1, K_1; s_1; \cdot \rangle \overline{\cong} \langle \perp \rangle \\
 &\quad \langle A_1 \uplus \alpha', \alpha''; \Gamma_1, {}^i \alpha''; [\cdot], E'_1, K_1; s_1; \cdot \rangle \{v_1/\alpha'\} \{\alpha/\alpha'\} \text{fldv}(\text{fldv}(\overline{\cong})) \langle \perp \rangle && (\text{UTFD3}) \\
 &\quad \langle A_1; \Gamma_1, {}^i v_1; [\cdot], E'_1, K_1; s_1; \cdot \rangle \text{fldv}(\text{fldv}(\overline{\cong})) \langle \perp \rangle && (\alpha', \alpha'' \notin \text{an}(\Gamma_1, K_1, s_1, E'_1, v_1)) \\
 C'_1 &= \langle A_1; \Gamma_1, {}^i v_1; E'_1[[\cdot], K_1; s_1; \cdot] \rangle \text{flde}(\text{fldv}(\text{fldv}(\overline{\cong}))) \langle \perp \rangle && (\text{UTFE2})
 \end{aligned}$$

* Otherwise we have:

$$\begin{aligned}
C_4 &= \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; w_2 \rangle \\
&\xrightarrow{\text{ret}(j)} \langle A_2; \Gamma_2, {}^j w_2; E_2, K_2; s'_2; \cdot \rangle \\
&\xrightarrow{\text{ret}(\alpha')} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j w_2; K_2; s'_2; E_2[\alpha'] \rangle \quad (\alpha' \notin \text{an}(\Gamma_2, K_2, w_2)) \\
&\xrightarrow{\tau} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j w_2; K_2; s''_2; E'_2[\alpha' w'_2] \rangle \\
&\xrightarrow{\text{app}(\alpha', j')} \langle A_1 \uplus \alpha'; \Gamma_2, {}^j w_2, {}^{j'} w'_2; E'_2, K_2; s''_2; \cdot \rangle \\
&\xrightarrow{\text{app}(j, \alpha'')} \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^j w_2, {}^{j'} w'_2; E'_2, K_2; s''_2; w_2 \alpha'' \rangle \quad (\alpha'' \notin \text{an}(w'_2)) \\
&\xrightarrow{\tau} \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^j w_2, {}^{j'} w'_2; E'_2, K_2; s''_2; E''_2[\alpha v_2] \rangle \\
&\xrightarrow{\text{app}(\alpha, i)} \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^j w_2, {}^{j'} w'_2, {}^i v'_2; E'_2, E''_2, K_2; s''_2; \cdot \rangle = C'_4
\end{aligned}$$

Using Lem. 60 and LTS rule TAU, we can derive the following transitions from C_2 :

$$\begin{aligned}
C_2 &= \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; K_2; s'_2; E_2[w_2] \rangle \\
&= \langle A_2 \uplus \alpha'; \Gamma_2; K_2; s'_2; E_2[\alpha'] \rangle \{w_2/\alpha'\} \quad (\alpha' \notin \text{an}(w_2)) \\
&\xrightarrow{\tau} \langle A_2 \uplus \alpha'; \Gamma_2; K_2; s''_2; E'_2[\alpha' w'_2] \rangle \{w_2/\alpha'\} \\
&= \langle A_1 \uplus \alpha', \alpha''; \Gamma_2; K_2; s''_2; E'_2[w_2 \alpha''] \rangle \{w'_2/\alpha''\} \{w_2/\alpha'\} \quad (\alpha'' \notin \text{an}(w'_2)) \\
&\xrightarrow{\tau} \langle A_1 \uplus \alpha', \alpha''; \Gamma_2; K_2; s''_2; E'_2[E''_2[\alpha v_2]] \rangle \{w'_2/\alpha''\} \{w_2/\alpha'\} \\
&\xrightarrow{\text{app}(\alpha, i)} \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^i v_2; E'_2[E''_2], K_2; s''_2; \cdot \rangle \{w'_2/\alpha''\} \{w_2/\alpha'\} = C'_2
\end{aligned}$$

We also have

$$\begin{aligned}
C'_3 &= \langle A_1 \uplus \alpha', \alpha''; \Gamma_1, {}^j \alpha, {}^{j'} v_1, {}^i \alpha''; [\cdot], E'_1, K_1; s_1; \cdot \rangle \\
&\xrightarrow{\cong} \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^j w_2, {}^{j'} w'_2, {}^i v'_2; E''_2, E'_2, K_2; s''_2; \cdot \rangle = C'_4
\end{aligned}$$

By rule UTFv

$$\begin{aligned}
&\langle A_1 \uplus \alpha', \alpha''; \Gamma_1, {}^i \alpha''; [\cdot], E'_1, K_1; s_1; \cdot \rangle \{v_1/\alpha''\} \{\alpha/\alpha'\} \\
&\text{fldv}(\text{fldv}(\cong)) \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^i v'_2; E''_2, E'_2, K_2; s''_2; \cdot \rangle \{w'_2/\alpha''\} \{w_2/\alpha'\}
\end{aligned}$$

Because $\alpha', \alpha'' \notin \text{an}(\Gamma_1, K_1, s_1, E'_1, v_1)$

$$\begin{aligned}
&\langle A_1; \Gamma_1, {}^i v_1; [\cdot], E'_1, K_1; s_1; \cdot \rangle \\
&\text{fldv}(\text{fldv}(\cong)) \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^i v'_2; E''_2, E'_2, K_2; s''_2; \cdot \rangle \{w'_2/\alpha''\} \{w_2/\alpha'\}
\end{aligned}$$

By rule UTFE2

$$\begin{aligned}
&\langle A_1; \Gamma_1, {}^i v_1; E'_1[\cdot], K_1; s_1; \cdot \rangle \\
&\text{flde}(\text{fldv}(\text{fldv}(\cong))) \langle A_1 \uplus \alpha', \alpha''; \Gamma_2, {}^i v'_2; E''_2[E''_2], K_2; s''_2; \cdot \rangle \{w'_2/\alpha''\} \{w_2/\alpha'\}
\end{aligned}$$

Therefore $C'_1 \text{ flde}(\text{fldv}(\text{fldv}(\cong))) C'_2$ as required.

- $e_1 = v_1$, $E_1 = E'_1[\alpha[\cdot]]$ and $C'_1 = \langle A_1; \Gamma_1, {}^i v_1; E'_1; K_1; s_1; \cdot \rangle$: In this case configuration C_3 can perform the transitions:

$$\begin{aligned}
 C_3 &\xrightarrow{\text{ret}(j)} \langle A_1; \Gamma_1, {}^j v; E'_1[\alpha[\cdot]], K_1; s_1; \cdot \rangle \\
 &\xrightarrow{\text{ret}(\alpha')} \langle A_1 \uplus \alpha'; \Gamma_1, {}^j v; K_1; s_1; E'_1[\alpha \alpha'] \rangle \quad (\alpha' \notin \text{an}(\Gamma_1, v_1, K_1, s_1, E'_1)) \\
 &\xrightarrow{\text{app}(\alpha, i)} \langle A_1 \uplus \alpha'; \Gamma_1, {}^j v, {}^i \alpha'; E'_1, K_1; s_1; \cdot \rangle = C'_3
 \end{aligned}$$

Because $C_3 \sqsubseteq C_4$, there exists C'_4 such that $C_4 \xrightarrow{\text{ret}(j)} \xrightarrow{\text{ret}(\alpha')} \xrightarrow{\text{app}(\alpha, i)} C'_4$ and $C'_3 \sqsubseteq C'_4$. By analysis of the transitions we have

$$\begin{aligned}
 C_4 &= \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; v_2 \rangle \\
 &\xrightarrow{\text{ret}(j)} \langle A_2; \Gamma_2, {}^j v_2; E_2, K_2; s'_2; \cdot \rangle \\
 &\xrightarrow{\text{ret}(\alpha')} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j v_2; K_2; s'_2; E_2[\alpha'] \rangle \quad (\alpha' \notin \text{an}(\Gamma_2, v_2, K_2, s'_2, E'_2)) \\
 &\xrightarrow{\tau} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j v_2; K_2; s''_2; E'_2[\alpha w_2] \rangle \\
 &\xrightarrow{\text{app}(\alpha, i)} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j v_2, {}^i w_2; E'_2, K_2; s''_2; \cdot \rangle = C'_4
 \end{aligned}$$

Using Lem. 60 and LTS rule TAU, we can derive the following transitions from C_2 :

$$\begin{aligned}
 C_2 &= \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; K_2; s'_2; E_2[v_2] \rangle \\
 &= \langle A_2 \uplus \alpha'; \Gamma_2; K_2; s'_2; E_2[\alpha'] \rangle \{v_2/\alpha'\} \xrightarrow{\tau} \langle A_2 \uplus \alpha'; \Gamma_2; K_2; s''_2; E'_2[\alpha w_2] \rangle \{v_2/\alpha'\} \\
 &\xrightarrow{\text{app}(\alpha, i)} \langle A_2 \uplus \alpha'; \Gamma_2, {}^i w_2; E'_2, K_2; s''_2; \cdot \rangle \{v_2/\alpha'\} = C'_2
 \end{aligned}$$

We also have

$$C'_3 = \langle A_1 \uplus \alpha'; \Gamma_1, {}^j v, {}^i \alpha'; E'_1, K_1; s_1; \cdot \rangle \sqsubseteq \langle A_2 \uplus \alpha'; \Gamma_2, {}^j v_2, {}^i w_2; E'_2, K_2; s''_2; \cdot \rangle = C'_4$$

And by rule UTFv:

$$\begin{aligned}
 C'_1 &= \langle A_1; \Gamma_1, {}^i v_1; E'_1, K_1; s_1; \cdot \rangle \\
 &= \langle A_1 \uplus \alpha'; \Gamma_1, {}^i \alpha'; E'_1, K_1; s_1; \cdot \rangle \{v_1/\alpha'\} \text{fldv}(\sqsubseteq) \langle A_2 \uplus \alpha'; \Gamma_2, {}^i w_2; E'_2, K_2; s''_2; \cdot \rangle \{v_2/\alpha'\} = C'_2
 \end{aligned}$$

- ? and PROPRET: here we show only the case for the latter rule. We have $\eta = \text{ret}(i)$ and $E_1 = [\cdot]$ and $C'_1 = \langle A_1; \Gamma_1, {}^i v_1; K_1; s_1; \cdot \rangle$. Configuration C_3 can perform the transitions:

$$\begin{aligned}
 C_3 &\xrightarrow{\text{ret}(j)} \langle A_1; \Gamma_1, {}^j v_1; [\cdot], K_1; s_1; \cdot \rangle \\
 &\xrightarrow{\text{ret}(\alpha')} \langle A_1 \uplus \alpha'; \Gamma_1, {}^j v_1; K_1; s_1; [\alpha'] \rangle \quad (\alpha' \notin \text{an}(\Gamma_1, v_1, K_1, s_1)) \\
 &\xrightarrow{\text{ret}(i)} \langle A_1 \uplus \alpha'; \Gamma_1, {}^j v_1, {}^i \alpha'; K_1; s_1; \cdot \rangle = C'_3
 \end{aligned}$$

Because $C_3 \sqsubseteq C_4$, there exists C'_4 such that $C_4 \xrightarrow{\text{ret}(j)} \xrightarrow{\text{ret}(\alpha')} \xrightarrow{\text{ret}(i)} C'_4$ and $C'_3 \sqsubseteq C'_4$. By analysis of the transitions we have

$$\begin{aligned}
C_4 &= \langle A_2; \Gamma_2; E_2, K_1; s_1; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_1; s'_1; v_2 \rangle \\
&\xrightarrow{\text{ret}(j)} \langle A_2; \Gamma_2, {}^j v_2; E_2, K_2; s'_2; \cdot \rangle \\
&\xrightarrow{\text{ret}(\alpha')} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j v_2; K_2; s'_2; E_2[\alpha'] \rangle \quad (\alpha' \notin \text{an}(\Gamma_2, v_2, K_2, s'_2, E_2)) \\
&\xrightarrow{\tau} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j v_2; K_2; s''_2; w_2 \rangle \\
&\xrightarrow{\text{ret}(i)} \langle A_2 \uplus \alpha'; \Gamma_2, {}^j v_2, {}^i w_2; K_2; s''_2; \cdot \rangle = C'_4
\end{aligned}$$

Using Lem. 60 and LTS rule TAU, we can derive the following transitions from C_2 :

$$\begin{aligned}
C_2 &= \langle A_2; \Gamma_2; K_1; s_1; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; K_1; s'_1; E_2[v_2] \rangle \\
&= \langle A_2 \uplus \alpha'; \Gamma_2; K_2; s'_2; E_2[\alpha'] \rangle \{v_2/\alpha'\} \xrightarrow{\tau} \langle A_2 \uplus \alpha'; \Gamma_2; K_2; s''_2; w_2 \rangle \{v_2/\alpha'\} \\
&\xrightarrow{\text{ret}(i)} \langle A_2 \uplus \alpha'; \Gamma_2, {}^i w_2; K_2; s''_2; \cdot \rangle \{v_2/\alpha'\} = C'_2
\end{aligned}$$

Moreover,

$$\begin{aligned}
C'_1 &= \langle A_1; \Gamma_1, {}^i v_1; K_1; s_1; \cdot \rangle \\
&= \langle A_1 \uplus \alpha'; \Gamma_1, {}^i \alpha'; K_1; s_1; \cdot \rangle \{v_1/\alpha'\} \text{fldv}(\sqsubseteq) \langle A_2 \uplus \alpha'; \Gamma_2, {}^i w_2; K_2; s''_2; \cdot \rangle \{v_2/\alpha'\} = C'_2
\end{aligned}$$

- TAU: We have $\eta = \tau$ and $C'_1 = \langle A_1; \Gamma_1; K_1; s'_1; e'_1 \rangle$ and $\langle s_1; E_1[e_1] \rangle \rightarrow \langle s'_1; e'_1 \rangle$. By the reduction rule we get $E_1[e_1] = E[e]$ and $\langle s_1; e \rangle \hookrightarrow \langle s'_1; e' \rangle$ and $e'_1 = E[e']$, for some E, e, e' .

We proceed by cases on $E_1[e_1] = E[e]$:

- $e_1 = E_{e_1}[e]$ and $E = E_1[E_{e_1}[\cdot]]$ and $e'_1 = E_1[E_{e_1}[e']]$. In this case C_3 can perform the same transition:

$$C_3 = \langle A_1; \Gamma_1; E_1, K_1; s_1; E_{e_1}[e] \rangle \xrightarrow{\tau} \langle A_1; \Gamma_1; E_1, K_1; s'_1; E_{e_1}[e'] \rangle = C'_3$$

Because $C_3 \sqsubseteq C_4$,

$$C_4 = \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; e'_2 \rangle = C'_4$$

and $C'_3 \sqsubseteq C'_4$. Therefore

$$C_2 = \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; K_2; s'_2; E_2[e'_2] \rangle = C'_2$$

and from rule UTFE1, $C'_1 \text{flde}(\sqsubseteq) C'_2$.

- $e_1 = v_1$ and $E_1 = E[F]$ and $\langle s_1; F[v_1] \rangle \hookrightarrow \langle s'_1; e' \rangle$, where F is one of the following contexts: $(op(\vec{c}_{11}, [\cdot], \vec{c}_{12}))$ or $([\cdot]v)$ or $(v[\cdot])$ or $(l := [\cdot])$ or $(\text{if}[\cdot] \text{ then } e_{11} \text{ else } e_{12})$. We proceed by cases on F :

- * $F = op(\vec{c}_{11}, [\cdot], \vec{c}_{12})$: Here it must be $v_1 = c$ and $e' = c' = op(\vec{c}_{11}, c, \vec{c}_{12})$. Thus C_3 can perform the transitions

$$\begin{aligned} C_3 &= \langle A_1; \Gamma_1; E[F], K_1; s_1; c \rangle \xrightarrow{\text{ret}(c)} \langle A_2; \Gamma_1; E[F], K_1; s_1; \cdot \rangle \\ &\xrightarrow{\text{ret}(c)} \langle A_2; \Gamma_1; K_1; s_1; E[F[c]] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_1; K_1; s_1; E[c'] \rangle = C'_3 = C'_1 \end{aligned}$$

Because $C_3 \sqsubseteq C_4$,

$$\begin{aligned} C_4 &= \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; c \rangle \xrightarrow{\text{ret}(c)} \langle A_2; \Gamma_2; E_2, K_2; s'_2; \cdot \rangle \\ &\xrightarrow{\text{ret}(c)} \langle A_2; \Gamma_2; K_2; s'_2; E_2[c] \rangle \xrightarrow{\tau} C'_4 \end{aligned}$$

and $C'_3 \sqsubseteq C'_4$. Moreover we derive the transitions:

$$C_2 = \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; K_2; s'_2; E_2[c] \rangle \xrightarrow{\tau} C'_4 = C'_2$$

We also derive $C'_1 \xrightarrow{f^0(\sqsubseteq)} C'_2$ as needed.

- * $F = [\cdot]v$: Here it must be $v_1 = \text{fix}f(x).e_{11}$ and $e' = e_{11}[v/x]$. By the LTS and $C_3 \sqsubseteq C'_3$ we have:

$$\begin{aligned} C_3 &= \langle A_1; \Gamma_1; E[F], K_1; s_1; \text{fix}f(x).e_{11} \rangle \\ &\xrightarrow{\text{ret}(i)} \langle A_2; \Gamma_1; {}^i\text{fix}f(x).e_{11}; E[F], K_1; s_1; \cdot \rangle \\ &\xrightarrow{\text{ret}(\alpha)} \langle A_1 \uplus \alpha; \Gamma_1; {}^i\text{fix}f(x).e_{11}; K_1; s_1; E[\alpha v] \rangle \quad (\alpha \notin \text{an}(A_1, \Gamma_1, e_{11}, K_1, s_1, E)) \\ &\xrightarrow{\text{app}(\alpha, j)} \langle A_1 \uplus \alpha; \Gamma_1; {}^i\text{fix}f(x).e_{11}, {}^jv; E, K_1; s_1; \cdot \rangle \\ &\xrightarrow{\text{app}(i, \alpha')} \langle A_1 \uplus \alpha; \Gamma_1; {}^i\text{fix}f(x).e_{11}, {}^jv; E, K_1; s_1; e_{11}[\alpha'/x] \rangle = C'_3 \quad (\alpha' \notin \text{an}(A_1, \Gamma_1, e_{11}, K_1, s_1, E)) \end{aligned}$$

$$\begin{aligned} C_4 &= \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; v_2 \rangle \\ &\xrightarrow{\text{ret}(i)} \langle A_2; \Gamma_2; {}^i v_2; E_2, K_2; s'_2; \cdot \rangle \\ &\xrightarrow{\text{ret}(\alpha)} \langle A_2 \uplus \alpha; \Gamma_2; {}^i v_2; K_2; s'_2; E_2[\alpha] \rangle \quad (\alpha \notin \text{an}(A_2, \Gamma_2, v_2, K_2, s'_2, E_2)) \\ &\xrightarrow{\tau} \langle A_2 \uplus \alpha; \Gamma_2; {}^i v_2; K_2; s''_2; E'_2[\alpha v'_2] \rangle \\ &\xrightarrow{\text{app}(\alpha, j)} \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; {}^i v_2, {}^j v'_2; E'_2, K_2; s''_2; \cdot \rangle \\ &\xrightarrow{\text{app}(i, \alpha')} \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; {}^i v_2, {}^j v'_2; E'_2, K_2; s''_2; e'_2 \rangle \quad (\alpha' \notin \text{an}(A_2, \Gamma_2, v_2, v'_2, E'_2, K_2, s''_2)) \\ &= \begin{cases} \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; {}^i v_2, {}^j v'_2; E'_2, K_2; s''_2; \alpha'' \alpha' \rangle & \text{if } v_2 = \alpha'' \\ \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; {}^i v_2, {}^j v'_2; E'_2, K_2; s''_2; e'_2[\alpha'/x] \rangle & \text{if } v_2 = \text{fix}f(x).e'_2 \end{cases} \\ &\xrightarrow{\tau} \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; {}^i v_2, {}^j v'_2; E'_2, K_2; s'''_2; e''_2 \rangle = C'_4 \quad C'_3 \sqsubseteq C'_4 \end{aligned}$$

$$\begin{aligned}
C_2 &= \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; E_2[v_2] \rangle \\
&= \langle A_2 \uplus \alpha; \Gamma_2; K_2; s'_2; E_2[\alpha] \rangle \{v_2/\alpha\} \xrightarrow{\tau} \langle A_2 \uplus \alpha; \Gamma_2; K_2; s''_2; E'_2[\alpha v'_2] \rangle \{v_2/\alpha\} \\
&= \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; K_2; s''_2; E'_2[\alpha \alpha'] \rangle \{v'_2/\alpha'\} \{v_2/\alpha\} \\
&\begin{cases} = \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; K_2; s''_2; E'_2[\alpha'' \alpha'] \rangle \{v'_2/\alpha'\} \{v_2/\alpha\} & \text{if } v_2 = \alpha'' \\ \xrightarrow{\tau} \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; K_2; s''_2; E'_2[e'_2[\alpha'/x]] \rangle & \text{if } v_2 = \text{fix}f(x).e'_2 \end{cases} \\
&\xrightarrow{\tau} \langle A_2 \uplus \alpha, \alpha'; \Gamma_2; K_2; s''_2; E'_2[e'_2] \rangle \{v'_2/\alpha'\} \{v_2/\alpha\} = C'_2
\end{aligned}$$

Moreover: $C'_1 \text{ fldv}(\underline{\cong}) C'_2$, as required.

* $F = v[\cdot]$: Here it must be $v = \text{fix}f(x).e_{11}$ and $e' = e_{11}[v_1/x]$. By the LTS and $C_3 \underline{\cong} C'_3$ we have:

$$\begin{aligned}
C_3 &= \langle A_1; \Gamma_1; E[F], K_1; s_1; v_1 \rangle \\
&\xrightarrow{\text{ret}(i)} \langle A_2; \Gamma_1, {}^i v_1; E[F], K_1; s_1; \cdot \rangle \\
&\xrightarrow{\text{ret}(\alpha)} \langle A_1 \uplus \alpha; \Gamma_1, {}^i v_1; K_1; s_1; E[(\text{fix}f(x).e_{11}) \alpha] \rangle \\
&\xrightarrow{\tau} \langle A_1 \uplus \alpha; \Gamma_1, {}^i v_1; K_1; s_1; E[e_{11}[\alpha/x]] \rangle = C'_3 \quad (\alpha \notin \text{an}(A_1, \Gamma_1, v_1, K_1, s_1, E, v_1))
\end{aligned}$$

$$\begin{aligned}
C_4 &= \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; v_2 \rangle \\
&\xrightarrow{\text{ret}(i)} \langle A_2; \Gamma_2, {}^i v_2; E_2, K_2; s'_2; \cdot \rangle \\
&\xrightarrow{\text{ret}(\alpha)} \langle A_2 \uplus \alpha; \Gamma_2, {}^i v_2; K_2; s'_2; E_2[\alpha] \rangle \quad (\alpha \notin \text{an}(A_2, \Gamma_2, v_2, K_2, s'_2, E_2)) \\
&\xrightarrow{\tau} \langle A_2 \uplus \alpha; \Gamma_2, {}^i v_2; K_2; s''_2; e'_2 \rangle = C'_4 \quad C'_3 \underline{\cong} C'_4
\end{aligned}$$

$$\begin{aligned}
C_2 &= \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; E_2[v_2] \rangle \\
&= \langle A_2 \uplus \alpha; \Gamma_2; K_2; s'_2; E_2[\alpha] \rangle \{v_2/\alpha\} \xrightarrow{\tau} \langle A_2 \uplus \alpha; \Gamma_2; K_2; s''_2; e'_2 \rangle \{v_2/\alpha\} = C'_2
\end{aligned}$$

Moreover, $C'_1 \text{ fldv}(\underline{\cong}) C'_2$.

* $F = l := [\cdot]$: Here $e' = \text{tt}$ and $s'_1 = s_1[l \mapsto v_1]$. By the LTS and $C_3 \underline{\cong} C'_3$ we have:

$$\begin{aligned}
C_3 &= \langle A_1; \Gamma_1; E[F], K_1; s_1; v_1 \rangle \\
&\xrightarrow{\text{ret}(i)} \langle A_2; \Gamma_1, {}^i v_1; E[F], K_1; s_1; \cdot \rangle \\
&\xrightarrow{\text{ret}(\alpha)} \langle A_1 \uplus \alpha; \Gamma_1, {}^i v_1; K_1; s_1; E[l := \alpha] \rangle \\
&\xrightarrow{\tau} \langle A_1 \uplus \alpha; \Gamma_1, {}^i v_1; K_1; s_1[l \mapsto \alpha]; E[\text{tt}] \rangle = C'_3 \quad (\alpha \notin \text{an}(A_1, \Gamma_1, v_1, K_1, s_1, E, v_1))
\end{aligned}$$

$$\begin{aligned}
C_4 &= \langle A_2; \Gamma_2; E_2, K_2; s_2; e_2 \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; v_2 \rangle \\
&\xrightarrow{\text{ret}(i)} \langle A_2; \Gamma_2, {}^i v_2; E_2, K_2; s'_2; \cdot \rangle \\
&\xrightarrow{\text{ret}(\alpha)} \langle A_2 \uplus \alpha; \Gamma_2, {}^i v_2; K_2; s'_2; E_2[\alpha] \rangle \quad (\alpha \notin \text{an}(A_2, \Gamma_2, v_2, K_2, s'_2, E_2)) \\
&\xrightarrow{\tau} \langle A_2 \uplus \alpha; \Gamma_2, {}^i v_2; K_2; s''_2; e'_2 \rangle = C'_4 \quad C'_3 \sqsubseteq C'_4
\end{aligned}$$

$$\begin{aligned}
C_2 &= \langle A_2; \Gamma_2; K_2; s_2; E_2[e_2] \rangle \xrightarrow{\tau} \langle A_2; \Gamma_2; E_2, K_2; s'_2; E_2[v_2] \rangle \\
&= \langle A_2 \uplus \alpha; \Gamma_2; K_2; s'_2; E_2[\alpha] \rangle \{v_2/\alpha\} \xrightarrow{\tau} \langle A_2 \uplus \alpha; \Gamma_2; K_2; s''_2; e'_2 \rangle \{v_2/\alpha\} = C'_2
\end{aligned}$$

Moreover, $C'_1 \text{ fldv}(\sqsubseteq) C'_2$.

- * $F = \text{if}[\cdot] \text{ then } e_{11} \text{ else } e_{12}$: Here $v_1 = \text{tt}$ or $v_1 = \text{ff}$, and $e' = e_{11}$ or $e' = e_{12}$, respectively. In both sub-cases, the proof proceeds as in the case where $F = \text{op}(\vec{c}_{11}, [\cdot], \vec{c}_{12})$.

UTFE2: In this case we have: $C_1 = \langle A_1; \Gamma_1; K_{11}, E'_1[E_1], K_{12}; s_1; \hat{e}_1 \rangle$ and $C_2 = \langle A_2; \Gamma_2; K_{21}, E'_2[E_2], K_{22}; s_2; \hat{e}_2 \rangle$ and $C_3 = \langle A_1; \Gamma_1; K_{11}, E_1, E'_1, K_{12}; s_1; \hat{e}_1 \rangle$ and $C_4 = \langle A_2; \Gamma_2; K_{21}, E_2, E'_2, K_{22}; s_2; \hat{e}_2 \rangle$ and $|K_{11}| = |K_{21}|$ and $C_3 \sqsubseteq C_4$. When $|K_{11}| > 0$, because C_1 (C_3) and C_2 (resp. C_4) have the same expressions, the proof requires a simple simulation diagram chasing, with the resulting configurations related in $\text{flde}(\sqsubseteq)$ via rule UTFE2. Similarly when $\eta \notin \{\text{ret}(c), \text{ret}(\alpha) \mid \text{any } c, \alpha\}$. When $\eta \in \{\text{ret}(c), \text{ret}(\alpha) \mid \text{any } c, \alpha\}$, the simulation diagram is similar, but completed by relating the resulting configurations in $\text{flde}(\sqsubseteq)$ via rule UTFE1. Note that a \downarrow -transition is not possible from C_1 .

UTFD1, UTFD2: The proof in these cases proceeds as the corresponding cases above, with the simplification that the right-hand side configurations (C_2 and C_4) perform the required transitions via the LTS rule RESPONSE.

Lemma 65. *Let $f = \text{flde} \sqcup \text{fldv}$ and $g = \text{perm}$; then $\text{fldv} \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \rightsquigarrow (f \sqcup g)^\omega \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})}$.*

Proof. We need to show $\text{fldv} \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \circ \mathbf{wp}(\mathcal{R}) \sqsubseteq \mathbf{wp} \circ (f \sqcup g)^\omega \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})}(\mathcal{R})$. Because $\mathbf{c}_{\text{gfp}(\mathbf{wp})}$ is the constant function mapping its argument to (\sqsubseteq) , we need to show $\text{fldv}(\sqsubseteq) \sqsubseteq \mathbf{wp} \circ (f \sqcup g)^\omega(\sqsubseteq)$.

Let $C_1 \text{ fldv}(\sqsubseteq) C_2$, and $C_1 \xrightarrow{\eta} C'_1$. If $C_1 \xrightarrow{\eta} C'_1$ is produced by rule RESPONSE then the proof is trivial as C_2 can perform the same transition and $C'_1 = \langle \perp \rangle \text{id}(\sqsubseteq) \langle \perp \rangle = C'_2$. We thus consider only non-RESPONSE transitions from C_1 . By case analysis, the derivation $C_1 \text{ fldv}(\sqsubseteq) C_2$ can be produced by the UTFV or the UTFD3 rules. We show the former; the proof of the latter is similar but simpler.

UTFV: Here we have $C_1 = C_3 \kappa_1$ and $C_2 = C_4 \kappa_2$ and $C_3 = \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \hat{e}_3 \rangle$ and $C_4 = \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \hat{e}_4 \rangle$ and $C_3 \sqsubseteq C_4$ and $\alpha \notin \text{an}(v_3, v_4)$ and $\kappa_1 = \{v_3/\alpha\}$ and $\kappa_2 = \{v_4/\alpha\}$. We proceed by cases on the transition from C_1 . Cases ?, PROPRET, ?, OPRET, and TERM are straightforward and are proved using Lem. 60.

- ?, PROPAPP: Both cases are similar; here we show the latter. We have $\eta = \text{app}(\alpha', j)$ and $\alpha \in A_3$ ($\alpha \neq \alpha'$) and $\hat{e}_3\kappa_1 = E_1[\alpha' u_1]$ and

$$C_1 = \langle A_3; \Gamma_3\kappa_1; K_3\kappa_1; s_3\kappa_1; E_1[\alpha' u_1] \rangle \xrightarrow{\text{app}(\alpha', j)} \langle A_3; \Gamma_3\kappa_1, j u_1; E_1, K_3\kappa_1; s_3\kappa_1; \cdot \rangle = C'_1$$

We choose $j' \notin \text{dom}(\Gamma_3) \uplus i$ and create the permutation $\pi = (j \leftrightarrow j')$. We derive

$$C_1 = \langle A_3; \Gamma_3\kappa_1; K_3\kappa_1; s_3\kappa_1; E_1[\alpha' u_1] \rangle \xrightarrow{\text{app}(\alpha', j')} \langle A_3; \Gamma_3\kappa_1, j' u_1; E_1, K_3\kappa_1; s_3\kappa_1; \cdot \rangle = C'_1\pi$$

We consider cases on $\hat{e}_3\kappa_1 = E_1[\alpha' u_1]$:

- $\hat{e}_3 = E_3[\alpha' u_3]$ and $E_1 = E_3\kappa_1$ and $u_1 = u_3\kappa_1$. In this case we have

$$C_3 = \langle A_3 \uplus \alpha; \Gamma_3, i v_3; K_3; s_3; E_3[\alpha' u_3] \rangle \xrightarrow{\text{app}(\alpha', j')} \langle A_3 \uplus \alpha; \Gamma_3, i v_3, j' u_3; E_3, K_3; s_3; \cdot \rangle = C'_3$$

By $C_3 \sqsubseteq C_4$ we have $C_4 \xrightarrow{\text{app}(\alpha', j')} C_4$ and $C'_3 \sqsubseteq C'_4$.

We have two cases. The first is when $C'_4 = \langle \perp \rangle$. This case is straightforward using rule UTFD3 and relating the resulting configurations in $\text{perm}(\text{fldv}(\sqsubseteq))$. The other case is as follows.

$$\begin{aligned} C_4 &= \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s_4; \hat{e}_4 \rangle \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s'_4; E_4[\alpha' u_4] \rangle \\ &\xrightarrow{\text{app}(\alpha', j')} \langle A_4 \uplus \alpha; \Gamma_4, i v_4, j' u_4; E_4, K_4; s'_4; \cdot \rangle = C'_4 \quad \text{and } C'_3 \sqsubseteq C'_4 \end{aligned}$$

$$C_2 = \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s_4; \hat{e}_4 \rangle \kappa_2 \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s'_4; E_4[\alpha' u_4] \rangle \kappa_2 \quad (60)$$

$$\begin{aligned} &\xrightarrow{\text{app}(\alpha', j')} \langle A_4 \uplus \alpha; \Gamma_4, i v_4, j' u_4; E_4, K_4; s'_4; \cdot \rangle \kappa_2 \\ C_2 &\xrightarrow{\text{app}(\alpha', j)} C'_4 \kappa_2 \pi = C'_2 \end{aligned} \quad (41)$$

Moreover, $C'_1 \text{ perm}(\text{fldv}(\sqsubseteq)) C'_2$, as required.

- $\hat{e}_3 = E_3[\alpha u_3]$ and $E_1 = E_3\kappa_1$ and $u_1 = u_3\kappa_1$ and $\kappa_1 = \{\alpha'/\alpha\}$ and $v_3 = \alpha$. In this case we have

$$C_3 = \langle A_3 \uplus \alpha; \Gamma_3, i v_3; K_3; s_3; E_3[\alpha u_3] \rangle \xrightarrow{\text{app}(\alpha, j')} \langle A_3 \uplus \alpha; \Gamma_3, i v_3, j' u_3; E_3, K_3; s_3; \cdot \rangle = C'_3$$

By $C_3 \sqsubseteq C_4$ we have $C_4 \xrightarrow{\text{app}(\alpha, j')} C_4$ and $C'_3 \sqsubseteq C'_4$.

We have two cases. The first is when $C'_4 = \langle \perp \rangle$. This case is straightforward using rule UTFD3 and relating the resulting configurations in $\text{perm}(\text{fldv}(\sqsubseteq))$. The other case is as follows.

$$\begin{aligned} C_4 &= \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s_4; \hat{e}_4 \rangle \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s'_4; E_4[\alpha u_4] \rangle \\ &\xrightarrow{\text{app}(\alpha, j')} \langle A_4 \uplus \alpha; \Gamma_4, i v_4, j' u_4; E_4, K_4; s'_4; \cdot \rangle = C'_4 \quad \text{and } C'_3 \sqsubseteq C'_4 \end{aligned}$$

$$C_2 = \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s_4; \hat{e}_4 \rangle \kappa_2 \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, i v_4; K_4; s'_4; E_4[\alpha u_4] \rangle \kappa_2 \quad (60)$$

$$\begin{aligned} &\xrightarrow{\text{app}(\alpha', j')} \langle A_4 \uplus \alpha; \Gamma_4, i v_4, j' u_4; E_4, K_4; s'_4; \cdot \rangle \kappa_2 \\ C_2 &\xrightarrow{\text{app}(\alpha', j)} C'_4 \kappa_2 \pi = C'_2 \end{aligned} \quad (41)$$

Moreover, $C'_1 \text{ perm}(\text{fldv}(\underline{\cong})) C'_2$, as required.

- $?$, OPAPP : Both cases are similar; here we show the latter. We have $\eta = \underline{\text{app}}(i', \alpha')$ and $\alpha' \notin A_3$ and $\hat{e}_3 = \cdot$ and $\Gamma_3(i') = u_3$ and $u_3 \kappa_1 \alpha' \succ e'_1$ and

$$C_1 = \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \cdot \rangle \kappa_1 \xrightarrow{\underline{\text{app}}(i', \alpha')} \langle A_3 \uplus \alpha'; \Gamma_3 \kappa_1; K_3 \kappa_1; s_3 \kappa_1; e'_1 \rangle = C'_1$$

We choose $\alpha'' \notin A \uplus \alpha$ and create the permutation $\pi = (\alpha' \leftrightarrow \alpha'')$. We derive:

$$C_1 = \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \cdot \rangle \kappa_1 \xrightarrow{\underline{\text{app}}(i', \alpha'')} \langle A_3 \uplus \alpha''; \Gamma_3 \kappa_1; K_3 \kappa_1; s_3 \kappa_1; e'_1 \rangle = C'_1 \pi$$

with $u_3 \kappa_1 \alpha'' \succ e'_1$. We consider cases on u_3 and v_3 :

- $u_3 = \alpha$ and $v_3 = \text{fix}f(x).e'_3$: we have $e'_1 = e'_3[\alpha''/x] = e'_3[\alpha''/x] \kappa_1$ ($\alpha \notin \text{an}(v_3) \cup \alpha''$).

$$C_1 = \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \cdot \rangle \kappa_1 \xrightarrow{\underline{\text{app}}(i', \alpha'')} \langle A_3 \uplus \alpha, \alpha''; \Gamma_3, {}^i v_3; K_3; s_3; e'_3[\alpha''/x] \rangle \kappa_1 = C'_1 \pi$$

$$\begin{aligned} C_3 &= \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \cdot \rangle \xrightarrow{\underline{\text{app}}(i', \alpha'')} \langle A_3 \uplus \alpha, \alpha''; \Gamma_3, {}^i v_3; K_3; s_3; \alpha \alpha'' \rangle \\ &\xrightarrow{\underline{\text{app}}(\alpha, j)} \langle A_3 \uplus \alpha, \alpha''; \Gamma_3, {}^i v_3, {}^j \alpha''; [\cdot], K_3; s_3; \cdot \rangle \\ &\xrightarrow{\underline{\text{app}}(i, \alpha''')} \langle A_3 \uplus \alpha, \alpha'', \alpha'''; \Gamma_3, {}^i v_3, {}^j \alpha''; [\cdot], K_3; s_3; e'_3[\alpha'''/x] \rangle = C'_3 \end{aligned}$$

By $C_3 \underline{\cong} C_4$ we have two cases. The first is when C_4 weakly matches these transitions and becomes $C'_4 = \langle \perp \rangle$, due to an application of the RESPONSE rule. This case is proved using rule UTFD3 and relating the resulting configurations in $\text{perm}(\text{fde}(\text{fldv}(\text{fldv}(\underline{\cong}))))$. The other case is as follows.

By Lem. 62, $\hat{e}_4 = \cdot$ and $\Gamma_4(i') = u_4$ and we proceed by cases on u_4 and v_4 :

- * $u_4 = \alpha_4 \neq \alpha$: This is not possible because then C_4 would not be able to match the second transition from C_3 .
- * $u_4 = \alpha$ and $v_4 = \text{fix}f(x).e'_4$: we have:

$$\begin{aligned} C_4 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \cdot \rangle \xrightarrow{\underline{\text{app}}(i', \alpha'')} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \alpha \alpha'' \rangle \\ &\xrightarrow{\underline{\text{app}}(\alpha, j)} \langle A_4 \uplus \alpha, \alpha''; \Gamma_4, {}^i v_4, {}^j \alpha''; [\cdot], K_4; s_4; \cdot \rangle \\ &\xrightarrow{\underline{\text{app}}(i, \alpha''')} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j \alpha''; [\cdot], K_4; s_4; e'_4[\alpha'''/x] \rangle \\ &\xrightarrow{\tau} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j \alpha''; [\cdot], K_4; s'_4; e''_4 \rangle = C'_4 \end{aligned}$$

and $C'_3 \underline{\cong} C'_4$.

$$\begin{aligned} C_2 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \cdot \rangle \kappa_2 \xrightarrow{\underline{\text{app}}(i', \alpha'')} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j \alpha''; K_4; s_4; e''_4[\alpha'''/x] \rangle \kappa_2 \kappa_4 \\ &\hspace{15em} (\kappa_4 = \{\alpha''/\alpha'''\}) \\ &\xrightarrow{\tau} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j \alpha''; K_4; s'_4; e''_4 \rangle \kappa_2 \kappa_4 = C'_2 \pi \\ C_2 &\xrightarrow{\underline{\text{app}}(i', \alpha')} C'_2 \end{aligned}$$

Moreover we derive $C'_1 \text{ perm}(\text{flde}(\text{fldv}(\text{fldv}(\overline{\approx})))) C'_2$.

* $u_4 = \alpha$ and $v_4 = \alpha_4 \in A_4$: The proof is as before with transitions:

$$\begin{aligned} C_4 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \cdot \rangle \xrightarrow{\text{app}(i', \alpha'')} \text{app}(\alpha, j) \rightarrow \\ &\quad \xrightarrow{\text{app}(i, \alpha''')} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j \alpha''; [\cdot], K_4; s_4; \alpha_4 \alpha'''' \rangle = C'_4 \\ C_2 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \cdot \rangle \kappa_2 \xrightarrow{\text{app}(i', \alpha'')} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j \alpha''; K_4; s_4; \alpha_4 \alpha'''' \rangle \kappa_2 \kappa_4 = C'_2 \\ &\quad (\kappa_4 = \{\alpha''/\alpha'''\}) \\ C_2 &\xrightarrow{\text{app}(i', \alpha')} C'_2 \end{aligned}$$

and $C'_3 \overline{\approx} C'_4$. Moreover we derive $C'_1 \text{ perm}(\text{flde}(\text{fldv}(\text{fldv}(\overline{\approx})))) C'_2$.

* $u_4 = \text{fix}f(x).e_4$: In this case we derive:

$$\begin{aligned} C_4 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \cdot \rangle \xrightarrow{\text{app}(i', \alpha'')} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; e_4[\alpha''/x] \rangle \\ &\quad \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s'_4; E_4[\alpha w_4] \rangle \\ &\quad \xrightarrow{\text{app}(\alpha, j)} \langle A_4 \uplus \alpha, \alpha''; \Gamma_4, {}^i v_4, {}^j w_4; E_4, K_4; s_4; \cdot \rangle \\ &\quad \xrightarrow{\text{app}(i, \alpha''')} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j w_4; E_4, K_4; s_4; e'_4 \rangle \quad (v_4 \alpha'''' \succ \alpha_4) \\ &\quad \xrightarrow{\tau} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j w_4; E_4, K_4; s'_4; e''_4 \rangle = C'_4 \end{aligned}$$

and $C'_3 \overline{\approx} C'_4$.

$$\begin{aligned} C_2 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \cdot \rangle \kappa_2 \xrightarrow{\text{app}(i', \alpha'')} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; e_4[\alpha''/x] \rangle \kappa_2 \\ &\quad \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s'_4; E_4[\alpha w_4] \rangle \kappa_2 \\ &\quad (\xrightarrow{\tau} \cup =) \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j w_4; K_4; s_4; E_4[e'_4] \rangle \kappa_2 \kappa_4 \\ &\quad (\kappa_4 = \{w_4/\alpha'''\}, (v_4 \alpha'''' \succ e'_4 \kappa_2 \kappa_4)) \\ &\quad \xrightarrow{\tau} \langle A_4 \uplus \alpha, \alpha'', \alpha'''; \Gamma_4, {}^i v_4, {}^j w_4; K_4; s'_4; E_4[e''_4] \rangle \kappa_2 \kappa_4 = C'_2 \pi \\ C_2 &\xrightarrow{\text{app}(i', \alpha')} C'_2 \end{aligned} \tag{6}$$

Moreover we derive $C'_1 \text{ perm}(\text{flde}(\text{fldv}(\text{fldv}(\overline{\approx})))) C'_2$.

- $u_3 = \alpha$ and $v_3 = \alpha_3 \in A_3$: In this case we have $e''_1 = \alpha_3 \alpha'' = (\alpha_3 \alpha'') \kappa_1$ ($\alpha_3 \neq \alpha \neq \alpha''$), and derive the transitions:

$$\begin{aligned} C_1 &= \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \cdot \rangle \kappa_1 \xrightarrow{\text{app}(i', \alpha'')} \langle A_3 \uplus \alpha, \alpha''; \Gamma_3, {}^i v_3; K_3; s_3; \alpha \alpha'' \rangle \kappa_1 = C'_1 \pi \\ C_3 &= \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \cdot \rangle \xrightarrow{\text{app}(i', \alpha'')} \langle A_3 \uplus \alpha, \alpha''; \Gamma_3, {}^i v_3; K_3; s_3; \alpha \alpha'' \rangle = C'_3 \end{aligned}$$

By $C_3 \overline{\approx} C_4$ we have two cases. The first is when C_4 weakly matches these transitions and becomes $C'_4 = \langle \perp \rangle$. This case is proved using rule

UTFD3 and relating the resulting configurations in $\text{perm}(\text{fldv}(\underline{\cong}))$. The other case is as follows: By Lem. 62, $\hat{e}_4 = \cdot$ and $\Gamma_4(i') = u_4$ and

$$\begin{aligned} C_4 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s_4; \cdot \rangle \xrightarrow{\text{app}(i', \alpha'')} \langle A_4 \uplus \alpha, \alpha''; \Gamma_4, {}^i v_4; K_4; s_4; e_4 \rangle \xrightarrow{\tau} C'_4 \quad ((u_4 \alpha'') \succ e_4) \\ C_2 &= C_4 \kappa_2 \xrightarrow{\text{app}(i', \alpha'')} C'_4 \kappa_2 = C_2 \pi \quad (\text{as above}) \\ C_2 &= C_4 \kappa_2 \xrightarrow{\text{app}(i', \alpha')} C'_4 \kappa_2 \pi = C_2 \end{aligned}$$

Moreover $C'_3 \underline{\cong} C'_4$ and thus $C'_1 \text{perm}(\text{fldv}(\underline{\cong})) C'_2$.

- $v_1 = \alpha_3 \in A_3$: Here $\alpha_3 \neq \alpha$ and the proof proceeds as in the previous case. The resulting configurations are again related in $\text{perm}(\text{fldv}(\underline{\cong}))$.
 - $u_3 = \text{fix}f(x).e_3$: similarly.
- TAU: Here we have $\eta = \tau$ and

$$C_1 = \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; \hat{e}_3 \rangle \kappa_1 \xrightarrow{\tau} \langle A_3; \Gamma_3 \kappa_1; K_3 \kappa_1; s'_3 \kappa_1; e'_1 \rangle = C'_1$$

We distinguish the case when $\hat{e}_3 \kappa_1 \neq E_1[\text{fix}f(x).e'_1 u_1]$, for any $E_1, \text{fix}f(x).e'_1, u_1$. This case is straightforward and follows from Lem. 60. In the remaining case we proceed by case analysis of the equality $\hat{e}_3 \kappa_1 = E_1[\text{fix}f(x).e'_1 u_1]$:

- $\hat{e}_3 = E_3[\text{fix}f(x).e'_3 u_3]$ and $E_1 = E_3 \kappa_1$ and $\text{fix}f(x).e'_1 = \text{fix}f(x).e'_3 \kappa_1$ and $u_1 = u_3 \kappa_1$ and $s_3 = s'_3$. This case follows again from Lem. 60.
- $\hat{e}_3 = E_3[\alpha u_3]$ and $E_1 = E_3 \kappa_1$ and $v_3 = \text{fix}f(x).e'_1$ and $u_1 = u_3 \kappa_1$ and $s_3 = s'_3$. Here we have

$$\begin{aligned} C_3 &= \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3; K_3; s_3; E_3[\alpha u_3] \rangle \xrightarrow{\text{app}(\alpha, j)} \langle A_3 \uplus \alpha; \Gamma_3, {}^i v_3, {}^j u_3; E_3, K_3; s_3; \cdot \rangle \\ &\xrightarrow{\text{app}(i, \alpha')} \langle A_3 \uplus \alpha, \alpha'; \Gamma_3, {}^i v_3, {}^j u_3; E_3, K_3; s_3; e'_1[\alpha'/x] \rangle = C'_3 \end{aligned}$$

By $C_3 \underline{\cong} C_4$ we have two cases. The first is when C_4 weakly matches these transitions and becomes $C'_4 = \langle \perp \rangle$. This case is proved using rule UTFD3 and relating the resulting configurations in $\text{fide}(\text{fldv}(\text{fldv}(\underline{\cong})))$. The other case is as follows:

$$\begin{aligned} C_4 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s'_4; \hat{e}_4 \rangle \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s'_4; E_4[\alpha u_4] \rangle \\ &\xrightarrow{\text{app}(\alpha, j)} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4, {}^j u_4; E_4, K_4; s'_4; \cdot \rangle \\ &\xrightarrow{\text{app}(j, \alpha')} \langle A_4 \uplus \alpha, \alpha'; \Gamma_4, {}^i v_4, {}^j u_4; E_4, K_4; s'_4; e'_4 \rangle \quad (v_4 \alpha' \succ e'_4) \\ &\xrightarrow{\tau} C'_4 \end{aligned}$$

and $C'_3 \underline{\cong} C'_4$. As above we can derive:

$$\begin{aligned} C_2 &= \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s'_4; \hat{e}_4 \rangle \kappa_2 \xrightarrow{\tau} \langle A_4 \uplus \alpha; \Gamma_4, {}^i v_4; K_4; s'_4; E_4[\alpha u_4] \rangle \kappa_2 \\ &\quad (\xrightarrow{\tau} \cup =) \langle A_4 \uplus \alpha, \alpha'; \Gamma_4, {}^i v_4, {}^j u_4; K_4; s'_4; E_4[e'_4] \rangle \kappa_2 \kappa_4 \\ &\quad \quad \quad (\kappa_4 = \{u_4/\alpha'\}, (v_4 \alpha') \kappa_2 \kappa_4 \succ e'_4 \kappa_2 \kappa_4) \\ &\xrightarrow{\tau} C'_4 \kappa_2 \kappa_4 \end{aligned}$$

$$\begin{array}{c}
\frac{\langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle \mathcal{R} \langle A_2; \Gamma_2; K_2; s_2; \hat{e}_2 \rangle}{\vec{l} \notin \text{dom}(s_1) \sqcup \text{dom}(s_2) \quad \text{fl}(\vec{v}) \sqsubseteq \{\vec{l}\} \quad \text{an}(\vec{v}) \sqsubseteq A_1 \cap A_2} \text{UPToCxt}_s \\
\langle A_1; \Gamma_1; K_1; s_1[\vec{l} \mapsto \vec{v}]; \hat{e}_1 \rangle \text{cxt}_{\vec{l}}(\mathcal{R}) \langle A_2; \Gamma_2; K_2; s_2[\vec{l} \mapsto \vec{v}]; \hat{e}_2 \rangle \\
\\
\frac{\langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle \text{cxt}_{\vec{l}}(\mathcal{R}) \langle A_2; \Gamma_2; K_2; s_2; \hat{e}_2 \rangle}{i \notin \text{dom}(\Gamma_1) \sqcup \text{dom}(\Gamma_2) \quad \text{fl}(v) \sqsubseteq \{\vec{l}\} \quad \text{an}(v) \sqsubseteq A_1 \cap A_2} \text{UPToCxt}_\Gamma \\
\langle A_1; \Gamma_1; {}^i v; K_1; s_1; \hat{e}_1 \rangle \text{cxt}_{\vec{l}}(\mathcal{R}) \langle A_2; \Gamma_2; {}^i v; K_2; s_2; \hat{e}_2 \rangle \\
\\
\frac{\langle A_1; \Gamma_1; K_{11}, K_{12}; s_1; \hat{e}_1 \rangle \text{cxt}_{\vec{l}}(\mathcal{R}) \langle A_2; \Gamma_2; K_{21}, K_{22}; s_2; \hat{e}_2 \rangle}{|K_{11}| = |K_{21}| \quad \text{fl}(E) \sqsubseteq \{\vec{l}\} \quad \text{an}(E) \sqsubseteq A_1 \cap A_2} \text{UPToCxt}_K \\
\langle A_1; \Gamma_1; K_{11}, E, K_{12}; s_1; \hat{e}_1 \rangle \text{cxt}_{\vec{l}}(\mathcal{R}) \langle A_2; \Gamma_2; K_{21}, E, K_{22}; s_2; \hat{e}_2 \rangle \\
\\
\frac{\langle A_1; \Gamma_1; K_1; s_1; \cdot \rangle \text{cxt}_{\vec{l}}(\mathcal{R}) \langle A_2; \Gamma_2; K_2; s_2; \cdot \rangle \quad \text{fl}(e) \sqsubseteq \{\vec{l}\} \quad \text{an}(e) \sqsubseteq A_1 \cap A_2}{\langle A_1; \Gamma_1; K_1; s_1; e \rangle \text{cxt}_{\vec{l}}(\mathcal{R}) \langle A_2; \Gamma_2; K_2; s_2; e \rangle} \text{UPToCxt}_e \\
\\
\text{cxt} = \bigcup_{\vec{l}} \text{cxt}_{\vec{l}}
\end{array}$$

Fig. 7. Up-to context.

Moreover we derive $C'_1 \text{flde}(\text{fldv}(\text{fldv}(\frac{\sqsubseteq}{\approx}))) C'_2$.

Proposition 66. *Functions flde and fldv are sound up-to techniques.*

Proof. Consider $f = \text{flde} \sqcup \text{fldv}$. It suffices to show that f is **wp**-sound. By Lem. 37 (3), it is sufficient to show

$$\begin{array}{l}
\text{flde} \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \rightsquigarrow (f \sqcup g_1)^\omega \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \quad \text{and} \\
\text{fldv} \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \rightsquigarrow (f \sqcup g_2)^\omega \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})}
\end{array}$$

where $g_1 = \emptyset$ and $g_2 = g = \text{perm}$. We have $g_i \sqsubseteq \mathbf{t}$ by Lem. 45. We finally establish the needed progressions by Lem.(s) 64 and 65.

E.5 Up to Context

Lemma 67. *Let $g = \text{id} \sqcup \text{perm} \sqcup \text{weak} \sqcup \text{gc} \sqcup \text{flde} \sqcup \text{fldv}$; then $\text{cxt}_{\vec{l}} \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \rightsquigarrow (\text{cxt} \sqcup g)^\omega \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})}$.*

Proof. We need to show $\text{cxt}_{\vec{l}} \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})} \circ \mathbf{wp}(\mathcal{R}) \sqsubseteq \mathbf{wp} \circ (\text{cxt} \sqcup g)^\omega \circ \mathbf{c}_{\text{gfp}(\mathbf{wp})}(\mathcal{R})$. Because $\mathbf{c}_{\text{gfp}(\mathbf{wp})}$ is the constant function mapping its argument to $(\frac{\sqsubseteq}{\approx})$, we need to show $\text{cxt}_{\vec{l}}(\frac{\sqsubseteq}{\approx}) \sqsubseteq \mathbf{wp} \circ (\text{cxt} \sqcup g)^\omega(\frac{\sqsubseteq}{\approx})$.

We proceed by induction on the derivation of $C_1 \text{cxt}_{\vec{l}}(\frac{\sqsubseteq}{\approx}) C_2$.

UPToCxt_s: We have $C_1 = \langle A_1; \Gamma_1; K_1; s_1[\vec{l} \mapsto \vec{v}]; \hat{e}_1 \rangle$ and $C_2 = \langle A_2; \Gamma_2; K_2; s_2[\vec{l} \mapsto \vec{v}]; \hat{e}_2 \rangle$ and $C_3 = \langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle$ and $C_4 = \langle A_2; \Gamma_2; K_2; s_2; \hat{e}_2 \rangle$ and $C_3 \approx C_4$

C_4 and $\vec{l} \notin \text{dom}(s_1) \sqcup \text{dom}(s_2)$ and $\text{fl}(\vec{v}) \sqsubseteq \{\vec{l}\}$ and $\text{an}(\vec{v}) \sqsubseteq A_1 \cap A_2$. By definition 40, $C_1 \asymp C_3$ and $C_2 \asymp C_4$. Therefore $C_1 \text{gc}(\overline{\approx}) C_2$. By Lem. 47, $C_1 \overline{\approx} C_2$, thus $C_1 \text{wp} \circ \text{id}(\overline{\approx}) C_2$, and finally $C_1 \text{wp} \circ (\text{cxt} \cup g)^\omega(\overline{\approx}) C_2$.

UPToCXT_T: We have $C_1 = \langle A_1; \Gamma_1, {}^i v; K_1; s_1; \hat{e}_1 \rangle$ and $C_2 = \langle A_2; \Gamma_2, {}^i v; K_2; s_2; \hat{e}_2 \rangle$ and $C_3 = \langle A_1; \Gamma_1; K_1; s_1; \hat{e}_1 \rangle$ and $C_4 = \langle A_2; \Gamma_2; K_2; s_2; \hat{e}_2 \rangle$ and $C_3 \text{cxt}_{\vec{l}}(\mathcal{R}) C_4$ and $i \notin \text{dom}(\Gamma_1) \cup \text{dom}(\Gamma_2)$ and $\text{fl}(v) \sqsubseteq \{\vec{l}\}$ and $\text{an}(v) \sqsubseteq A_1 \cap A_2$.

Let $C_1 \text{cxt}_{\vec{l}}(\overline{\approx}) C_2$ and $C_1 \xrightarrow{\eta} C'_1$. We need to show that there exists C'_2 such that $C_2 \xrightarrow{\eta} C'_2$ and $C'_1 \mathcal{S} C'_2$, where $\mathcal{S} = (\text{cxt} \cup g)^\omega(\overline{\approx})$.

The proof is similar in the remaining two cases. □

Lemma 68. $e_1 \equiv e_2$ iff $\text{fix}f(x).e_1 \equiv \text{fix}f(x).e_2$

Proof. See Theorem 3.2 in [16].

Theorem 69 (Soundness of \approx). $e_1 \approx e_2$ implies $e_1 \equiv e_2$.

Proof. It suffices to show that (*bisimil*) is a congruence. By the above lemma, it suffices to show that (*bisim*) is a congruence for values. Let $v_1 \approx v_2$ and context D . We have

$$\begin{aligned}
 v_1 &\approx v_2 \\
 \langle \cdot; {}^i v_1; \cdot; \cdot; \cdot \rangle &\approx \langle \cdot; {}^i v_1; \cdot; \cdot; \cdot \rangle && \text{(by bisimulation def. and PROPRT transition)} \\
 \langle \alpha; {}^i v_1; \cdot; \cdot; \cdot \rangle &\approx \langle \alpha; {}^i v_1; \cdot; \cdot; \cdot \rangle && \text{(by soundness of gc}(\approx)\text{)} \\
 \langle \alpha; {}^i v_1; \cdot; \cdot; D[\alpha] \rangle &\approx \langle \alpha; {}^i v_2; \cdot; \cdot; D[\alpha] \rangle && \text{(by soundness of cxt}(\approx)\text{)} \\
 \langle \cdot; \cdot; \cdot; \cdot; D[v_1] \rangle &\approx \langle \cdot; \cdot; \cdot; \cdot; D[v_2] \rangle && \text{(by soundness of fldv}(\approx)\text{)} \\
 D[v_1] &\approx D[v_2] && \text{(by definition)}
 \end{aligned}$$

□

F Completeness of (\approx)

Let us denote by \mathcal{A} and \mathcal{I} the countably infinite sets of abstract and index names respectively. In this section we present the proof of the following result.

Theorem 70 (Completeness). *For any two doubly closed expressions e_1 and e_2 , if $e_1 \equiv e_2$ then $e_1 \approx e_2$.*

We start off with a few auxiliary results.

Lemma 71. *Let C be a configuration with state s , and r a location such that $s(r) = \text{ff}$ and:*

- all the assignments of r occurring in C are of the form $r := \text{ff}$
- C contains a subterm (in one of its Γ, K, s, \hat{e}) of the form $\text{if } !r \text{ then } e_{\text{tt}} \text{ else } e_{\text{ff}}$.

Then, $C \approx C'$, where C' is obtained from C by replacing the subterm above with e_{ff} .

Proof. Let us define the relation $C \succ_r C'$ to hold for each pair of C, C' as above. The statement then follows by showing that $\mathcal{R} = \{(C, C') \mid C \succ_r^* C'\}$ is a weak bisimulation up to beta.

Lemma 72. *Let $C = \langle A; \Gamma; K; s; e \rangle$ be a configuration with $\alpha \in A$, $s(l) = \lambda z. \alpha z$ for some location l that has no assignments in C , and:*

- C' be obtained from C by replacing an occurrence of α with $\lambda z. \alpha z$; or
- C' be obtained from C by replacing an occurrence of α with $!l$ (other than in $s(l) = \lambda z. \alpha z$).

Then, $C \approx C'$.

Proof. Let us define the relation $C \succ_\alpha C'$ to hold for each pair of C, C' as above. The statement then follows by showing that $\mathcal{R} = \{(C, C') \mid C \succ_\alpha^* C'\}$ is a weak bisimulation up to beta.

Lemma 73. *Let $C = \langle A; \Gamma; K; s; e \rangle$ be a configuration with $\alpha \in A$, $s(l) = \lambda z. cz$ for some constant c and location l that has no assignments in C , and:*

- C' be obtained from C by replacing an occurrence of α with $\lambda z. cz$; or
- C' be obtained from C by replacing an occurrence of α with $!l$ (other than in $s(l) = \lambda z. cz$).

Then, $C' \sqsubseteq_{\approx} C$.

Proof. Let us define the relation $C \succ_c C'$ to hold for each pair of C, C' as above. The statement then follows by showing that $\mathcal{R} = \{(C', C) \mid C \succ_c^* C'\}$ is a weak simulation up to beta.

Definition 74 (Traces). *For any C , we let $\text{Traces}(C) \stackrel{\text{def}}{=} \{ t \mid \tau, \downarrow \notin t \wedge C \xrightarrow{t} * \xrightarrow{\downarrow} \langle \perp \rangle \}$.*

Lemma 75. *For any two configurations C_1 and C_2 , $C_1 \sqsubseteq_{\approx} C_2$ iff $\text{Traces}(C_1) \subseteq \text{Traces}(C_2)$.*

Proof. For the right-to-left direction, let \mathcal{R} be the relation:

$$\mathcal{R} \stackrel{\text{def}}{=} \{(C_1, C_2) \mid \text{Traces}(C_1) \subseteq \text{Traces}(C_2)\}.$$

We claim that \mathcal{R} is a simulation. Take $C_1 \mathcal{R} C_2$. Let $C_1 \xrightarrow{\eta} C'_1$ and suppose that $C'_1 \neq \langle \perp \rangle$. If $\eta = \tau$ then, since the operational semantics is deterministic and transitions to $\langle \perp \rangle$ do not contribute traces, $\text{Traces}(C_1) = \text{Traces}(C'_1)$ and therefore $C'_1 \mathcal{R} C_2$. If η is one of $\text{app}(\alpha, v), \text{ret}(v)$, for some α, v , then there are two cases:

- If there exists some $\eta t \in \text{Traces}(C_1)$ then equality of traces implies that $C_2 \xrightarrow{\eta} C'_2$, for some $C'_2 \neq \langle \perp \rangle$. By determinacy of the LTS outside $\langle \perp \rangle$'s, we have that $\text{Traces}(C'_1) = \text{Traces}(C'_2)$.

- Otherwise, $C_2 \xrightarrow{\eta} \langle \perp \rangle$ and $\text{Traces}(C'_1) = \text{Traces}(\langle \perp \rangle) = \emptyset$.

Therefore, in both cases $C'_1 \mathcal{R} C'_2$. Similarly if η is one of $\mathbf{app}(i, v)$, $\mathbf{ret}(v)$, for some i, v . Finally, let $C_1 \xrightarrow{\eta} \langle \perp \rangle$. If $\eta = \downarrow$ then $\epsilon \in \text{Traces}(C_i)$ and therefore $C_2 \xRightarrow{\downarrow} \langle \perp \rangle$; if $\eta \neq \downarrow$ then $C_2 \xrightarrow{\eta} \langle \perp \rangle$. In both cases, we conclude by noting that $\langle \perp \rangle \mathcal{R} \langle \perp \rangle$.

Conversely, suppose $C_1 \sqsubseteq C_2$ and let $t = \eta_1 \cdots \eta_n \in \text{Traces}(C_1)$. Then, there are $C_1 = C_1^0, C_1^1, \dots, C_1^n$ such that $C_1^{i-1} \xrightarrow{\eta_i} C_1^i$, for each $1 \leq i \leq n$, and $C_1^n \xRightarrow{\downarrow} \langle \perp \rangle$. Since $C_1 \sqsubseteq C_2$, C_2 can simulate these transitions and produce the same trace t .

Corollary 76. *For any two doubly closed expressions e_1 and e_2 , $e_1 \approx e_2$ iff $\text{Traces}(C_{e_1}) = \text{Traces}(C_{e_2})$, where $C_{e_i} = \langle \cdot; \cdot; \cdot; \cdot; e_i \rangle$.*

Proof. Directly from the previous lemma.

We call a finite partial bijection $\phi : \mathcal{A} \cup \mathcal{I} \xrightarrow{\cong} \mathcal{A} \cup \mathcal{I}$ a *dualiser* if, for all $x \in \text{dom}(\phi)$, $x \in \mathcal{A} \iff \phi(x) \notin \mathcal{A}$. Given stacks K_1, K_2 , we define their (left) composition $K_1 \triangleright K_2$ recursively by:

$$K_1 \triangleright K_2 \stackrel{\text{def}}{=} \begin{cases} [\cdot] & \text{if } K_1 = K_2 = \cdot \\ (K_1 \triangleright K'_2)[E] & \text{if } K_2 = E, K'_2 \text{ and } |K_1| = |K_2| \\ (K'_1 \triangleright K_2)[E] & \text{if } K_1 = E, K'_1 \text{ and } |K_1| = |K_2| + 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Thus, if well defined, $K_1 \triangleright K_2$ is an evaluation context.

Definition 77. *For any two configurations C_1, C_2 , with $C_i = \langle A_i; \Gamma_i; K_i; s_i; \hat{e}_i \rangle$ and dualiser ϕ , we say that (C_1, ϕ) is a context for C_2 if:*

- $\phi : A_2 \cup \text{dom}(\Gamma_2) \xrightarrow{\cong} A_1 \cup \text{dom}(\Gamma_1)$
- $\text{dom}(s_1) \cap \text{dom}(s_2) = \emptyset$
- exactly one of \hat{e}_1, \hat{e}_2 is \cdot .
- $K_1 \triangleright K_2$ is well defined
- taking ψ to be the following map from $A_1 \cup A_2$ to terms: $\psi = (\Gamma_2 \circ (\phi^{-1} \upharpoonright A_1)) \cup (\Gamma_1 \circ (\phi \upharpoonright A_2))$, there is some k such that $\psi^k = \psi^{k+1}$ (note: $\psi^1 \stackrel{\text{def}}{=} \psi$ and $\psi^{n+1} \stackrel{\text{def}}{=} \alpha \mapsto (\psi^n(a))\psi$).

In such a case, we set $C_1[C_2]_\phi \stackrel{\text{def}}{=} \langle s_1 \cup s_2; (K_1 \triangleright K_2)[e] \rangle \psi^*$, where $e \in \{\hat{e}_1, \hat{e}_2\} \setminus \{\cdot\}$ and $\psi^* \stackrel{\text{def}}{=} \psi^k$.

We write $C_1 \diamond C_2$ when C_1 and C_2 have the same contexts. We call a configuration C *lost* if either $C = \langle \perp \rangle$ or there is no context (C', ϕ) for C such that $C'[C]_\phi \downarrow$. Finally, for moves η, η' , we let $\eta \succ_P \eta'$ hold if:

- η is in one of the forms $\mathbf{app}(\alpha, i)$ or $\mathbf{ret}(i)$, and
- η' is in one of the forms $\mathbf{app}(\alpha, c)$ or $\mathbf{ret}(c)$, respectively, for some constant c .

Theorem 78 (One-step definability). *Given C_0, C'_0, C', ϕ' and $\eta \neq \tau, \downarrow$ such that $C_0 \xrightarrow{\eta} C'_0$ and $C'[C'_0]_{\phi'} \Downarrow$, there is a context (C, ϕ) for C_0 such that:*

- $C[C_0]_{\phi} \Downarrow$ and
- for all $C''_0 \diamond C_0$, $C[C''_0]_{\phi} \Downarrow$ implies that $C''_0 \xrightarrow{\eta'}$ for some $\eta \succ_P \eta'$.

Proof. Let us assume $C_0 = \langle A_0; \Gamma_0; K_0; s_0; \hat{e}_0 \rangle$, $C' = \langle A'; \Gamma'; K'; s'; \hat{e}' \rangle$ and build a configuration C and a dualiser $\phi' \subseteq \phi$. We do case analysis on η .

If $\eta = \text{app}(\alpha, i)$, let $i_\alpha = \phi'(\alpha) \in \text{dom}(\Gamma')$ and $\alpha_i = \phi'(i) \in A'$. We have $\hat{e}' = e'$ and take $\phi = \phi' \setminus \{(i, \alpha_i)\}$ and $C = \langle A' \setminus \{\alpha_i\}; \Gamma; K; s; e \rangle \cdot [!l/\alpha_i]$ with $s = s' \uplus [r \mapsto \text{tt}, l \mapsto \text{tt}]$. We also let $\text{dom}(\Gamma) = \text{dom}(\Gamma')$ and for each $j \in \text{dom}(\Gamma')$, assuming $\Gamma'(j) = \lambda x.e_j$:

$$\Gamma(j) \stackrel{\text{def}}{=} \begin{cases} \lambda x. \text{if } !r \text{ then } \perp \text{ else } e_j & \text{if } j \neq i_\alpha \\ \lambda x. \text{if } !r \text{ then } (r := \text{ff}; l := \lambda z.xz; e') \text{ else } e_j & \text{if } j = i_\alpha \end{cases}$$

Moreover, if K' is empty then $K \stackrel{\text{def}}{=} K'$; and if $K' = E', K_1$ then $K \stackrel{\text{def}}{=} E'[(\lambda x. \text{if } !r \text{ then } \perp \text{ else } x)[\cdot], K_1]$. Setting Γ'', K'', s'', e'' to be $\Gamma[!l/\alpha_i], K[!l/\alpha_i], s'[!l/\alpha_i], e'[!l/\alpha_i]$ respectively, the non-blocking transitions from C are $C \xrightarrow{\text{app}(i_\alpha, \alpha_i)} C'' = \langle A'; \Gamma''; K''; s'' \uplus [r \mapsto$

$\text{ff}, l \mapsto \lambda z.\alpha_i z]; e'' \rangle$ (modulo renamings of α_i) and $C \xrightarrow{\text{app}(i, c)} C'_c = \langle A; \Gamma''; K''; s'' \uplus [r \mapsto$

$\text{ff}, l \mapsto \lambda z.cz]; e'' \rangle$, for all constants c . Let us now consider $C[C_0]_{\phi} = \langle s_0 \uplus s; (K \triangleright K_0)[E[\alpha v]] \rangle \psi^*$, with ψ defined as above. By construction, $C[C_0]_{\phi} \rightarrow^* C''[C'_0]_{\phi'}$ and thus, by Lem.(s) 71 and 72, $C[C_0]_{\phi} \Downarrow$. Take now some $C''_0 \diamond C_0$. By construction, in order for $C[C''_0]_{\phi}$ to take a transition, it must be an internal call to i_α

with an arbitrary argument, i.e. we need $C''_0 \xrightarrow{\eta'}$ for some $\eta \succ_P \eta'$.

If $\eta = \text{ret}(i)$, let $\alpha_i = \phi'(i) \in A'$. We have $\hat{e}' = e'$ and take $C = \langle A' \setminus \{\alpha_i\}; \Gamma; E, K'; s; e \rangle \cdot [!l/\alpha_i]$ with $s = s' \uplus [r \mapsto \text{tt}, l \mapsto \text{tt}]$, and $\phi = \phi' \setminus \{(i, \alpha_i)\}$. We also let $\text{dom}(\Gamma) = \text{dom}(\Gamma')$ and set $\Gamma(j) \stackrel{\text{def}}{=} \lambda x. \text{if } !r \text{ then } \perp \text{ else } e_j$ (where $\Gamma(j) = \lambda x.e_j$) and $E \stackrel{\text{def}}{=} (\lambda x.r := \text{ff}; l := \lambda z.xz; e)[\cdot]$. The argument then follows that of the previous case.

If $\eta = \text{app}(\alpha, c)$, let $i_\alpha = \phi'(\alpha) \in \text{dom}(\Gamma')$. We have $\hat{e}' = e'$ and take $C = \langle A'; \Gamma; K; s; \cdot \rangle$ with $s = s' \uplus [r \mapsto \text{tt}]$, and $\phi = \phi'$. We also let $\text{dom}(\Gamma) = \text{dom}(\Gamma')$ and for each $j \in \text{dom}(\Gamma')$, assuming $\Gamma'(j) = \lambda x.e_j$:

$$\Gamma(j) \stackrel{\text{def}}{=} \begin{cases} \lambda x. \text{if } !r \text{ then } \perp \text{ else } e_j & \text{if } j \neq i_\alpha \\ \lambda x. \text{if } !r \text{ then } (r := \text{ff}; \text{if } x == c \text{ then } e \text{ else } \perp) \text{ else } e_j & \text{if } j = i_\alpha \end{cases}$$

Moreover, if K' is empty then $K \stackrel{\text{def}}{=} K'$; and if $K' = E', K_1$ then $K \stackrel{\text{def}}{=} E'[(\lambda x. \text{if } !r \text{ then } \perp \text{ else } x)[\cdot], K_1]$. The only non-blocking transition from C is $C \xrightarrow{\text{app}(i_\alpha, c)}$

$C'' = \langle A'; \Gamma; K; s' \uplus [r \mapsto \text{ff}]; e \rangle$. Let us now consider $C[C_0]_{\phi} = \langle s_0 \uplus s; (K \triangleright K_0)[E[\alpha c]] \rangle \psi^*$. By construction, $C[C_0]_{\phi} \rightarrow^* C''[C'_0]_{\phi'}$ and thus, by Lem. 71, $C[C_0]_{\phi} \Downarrow$. Take now some $C''_0 \diamond C_0$. By construction, in order for $C[C''_0]_{\phi}$ to take

a transition, it must be an internal call to i_α with argument c , i.e. we need $C''_0 \xrightarrow{\eta}$.

If $\eta = \text{ret}(c)$, we have $\hat{e}' = e'$ and take $C = \langle A'; \Gamma; E, K'; s; \cdot \rangle$ with $s = s' \uplus [r \mapsto \text{tt}]$, and $\phi = \phi'$. We also let $\text{dom}(\Gamma) = \text{dom}(\Gamma')$ and for each $j \in \text{dom}(\Gamma')$, $\Gamma(j) \stackrel{\text{def}}{=} \lambda x. \text{if } !r \text{ then } \perp \text{ else } e_j$ (where $\Gamma(j) = \lambda x. e_j$) and $E \stackrel{\text{def}}{=} (\lambda x. r := \text{ff}; \text{if } x == c \text{ then } e \text{ else } \perp)[\cdot]$. We conclude as in the previous case.

If $\eta = \underline{\text{app}}(\alpha, v)$, we must have $K' = E, K$. We take $C = \langle A'; \Gamma' \setminus \{v\}; K; s'; E[\phi(\alpha)v'] \rangle$, with $v' = \Gamma'(v)$ if $v \in \mathcal{I}$, and $v' = v$ otherwise. Moreover, we set $\phi = \phi' \setminus \{(v, \phi'(v))\}$ if $v \in \mathcal{I}$, and $\phi = \phi'$ otherwise. We can see that $C[C_0]_\phi \rightarrow C'[C'_0]_{\phi'}$ and, for any $C''_0 \diamond C_0$, if $C[C''_0]_\phi$ is well formed then $C''_0 \xrightarrow{\eta}$. Similarly, if $\eta = \underline{\text{ret}}(v)$, we construct $C = \langle A'; \Gamma' \setminus \{v\}; K'; s'; v' \rangle$ and ϕ , with v' and ϕ defined as above.

Proof (Proof of Thm. 70). It suffices to show that the following relation is a bisimulation.

$$\mathcal{R} \stackrel{\text{def}}{=} \{(C_1, C_2) \mid C_1, C_2 \text{ lost} \vee (C_1 \diamond C_2 \wedge \forall (C, \phi) \in \text{Cxt}(C_i). C[C_1]_\phi \Downarrow \iff C[C_2]_\phi \Downarrow)\}$$

Let $C_1 \mathcal{R} C_2$ and suppose $C_1 \xrightarrow{\eta} C'_1$. If C'_1 is lost then $C_2 \xrightarrow{\eta} \langle \perp \rangle$, as required. Otherwise:

- If $\eta = \tau$ then, for all C, ϕ , $C[C_1]_\phi \Downarrow$ iff $C[C'_1]_\phi \Downarrow$, so $C'_1 \mathcal{R} C_2$.
- If $\eta = \underline{\text{app}}(\alpha, v)$ then let (C', ϕ') be a context such that $C'[C'_1]_{\phi'} \Downarrow$. By Thm. 78, there is a context (C, ϕ) such that $C[C_1]_\phi \Downarrow$ and, for all $C''_1 \diamond C_1$, if $C[C''_1]_\phi \Downarrow$ then $C''_1 \xrightarrow{\eta'}$, for some $\eta \succ_P \eta'$. By hypothesis, $C[C_2]_\phi \Downarrow$, thus $C_2 \xrightarrow{\eta'} C'_2$ and $\eta \succ_P \eta'$. If $\eta' \neq \eta$ then we repeat the same argument (swapping the roles of C_1 and C_2) to conclude that $C_1 \xrightarrow{\eta''} C''_1$ for some $\eta'' \succ_P \eta'$, which contradicts determinacy of our LTS. Thus, $\eta = \eta'$ and, again by hypothesis, $C'_1 \mathcal{R} C'_2$. The case for $\eta = \underline{\text{ret}}(v)$ is treated similarly.
- If $\eta = \underline{\text{app}}(i, v)$ then let (C', ϕ') be a context such that $C'[C'_1]_{\phi'} \Downarrow$. By Thm. 78, there is a context (C, ϕ) such that $C[C_1]_\phi \Downarrow$ and, for all $C''_1 \diamond C_1$, if $C[C''_1]_\phi \Downarrow$ then $C''_1 \xrightarrow{\eta}$. By hypothesis, $C[C_2]_\phi \Downarrow$, thus $C_2 \xrightarrow{\eta} C'_2$. Now pick any context (C'_0, ϕ'_0) such that $C'_0[C'_1]_{\phi'_0} \Downarrow$, say $C'_0 = \langle A_0; \Gamma_0; E, K_0; s_0; \cdot \rangle$. Taking $C'_0 = \langle A_0; \Gamma_0 \setminus \{\phi'_0(v)\}; K_0; s_0; E[\phi'_0(i)v'] \rangle$, where $v' = v$ if the latter is a constant and $v' = \Gamma_0(\phi'_0(v))$ otherwise, and $\phi_0 = \phi$ if v is a constant and $\phi_0 = \phi' \setminus \{(v, \phi'_0(v))\}$ otherwise, we have $C'_0[C_n]_{\phi_0} \rightarrow C_0[C'_n]_{\phi'_0}$, for $n = 1, 2$. Thus, $C'_0[C_n]_{\phi_0} \Downarrow$ and hence $C_0[C'_2]_{\phi'_0} \Downarrow$. The case of $\eta = \underline{\text{ret}}(v)$ is addressed similarly.

G Examples

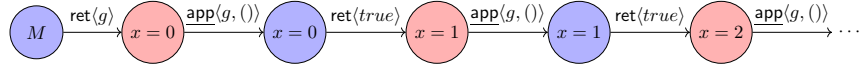
G.1 Simple Invariants

Up to invariants states that values stored in references can be abstracted if they validate a predicate. Consider Ex. 79.

Example 79.

```
M = ref x = 0 in fun () -> x++; !x > 0
N = fun () -> true
```

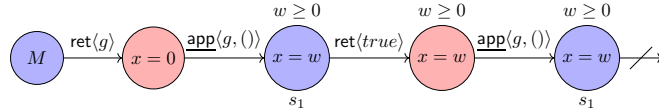
This example is also like the previous two. This time, the reference x is incremented in M and the function checks whether x holds a positive integer.



We can see above that each call to g increments the state, which makes it hard to find cycles. We annotate the function with an invariant as follows:

```
M = ref x = 0 in fun () { w | x as w | w >= 0 } -> x++; !x > 0
```

The invariant—shown in the curly braces—states that the value in x can be abstracted by any w such that $w \geq 0$, so long as $x \geq 0$ is also valid.



Shown above, we see that the states labelled s_1 are identical, which lets us prune it, e.g. via memoisation, to end the game.

G.2 Landin's Fixpoint

Example 80. The following equivalence relates Landin's imperative fixpoint operator with a fixpoint with letrec. The type of the two expressions is $((\text{int} \rightarrow \text{int}) \rightarrow \text{int} \rightarrow \text{int}) \rightarrow \text{int} \rightarrow \text{int}$.

```
M = let landinsfixpoint f =          N = let rec fix f =
      ref x = fun z -> z in          ( fun y -> f (fix f) y )
      x := ( fun y {} -> f !x y ); !x in landinsfixpoint
```

In this example, up to separation removes the outer functions from the Γ environments, thus they are only applied once. However the inner functions $(\text{fun } y \{ \} \rightarrow f !x y)$ and $(\text{fun } y \rightarrow f (\text{fix } f) y)$, which are provided as arguments to opponent function f , cannot be removed from the Γ environments by up to separation because of the access to location x , and are arbitrarily nested in the bisimulation transition system. The up to re-entry technique removes the need for this nesting. The syntax $\{ \}$ serves as the flag to apply this technique to these inner functions. \square

G.3 Full Example 23

Example 81. The following is the full example by Meyer and Sieber [21] featuring location passing, adapted to λ^{imp} where locations are local.


```

M = let loc_eq loc1loc2 =
    let (l1,l2) = loc1loc2 in
    let (r1,w1) = l1 in
    let (r2,w2) = l2 in
    let val1 = r1 () in let val2 = r2 () in
    w2(val2+1);
    let res = if r1() = val1+1 then true else false
    in w1(val1); w2(val2); res in
  fun q ->
    ref x = 0 in
    let locx = (fun () -> !x) , (fun v -> x := v) in
    let almostadd_2 locz {w | x as w | w mod 2 == 0} =
      if loc_eq (locx,locz) then x := 1 else x := !x + 2
    in q almostadd_2; if !x mod 2 = 0 then _bot_ else ()

N = fun q -> _bot_

```