# The Yamabe Problem on Non-Compact Manifolds of Negative Curvature Type


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# 1

# Introduction

We are interested in the Yamabe problem: given some Riemannian manifold $(M, g)$ of dimension $n \geq 3$, can we find a pointwise conformal metric whose scalar curvature is constant? We will write the conformal metric as

$$\tilde{g} := u^{\frac{4}{n-2}} g$$

where $u$ is some strictly positive smooth function on $M$. The scalar curvature changes according to the following equation

$$S_{\tilde{g}} = u^{-\frac{n+2}{n-2}} \left(-c_n \Delta_g u + S_g u\right)$$

where $c_n := \frac{4(n-1)}{n-2}$ and $S_g$ and $S_{\tilde{g}}$ refer to the scalar curvatures of the corresponding metrics. Asserting, as we would like, that $S_{\tilde{g}} \equiv K$ for some constant scalar curvature $K$, we obtain the Yamabe equation

$$-c_n \Delta_g u + S_g u = K u^{\frac{n+2}{n-2}} \, . \tag{1.1}$$

The analysis of this semilinear PDE will be central to our understanding of the Yamabe problem. We refer to the operator $-c_n \Delta_g + S_g$ as the *conformal Laplacian.*

The Yamabe problem was first considered by Hidehiko Yamabe in his paper "On a deformation of Riemannian structures on compact manifolds" in 1960 ([Yam60]). Since then, the Yamabe problem, as well as its many variations and related questions, has been the subject of a great deal of research and represents an important area of geometric analysis and the study of nonlinear PDEs. The breadth of the progress made on the topic of the Yamabe problem is too vast for a single introductory survey and so we direct the reader to the surveys of [LP87, Aub98, KMS09, BM11] and the references therein.

In the case that the manifold $(M, g)$ in question is compact, the problem was eventually resolved over a number of years following Yamabe's work (see [Yam60], [Tru68], [Aub76] and [Sch84]). This DPhil project focuses on the case that $(M, g)$ is a *non-compact* manifold. This setting for the Yamabe problem has seen significant progress in the literature (for example, see [LN74, AM88, CGS89, ACF92, Mar08, AILA18] and references therein); however, the overall understanding of the Yamabe problem in the non-compact setting is far from complete and remains a topic of active research.

Throughout this section, we will give an introduction to the Yamabe problem and the work of this DPhil project. In Section 1.1, we will provide a brief discussion of the much celebrated solution of the Yamabe problem for compact manifolds. In Section 1.2, we will discuss the Yamabe problem in the non-compact setting and review progress in the literature on non-compact manifolds of negative curvature type, which will be the focus of our studies in this thesis. We then make clear the particular goals and motivating questions which guided the work of this project in Section 1.3. Finally, having made our goals for this project clear, we provide an overview of a number of new results that we prove in this thesis in Section 1.4; there, we will discuss the ways that these results address our motivating questions and how they contribute to the wider literature on the Yamabe problem for non-compact manifolds.

# 1.1

# The Yamabe Invariant and the Resolution of the Yamabe Problem in the Compact Setting

In the case that $M$ is a compact manifold, the Yamabe problem has been resolved. Over a number of years, the combined work of Yamabe [Yam60], Trudinger [Tru68], Aubin [Aub76] and Schoen [Sch84] eventually proved that, given any compact Riemannian manifold $(M, g)$, one may always find a pointwise conformal metric to $g$ which has constant scalar curvature.

A key observation in the compact case is that we can realise the Yamabe equation as the Euler-Lagrange equation for the normalised Hilbert-Einstein functional

$$Q(\tilde{g}) = \frac{\int_M S_{\tilde{g}} \, dV_{\tilde{g}}}{\left(\int_M \, dV_{\tilde{g}}\right)^{\frac{n-2}{n}}} \, .$$

We call the infimum of $Q$ over all conformal metrics $\tilde{g}$ of $g$ the *Yamabe invariant* $\lambda(M, g)$. We note that the value of $\lambda(M, g)$ depends on the conformal class of $g$, rather than the metric $g$ itself. Understanding the properties of this invariant plays a fundamental role in the solution of the Yamabe problem in the compact setting. A simple but fundamental fact about the Yamabe invariant due to Aubin (see [Aub76]) is that $\lambda(M, g) \leq \lambda(\mathbb{S}^n)$, where $\mathbb{S}^n$ is the round sphere.

It has been well noted that the Yamabe invariant carries important information about the conformal class of $g$. For example, on any compact manifold $(M, g)$, one can show that there always exists a representative from the conformal class of $g$ whose scalar curvature has the same sign as the Yamabe invariant (see, for example, [Sch89]). This fact will provide important perspective on the non-compact setting, where the availability of an equivalent formulation of a Yamabe invariant is less clear.

Following the initial attempt at a proof by Yamabe, a study of the Yamabe invariant became of central importance to solving the Yamabe problem. First progress was made by Trudinger in [Tru68], who was able to complete Yamabe's proof in the cases that $\lambda(M, g) < 0$ or that $\lambda(M, g) = 0$.

The first progress in the case that $\lambda(M, g) > 0$ was made by Aubin in [Aub76] who was able to show that, if $\lambda(M, g)$ is less than the Yamabe invariant of the round $n$-sphere, then the Yamabe problem is solvable. Furthermore, he demonstrated that this strict inequality holds if $n \geq 6$ and $(M, g)$ is not locally conformally flat.

The final steps in solving the Yamabe problem were made by Schoen in [Sch84]; he was able to show that $\lambda(M, g) < \lambda(\mathbb{S}^n)$ (unless $(M, g)$ is actually conformal to $\mathbb{S}^n$) in the remaining cases untreated by the work of Aubin. As a consequence of these two works above, $\lambda(M, g) = \lambda(\mathbb{S}^n)$ if and only if $(M, g)$ is conformally equivalent to the sphere, thereby completing the solution of the Yamabe problem in the compact case.

As can be seen from the timeline above, the problem splits into the now more straightforward cases in which $\lambda(M, g) < 0$ and $\lambda(M, g) = 0$ and the more involved case where $\lambda(M, g) > 0$. For further detail on the solution in the compact case, see the survey by Lee and Parker [LP87].

## 1.2 The Yamabe Problem on Non-Compact Manifolds

As discussed, the focus of this thesis has been on the Yamabe problem in the case that the manifold in question is non-compact, where more work remains to be done. Early examples of the Yamabe problem being posed for non-compact manifolds include [Yau82] and [Kaz85].

This non-compact setting demands further assumptions on the manifold as, unlike in the compact case, there exist counter-examples to the existence of solutions to the non-compact Yamabe problem. Some such counter-examples are constructed in [Jin88], [Max05]; we point out here how these examples highlight some ways in which the non-compact case differs from the compact case.

We will explore the counter-examples due to Jin in [Jin88]. Jin constructs a set of examples of non-compact manifolds which arise via puncturing a compact manifold $(M_0, g_0)$ by the removal of a finite number of distinct points $\{p_1, \ldots, p_k\} \in M_0$ to define the manifold $M := M_0 \setminus \{p_1, \ldots, p_k\}$. We may then construct a complete metric on $M$ in the conformal class of $g_0$ via some conformal factor which blows up sufficiently rapidly at the puncture points. Jin is able to show that, for these examples, if the underlying compact manifold has a Yamabe invariant satisfying $\lambda(M_0, g_0) < 0$, then there exists no *complete* metric conformal to $g$ on $M$ which has constant scalar curvature.

An important point to note from these counter-examples involves the particular modes by which the Yamabe problem fails to be true. In the case that the target constant scalar curvature is negative, central to the proof is a removable singularity result (see [Vér81] and also [Avi82]) which is used to show that any solution of the Yamabe equation (1.1) on the punctured manifold $M$ must extend continuously to $M_0$; consequently, the corresponding conformal metric fails to be complete on $M$. In the case that the target constant scalar curvature is non-negative, Jin shows that the Yamabe equation has no positive solution at all. Thus, the example demonstrates the importance in the non-compact setting of the Yamabe problem of *both* the solvability of the Yamabe equation itself as well as the completeness requirement; the latter obstacle is not present in the compact setting, where completeness is automatic.

The nature of the Yamabe problem with a prescribed singular set as in the above example has seen significant attention in the literature. The so-called singular Yamabe problem considers compact manifolds with a prescribed singular set where the solution of the Yamabe equation is required to blow up (in the case that this manifold is a domain in Euclidean space, this is also known as the Loewner-Nirenberg problem). It has been shown that the existence of a solution to the problem is dependent on the dimension $d$ of the singular set; in particular, in the negative constant scalar curvature case, in [LN74], Loewner and Nirenberg proved that a solution exists provided that $d > \frac{n-2}{2}$ and that there is no solution if $d < \frac{n-2}{2}$ (thus explaining the observations in Jin's example above). Loewner and Nirenberg also conjectured that no solution existed in the critical case $d = \frac{n-2}{2}$ which was shown to be true in [Vér81] and [Avi82]. As for the positive curvature case, the work of Schoen and Yau in [SY88] combined with the work of Mazzeo and Pacard in [MP96] showed that, in contrast, a solution exists in this case if and only if $d < \frac{n-2}{2}$.

In the compact case, as discussed earlier, it has been well noted that the Yamabe problem splits on the sign of the Yamabe invariant $\lambda(M, g)$ into the cases $\lambda(M, g) < 0$ and $\lambda(M, g) = 0$ which are easier to solve and then the more difficult case that $\lambda(M, g) > 0$. We note again that, in the compact case, there always exists a representative from the conformal class of $g$ with scalar curvature of everywhere the same sign as $\lambda(M, g)$. In the non-compact case, while it is difficult to make a precise definition corresponding to "the sign of $\lambda(M, g)$", we can certainly continue to make assumptions on the sign of the scalar curvature of a given representative from the conformal class and so we shift to this perspective from here on.

As the negative case was the first case to be solved in the compact setting of the Yamabe problem, and paved the way for the eventual solution of the problem in the remaining cases, we are motivated to study the Yamabe problem for non-compact manifolds of negative curvature type (in a sense we make precise in the following) which still remains unresolved in full generality.

We do not attempt to address the Yamabe problem on any non-compact manifolds of positive curvature type in this thesis and so it will not be mentioned further. There, the analysis is of a different nature, for example note the difference in the condition on the dimension of the singular set in the singular Yamabe problem mentioned earlier. To highlight progress in the area, we direct the reader to the works [CGS89, Mar08, XZ20] and references therein.

We will be interested in obtaining conformal changes to constant *negative* scalar curvature and so, asserting that $S_{\tilde{g}} \equiv -n(n-1)$, we get the specific case of the Yamabe equation (1.1)

$$-c_n\Delta_g u + S_g u = -n(n-1)u^{\frac{n+2}{n-2}} \ . \tag{Ya}$$

From here on in our discussion of the Yamabe problem, we will be focused, in particular, on finding a solution of the equation (Ya) above for which the corresponding conformal metric is complete.

A number of important existence results for non-compact manifolds of negative curvature type have been established in the literature (see Allen, Isenberg, Lee and Allen [AILA18], Andersson, Chruściel and Friedrich [ACF92], Aviles and McOwen [AM85, AM88], Chruściel and Pollack [CP08], Finn [Fin99], Jin [Jin88], Loewner and Nirenberg [LN74], Mazzeo and Pacard [MP99, MP01], Ni [Ni82] and references therein). We now provide a brief overview of some of the progress made which is of particular relevance to our discussion in this introduction; we defer a more detailed review of these important works to the introductions of Chapters 3 and 4.

We first discuss the work of Aviles and McOwen in [AM88]. There, they established the existence of complete conformal metrics of constant negative scalar curvature on arbitrary non-compact manifolds provided that $S_g \leq -\varepsilon < 0$ outside of some compact set and either:

1. $S_g \leq 0$ globally on $M$ or

2. There exists a negative first eigenvalue for the conformal Laplacian on some compact domain in $M$.

They additionally demonstrated, via their Example 6.1 of [AM88], that there exist manifolds for which the Yamabe problem cannot be solved that satisfy $S_g \leq -\varepsilon < 0$ outside of a compact set but not conditions 1 or 2 above. That is to say, it is insufficient to assume only that $S_g \leq -\epsilon < 0$ outside of some compact set to conclude that there exists a solution to the Yamabe problem.

In addition, they were able to address the case that the scalar curvature is asymptotically negative but decaying to zero via additional restrictions on the Ricci curvature. However, this is not directly related to our main goals for this project (detailed in the next section) and so we do not discuss this result further.

A great deal of the progress made in the non-compact Yamabe problem in this negative curvature setting has been in the context of manifolds which admit a conformal compactification (for a definition of this notion see Chapter 3, we also refer the reader to [LeB82, FG85, PR86] for further discussion). A fundamental and pioneering work in this setting is of Loewner and Nirenberg in [LN74]. Among other topics, the authors addressed the case that the conformal compactification of the manifolds in question can be realised as a bounded domain in Euclidean space, in which case their work provides us with an existence result for the Yamabe problem. The progress made in this work laid the foundation for further progress in the setting of conformally compactifiable manifolds; in particular, those manifolds which are asymptotic to the hyperbolic space.

A major milestone in the study of the Yamabe problem on asymptotically hyperbolic manifolds is the work [ACF92] of Andersson, Chruściel and Friedrich. In their paper, the authors were able to provide an existence theorem for a wider class of asymptotically hyperbolic manifolds where the conformal compactification needn't be realised as a domain in Euclidean space. Furthermore, they provided a highly detailed understanding of the asymptotic behaviour of the conformal factor arising as the solution of (Ya) in their setting. We additionally highlight the more recent work of Allen, Isenberg, Lee and Stavrov Allen in [AILA18] which extended the progress made above by weakening the restrictions needed on the regularity of the conformal compactification.

As we have seen, due to the existence of counter-examples, an understanding of the solvability of the Yamabe problem in the non-compact setting demands some kind of additional geometric condition on the Riemannian manifold. Furthermore, in solving the problem we need not only find a solution to the Yamabe equation, but also to understand the asymptotic behaviour of the solution so as to ensure that the corresponding conformal metric is complete. Despite these challenges, substantial progress has been made on the Yamabe problem in the non-compact setting, as detailed above. However, the progress towards a full understanding of the non-compact Yamabe problem, even in this negative curvature setting, has stalled. Over the last 20–30 years, a satisfying set of results have been achieved for the asymptotically hyperbolic case; however, outside of this specific setting, some work has been done (see for example [ZX04], [Zha05]) but limited progress has been made since the paper [AM88] of Aviles and McOwen. We hope via our exploration of the problem, we may be able to uncover some new perspective that may offer a direction for new progress.

## 1.3 Goals and Motivating Questions of this DPhil Project

As mentioned previously, our focus will be on the Yamabe problem on non-compact manifolds of negative curvature type; before discussing our goals and motivating questions, we make this notion clear.

In light of our discussion regarding the Yamabe invariant in the compact case and its equivalence to a sign condition on the scalar curvature, our natural starting point is to consider those manifolds $(M, g)$ which admit a representative $\tilde{g}$ from the conformal class of $g$ which is complete and has asymptotically negative scalar curvature. To be precise, we mean that they satisfy a condition of the type

$$\limsup S_{\tilde{g}} \leq -\varepsilon < 0$$

for some $\varepsilon > 0$ and where the limit is taken along any divergent sequence in the manifold. As we have seen in the literature review of the section above, such a condition is true both in the work of Aviles and McOwen in [AM88] and in the work on asymptotically hyperbolic manifolds. However, in both cases additional requirements are needed in order to establish the existence of a solution to the Yamabe problem.

Throughout the document, we simply assume that the metric $g$ *itself* is a representative of the conformal class which satisfies the asymptotic negative scalar curvature condition. The question of when, given an arbitrary metric, the conformal class admits such a representative satisfying the negativity condition is an interesting one, but it will not be the main focus of our work. However, we are naturally led to a result of this type in Section 3.3 where we study the conformal classes of warped product metrics.

Our goal will be to try and understand what additional requirements are needed on top of the asymptotic negativity of the scalar curvature in order to solve the Yamabe problem. As mentioned in the previous sections, Example 6.1 of [AM88] demonstrates that some additional condition must be required in general. We outline here some motivating questions we feel would be useful in providing insight into the nature of this gap.

- Considering a restricted class of manifolds which have a negative scalar curvature end described as a warped product, can we find a condition on the warping function such that the Yamabe problem can be solved? Conversely, can we find a corresponding condition for non-existence? To what extent can we expand our analysis to include metrics which asymptote to reference metrics in this warped product class? Do these conditions provide any insight into the restrictions on the existence of a solution in the general case?

- It is known that a negative first eigenvalue for the conformal Laplacian on some compact domain can be used to obtain a solution to the Yamabe equation on the entire manifold. Given the relationship between the first eigenvalue of the Laplacian and the isoperimetric inequality, can we find a similar type of volume ratio type condition that implies the existence of a negative first eigenvalue and so a solution to the Yamabe problem? If we can find such a condition and corresponding existence result, can we establish a sharp version?

- Can we find a weaker condition on the first eigenvalue? For example, what can we say if the first eigenvalue of the conformal Laplacian is less than that of the first eigenvalue of the conformal Laplacian for the hyperbolic space?

- Can we deduce existence for the problem given, in addition to the asymptotic negativity of the scalar curvature, a bound (possibly pinched) on the Ricci curvature outside of a compact set?

In the work of this thesis, we tackle the first two points in the list above, with the first point being our main focus in Chapter 3 and the second point being the focus of Chapter 4; an overview of our progress on each is given in the next section, however we briefly expand on the motivation behind these two questions below.

Our first question is motivated by an attempt to address our main goal of understanding what additional requirements are needed alongside the asymptotic negativity of the scalar curvature, but in a restricted setting. Many of the results discussed from the literature on asymptotically hyperbolic manifolds have an asymptotically warped product structure and so there is already progress in this direction. Additionally, the non-existence Example 6.1 of Aviles and McOwen in [AM88] gives an explicit warped product metric with asymptotic negative scalar curvature but on which we cannot solve the Yamabe problem. From this perspective, we may reframe our main goal of understanding what conditions are required on top of the asymptotic negativity of the scalar curvature in terms of conditions on the warped product reference metrics and the corresponding asymptotic decay.

The second question is more open ended than the first and there does not appear to be much work in the literature focusing on this issue. In particular, the connection of the negativity of the first eigenvalue of the conformal Laplacian to the solvability of the Yamabe problem, as demonstrated by Aviles and McOwen in [AM88], begs the question of what kind of a geometric condition might guarantee the negativity of the eigenvalue on some compact region. A geometric condition which could establish a result in this direction may provide further insight into the Yamabe problem for non-compact manifolds and guide further study.

# 1.4

# A Summary of New Results

We now overview and discuss our main results of the thesis. Our work splits into two parts corresponding broadly to the two main questions discussed in the previous section.

### 1.4.1 The Yamabe Problem on Asymptotically Locally Hyperbolic Manifolds

In our first setting, we study those manifolds with an asymptotically warped product structure at infinity and which satisfy our asymptotic negativity condition on the scalar curvature. A particular class of the above type are asymptotically locally hyperbolic manifolds; these manifolds have received significant attention in the literature independently of the Yamabe problem, see for example [CH03], [CGNP18] and references therein. As discussed, significant progress on the Yamabe problem has been made in this direction via the study of conformal compactifications.

In our approach, we choose to focus on a warped product structure and intrinsic definition of asymptotic hyperbolicity instead of the more standard conformal compactification approach for two reasons. The first reason is in order to mesh our discussion more clearly with our goal of understanding the Yamabe problem on the wider family of asymptotically warped product manifolds. The second reason is to make as few extra assumptions on top of the asymptotic negativity of the scalar curvature on the manifold as possible; the conformal compactification approaches already existing in the literature make additional requirements on the curvature of the manifold. We will discuss this difference in approach in detail in Chapter 3.

To make things precise, our model spaces will be Riemannian manifolds $(M, g)$ satisfying the following conditions:

- $M$ is the union of a compact interior region and an exterior region $M^+ = \mathbb{R}_{\geq 0} \times N$, with $N$ some compact Riemannian manifold,
- $g$ is asymptotic (in a way made precise in Chapter 3) to a locally hyperbolic metric $\mathring{g}$ which is assumed to have the form

$$\mathring{g} = dr^2 + f_k^2(r + r_0)\mathring{h}$$

  for some $r_0 > 0$ where $\mathring{h}$ is a metric on $N$ of constant scalar curvature $(n-1)(n-2)k$ for $k \in \{-1, 0, 1\}$ and $f_k$ is such that $S_{\mathring{g}} = -n(n-1)$.

We refer to such metrics $g$ as *asymptotically locally hyperbolic.* Given an arbitrary positive warping function $f$ in place of $f_k$, we then refer to the metric $\mathring{g}$ as a *warped product metric*. We provide full details of our definitions, notation and terminology in Section 3.1.

We will now overview the results of Chapter 3; more detailed discussion of the results can be found in Section 3.1.3.

Our first theorem is our main result for asymptotically locally hyperbolic manifolds. The theorem is the culmination of new existence results and results about the behaviour of the conformal factor.

**Theorem A.** *Suppose* $(M, g)$ *is an asymptotically locally hyperbolic manifold of order* $\alpha \in (0, n]$ *in the sense of Definition 3.1.1. If the scalar curvature satisfies*

$$S_g \leq -n(n-1) + Ce^{-\alpha r} \text{ on } M \tag{1.2}$$

*for some constant* $C > 0$*, then there exists a positive smooth solution* $u$ *to* (Ya) *on* $M$ *satisfying* $u \geq 1 - \mathcal{O}(e^{-\alpha r})$ *if* $\alpha \in (0, n)$ *and* $u \geq 1 - \mathcal{O}(re^{-nr})$ *if* $\alpha = n$*. Therefore, there exists a complete conformal metric* $\tilde{g}$ *such that* $S_{\tilde{g}} \equiv -n(n-1)$ *on* $M$*.*

*In addition, if the scalar curvature satisfies the stronger condition*

$$|S_g + n(n-1)| \leq Ce^{-\alpha r} \text{ on } M, \tag{1.3}$$

*then* $u = 1 + \mathcal{O}(e^{-\alpha r})$ *if* $\alpha \in (0, n)$ *and* $u = 1 + \mathcal{O}(re^{-nr})$ *if* $\alpha = n$ *and* $u$ *is maximal in that any solution* $\tilde{u}$ *of* (Ya) *satisfies* $\tilde{u} \leq u$*. Furthermore, if* $\alpha \in (0, n)$*, the corresponding conformal manifold* $(M, \tilde{g})$ *is also asymptotically locally hyperbolic of the same order* $\alpha$*.*

We will now briefly discuss the ways in which this result contributes to the existing literature and addresses our main goals of the project.

The major theme of the contribution of Theorem A is the significantly weaker asymptotic requirements on the curvature of the manifold. In particular, we see that our result only require conditions on the scalar curvature; this is in contrast to the existing literature (see [ACF92, AILA18]) where the various conformal compactification approaches all have in common the higher order decay of all sectional curvatures to a negative constant. Additionally, we demonstrate that, in order to solve the Yamabe problem, one need only control the scalar curvature decay from one side (as in (1.2)).

We are also able to make some progress in reproducing similar uniqueness results as established elsewhere in the literature but in our weaker setting. We establish an a priori upper bound on the behaviour of the conformal factor at infinity which allows us to conclude that the solution obtained in Theorem A is maximal. Though this represents progress in that direction, a full uniqueness result was not obtainable within the time constraints of the project.

We recall our main goal for this chapter of understanding what additional requirements are necessary to obtain existence for the Yamabe problem in the case that our manifold is asymptotically of warped product type. In summary, our Theorem A addresses the specific case that our manifold is asymptotically locally hyperbolic and obtains existence under weaker requirements than in the existing literature. In particular, the theorem demonstrates that, in this case, we need only control the behaviour of the scalar curvature in order to obtain existence. We now turn to the more general setting of asymptotically warped product manifolds.

We allow, in place of $f_k$ in the definition of asymptotically locally hyperbolic manifolds above, an arbitrary warping function $f$ and allow an arbitrary compact cross section $(N, h)$. We refer to such manifolds as *asymptotically warped product manifolds.* Our work in this area is most clearly split into two goals; firstly, a study of the conformal classes of the reference warped product metrics and, secondly, a study of the asymptotic requirements for solvability of the Yamabe problem on metrics which asymptote to one of these reference metrics.

Before attempting to generalise our approach in Theorem A to the wider class of asymptotically warped product manifolds, it is important to understand when the conformal class of the reference warped product metric

$$\mathring{g}_f = dr^2 + f^2(r)h$$

admits a representative which is locally hyperbolic. In this case, we may perform a conformal change and then apply our Theorem A directly. We are thus led to our next main result of Chapter 3.

**Theorem** (Theorem 3.3.1)**.** *A metric $\mathring{g}_f$ with a warped product end is conformal to a metric with a locally hyperbolic end if and only if*

$$\int_0^\infty \frac{1}{f(s)}\, ds < \infty. \tag{1.4}$$

We are, furthermore, able to show that those warped product metrics not satisfying (1.4) are conformal to a complete metric of finite volume. As these finite volume metrics are very different in character from the asymptotically locally hyperbolic case, we focus our efforts of the rest of this chapter in applying our Theorem A to those asymptotically warped product manifolds whose reference metric has warping factor satisfying (1.4); evidently, there are a very large class of warped product metrics satisfying this condition. We contrast this, for example, to Example 6.2 of [AM88] which proves existence for exactly warped product manifolds under the much stronger conditions that $f$ is strictly increasing and

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} \frac{f'(r)}{f(r)} = \lim_{r\to\infty} \frac{f''(r)}{f(r)} = +\infty.$$

For our final main result of this chapter, we study the particular conformal factor obtained in the previous theorem above. We will use this conformal factor to provide conditions for which a solution to the Yamabe equation exists on those manifolds whose metrics asymptote to one of the reference warped product metrics. In particular, defining the quantity $H(z) := \int_z^\infty \frac{1}{f}$, we consider asymptotically warped product metrics

$$g_f = \mathring{g}_f + \varepsilon_{za} dz d\theta^a + \varepsilon_{ab} d\theta^a d\theta^b$$

on some exterior region $\mathbb{R}_{\geq 0} \times N$, with perturbation coefficients satisfying

$$\begin{aligned} &\varepsilon_{ab} = \mathcal{O}\left(f^2 H^\alpha\right), && \varepsilon_{za} = \mathcal{O}\left(f H^\alpha\right), \\ &\partial_z \varepsilon_{ab} = \mathcal{O}\left(\left(f^3 H + f' f + f H^{-1}\right) H^\alpha\right), && \partial_z \varepsilon_{za} = \mathcal{O}\left(\left(f^2 H + f' + H^{-1}\right) H^\alpha\right), \\ &\partial_c \varepsilon_{ab} = \mathcal{O}\left(f^2 H^\alpha\right), && \partial_c \varepsilon_{za} = \mathcal{O}\left(f H^\alpha\right) \end{aligned} \tag{1.5}$$

and warping functions satisfying

$$\left|\frac{1}{fH}\right| + \left|\frac{f'}{f}\right| \leq C \text{ and } \left|\frac{f''}{f}\right| \leq C H^{-\alpha}. \tag{1.6}$$

We are than able to apply our Theorem A to prove:

**Theorem** (Theorem 3.3.9)**.** *Let* $(M, g_f)$ *be a manifold with an asymptotically warped product end with perturbation coefficients satisfying* (1.5) *for some* $\alpha \in (0, n)$*. Suppose additionally that the warping function satisfies* (1.4) *and* (1.6)*.*

*If the scalar curvature satisfies*

$$S_{g_f} \leq S_{\mathring{g}_f} + C\frac{H^{\alpha-2}}{f^2} \quad \text{on } M^+$$

*for some constant* $C > 0$*, then there exists a positive smooth solution* $u_f$ *of the Yamabe equation for* $g_f$ *on* $M$ *satisfying*

$$\liminf_{r\to\infty} \left(u_f - \frac{1}{fH}\right) \geq 0$$

*and the corresponding conformal metric* $\tilde{g}$ *is complete and has constant scalar curvature* $S_{\tilde{g}} \equiv -n(n-1)$ *on* $M$*.*

*In addition, if the scalar curvature satisfies the stronger condition*

$$|S_{g_f} - S_{\mathring{g}_f}| \leq C\frac{H^{\alpha-2}}{f^2} \quad \text{on } M^+,$$

*then*

$$\left|u_f - \frac{1}{fH}\right| \longrightarrow 0 \text{ as } r \to \infty$$

*and* $u_f$ *is maximal in that any solution* $\tilde{u}_f$ *of the Yamabe equation for* $g_f$ *on* $M$ *satisfies* $\tilde{u}_f \leq u_f$*. Furthermore, the corresponding conformal manifold* $(M, \tilde{g})$ *is asymptotically locally hyperbolic of order* $\alpha$*.*

This theorem represents the culmination of our work on asymptotically warped product manifolds. We have been able to produce a number of additional constraints on the warping function and asymptotic behaviour in order to guarantee the existence of a solution to the Yamabe equation. As we have discussed in the case of an asymptotically locally hyperbolic manifold, we are able to significantly weaken the requirements for existence of a solution to the Yamabe problem, focusing on conditions only involving the scalar curvature.

### 1.4.2 Volume Ratio Conditions

In this sub-section we provide a brief overview of our main results of Chapter 4. Once again, a more detailed discussion may be found throughout the introduction of the chapter itself in Section 4.1.

We recall our second main question which motivates our work in Chapter 4. We ask what conditions, in addition to the asymptotic negativity of the scalar curvature, are required to deduce the existence of some compact domain on which the first eigenvalue of the conformal Laplacian is negative. As discussed, if we can find such a domain with negative first eigenvalue, we may then argue the existence of a solution to the Yamabe problem.

To this end, we explore possible geometric conditions which may be natural to consider in order to establish the existence of such a negative eigenvalue. When considering the first eigenvalue for the Laplacian, it has been well noted in the literature that an isoperimetric inequality is equivalent to a lower bound on the first Dirichlet eigenvalue for the Laplacian (see for example [Oss78]). We are consequently lead to consider volume ratio type conditions which are similar in feeling to conditions one may see when considering isoperimetry. Our main existence result of this chapter is the following:

**Theorem C.** *Let $(M, g)$ be a Riemannian manifold and suppose there exist two open sets $\Omega_1 \subset \Omega_2$ with $C^1$ boundary which satisfy, for some $R > 0$, that*

$$d_g(x, \partial\Omega_2) = R \text{ for each } x \in \partial\Omega_1,$$

*that*

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)} \leq \sinh^2\left(\frac{\sqrt{n(n-2)}}{2}R\right)$$

*and that the scalar curvature satisfies $S_g \leq -n(n-1)$ on $\Omega_2$. Suppose, furthermore, that $S_g \leq -\varepsilon < 0$ everywhere outside of some compact set for some constant $\varepsilon > 0$. Then there exists a complete metric $\tilde{g}$ conformal to $g$ on $M$ with constant scalar curvature $-n(n-1)$.*

We highlight that the above existence theorem does not make any assumptions on the asymptotic structure of the manifold, in contrast to the results of the previous chapter.

As discussed, we arrive at our main existence result by a study of the role of these volume ratio type conditions in estimating the first eigenvalue of the conformal Laplacian. In particular, we have our second main theorem of the chapter:

**Theorem B.** *Let $(M, g)$ be a Riemannian manifold and suppose there exist two open sets $\Omega_1 \subset \Omega_2$ with $C^1$ boundary which satisfy the conditions of the previous theorem. Then, the conformal Laplacian $-c_n\Delta_g + S_g$ for $(M, g)$ has a negative first eigenvalue on $\Omega_2$.*

The goal of the remainder of our work in this chapter is to establish a concrete set of questions and supporting examples to motivate further work into this alternative perspective which does not appear to have received much attention in the literature thus far. We achieve this through the study of a class of multiply warped product manifolds which generalise the warped product manifolds discussed in the previous chapter.

Through this study, which can be found in Section 4.3, we provide a large class of manifolds to which we can apply our existence theorem, thus demonstrating that the volume ratio condition is not vacuous. Furthermore, this set of examples will contain manifolds which fall outside the scope of the existence results of Chapter 3.

This study of multiply warped products also yields a class of examples which demonstrate the fact that the volume ratio condition

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)} \leq \sinh^2\left(\frac{\sqrt{n(n-2)}}{2}R\right)$$

required for the proof of Theorem B is actually *sharp* for the existence of a negative first eigenvalue for the conformal Laplacian on $\Omega_2$ (which was unexpected to the author, see Remark 4.3.4). More detail on the precise nature of this sharpness will be discussed in Chapter 4.

In summary, we are able to find a geometric condition which provides us with the desired negativity of the first eigenvalue and, in conjunction with our natural asymptotic negative scalar curvature condition, shows the existence of solutions to the Yamabe problem. This result is quite different to much of the existing literature on the non-compact Yamabe problem in simultaneously lacking any global requirements on $(M, g)$ whilst making no asymptotic restrictions other than on the scalar curvature itself. We have provided an exploratory set of examples and questions which we hope will motivate further attention in the study of these volume ratio type conditions and their implications for the Yamabe problem on non-compact manifolds of negative curvature type.

## 1.5

## Structure of the Thesis

We overview the structure of this thesis for the convenience of the reader. Chapter 2 will review the technical details of work from the literature that is used in our approach. We split our own work into two main chapters as already mentioned; Chapter 3 will cover our work and new results in the study of asymptotically locally hyperbolic manifolds and Chapter 4 will cover our new approach to the Yamabe problem via a study of volume ratio conditions. In Chapter 5 we conclude our work by overviewing some further questions we feel are interesting or important but which fell outside the scope of this DPhil project. Finally, we include a number of appendices containing further literature review or additional work supplementary to the main body of the thesis.

# 2

# An Existence Theorem of Aviles and McOwen

In this chapter, we review a particular part of the work of Aviles and McOwen in [AM88] which is of importance in both of our main Chapters 3 and 4. In particular, we will detail the proof of the following theorem:

**Theorem** (Aviles, McOwen [AM88])**.** *If $(M, g)$ is a complete Riemannian manifold with scalar curvature $S_g$ satisfying*

$$S_g(x) \leq -\varepsilon < 0 \qquad on \;\; M, \tag{2.1}$$

*then there is a complete conformal metric $\tilde{g}$ with $S_{\tilde{g}} \equiv -1$.*

Of particular importance in our work is the sub- and super-solution method that Aviles and McOwen employ to establish, from the existence of a suitable sub-solution, the existence of a smooth, positive solution of the Yamabe equation on a non-compact manifold. For the convenience of the reader, we summarise their argument here:

1. Firstly, one establishes the existence of a global sub- solution $u_- \in H^1(M)$.
2. One reduces to solving a problem on a bounded domain by considering an exhaustion $M = \bigcup_{k=1}^{\infty} \Omega_k$, where each $\Omega_k$ is bounded and $\overline{\Omega}_k \subset \Omega_{k+1}$.

3. One finds a weak solution $u_k > 0$ of (Ya) on each compact domain $\Omega_k$ using the method of sub- and super-solutions.

4. One uses elliptic regularity theory to show that each $u_k \in C^\infty$.

5. One establishes an a priori uniform interior supnorm bound for solutions of (Ya) on compact domains.

6. Fixing a compact domain $\Omega_i$ on which we would like to gain convergence, one considers a sequence of solutions above $u_k = u_{i,k} = u_k|_{\Omega_i}$ and uses the a priori bound to get uniform control on the supnorms on the larger domain $\Omega_{i+2}$ (and, consequently, uniform $L^p$ control).

7. One then uses $L^p$ estimates for elliptic equations to obtain $W^{2,p}$ control on $\Omega_{i+1}$.

8. One can then use Sobolev embeddings, choosing $p > n$, to obtain $C^1$ control on $\Omega_{i+1}$. Then, one can use Schauder interior estimates and the Arzela-Ascoli theorem to obtain a $C^2$ convergent subsequence $u_{i,k} \to u_i$, where $u_i$ solves the PDE on $\Omega_i$.

9. One now considers a diagonal sequence $u_{i,i}$ and defines

$$u(x) := \lim_i \ u_{i,i}(x) = \lim_i \ u_i|_{\Omega_i}(x) \tag{2.2}$$

which, using the $C^2$ convergence, provides a solution to (Ya) on all of $M$.

10. Finally, one uses a lower bound on $u$ to deduce completeness of the corresponding $\tilde{g}$.

We note that the role of the global non-positivity assumption in Aviles and McOwen's theorem is in obtaining a sub-solution with which to begin the above argument. Much of the nature of the existence parts of our work in this thesis will involve finding weaker conditions under which we can still find a suitable sub-solution with which to begin the argument above.

For the sake of the completeness of this thesis, in Sections 2.1 and 2.2 we will carefully review the details and provide proofs of the techniques used in the argument of Aviles and McOwen outlined above. However, these techniques are, by now, quite standard in the literature and so *the reader may safely skip the remainder of this chapter and proceed to Chapter 3.*

# 2.1

# The Monotone Iteration Scheme

In this section we provide details of the monotone iteration scheme used in the sub- and super-solution approach of Aviles and McOwen discussed in the next section. In particular, we will prove:

**Proposition 2.1.1.** *Let $\Omega$ be a bounded open subset of $M$ where $(M, g)$ is some Riemannian manifold. Suppose there exist $u_-, u_+ \in H^1(\Omega)$ with $u_- \leq u_+$ that are weak sub- and super- solutions resp. to the PDE problem*

$$\begin{cases} \Delta_g u = f(u, x) & \text{in } \Omega \\ u = h & \text{on } \partial\Omega \text{ in the trace sense} \end{cases} \tag{2.3}$$

*where $h \in H^1(\Omega)$ and $f \in C^1(\mathbb{R} \times \Omega)$ satisfying $|f_z| < C$ for some constant $C$. Then there exists $u \in H^1(\Omega)$ solving (2.3) satisfying $u_- \leq u \leq u_+$.*

We have chosen to state the proposition with fairly restrictive properties on $f$ and limited additional conclusions regarding the solution $u$ in order to encapsulate the fundamental idea. More general equations can fit into these constraints with adjustments made to the problem itself (see approach in part $\boxed{1}$ of the proof of Theorem 2.2.1) but, without somehow being able to bound the derivative of $f$, the argument below fails.

*Proof.* Define $F$ by

$$F(z,x) := f(z,x) - Cz \tag{2.4}$$

so that $F$ satisfies $F_z < 0$. Problem (2.3) is then equivalent to

$$\begin{cases} \Delta_g u - Cu = F(u,x) & \text{in } \Omega \ , \\ u = h & \text{on } \partial\Omega \ . \end{cases} \tag{2.5}$$

Now consider the operator $T : L^2(\Omega) \to H^1(\Omega)$ defined by $Tu = (\Delta_g - C)_h^{-1}(F(u,x))$ where we define $(\Delta_g - C)_h^{-1}(\varphi)$ for $\varphi \in L^2$ to be the unique $H^1(\Omega)$ solution of

$$\begin{cases} \Delta_g w - Cw = \varphi & \text{in } \Omega \ , \\ w = h & \text{on } \partial\Omega \ . \end{cases} \tag{2.6}$$

A fixed point $Tu = u$ implies that $u$ is a solution of our original equation.

We show that $T$ is monotone. For $u_1 \le u_2$, we obtain

$$(\Delta_g - C)(Tu_2 - Tu_1) = F(u_2) - F(u_1) \le 0. \tag{2.7}$$

We may then apply the weak maximum principle [GT01, Theorem 8.1 p.179], noting that $C > 0$ implies the condition (8.8) of [GT01] in our particular case, to conclude that

$$Tu_2 \ge Tu_1 \tag{2.8}$$

and so $T$ is monotone.

We now construct the desired fixed point $u$ as the limit of the sequence $u_{k+1} := Tu_k$ with $u_0 := u_-$.

We first establish that the sequence $u_k$ is monotone increasing. We have already shown that $T$ is monotone and so, by induction, we have that if $u_k \ge u_{k-1}$ then $u_{k+1} = Tu_k \ge Tu_{k-1} = u_k$. Thus, we need only show that $u_- = u_0 \le u_1$ to conclude that the sequence is monotone increasing. To see this, note that $u_-$ weakly satisfies

$$\Delta_g u_- - Cu_- \ge f(u_-, x) - Cu_- = F(u_-) \ . \tag{2.9}$$

Using this, we see that the difference $u_1 - u_-$ weakly satisfies

$$\begin{aligned}(\Delta_g - C)(u_1 - u_-) &= (\Delta_g - C)(\Delta_g - C)_h^{-1}(F(u_-)) - (\Delta_g - C)u_- \\ &= F(u_-) - (\Delta_g - C)u_- \\ &\leq F(u_-) - F(u_-) = 0\end{aligned}$$

and so, again by the weak maximum principle, we obtain that

$$u_1 - u_- \geq \inf_{\partial\Omega}(u_1 - u_-) \geq 0 \tag{2.10}$$

where the final inequality comes from the fact that $u_1 = h$ on $\partial\Omega$ and $u_- \leq h$ on $\partial\Omega$. Therefore we have shown that the sequence $u_k$ is monotone increasing.

We now show that the sequence is bounded above by $u_+$. We assume inductively that $u_{k-1} \leq u_+$, noting that by assumption we have $u_0 = u_- \leq u_+$. Taking a similar approach to the above we have that the difference $u_+ - u_k$ satisfies

$$\begin{aligned}(\Delta_g - C)(u_+ - u_k) &= (\Delta_g - C)u_+ - (\Delta_g - C)(\Delta_g - C)_h^{-1}(F(u_{k-1})) \\ &= (\Delta_g - C)u_+ - F(u_{k-1}) \\ &\leq F(u_+) - F(u_{k-1}) \leq 0\end{aligned}$$

where the last inequality comes from the fact that $F$ is non-increasing. Then we conclude that $u_+ \geq u_k$ in the same way as above.

We have now shown

$$u_0 \leq u_1 \leq ... \leq u_+ \tag{2.11}$$

and so we can define the pointwise limit

$$u(x) := \lim_k u_k(x) \ . \tag{2.12}$$

Certainly $u_- \leq u \leq u_+$. We improve our pointwise convergence by using the fact that $|u - u_k|^2 \leq 2\max(|u_-|^2, |u_+|^2)$; the RHS of the latter is integrable so we can apply the dominated convergence theorem to conclude that $u_k \to u$ in $L^2$. As $T : L^2(\Omega) \to H^1(\Omega)$ is continuous, $Tu_k \to Tu$ in $H^1$. On the other hand, $Tu_k = u_{k+1} \to u$ in $L^2$.

We conclude that $Tu = u$ as required and so obtain a solution $u \in H^1(\Omega)$ of our problem satisfying $u_- \leq u \leq u_+$.

$\square$

## 2.2

## Aviles and McOwen's Theorem for Existence on Non-Compact Manifolds of Negative Scalar Curvature

In this section we provide a detailed review of the sub- and super-solution argument of Aviles and McOwen in [AM88] as their approach is of special relevance to our work.

We recall that we will present a detailed proof of the following theorem which solves the Yamabe problem in the most restrictive setting of [AM88] (a small simplification of Theorem A in [AM88]) which we feel demonstrates with most clarity the fundamentals of their argument.

**Theorem 2.2.1** (Aviles, McOwen [AM88]). *If $(M, g)$ is a complete Riemannian manifold with scalar curvature $S_g$ satisfying*

$$S_g(x) \leq -\varepsilon < 0 \qquad on \;\; M, \tag{2.13}$$

*then there is a complete conformal metric $\tilde{g}$ with $S_{\tilde{g}} \equiv -1$.*

To reiterate, our goal is to find a smooth solution $u > 0$ of

$$c_n \Delta_g u = u^{\frac{n+2}{n-2}} + S_g u \tag{2.14}$$

on all of $M$ such that $\tilde{g} = u^{\frac{4}{n-2}} g$ is complete. As discussed in the overview at the start of this chapter, we will use the iteration scheme of the previous section with a sub-solution of the equation above to solve the equation on compact domains which form an exhaustion of $M$. The remaining work is in showing convergence to a solution on the entire manifold.

*Proof.*

1 Existence of a positive weak solution on some bounded $\Omega \subset M$.

We apply the approach of sub- and super-solutions proved in the previous Section 2.1. To apply the result, we need to adjust our $f$ to satisfy the bounded derivative requirement and to find suitable sub- and super-solutions and to decide an appropriate boundary condition.

For our $f(z,x) = C[z^{\frac{n+2}{n-2}} + S_g(x)z]$ it is clear that we can only hope for the derivative in $z$ to be bounded if we restrict $|z|$ to be bounded (note that $S_g$ is smooth thus bounded on $\Omega$ meaning the second term is unproblematic). In other words, we would like to ensure that the solution we would obtain would be bounded in absolute value, this would certainly be the case if we can find bounded $u_-$ and $u_+$. If we can find such sub- and super-solutions, we can then cut-off $f$ in the $z$ variable and solve the equivalent problem via the proposition above directly.

To obtain the sub- and super-solutions we desire, we analyse directly the RHS $f(u,x)$. In particular, utilising the fact that $-m \leq S_g(x) \leq -\epsilon$ on $\Omega$ (using in the first inequality that $S_g$ is smooth and in the second our main assumption) we get

$$C[u^{\frac{n+2}{n-2}} - mu] \leq f(u,x) \leq C[u^{\frac{n+2}{n-2}} - \epsilon u] \ . \tag{2.15}$$

Thus, we can choose a constant $u_-$ ($u_+$) with $|u_-|$ sufficiently small (resp. large) so that $0 \geq f(u_-,x)$ (resp. $f(u_+,x) \geq 0$) with $u_- < u_+$ from which we find

$$\int_\Omega \nabla u_- \cdot \nabla v = 0 \leq \int_\Omega f(u_-,x)v \tag{2.16}$$

(and vice versa). Picking $h(x) = u_-$ as our boundary condition we conclude that $u_-$, $u_+$ are valid sub and super solutions and, of course, satisfy the boundedness conditions we required to cut-off $f$. Thus we obtain a solution to the problem $u_- \leq u \leq u_+$ with $u \in H^1(\Omega)$. In the above it is clear that we are free to choose $u_- > 0$ and so we have, furthermore, that $u$ is positive.

$\boxed{2}$ Bootstrapping with elliptic regularity to deduce smoothness of $u$

We use standard elliptic regularity theory on $\Omega$ to conclude higher regularity on $u$. In particular, we use Theorem 8.12 of [GT01, p. 186] with $k = 1$, noting that $f(z, x) \in C^\infty(\mathbb{R} \times M)$ provided $z > 0$ so that, in particular, $f(u, \cdot) \in H^1(\Omega)$ as $u > 0$ and $u$ is bounded. Thus, Theorem 8.10 gives us that $u \in H^3(\Omega)$ and we can repeat the above inductively with $k = 3, 5, ...$ to obtain $u \in C^\infty(\Omega)$.

$\boxed{3}$ Uniform boundedness for an approximating sequence

Let $\{\Omega_k\}_{k=1}^\infty$ be an exhaustion of $M$ (that is, $\cup_{k=1}^\infty \Omega_k = M$) with $\Omega_k$ open, bounded, with $C^2$ boundary and $\overline{\Omega}_k \subset \Omega_{k+1}$. Step $\boxed{1}$ provides, on each $\Omega_k$, a positive smooth solution $u_k$ to (Ya). Our goal will be to show that, as $k \to \infty$, the sequence $u_k$ converges to some smooth $u$ locally in $C^2$. This $u$ will be our desired solution to (Ya) on all of $M$. We note that the upper bounds provided by construction in $\boxed{1}$ for each $u_k$ may deteriorate as $k \to \infty$. The following provides the required local uniform boundedness of $u_k$.

**Proposition 2.2.2.** *Let $(M, g)$ be a Riemannian manifold and $\Omega \subset M$ some bounded domain. For every compact $X \subset \Omega$, there a exists a constant $C_0$ such that for any non-negative weak solution $u \in H^1(\Omega)$ of $\Delta_g u \geq u^\alpha + S_g u$ in $\Omega$ for $\alpha > 1$,*

$$\sup_X \ u \leq C_0 \ . \tag{2.17}$$

*Proof.* Let $\{B_R(y_i)\}_{i=1}^m$ be a finite cover of $X$, where we choose $R > 0$ sufficiently small so that each $B_{2R}(y_i)$ is contained in a chart. A key result will be the de Giorgi-Moser-Nash Theorem (see, for example, Theorem 8.17 of [GT01, p. 194]). By smoothness of $S_g$, we have that $\Delta_g u \geq -\left(\sup_{B_{2R}(y_i)} S_g\right) u$ on $B_{2R}(y_i)$ and so the de Giorgi-Moser-Nash Theorem gives us that,

$$\sup_X \ u \leq \sup_{B_R(y_i)} \ u \leq C R^{-n/p} \|u\|_{L^p(B_{2R}(y_i))} \tag{2.18}$$

where the first inequality is true for at least one $y_i$, $p > 1$ and $C$ depends only on $n$, $p$, $\sup_{B_{2R}(y_i)} S_g$ and the ellipticity constants for $\Delta_g$ in the corresponding local charts.

We would now like to use that $u$ weakly satisfies $\Delta_g u \geq u^\alpha + S_g u$. We note that, for $\varphi \in C_0^\infty(\Omega)$ and $\varphi \equiv 1$ on $B_{2R}(y_i)$, $\varphi \geq 0$,

$$\|u\|^p_{L^p(B_{2R}(y_i))} \leq \int_\Omega u^p \varphi^q \tag{2.19}$$

and so we test against $u^{p-\alpha}\varphi^q$ for some $q \geq 0$ to be fixed to obtain,

$$\int_\Omega u^p \varphi^q \leq -\int_\Omega \nabla u \cdot \nabla(u^{p-\alpha}\varphi^q) + M \int_\Omega u^{p-\alpha+1}\varphi^q =: A \tag{2.20}$$

where we used that $S_g(x) \geq -M$ once again.

In the computations that follow, $p$ could be taken as any value satisfying $p \geq \alpha+1$. For ease of exposition we take $p = \alpha + 1$. Expanding the term $\nabla(u^{p-\alpha}\varphi^q)$ we obtain

$$A = -q \int_\Omega u\varphi^{q-1}\nabla u \cdot \nabla\varphi - \int_\Omega \varphi^q |\nabla u|^2 + M \int_\Omega u^2 \varphi^q \ . \tag{2.21}$$

By the Cauchy-Schwarz inequality we can bound the first term on the RHS of (2.21) as follows

$$-q \int_\Omega u\varphi^{q-1}\nabla u \cdot \nabla\varphi \leq \int_\Omega \varphi^q |\nabla u|^2 + \frac{q^2}{4} \int_\Omega u\varphi^{q-2}|\nabla\varphi|^2 \ . \tag{2.22}$$

Hence

$$A \leq \frac{q^2}{4} \int_\Omega u^2 \varphi^{q-2}|\nabla\varphi|^2 + M \int_\Omega u^2 \varphi^q \ . \tag{2.23}$$

We next use Hölder's inequality to absorb the remaining terms containing $u$ into the LHS of (2.20). We choose $q = \frac{2(\alpha+1)}{\alpha-1}$ so that

$$(u^2\varphi^{q-2})^{\frac{q}{q-2}} = u^{\frac{2q}{q-2}}\varphi^q = u^p \varphi^q \ . \tag{2.24}$$

We hence have

$$A \leq \left(\int_\Omega u^p \varphi^q\right)^{\frac{q-2}{q}} \left[\frac{q^2}{4}\left(\int_\Omega |\nabla\varphi|^q\right)^{\frac{2}{q}} + M\left(\int_\Omega \varphi^q\right)^{\frac{2}{q}}\right] \ . \tag{2.25}$$

Putting back into (2.20), we get

$$\int_\Omega u^p\varphi^q \le \left(\frac{q^2}{4}\left(\int_\Omega |\nabla\varphi|^q\right)^{\frac{2}{q}} + M\left(\int_\Omega \varphi^q\right)^{\frac{2}{q}}\right)^{\frac{q}{2}} =: C_1\ . \tag{2.26}$$

This, together with (2.18) and (2.19) gives

$$\sup_X\ u \le CR^{-n/p}\|u\|_{L^p(B_{2R}(y_i))} \le CR^{-\frac{n}{p}}\left(\int_\Omega u^p\varphi^q\right)^{\frac{1}{p}} \le CC_1^{\frac{1}{p}}R^{-\frac{n}{p}} =: C_0$$

as required.

□

**4** Take a sequence of solutions on an exhaustion of M

We restrict to each $\Omega_i$ and consider the solutions $u_k$ restricted to $\Omega_i$ for $k \ge i+3$. By Proposition 2.2.2 we have

$$\sup_k \|u_k\|_{L^\infty(\overline{\Omega}_{i+2})} \le C_i\ , \tag{2.27}$$

where we have taken $X = \overline{\Omega}_{i+2}$.

Note that our sequence of solutions $u_k$ are all smooth on $\Omega_{i+2}$ and solve (Ya) with $K \equiv -1$, which can be written in the form

$$(Lu_k =)c_n\Delta_g u_k - S_g(x)u_k = u_k^{\frac{n+2}{n-2}}(= f_k)\ . \tag{2.28}$$

As the $u_k$ are smooth, uniformly bounded and $x \mapsto x^{\frac{n+2}{n-2}}$ is $C^1$, the $f_k$ are uniformly bounded in $L^p$ for any $p$. Thus we can apply Theorem 9.11 of [GT01] (the coefficients of $L$ more than satisfy the requirements) to obtain

$$\|u_k\|_{W^{2,p}(\Omega_{i+1})} \le C(\|u_k\|_{L^p(\Omega_{i+2})} + \|f_k\|_{L^p(\Omega_{i+2})}) \le C_i\ . \tag{2.29}$$

Furthermore, taking $p > n$, Sobolev embeddings allow us to conclude that $\|u_k\|_{C^{1,\alpha}(\Omega_{i+1})} \le C_i$ for some $\alpha \in (0,1)$.

Using that $\|f_k\|_{C^{0,\alpha}(\Omega_{i+1})} \le \|f_k\|_{C^1(\Omega_{i+1})} \le C_i$ (as $u_k$ are uniformly $C^1$ bounded), now we can apply the Schauder interior estimate of Theorem 6.2 in [GT01] to obtain

$$\|u_k\|_{C^{2,\alpha}(\Omega_i)} \le C(\|u_k\|_{C^{0,\alpha}(\Omega_{i+1})} + \|f_k\|_{C^{0,\alpha}(\Omega_{i+1})}) \le C_i\ . \tag{2.30}$$

Now by the Arzela-Ascoli theorem we have shown that, on any $\Omega_i$, we can take a $C^2$ convergent subsequence of the $u_k$ (whose limit necessarily solves (Ya) there). We now construct our solution via a diagonal argument. In particular, we define our global solution $u$ successively on our exhaustion as follows: First we take a subsequence $u_{k_j}$ as above on $\Omega_1$, writing

$$u_j^1 := u_{k_j} \xrightarrow{C^2(\Omega_1)} u^1 =: u|_{\Omega_1}$$

and continue inductively taking more subsequences so that

$$u_j^i := u_{k_j}^{i-1} \xrightarrow{C^2(\Omega_i)} u^i =: u|_{\Omega_i}$$

noting that $u^i$ and $u^{i-1}$ agree on $\Omega_{i-1}$ and so $u$ is well defined. We then define

$$u(x) := \lim_i u_i^i(x) \tag{2.31}$$

we can eventually use the $C^2$ convergence obtained for each $\Omega_i$ to deduce that the above limit is well defined anywhere in $M$ and that $u \in C^2(M)$ solves (Ya) on all of $M$. As $u_k \geq u_- > 0$, as shown in $\boxed{1}$, the limit $u$ is strictly positive via the $C^2$ convergence.

$\boxed{5}$ <u>Completeness of $\tilde{g}$</u>

To see, finally, that $\tilde{g} = u^{\frac{4}{n-2}} g \geq u_-^{\frac{4}{n-2}} g = Cg$ is complete, we note that the length of a curve $\gamma$ satisfies $\mathrm{L}_{\tilde{g}}(\gamma) \geq C' \mathrm{L}_g(\gamma)$. Thus, for any divergent $\gamma$, completeness of $g$ implies $\gamma$ has infinite length with respect to $g$ and so also with respect to $\tilde{g}$, thus $\tilde{g}$ is complete. $\square$

To conclude this section, we remark here that the above argument depended on the strict negativity condition (2.13) only in parts $\boxed{1}$ , in determining the existence of a subsolution, and $\boxed{5}$, in establishing completeness of the resulting $\tilde{g}$. We will see in the remaining sections of this document that these two goals will be the main obstacles in weakening the heavy restrictions required for existence theorem above.

# 3

# The Yamabe Problem on Asymptotically Hyperbolic Manifolds

In this chapter, we study solvability and uniqueness for the Yamabe problem on asymptotically hyperbolic manifolds and then go on to relate this model case to the wider question of solvability on manifolds which have an asymptotically warped product end. We recall that our goal in this section is to try and understand when we can solve the Yamabe problem without any assumptions on the metric in an arbitrary interior region and with the scalar curvature being negative in the exterior region; in this chapter we add the simplifying assumption that the exterior region can be expressed in a warped product structure.

In Section 3.1 we provide an introduction of the particular model spaces we will use, give a review of relevant literature and overview our new results and their implications regarding the wider goals and questions of the thesis. In Section 3.2, we obtain new existence results in the well studied asymptotically locally hyperbolic setting, in particular we weaken the curvature requirements of the asymptotically locally hyperbolic end to involve *only* the scalar curvature. Finally, in Section 3.3, we provide supplementary results relating a wider class of asymptotically warped product manifolds satisfying a particular criterion on the warping factor to the asymptotically locally hyperbolic setting.

# 3.1 Introduction

We recall from Chapter 1 that we consider complete non-compact manifolds $(M, g)$ which may be separated into a compact interior region $M_0$ and an exterior region $M^+$ and that we will not make any assumptions about the behaviour of $g$ on $M_0$. Our goal is to find a conformal metric to $g$ which is complete and has constant negative scalar curvature on all of $M$. As discussed in the introduction of this thesis, we make the natural additional assumption that the scalar curvature $S_g$ satisfies a negativity condition of the type

$$\limsup S_g \leq -\varepsilon < 0$$

for some $\varepsilon > 0$ in the exterior region $M^+$.

In this chapter, we aim to provide some insight into the Yamabe problem in the setting above with some simplifying assumptions on the structure of $M^+$. In particular, we suppose that $M^+$ can be written as the product manifold $\mathbb{R}_{\geq 0} \times N$ where $N$ is some compact manifold. Additionally, we consider those metrics which are asymptotic to a warped product of the radial factor above and some fixed metric $\mathring{h}$ on $N$. We consider various such metrics throughout the chapter and make their definitions precise in due course.

We note here that we may choose, without loss of generality, that the metric $\mathring{h}$ has constant scalar curvature on $N$; this is a consequence of the already mentioned fact that the Yamabe problem has been solved in the affirmative for any compact Riemannian manifold $(N, \mathring{h})$ and so we may always take a representative in the conformal class of $(M^+, g)$ which has a constant scalar curvature cross-section $(N, \mathring{h})$.

To summarise, on a given non-compact Riemannian manifold $(M, g)$ which has an exterior region on which the metric is asymptotic to a warped product metric, we would like to find a complete metric $\tilde{g}$ conformal to $g$ with constant negative scalar curvature. If we write the metric as $\tilde{g} = u^{\frac{4}{n-2}} g$ for some smooth function $u > 0$, then $u$ must satisfy the Yamabe equation

$$-c_n \Delta_g u + S_g u = -n(n-1) u^{\frac{n+2}{n-2}} \qquad \text{on } M. \tag{Ya}$$

Our goal in making the above simplifying assumptions is to first try and find some condition on the warping function of the asymptotically warped product end which will allow us to conclude existence of a solution to the Yamabe problem. We note that, as we have seen in the discussion regarding Example 6.1 of [AM88], the asymptotically negative scalar curvature alone is insufficient to conclude that a solution to the Yamabe problem exists. Once we have found such an additional condition on the warping function, we can then hope that a study of this condition may yield some further insight into the Yamabe problem in a wider setting.

Our explorations in this setting lead us to consider the particular class of such warped product manifolds which are asymptotically locally hyperbolic. The Yamabe problem on asymptotically hyperbolic manifolds is well studied in the literature as discussed in Chapter 1; we review relevant details of the significant progress already made later in the introduction. We highlight here that our goal above leads us to diverge from the definitions of asymptotic local hyperbolicity found in the literature. In particular, we avoid requiring decay of the full curvature tensor to $-1$ and rely only on a negativity condition on the scalar curvature.

In Section 3.1.1, we define our notion of asymptotically locally hyperbolic metrics and make clear the relevant notation and terminology. In addition, Section 3.1.1 also provides a comparison of the conformally compact models used in the literature for asymptotically hyperbolic manifolds with our own weaker definition. We then provide a literature review of the existing progress on the problem in various related settings in Section 3.1.2. Finally, we conclude the introductory section with an overview of our new results in Section 3.1.3 and discuss their place in the literature.

### 3.1.1 Definition of Asymptotic (Local) Hyperbolicity

We will consider manifolds which may be decomposed as a union $M = M_0 \cup M^+$, where $M_0$ is some compact interior region, $M^+$ is a non-compact exterior region and both parts are disjoint apart form their common boundary. We assume further that we may express $M^+ = \mathbb{R}_{\geq 0} \times N$ where $N$ is some $(n-1)$-dimensional compact manifold. On the end $\mathbb{R}_{\geq 0} \times N$, we denote by $r$ a coordinate on the $\mathbb{R}_{\geq 0}$ fibre. Additionally, we denote the coordinates on any local (angular) chart on $N$ with a $\theta^a$, where $a = 1, \ldots, n-1$, and we use $a, b, c, \ldots$ to index angular coordinates. When referring to the full set of coordinates on $M^+$ we use the notation $x^1 = \theta^1, \ldots, x^{n-1} = \theta^{n-1}$, $x^n = r$ and we use $i, j, k, \ldots$ to index over all coordinates.

We define a reference locally hyperbolic metric $\mathring{g}$ on the exterior region $M^+$ to be

$$\mathring{g} = dr^2 + f_k^2(r + r_0)\mathring{h} \tag{3.1}$$

for some $r_0 > 0$ where $\mathring{h}$ is a metric on $N$ of constant scalar curvature $(n-1)(n-2)k$ for $k \in \{-1, 0, 1\}$ and

$$f_k(r) = \begin{cases} \sinh(r) & k = 1, \\ e^r & k = 0, \\ \cosh(r) & k = -1. \end{cases}$$

In particular, when $k = 1$ and $N = \mathbb{S}^{n-1}$ in the above, one recovers the standard hyperbolic metric from $\mathring{g}$.

Note that the scalar curvature of a warped product metric like $\mathring{g}$ may be computed via the formula

$$S_{\mathring{g}} = -2(n-1)\frac{f_k''}{f_k} - (n-1)(n-2)\left(\frac{f_k'}{f_k}\right)^2 + \frac{S_{\mathring{h}}}{f_k^2}.$$

From this we may readily compute that $S_{\mathring{g}} \equiv -n(n-1)$ for each $k$ in the above definition.

For the next definition below, we choose a finite set of preferred charts $U_i$ covering $N$, each with a preferred choice of local coordinates $\{\theta^1, \ldots, \theta^{n-1}\}$. We extend these charts to $M^+$ by defining $V_i = \mathbb{R}_{\geq 0} \times U_i$ with coordinates $\{r, \theta^1, \ldots, \theta^{n-1}\}$ and fix them from hereon. For a function $\varphi$ on the end $\mathbb{R}_{\geq 0} \times N$ we use the notation $\varphi = \mathcal{O}_k(e^{-\alpha r})$ to indicate that $\varphi$ and all of its first $k$ derivatives in the coordinates defined above decay as $e^{-\alpha r}$, that is there exists a constant $C > 0$ such that $\varphi$ satisfies

$$|\varphi| + |\partial_\beta \varphi| \leq Ce^{-\alpha r}$$

where $\beta$ indicates any multi-index with $|\beta| \leq k$.

We may now state our definition of asymptotically locally hyperbolic manifolds which is in line with other definitions in the literature (for example, see [CH03]); we will highlight important comparisons between the setting we use and those seen in the literature later in this section.

**Definition 3.1.1** (Asymptotically locally hyperbolic). *We say a Riemannian manifold $(M, g)$ is* asymptotically locally hyperbolic of order $\alpha$ *for some $\alpha > 0$ if we can write $M = M_0 \cup (\mathbb{R}_{\geq 0} \times N)$ and we can write the metric $g$ on $\mathbb{R}_{\geq 0} \times N$ as*

$$g_{rr} = \mathring{g}_{rr} = 1 \tag{3.2}$$

$$g_{ab} = \mathring{g}_{ab} + \mathcal{O}_1(e^{-(\alpha-2)r}) \tag{3.3}$$

$$g_{ra} = \mathcal{O}_1(e^{-(\alpha-1)r}) \tag{3.4}$$

*where $\mathring{g}$ is a reference locally hyperbolic metric defined in (3.1). If $(N, \mathring{h})$ is the round sphere then we drop the word locally and simply say that $g$ is asymptotically hyperbolic.*

It is clear that if $(M, g)$ is asymptotically locally hyperbolic of order $\alpha$ then it is also asymptotically locally hyperbolic of order $\alpha'$ for any $0 < \alpha' < \alpha$.

We comment here that one interpretation of the above asymptotic conditions is as imposing a type of "asymptotic orthogonality". In particular, considering the frame on the exterior region $M^+$ defined by $\{\partial_r, e^{-r}\partial_1, ..., e^{-r}\partial_{n-1}\}$ which is, in a sense, "asymptotically of unit size" with respect to the reference metric $\mathring{g}$ in each direction, the conditions (3.2)–(3.4) ensure that if $\{\partial_r, e^{-r}\partial_1, ..., e^{-r}\partial_{n-1}\}$ are orthogonal with respect to $\mathring{g}$ then they "remain" orthogonal with respect to the perturbed metric $g$ up to an error term of size $\mathcal{O}(e^{-\alpha r})$.

In the remainder of this document, we will make regular use of the coordinate function $r$ corresponding to the $\mathbb{R}_{\geq 0}$ fibre of the exterior region $M^+$. We note that the particular choice of $r$ is not unique, in that the reference metric $\mathring{g}$ defined above may be expressed in the form (3.1) for arbitrarily many choices of coordinate function $r$ via diffeomorphism of $M^+$ or by an altogether different choice of splitting of $M$ into the interior and exterior regions $M_0$ and $M^+$. To avoid this frustration, whenever we speak of an asymptotically locally hyperbolic manifold as defined above, we implicitly assume that there is a pre-chosen $r$.

Before we compare our definition with those found in the literature, we provide the following computational lemma establishing the corresponding decay of the metric inverse and Christoffel symbols which will be useful to us later in this chapter.

**Lemma 3.1.2.** *Suppose $(M, g)$ is an asymptotically locally hyperbolic manifold of order $\alpha$. Then,*

$$\begin{aligned} g^{rr} &= 1 + \mathcal{O}_1(e^{-2\alpha r}) \\ g^{ra} &= \mathcal{O}_1(e^{-(\alpha+1)r}) \\ g^{ab} &= \mathring{g}^{ab} + \mathcal{O}_1(e^{-(\alpha+2)r}) \end{aligned}$$

*and*

$$\begin{aligned} \Gamma^r_{rr} &= \mathcal{O}\left(e^{-\alpha r}\right), & \Gamma^a_{br} &= \mathring{\Gamma}^a_{br} + \mathcal{O}\left(e^{-\alpha r}\right), \\ \Gamma^a_{rr} &= \mathcal{O}\left(e^{-(\alpha+1)r}\right), & \Gamma^r_{ab} &= \mathring{\Gamma}^r_{ab} + \mathcal{O}\left(e^{-(\alpha-2)r}\right), \\ \Gamma^r_{ar} &= \mathcal{O}\left(e^{-(\alpha-1)r}\right), & \Gamma^a_{bc} &= \mathring{\Gamma}^a_{bc} + \mathcal{O}\left(e^{-\alpha r}\right), \end{aligned}$$

*where we use the notation $\mathring{\Gamma}$ to denote the Christoffel symbols of the reference locally hyperbolic metric $\mathring{g}$ which satisfy*

$$\mathring{\Gamma}^r_{ab} = \mathcal{O}\left(e^{2r}\right), \qquad \mathring{\Gamma}^a_{br} = \mathcal{O}\left(1\right), \qquad \mathring{\Gamma}^a_{bc} = \mathcal{O}\left(1\right).$$

*Proof.* As $\mathring{h}$ is positive definite, there exists a non-singular $B$ such that $\mathring{h} = B^T B$ and so we may write $\mathring{g} = A^T A$ where

$$A = \left( \begin{array}{c|c} 1 & 0 \\ \hline 0 & f_k(r)B \end{array} \right).$$

We may then write $g = A^T(I+\varepsilon)A$ where

$$\varepsilon = \left( \begin{array}{c|c} 0 & \mathcal{O}_1(e^{-\alpha r}) \\ \hline \mathcal{O}_1(e^{-\alpha r}) & \mathcal{O}_1(e^{-\alpha r}) \end{array} \right).$$

Consequently, we may write

$$\begin{aligned} g^{-1} &= A^{-1}\left(I+\varepsilon\right)^{-1}(A^{-1})^T = A^{-1}(I - \varepsilon + \mathcal{O}(\varepsilon^2))\,(A^{-1})^T \\ &= \mathring{g}^{-1} - A^{-1}\varepsilon(A^{-1})^T + A^{-1}\mathcal{O}(\varepsilon^2)(A^{-1})^T \end{aligned}$$

from which the decay of the inverse components follows. Having established the decay of the inverse, the decay of the Christofell symbols can be computed directly. □

We now relate our definition above to definitions of asymptotically locally hyperbolic manifolds found in the literature, for example in [CH03]. In the definition found there, the manifold $M$ is defined in an equivalent way with a corresponding reference metric on the exterior region defined as

$$\mathring{g} = \frac{dx^2}{(x+x_0)^2+k} + (x+x_0)^2\mathring{h}$$

where $k \in \{-1, 0, 1\}$ corresponds to the sign of the constant scalar curvature metric $\mathring{h}$ and we use $x$ to denote their alternative coordinate on $\mathbb{R}_{\geq 0}$. The equivalence of the two definitions remains only to be seen in the change between the 'radial' coordinates $r$ and $x$. We define $x = f_k(r+r_0) - x_0$ so that $dx = f_k'(r+r_0)dr$ and we obtain

$$\mathring{g} = \frac{(f_k'(r+r_0))^2}{f_k^2(r+r_0)+k}dr^2 + f_k^2(r+r_0)\mathring{h}.$$

For the two definitions to agree, we require that $(f_k'(r))^2 = f_k^2(r) + k$, the solutions of which correspond directly to the $f_k$ in (3.1).

Elsewhere in the literature, asymptotically hyperbolic manifolds are often defined in terms of a sufficiently regular conformal compactification of the metric, often $C^{1,1}$ or better. In particular, in many works (for example in [ACF92]), the error terms (3.3) and (3.4) involve some sense of decay in the second derivatives. Even when the regularity of the compactification is allowed to be weaker, for example as in [AILA18], the definitions used still require a decay of the full curvature tensor to $-\mathrm{Id}$. In our definition, as we choose a warped product model and make our definition of asymptotically locally hyperbolic purely intrinsic, we do not require direct assumptions on the regularity of such a conformal compactification. In particular, we require no control of the second derivatives of the metric components at infinity and, furthermore, do not directly impose any conditions on decay of the curvature tensor aside from on the scalar curvature.

Additionally, we note that we do not assume $\mathring{h}$ is Einstein (and so neither is $\mathring{g}$) and so $g$ does not asymptote to an Einstein metric as required in other definitions used in the literature, for example in [ACF92] and [AILA18].

We provide an example of an asymptotically locally hyperbolic manifold in the sense of Definition 3.1.1 which does not fit into the usual definitions taken in the literature concerning the Yamabe problem in this setting.

**Example 3.1.3.** *Consider the warped product manifold* $M^3 = \mathbb{R}_{\geq 0} \times \mathbb{T}^2$ *where* $\mathbb{T}^2$ *is the* 2*-dimensional flat torus with metric* $(dx^1)^2 + (dx^2)^2$ *and standard coordinates* $\{x^a\}$*. We endow* $M^3$ *with the diagonal metric*

$$g = dr^2 + e^{2r}\left(p(r)(dx^1)^2 + p^{-1}(r)(dx^2)^2\right).$$

*This metric is asymptotically locally hyperbolic in the sense of Definition 3.1.1 provided, for example,*

$$p(r) = 1 + \mathcal{O}(e^{-\alpha r})$$
$$p'(r) = \mathcal{O}(e^{-\alpha r})$$

*which we assume in this example. The metric g has Ricci curvature*

$$\begin{aligned}
R_{rr} &= -2 - \frac{1}{2}\left(\frac{p'}{p}\right)^2, \\
R_{11} &= e^{2r}p\left(-2 + \frac{1}{2}\left(\frac{p'}{p}\right)^2 - \frac{p'}{p} - \frac{1}{2}\left(\frac{p''}{p}\right)\right), \\
R_{22} &= e^{2r}p^{-1}\left(-2 - \frac{1}{2}\left(\frac{p'}{p}\right)^2 + \frac{p'}{p} + \frac{1}{2}\left(\frac{p''}{p}\right)\right).
\end{aligned}$$

*We see that the Ricci curvature does not necessarily decay to a constant multiple of the metric (note the presence of the $p''$ term in $R_{11}$ and $R_{22}$). In contrast, the metric $g$ has scalar curvature*

$$S_g = -6 - \frac{1}{2}\left(\frac{p'}{p}\right)^2 = -6 - \mathcal{O}(e^{-\alpha r}). \tag{3.5}$$

The example above demonstrates a class which does not satisfy the requirements (discussed in more detail in the following section) in [ACF92] or [AILA18] but which falls under Definition 3.1.1. There are certainly many such $p$ which behave wildly in $C^2$ and so have poor behaviour of the Ricci curvature, for example take $p(r) = 1 + e^{-2\alpha r}\sin(e^{\alpha r})$.

The question in general of when intrinsic definitions of non-compact asymptotically locally hyperbolic manifolds imply the existence of a $C^{1,1}$ conformal compactification is addressed, for example, in [BG11] and [Gic13]. However, in these papers the various intrinsic assumptions required impose stronger conditions than we will require, for example again decay to a negative constant of all sectional curvatures.

### 3.1.2 Literature Review

We now overview existence theorems known in the literature. We first briefly discuss the work of [AM88] in this context. We then review the significant progress made regarding the Yamabe problem in the specific case of asymptotically locally hyperbolic manifolds via a conformal compactification approach.

#### 3.1.2.1 The Sub- and Super-Solution Argument of Aviles and McOwen

As already mentioned in Chapter 1, Aviles and McOwen [AM88] provided existence results on a broad class of negatively curved manifolds. For example, they are able to prove existence given either:

1. $S_g \leq 0$ globally on $M$ and $S_g \leq -\epsilon < 0$ outside of some compact set,
2. There exists a negative first eigenvalue for the conformal Laplacian on some compact domain in $M$.

Though their results are robust enough to apply to a large class of manifolds, they are not well suited to asymptotically hyperbolic manifolds as condition 1 requires a global assumption of at least non-positivity of the scalar curvature and it is known that condition 2 cannot be true in the model hyperbolic space itself. For more discussion into the first eigenvalue approach, see Chapter 4 of this thesis.

Central to our existence proof will be certain techniques developed by Aviles and McOwen in [AM88] covered in Chapter 2 where we provided an overview of the sub- and super-solution argument used in their proof. Their argument establishes the existence of a positive smooth solution to the Yamabe equation on a non-compact Riemannian manifold given the existence of a non-negative sub-solution and a global non-positivity condition on the scalar curvature.

We take a moment to again highlight the key difference between this result and results for asymptotically hyperbolic manifolds to be discussed next, is that the result above relies upon the *global* condition on the non-positivity of the scalar curvature. In removing this global non-positivity of the scalar curvature, the obstacle to directly applying the argument from Chapter 2 is in establishing the existence of a sub-solution to (Ya). This will be the focus of the existence part of our Theorem A.

#### 3.1.2.2 Conformal Compactification and Existence Results for Asymptotically Locally Hyperbolic Manifolds

We now briefly review major milestones in the progress toward understanding the Yamabe problem for asymptotically locally hyperbolic manifolds, from early pioneering work through to recent advancements.

As mentioned in Chapter 1, an important notion involved in the study of asymptotically hyperbolic manifolds is that of a conformal compactification model (see e.g. [FG85], [GL91]). In this case, $g$ is assumed to be conformal to an underlying metric $\bar{g}$ on a compact manifold $\overline{M}$ with boundary via some *defining function for the boundary* $\rho$, namely we write $g = \rho^{-2}\bar{g}$. Here, a defining function is some non-negative function $\rho$ whose zero set coincides with $\partial\overline{M}$ and which satisfies $|d\rho| \neq 0$. From this perspective, one can view the Yamabe problem as equivalent to finding a conformal metric $\tilde{g}$ of constant negative scalar curvature such that the conformal factor $\tilde{u}$ on $\overline{M}$ blows up at the boundary $\partial\overline{M}$.

In the case that the conformal compactification $\overline{M}$ above is a Euclidean domain, the existence of such a $\tilde{u}$ was established in an early pioneering work of Loewner and Nirenberg, see [LN74, Sections 2–5]. In their work, they demonstrated that the condition that $\tilde{u}$ blows up at the boundary of $\overline{M}$ is related to a notion of regularity of the boundary; this notion of regularity at the boundary is in fact shown to depend solely on the Hausdorff dimension of the boundary.

In the case that $g$ exhibits a $C^2$ conformal compactification (i.e. the underlying metric $\bar{g}$ is a $C^2$ metric on $\overline{M}$), Andersson, Chruściel and Friedrich solved the existence problem in [ACF92]. In comparison to Definition 3.1.1, we note that the existence of such a conformal compactification requires at least boundedness of all radial derivatives of the metric components as we approach the conformal boundary (in our definition, as $r \to \infty$).

A further extension of the work above is in the recent work [AILA18] of Allen, Isenberg, Lee and Stavrov Allen. In this work, similar results are obtained under weaker conditions on the regularity of the conformal compactification. In particular, their *Weakly Asymptotically Hyperbolic* manifolds have a $C^0$ conformal compactification, the metric components are bounded in some suitably weighted $C^2$ norm which in particular implies that decay of the scalar curvature is equivalent to decay of all of the sectional curvatures of $g$ decaying to $-1$.

We will compare our results to those overviewed above in the following section; however, we briefly note here that our notion of asymptotically locally hyperbolic in Definition 3.1.1 is equivalent to a $C^1$ conformal compactification. However, for our existence theorem we make no assumptions on the decay of any curvatures, as pointed out earlier we require only a one sided inequality on the scalar curvature

$$S_g \leq -n(n-1) + Ce^{-\alpha r}.$$

### 3.1.3 Overview and Discussion of Main Results

We now overview the results of this chapter and discuss their implications and place in the literature.

Our main result focuses on extending progress made in the case that the manifold in question is asymptotically locally hyperbolic. In particular, we show

**Theorem A.** *Suppose $(M, g)$ is an asymptotically locally hyperbolic manifold of order $\alpha \in (0, n]$. If the scalar curvature satisfies*

$$S_g \leq -n(n-1) + Ce^{-\alpha r} \text{ on } M \tag{3.6}$$

*for some constant $C > 0$, then there exists a positive smooth solution $u$ to* (Ya) *on $M$ satisfying $u \geq 1 - \mathcal{O}(e^{-\alpha r})$ if $\alpha \in (0, n)$ and $u \geq 1 - \mathcal{O}(re^{-nr})$ if $\alpha = n$. Therefore, there exists a complete conformal metric $\tilde{g}$ such that $S_{\tilde{g}} \equiv -n(n-1)$ on $M$.*

*In addition, if the scalar curvature satisfies the stronger condition*

$$|S_g + n(n-1)| \leq Ce^{-\alpha r} \text{ on } M, \tag{3.7}$$

*then* $u = 1 + \mathcal{O}(e^{-\alpha r})$ *if* $\alpha \in (0, n)$ *and* $u = 1 + \mathcal{O}(re^{-nr})$ *if* $\alpha = n$ *and* $u$ *is maximal in that any solution* $\tilde{u}$ *of* (Ya) *satisfies* $\tilde{u} \leq u$. *Furthermore, if* $\alpha \in (0, n)$, *the corresponding conformal manifold* $(M, \tilde{g})$ *is also asymptotically locally hyperbolic of the same order* $\alpha$.

The result above adds to the existing literature in a number of ways. A central theme throughout these contributions is the weaker setting in which they are established where curvature restrictions are made only on the scalar curvature. We contrast this to the existence results in the similar settings of [ACF92] and [AILA18] where, as discussed in the previous Section 3.1.2.2, their definition of asymptotically locally hyperbolic manifolds means the scalar curvature decay is equivalent to full decay of the curvature tensor to $-\mathrm{Id}$. We highlight that Example 3.1.3 demonstrates that there is a wide class of asymptotically locally hyperbolic manifolds in the sense of Definition 3.1.1 which have decay of the scalar curvature to $-n(n-1)$ sufficient to apply our Theorem A, but which fall outside the scope of the results in [ACF92] and [AILA18]; in particular, Example 3.1.3 demonstrates this via the poor behaviour of the Ricci curvature.

In Theorem A, we first establish the existence of solutions to the Yamabe problem in this weaker setting and furthermore show that only an upper bound on the $\limsup$ of the scalar curvature is needed.

Secondly, we make progress towards establishing uniqueness of solutions in our weaker setting, in line with what has been observed elsewhere in the literature, by showing maximality of the obtained solution. We show this via an a priori upper bound on solutions to the Yamabe equation for this class of asymptotically locally hyperbolic manifolds which we are, again, able to establish while making assumptions only on the behaviour of the scalar curvature.

Lastly, we establish higher order decay in the second derivatives of the conformal factor obtained in solving the Yamabe problem which allows us to conclude that the conformal metric of constant scalar curvature remains asymptotically locally hyperbolic.

Having shown this main result, we supplement our work in view of our initial motivating goal by studying the wider class of asymptotically warped product ended manifolds (the precise definition of which we make in Section 3.3). In particular, we aim to study the conformal class of such manifolds in the hope of finding a conformal asymptotically locally hyperbolic metric to which we may then apply Theorem A to address the Yamabe problem.

To this end, we first study the reference warped product metrics

$$\mathring{g}_f = dz^2 + f^2(z)\mathring{h}$$

generalising the reference locally hyperbolic metrics $\mathring{g}$ in (3.1). In particular, we first address the question of when these reference warped product metrics are conformal to locally hyperbolic metrics. We prove:

**Theorem** (Theorem 3.3.1)**.** *A metric $\mathring{g}_f$ with a warped product end is conformal to a metric with a locally hyperbolic end if and only if*

$$\int_0^\infty \frac{1}{f(s)}\, ds < \infty. \tag{3.8}$$

We take a moment to compare this to Example 6.2 of [AM88] which also considers manifolds with warped product ends and notes that their existence result for the Yamabe problem does not, in general, apply in this case (due to the lack of a global negativity condition on the scalar curvature). They briefly provide an existence proof using their sub- and super- solution technique under the conditions that $f$ is strictly increasing and

$$\lim_{r\to\infty} f(r) = \lim_{r\to\infty} \frac{f'(r)}{f(r)} = \lim_{r\to\infty} \frac{f''(r)}{f(r)} = +\infty$$

which are clearly much more restrictive than condition (3.8). However, in light of our Theorem 3.3.1 stated above, any such warped product metric is conformal to a metric with a locally hyperbolic end and so the existence of a solution to the Yamabe problem for these warped product ended manifolds follows from existence results in the literature (e.g. [ACF92]) while requiring only the condition (3.8).

We are then able to apply our Theorem A to the larger class of asymptotically warped product manifolds which have a metric satisfying

$$g_f = \mathring{g}_f + \varepsilon_{za} dz d\theta^a + \varepsilon_{ab} d\theta^a d\theta^b$$

on some exterior region $\mathbb{R}_{\geq 0} \times N$. We establish conditions on the perturbation coefficients above in order for $g_f$ to be conformally asymptotically locally hyperbolic in the sense of Definition 3.1.1 and then additional constraints on the warping function sufficient to allow us to control the scalar curvature of the conformally asymptotically locally hyperbolic manifold by controlling the scalar curvature of $g_f$. In particular, defining $H(z) := \int_z^\infty \frac{1}{f}$, we require

$$\begin{aligned} &\varepsilon_{ab} = \mathcal{O}\left(f^2 H^\alpha\right), && \varepsilon_{za} = \mathcal{O}\left(f H^\alpha\right), \\ &\partial_z \varepsilon_{ab} = \mathcal{O}\left(\left(f^3 H + f' f + f H^{-1}\right) H^\alpha\right), && \partial_z \varepsilon_{za} = \mathcal{O}\left(\left(f^2 H + f' + H^{-1}\right) H^\alpha\right), \\ &\partial_c \varepsilon_{ab} = \mathcal{O}\left(f^2 H^\alpha\right), && \partial_c \varepsilon_{za} = \mathcal{O}\left(f H^\alpha\right) \end{aligned} \tag{3.9}$$

and

$$\left|\frac{1}{fH}\right| + \left|\frac{f'}{f}\right| \leq C \text{ and } \left|\frac{f''}{f}\right| \leq C H^{-\alpha}. \tag{3.10}$$

We may then combine our Theorem 3.3.1 with our Theorem A to prove:

**Theorem** (Theorem 3.3.9)**.** *Let $(M, g_f)$ be a manifold with an asymptotically warped product end with perturbation coefficients satisfying* (3.9) *for some $\alpha \in (0, n)$. Suppose additionally that the warping function satisfies* (3.8) *and* (3.10)*.*

*If the scalar curvature satisfies*

$$S_{g_f} \leq S_{\mathring{g}_f} + C \frac{H^{\alpha-2}}{f^2} \text{ on } M^+ \tag{3.11}$$

*for some constant $C > 0$, then there exists a positive smooth solution $u_f$ of the Yamabe equation for $g_f$ on $M$ satisfying*

$$\liminf_{r \to \infty} \left(u_f - \frac{1}{fH}\right) \geq 0$$

*and the corresponding conformal metric* $\tilde{g}$ *is complete and has constant scalar curvature* $S_{\tilde{g}} \equiv -n(n-1)$ *on* $M$.

*In addition, if the scalar curvature satisfies the stronger condition*

$$|S_{g_f} - S_{\mathring{g}_f}| \leq C \frac{H^{\alpha-2}}{f^2} \text{ on } M^+, \tag{3.12}$$

*then*

$$\left| u_f - \frac{1}{fH} \right| \longrightarrow 0 \text{ as } r \to \infty$$

*and* $u_f$ *is maximal in that any solution* $\tilde{u}_f$ *of the Yamabe equation for* $g_f$ *on* $M$ *satisfies* $\tilde{u}_f \leq u_f$. *Furthermore, the corresponding conformal manifold* $(M, \tilde{g})$ *is asymptotically locally hyperbolic of order* $\alpha$.

This result ties our main Theorem A into our driving question regarding solvability for the Yamabe problem on asymptotically warped product manifolds. We note in particular that, in a similar way to in Theorem A, we see that the requirements on the perturbation coefficients and the warping function do not obstruct the Ricci curvature from having poor asymptotic behaviour.

# 3.2

# The Main Result

We recall that we write the conformal metrics of $g$ as $\tilde{g} = u^{\frac{4}{n-2}} g$ where $u$ is some positive smooth function. For $S_{\tilde{g}} \equiv -n(n-1)$ on $M$, $u$ must solve the Yamabe equation

$$-c_n \Delta_g u + S_g u = -n(n-1) u^{\frac{n+2}{n-2}} \text{ on M, where } c_n = 4\frac{n-1}{n-2}. \tag{Ya}$$

Our main result of this chapter is Theorem A which we restate here for the reader's convenience.

**Theorem A.** *Suppose* $(M, g)$ *is an asymptotically locally hyperbolic manifold of order* $\alpha \in (0, n]$*. If the scalar curvature satisfies*

$$S_g \leq -n(n-1) + Ce^{-\alpha r} \text{ on } M \tag{3.13}$$

*for some constant* $C > 0$*, then there exists a positive smooth solution* $u$ *to* (Ya) *on* $M$ *satisfying* $u \geq 1 - \mathcal{O}(e^{-\alpha r})$ *if* $\alpha \in (0, n)$ *and* $u \geq 1 - \mathcal{O}(re^{-nr})$ *if* $\alpha = n$*. Therefore, there exists a complete conformal metric* $\tilde{g}$ *such that* $S_{\tilde{g}} \equiv -n(n-1)$ *on* $M$*.*

*In addition, if the scalar curvature satisfies the stronger condition*

$$|S_g + n(n-1)| \leq Ce^{-\alpha r} \text{ on } M, \tag{3.14}$$

*then* $u = 1 + \mathcal{O}(e^{-\alpha r})$ *if* $\alpha \in (0, n)$ *and* $u = 1 + \mathcal{O}(re^{-nr})$ *if* $\alpha = n$ *and* $u$ *is maximal in that any solution* $\tilde{u}$ *of* (Ya) *satisfies* $\tilde{u} \leq u$*. Furthermore, if* $\alpha \in (0, n)$*, the corresponding conformal manifold* $(M, \tilde{g})$ *is also asymptotically locally hyperbolic of the same order* $\alpha$*.*

**Remark 3.2.1.** *Regarding uniqueness of the solution in Theorem A, we note that the completeness requirement is necessary to obtain uniqueness; for example, consider the disk model of the hyperbolic space endowed with the standard hyperbolic metric on a disk of larger radius. We were not able to establish whether completeness is a sufficient condition for uniqueness as at this time we require the additional restriction that* $\liminf_{|x| \to \infty} u(x) \geq 1$*. For further discussion, see sub-section 3.2.2.3.*

**Remark 3.2.2.** *In our Definition 3.1.1, if we were to assume additionally the decay of the second derivatives of the metric components,* (3.14) *automatically holds.*

We outline here the proof of Theorem A which will be carried out through a series of results throughout this section. As a base for our approach, we use the sub- and super-solution argument of Aviles and McOwen. Our work will use this approach and focus on points 1 and 10 of the sketch proof in the beginning of Chapter 2 which are, in a sense, interlinked. In particular, we must first be able to find such a sub-subsolution with which to begin the argument. For the point 10, to establish completeness we must use assumptions on the geometry of $(M, g)$ and the particular sub-solution. Provided we can establish these points, the argument of Aviles and McOwen yields a smooth solution $u$ of (Ya) satisfying $u_- \leq u$ on $M$ and a corresponding complete metric $\tilde{g}$ conformal to $g$. We produce a particular sub-solution under assumption (3.13) satisfying the requirements above in Lemma 3.2.3.

Having established the existence of a solution $u$ of (Ya) on $M$, to prove the first part of Theorem A it remains to ensure that $u$ is strictly positive and that the corresponding conformal metric $\tilde{g}$ is complete which, we prove in Lemma 3.2.5.

In order to prove the second part of Theorem A, we gain finer asymptotic control using a global super-solution to (Ya) which we produce in Lemma 3.2.12 and derive the corresponding asymptotics of the solution $u$ in Lemma 3.2.13. From there, we use elliptic theory to obtain decay in higher derivatives, shown in Lemma 3.2.15 and finally establish that this decay is sufficient to show that $\tilde{g}$ is asymptotically locally hyperbolic of the same order as $g$ in Lemma 3.2.16, completing the proof.

Our presentation of the proof of Theorem A is split into three parts as follows. Part I establishes the existence part of Theorem A when the scalar curvature satisfies (3.13). Part II studies an upper bound on solutions satisfying a lower bound on the scalar curvature. Finally, in part III we establish asymptotic decay and uniqueness of the solution established in the first part of the theorem under assumption (3.14).

### 3.2.1 Part I: Existence for the Yamabe Problem on Asymptotically Locally Hyperbolic Manifolds

In this sub-section, we establish the first part of Theorem A, namely the existence of a solution to the Yamabe problem. More specifically, we reiterate that we would like to establish the existence of a positive smooth solution $u$ of the Yamabe equation (Ya) such that the corresponding conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ is complete. As mentioned, we will apply the approach of Aviles and McOwen discussed in the previous sub-section. To this end, we start by constructing an explicit sub-solution.

We first consider the case that the manifold is asymptotically locally hyperbolic of order $\alpha < n$.

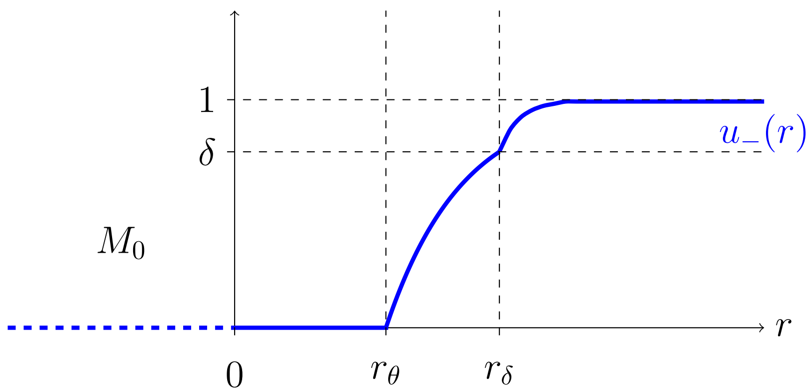
**Lemma 3.2.3.** *Let $(M, g)$ be asymptotically locally hyperbolic of order $\alpha < n$ and $S_g \leq -n(n-1) + Ce^{-\alpha r}$ for some constant $C$. For any $0 < \beta < \min(n-1, \alpha)$, there exist constants $0 < \delta < 1$, close to 1, and $\theta > 0$, large, such that the function $u_- \in H^1_{loc}(M)$ defined by*

$$u_- := \begin{cases} 1 - C(\theta, \delta)e^{-\alpha r} & \text{on } \{r_\delta \leq r\} \times N \\ 1 - \theta e^{-\beta r} & \text{on } \{r_\theta \leq r \leq r_\delta\} \times N \\ 0 & \text{on } M_0 \cup \{r \leq r_\theta\} \end{cases},$$

*where $r_\delta := \frac{1}{\beta} \log\left(\frac{\theta}{1-\delta}\right)$, $r_\theta := \frac{1}{\beta}\log(\theta)$ and*

$$C(\theta, \delta) = (1-\delta)\left(\frac{\theta}{1-\delta}\right)^{\frac{\alpha}{\beta}},$$

*is a subsolution of* (Ya) *on $M$ which is positive outside of some compact set.*




We note here that the choice of $C(\theta, \delta)$ ensures that $u_-$ is continuous and therefore belongs to $H^1_{\text{loc}}(M)$. We have a sub-solution of the form $u_- = u_-(r)$ so we are led to consider the ODE arising immediately from (Ya)

$$-c_n\left(1 + \mathcal{O}(e^{-2\alpha r})\right)u_-'' - c_n\left((n-1)\frac{f_k'}{f_k} + \mathcal{O}(e^{-\alpha r})\right)u_-' \\ - \left(n(n-1) + \mathcal{O}(e^{-\alpha r})\right)u_- \leq -n(n-1)u_-^{\frac{n+2}{n-2}}. \tag{3.15}$$

**Remark 3.2.4.** *We note that the addition of the change at $r_\delta$ is necessary in the proof to extend the result to include asymptotically locally hyperbolic ends of order $n-1 \leq \alpha < n$.*

*Proof.* Fix some $0 < \beta < \min(n-1, \alpha)$. To establish that $u_-$ is a sub-solution, we must show that (3.15) holds on $\{r_\theta \le r \le r_\delta\}$ and $\{r_\delta \le r\}$ and check the following transmission conditions at $r_\theta$ and at $r_\delta$, in this case that is to ensure that

$$\lim_{r \nearrow r_\theta} u'_-(r) \le \lim_{r \searrow r_\theta} u'_-(r) \tag{3.16}$$

and

$$\lim_{r \nearrow r_\delta} u'_-(r) \le \lim_{r \searrow r_\delta} u'_-(r). \tag{3.17}$$

See Appendix A for an appropriate formulation of the transmission conditions. Condition (3.16) is immediately clear as $\beta > 0$. Condition (3.17) holds true as $\beta < \alpha$.

To establish (3.15) it suffices to show, for some constant $C_1 > 0$ depending only on $g$, that there exist $\theta$ and $\delta$ (possibly depending on $\beta$) such that

$$\begin{aligned} L_- u_- := &- c_n u''_- - c_n \left( (n-1)\frac{f'_k}{f_k} - C_1 e^{-\alpha r} \right) u'_- \\ &- \left( n(n-1) - C_1 e^{-\alpha r} \right) u_- + C_1 e^{-2\alpha r} |u''_-| \\ &\le -n(n-1) u_-^{\frac{n+2}{n-2}} \end{aligned} \tag{3.18}$$

holds on $\{r_\theta < r < r_\delta\}$ and $\{r_\delta < r\}$. We note here that $u'_- \ge 0$ for all $r$.

In the following, we write $C$ to indicate a constant changing from line to line but depending only on $g$. For $\{r_\theta < r < r_\delta\}$, we have that $n(n-1)u_-^{\frac{n+2}{n-2}} \le n(n-1)\delta^{\frac{4}{n-2}} u_-$ as $0 \le u_- \le \delta$. We compute

$$\begin{aligned} L_- u_- + n(n-1)u_-^{\frac{n+2}{n-2}} < c_n \theta &\left( \beta^2 - (n-1)\beta + \frac{1}{4} n(n-2)(1 - \delta^{\frac{4}{n-2}}) \right) e^{-\beta r} \\ &+ C(\underbrace{e^{(\beta-\alpha)r} + \theta e^{-\alpha r} + \theta e^{-2r}}_{A}) e^{-\beta r} \end{aligned}$$

where we used that $\frac{f'_k(r+r_0)}{f_k(r+r_0)} > 1 - Ce^{-2r}$ (see Section 3.1.1) and $u'_- \ge 0$. Note first that, for $\delta$ close to 1, the first term of the RHS of the above is a negative multiple of $e^{-\beta r}$. In addition, as $\beta < \alpha$ we have, for $r \ge r_\theta$, that

$$|A(r)| \le A(r_\theta) \le C(\theta^{1-\frac{\alpha}{\beta}} + \theta^{1-\frac{2}{\beta}}).$$

Consequently, for $\theta$ sufficiently large, the first term on the RHS dominates and we have, for all $\delta$ sufficiently close to 1,

$$L_- u_- + n(n-1)u_-^{\frac{n+2}{n-2}} < 0 \text{ in } \{r_\theta < r < r_\delta\}.$$

We choose such a $\theta$ and fix it from here on.

On $\{r > r_\delta\}$, note that we have $\delta \leq u_- \leq 1$. As

$$\begin{aligned}\left| x^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}(x-1) \right| &= \left| \int_x^1 4\frac{n+2}{(n-2)^2} t^{-\frac{n-6}{n-2}}(x-t)\, dt \right| \\ &< 4\frac{n+2}{(n-2)^2}\delta^{-\frac{n-6}{n-2}}(x-1)^2\end{aligned}$$

for $\delta < x \leq 1$. We have,

$$\begin{aligned} n(n-1)u_-^{\frac{n+2}{n-2}} < n(n-1) &- c_n \frac{n}{4}(n+2)C(\theta,\delta)e^{-\alpha r} \\ &+ c_n \frac{n(n+2)}{(n-2)}\delta^{-\frac{n-6}{n-2}} C(\theta,\delta)^2 e^{-2\alpha r} \ . \end{aligned}$$

Using that, again, $\frac{f_k'(r+r_0)}{f_k(r+r_0)} > 1 - Ce^{-2r}$ and $u_-' \geq 0$, we obtain

$$\begin{aligned} L_- u_- + n(n-1)u_-^{\frac{n+2}{n-2}} \leq c_n e^{-\alpha r} C(\theta,\delta)\Big[\alpha^2 - (n-1)\alpha - n \\ + C\Big(\underbrace{\delta^{-\frac{n-6}{n-2}}C(\theta,\delta)e^{-\alpha r} + e^{-\alpha r} + e^{-2r} + \frac{1}{C(\theta,\delta)}}_{B}\Big)\Big]. \end{aligned} \tag{3.19}$$

As $\alpha < n$, $\alpha^2 - (n-1)\alpha - n < 0$.

We note that $B(r)$ is non-increasing. We have for $r > r_\delta$.

$$\begin{aligned} 0 < B(r) < B(r_\delta) = \Bigg[\delta^{-\frac{n-6}{n-2}}(1-\delta) + \left(\frac{\theta}{1-\delta}\right)^{-\frac{\alpha}{\beta}} + \left(\frac{\theta}{1-\delta}\right)^{-\frac{2}{\beta}} \\ + \frac{1}{C(\theta,\delta)}\Bigg] \to 0 \text{ as } \delta \nearrow 1. \end{aligned}$$

It follows that for $\delta$ close to 1,

$$L_- u_- + n(n-1)u_-^{\frac{n+2}{n-2}} < 0$$

as required. ∎

As seen in the proof above, if $\alpha = n$ the leading term in (3.19) vanishes and so we are unable to complete the proof in this case. Consequently, we now include the case that $\alpha = n$ in the following adjustment of Lemma 3.2.3.

**Lemma 3.2.3′.** *Let $(M, g)$ be asymptotically locally hyperbolic of order $\alpha = n$ and $S_g \leq -n(n-1) + Ce^{-nr}$ for some constant $C$. For any $0 < \beta < n-1$, there exist constants $0 < \delta < 1$, close to $1$, and $\theta > 0$, large, such that the function $u_- \in H^1_{loc}(M)$ defined by*

$$u_- := \begin{cases} 1 - C(\theta,\delta) r e^{-nr} & \text{on } \{r_\delta \leq r\} \times N \\ 1 - \theta e^{-\beta r} & \text{on } \{r_\theta \leq r \leq r_\delta\} \times N \\ 0 & \text{on } M_0 \cup \{r \leq r_\theta\} \end{cases},$$

*where $r_\delta := \frac{1}{\beta} \log\left(\frac{\theta}{1-\delta}\right)$, $r_\theta := \frac{1}{\beta}\log(\theta)$ and*

$$C(\theta, \delta) = \frac{\beta(1-\delta)^{\frac{\beta-n}{\beta}} \theta^{\frac{n}{\beta}}}{\log(\frac{\theta}{1-\delta})},$$

*is a subsolution of* (Ya) *on $M$ which is positive outside of some compact set.*

*Proof.* Again, we fix some $0 < \beta < n-1$. We follow the same proof as that of Lemma 3.2.3 to establish that (3.15) holds on $\{r_\theta \leq r \leq r_\delta\}$.

It remains to check that $u_-$ is a sub-solution on $\{r_\delta \leq r\}$ and that transmission conditions (3.16) and (3.17) hold at $r_\theta$ and at $r_\delta$.

Condition (3.16) is immediately clear as $\beta > 0$. Condition (3.17) holds true provided that $\beta < n - \frac{1}{r_\delta}$ which holds for all $1 > \delta > \delta_0$ for some $\delta_0$. In this case, we also have that $u_-' \geq 0$ for all $r$.

On $\{r > r_\delta\}$, note that we have $\delta \leq u_- \leq 1$. As

$$\begin{aligned} \left| x^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}(x-1) \right| &= \left| \int_x^1 4\frac{n+2}{(n-2)^2} t^{-\frac{n-6}{n-2}} (x-t)\, dt \right| \\ &< 4\frac{n+2}{(n-2)^2} \delta^{-\frac{n-6}{n-2}} (x-1)^2 \end{aligned}$$

for $\delta < x \leq 1$. We have,

$$\begin{aligned} n(n-1) u_-^{\frac{n+2}{n-2}} &< n(n-1) - c_n \frac{n}{4}(n+2) C(\theta,\delta) r e^{-nr} \\ &\quad + c_n \frac{n(n+2)}{(n-2)} \delta^{-\frac{n-6}{n-2}} C(\theta,\delta)^2 r^2 e^{-2nr}\ . \end{aligned}$$

Using that, again, $\frac{f_k'(r+r_0)}{f_k(r+r_0)} > 1 - Ce^{-2r}$ and $u_-' \geq 0$, we obtain

$$L_- u_- + n(n-1)u_-^{\frac{n+2}{n-2}} \leq c_n e^{-nr} C(\theta,\delta)\Big[ -(n+1) + C\underbrace{\Big(\delta^{-\frac{n-6}{n-2}} C(\theta,\delta) r^2 e^{-nr} + re^{-2r} + \frac{1}{C(\theta,\delta)}\Big)}_{B}\Big].$$

We note that $\lim_{\delta\nearrow 1} r_\delta = \infty$ and so, for $\delta$ close to 1, $B(r)$ is non-increasing. We have for $r > r_\delta$.

$$0 < B(r) < B(r_\delta) = \left[\delta^{-\frac{n-6}{n-2}}(1-\delta)\log\left(\frac{\theta}{1-\delta}\right) + \left(\frac{\theta}{1-\delta}\right)^{-\frac{2}{\beta}} \log\left(\frac{\theta}{1-\delta}\right) + \frac{1}{C(\theta,\delta)}\right] \to 0 \text{ as } \delta \nearrow 1.$$

It follows that for $\delta$ close to 1,

$$L_- u_- + n(n-1)u_-^{\frac{n+2}{n-2}} < 0$$

as required. □

We may now use the above sub-solutions to prove the first existence part of Theorem A.

**Proposition 3.2.5.** *Let $(M,g)$ be asymptotically locally hyperbolic of order $\alpha \leq n$ with $S_g \leq -n(n-1) + Ce^{-\alpha r}$ for some constant $C$. There exists a positive smooth solution $u$ of* (Ya) *on $M$ satisfying*

$$u \geq 1 - Ce^{-\alpha r} \ (u \geq 1 - Cre^{-\alpha r} \text{ if } \alpha = n)$$

*for some $C > 0$. Consequently, there exists a complete conformal metric $\tilde{g}$ such that $S_{\tilde{g}} \equiv -n(n-1)$ on $M$.*

*Proof.* By Lemma 3.2.3 and 3.2.3′ there exists a sub-solution $u_-$ of (Ya) which satisfies $u_- \geq 1 - Ce^{-\alpha r}$ (or $u_- \geq 1 - Cre^{-\alpha r}$ if $\alpha = n$). The sub- and super-solution argument of Aviles and McOwen ([AM88], also see Section 3.1.2.1) yields a smooth solution $u$ of (Ya) satisfying $u \geq u_-$ on $M$. It remains to show that $u$ is positive on all of $M$;

once this is established, we will have obtained a conformal metric $\tilde{g} = u^{\frac{4}{n-2}} g$ which is complete, from the sub-solution lower bound, and has constant scalar curvature.

As $u \geq u_-$, $u$ is non-negative and positive outside of some compact set. Let $B$ be a large ball on which $u \not\equiv 0$ and outside of which $u > 0$. On $B$, the scalar curvature $S_g \leq A$ for some constant $A > 0$. Consequently, taking a sufficiently large constant $C > 0$ such that $u$ satisfies

$$c_n \Delta_g u - Cu \leq [u^{\frac{n+2}{n-2}} + Au] - Cu \leq 0,$$

we can apply the strong maximum principle ([GT01, Theorem 3.5]) to deduce that, as $u \not\equiv 0$ on $B$, $u$ is strictly positive in $B$ and so on all of $M$. $\square$

**Remark 3.2.6.** *It appears to the author that, owing to the fact that we can construct sub-solutions which are identically zero everywhere off the asymptotic end, the above proof should carry over with minimal adaptation to the case that $(M, g)$ is a non-compact manifold with multiple different asymptotically locally hyperbolic ends. That is to say, the fact that $(M, g)$ has a single end does not appear to be fundamental in the above approach.*

### 3.2.2 Part II: An Upper Bound on Solutions at Infinity

Before establishing the second part of Theorem A, we turn to the question of the asymptotic behaviour of solutions to the Yamabe equation in the asymptotically locally hyperbolic setting. Our discussion is motivated by the aim of establishing uniqueness of solutions to the Yamabe problem, that is of solutions to (Ya) for which the resulting conformal metric is complete. In particular, we consider what conditions we may need to impose in addition to the completeness requirement of the conformal metric, if any, to conclude that our solution is unique. It has been well established in the literature (for example in [LN74] and [ACF92]) that solutions to the Yamabe problem for asymptotically hyperbolic manifolds of a more restrictive type to our Definition 3.1.1 must "asymptote to 1 at infinity" in a sense that we make precise in the following. We are thus led to consider the asymptotic behaviour of solutions along the asymptotically locally hyperbolic end in our weaker setting.

Asymptotic boundary conditions arise naturally in the study of PDE problems on non-compact manifolds; for example, in [SY94], Schoen and Yau establish uniqueness of harmonic functions on negatively curved manifolds when fixing the values on the so-called *geometric boundary* which they are able to construct as equivalence classes of diverging geodesic curves. In our setting, there is a natural candidate for the boundary "at infinity" via the warped product ended structure of the manifold. In particular, we say that a function $u$ achieves a particular boundary condition $\varphi \in C^\infty(N)$ if for any $\theta \in N$, we have

$$\lim_{r\to\infty} u(r,\theta) = \varphi(\theta).$$

The definition above coincides with the more general definition of the geometric boundary condition.

We will establish an a priori upper bound on solutions to the Yamabe equation for asymptotically locally hyperbolic manifolds in the sense of Definition 3.1.1. A key point for our discussion will be to compare the Laplacian of the perturbed metric $g$ to that of the reference metric $\mathring{g}$. In particular, where we use $\nabla$ and $\mathring{\nabla}$ to denote the covariant derivative with respect to $g$ and $\mathring{g}$ respectively, we compute directly that

$$\begin{aligned}\Delta_g\varphi &= g^{ij}\nabla_i\nabla_j\varphi = g^{ij}\mathring{\nabla}_i\mathring{\nabla}_j\varphi + g^{ij}(\nabla_i\nabla_j\varphi - \mathring{\nabla}_i\mathring{\nabla}_j\varphi)\\ &= \mathring{g}^{ij}\mathring{\nabla}_i\mathring{\nabla}_j\varphi + (g^{ij}-\mathring{g}^{ij})\mathring{\nabla}_i\mathring{\nabla}_j\varphi + g^{ij}(\Gamma^k_{ij} - \mathring{\Gamma}^k_{ij})\partial_k\varphi\\ &= \Delta_{\mathring{g}}\varphi + a^{ij}\mathring{\nabla}_{ij}\varphi + b^i\mathring{\nabla}_i\varphi \qquad (3.20)\end{aligned}$$

where $a^{ij} := g^{ij} - \mathring{g}^{ij}$ and $b^i := g^{jk}(\Gamma^i_{jk} - \mathring{\Gamma}^i_{jk})$ satisfy $|a^{ij}| = \mathcal{O}_1(e^{-(\alpha+2)r})$ and $|b^i| = \mathcal{O}(e^{-\alpha r})$ using the estimates for the Christoffel symbols found in the proof of Lemma 3.1.2.

In the following, we first address the simpler case that the $(M,g)$ is asymptotically hyperbolic (that is, $N = S^{n-1}$ and $\mathring{g}$ is the hyperbolic metric in Definition

3.1.1). Then, in the next sub-section 3.2.2.2, we prove the upper bound for an arbitrary asymptotically locally hyperbolic manifold. We choose to include the proof in the simpler asymptotically hyperbolic case so as to demonstrate the ideas motivating the both more general and more technical proof of Proposition 3.2.9 for general asymptotically locally hyperbolic manifolds.

We will make use of the following consequence of the strong maximum principle.

**Lemma 3.2.7.** *Let $u > 0$ be a bounded smooth solution to the Yamabe equation on $\Omega \subset M$ and $\bar{u} > 0$ be a smooth super–solution to the Yamabe equation on $\Omega$ satisfying $\bar{u}(x) \to \infty$ as $x \to \partial\Omega$. Then necessarily $u < \bar{u}$ on $\Omega$.*

*Proof.* Due to the facts that $\bar{u} \to \infty$ as $x \to \partial\Omega$, $u$ is bounded and both $u > 0$ and $\bar{u} > 0$, there must exist some $C > 0$ such that the difference $w_C := C\bar{u} - u$ satisfies that $w_C \geq 0$ and achieves 0 at some point in $\Omega$. We claim that $C < 1$. If $C \geq 1$, then $w_C$ would satisfy,

$$\begin{aligned} -c_n\Delta_g w_C - n(n-1)w_C &\geq -n(n-1)\left(C\bar{u}^{\frac{n+2}{n-2}} - u^{\frac{n+2}{n-2}}\right) \\ &\geq -n(n-1)\left((C\bar{u})^{\frac{n+2}{n-2}} - u^{\frac{n+2}{n-2}}\right) = -n(n-1)c(x)w_C. \end{aligned}$$

where

$$c(x) := \begin{cases} \frac{(C\bar{u})^{\frac{n+2}{n-2}} - u^{\frac{n+2}{n-2}}}{C\bar{u} - u} \text{ for } w_C \neq 0, \\ \frac{n+2}{n-2}(C\bar{u}(x))^{\frac{4}{n-2}} \text{ if } w_C = 0. \end{cases}$$

We could then apply the strong maximum principle to the linear differential inequality

$$-c_n\Delta_g w_C - n(n-1)\left(1 - c(x)\right) w_C \geq 0,$$

noting that the sign of $(1 - c(x))$ does not matter as the value of the minimum in question is 0, see [GT01, Section 3.2]. We would then have that $w_C \equiv 0$, a contradiction as $w_C \to \infty$ at the boundary. We conclude that $C < 1$ as claimed and so $u < \bar{u}$ in $\Omega$. □

#### 3.2.2.1 The Model Case of an Asymptotically Hyperbolic Manifold

This subsection addresses only the case that the $(M, g)$ is asymptotically hyperbolic for the purpose of motivating the ideas used to treat the general case in the next sub-section 3.2.2.2. We prove the following:

**Proposition 3.2.8.** *Suppose $(M, g)$ is asymptotically hyperbolic in the sense of Definition 3.1.1 (in particular, $(N, \mathring{h})$ is the round sphere) and satisfies, for some fixed $x_0 \in M$,*

$$\liminf_{r(x)\to\infty} S_g(x) \geq -n(n-1). \tag{3.21}$$

*Then all solutions $u$ of* (Ya) *on $(M, g)$ must satisfy*

$$\limsup_{r(x)\to\infty} u(x) \leq 1.$$

The idea in the following proof can be most clearly seen when $(M, g)$ is *exactly* the hyperbolic space on which we briefly provide an argument to show that any solution $u$ of the Yamabe equation must satisfy $u \leq 1$. We follow the ideas present in, for example, [LN74] and [ACF92]. We conveniently express the hyperbolic space as the disk model $(B_1, g_{\text{hyp}})$ where $B_1$ is the Euclidean unit ball. Suppose $u$ is a solution to the Yamabe equation for $(B_1, g_{\text{hyp}})$ i.e.

$$-c_n\Delta_{g_{\text{hyp}}} u - n(n-1)u = -n(n-1)u^{\frac{n+2}{n-2}} \text{ on } B_1.$$

Consider some ball $B_R \subset B_1$ and let $u_R$ be the conformal factor such that $u_R^{\frac{4}{n-2}} g_{\text{hyp}}$ is the standard hyperbolic metric on $B_R$. Necessarily $u_R \to \infty$ at the boundary $\partial B_R$ and $u_R$ solves

$$-c_n\Delta_{g_{\text{hyp}}} u_R - n(n-1)u_R = -n(n-1)u_R^{\frac{n+2}{n-2}} \text{ on } B_R.$$

We have that as $u_R \to \infty$ and so we may apply Lemma 3.2.7 to conclude that $u < u_R$ on $B_R$. We now take the limit $R \nearrow 1$ so that $u_R \to 1$ and so we conclude that $u \leq 1$.

In the following proof, we adapt the argument above to provide an asymptotic upper bound in the asymptotically hyperbolic case.

*Proof of Proposition 3.2.8.* Recall that the exterior region $M^+$ of $M$ is the product $\mathbb{R}_{\geq 0} \times N$. It suffices to show that, for every $R > 0$, $u$ satisfies

$$\limsup_{r(x)\to\infty} u(x) \leq \left(\frac{1}{\tanh R}\right)^{\frac{n-2}{2}}.$$

We first rewrite equation (Ya) in the exterior region as a perturbation of an elliptic equation with respect to the reference hyperbolic metric $\mathring{g}$, writing $\Delta_g$ as a perturbation of $\Delta_{\mathring{g}}$ as in (3.20). In particular, we have that $u$ solves

$$-c_n\Delta_{\mathring{g}}u + S_g u + n(n-1)u^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}u + b^i\mathring{\nabla}_i u = 0. \tag{3.22}$$

where $|a^{ij}| = \mathcal{O}_1(e^{-(\alpha+2)r})$ and $|b^i| = \mathcal{O}(e^{-\alpha r})$.

We now fix some arbitrary $R > 0$ and consider a geodesic ball $\mathring{B}_R(x_0)$ with respect to the reference metric $\mathring{g}$ at some point $x_0$ such that $\mathring{B}_R(x_0) \subset M^+$. Take standard reference coordinates $(z, \theta)$ on $\mathring{B}_R(x_0)$ in which

$$\mathring{g} = dz^2 + \sinh^2(z)ds^2_{n-1}$$

so that $\mathring{B}_R(x_0)$ corresponds to $\{z < R\}$. We note here that this step makes direct use of the homogeneity of the hyperbolic space.

Let $u_R$ be given by

$$u_R(z,\theta) := \left(\frac{\tanh R}{\tanh^2 R - \tanh^2 z}\right)^{\frac{n-2}{2}} \tag{3.23}$$

which is the unique solution to

$$\begin{cases} -c_n\Delta_{\mathring{g}}u_R - n(n-1)u_R + n(n-1)u_R^{\frac{n+2}{n-2}} = 0 \text{ on } \{z < R\}, \\ u_R(z,\theta) \to \infty \text{ as } z \to R. \end{cases} \tag{3.24}$$

We will now create a super-solution by rescaling $u_R$. Consider

$$\bar{u} := Au_R$$

for some $A > 0$ to be fixed. We can then compute directly that

$$\begin{aligned}
&-c_n\Delta_{\mathring{g}}\bar{u} + S_g\bar{u} + n(n-1)\bar{u}^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}\bar{u} + b^i\mathring{\nabla}_i\bar{u} \\
&\quad = -c_n\Delta_{\mathring{g}}\bar{u} - n(n-1)\bar{u} + n(n-1)\bar{u}^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}\bar{u} + b^i\mathring{\nabla}_i\bar{u} + (S_g + n(n-1))\bar{u} \\
&\quad = n(n-1)(A^{\frac{n+2}{n-2}} - A)u_R^{\frac{n+2}{n-2}} + A\left(a^{ij}\mathring{\nabla}_{ij}u_R + b^i\mathring{\nabla}_i u_R + (S_g + n(n-1))u_R\right) \\
&\quad \geq n(n-1)(A^{\frac{n+2}{n-2}} - A)u_R^{\frac{n+2}{n-2}} - A\varepsilon(r(x_0))\big(|u_R''| + |u_R'| + |u_R|\big)
\end{aligned}$$

where $\varepsilon(r) \geq 0$ and $\lim_{r\to\infty}\varepsilon(r) = 0$; here we have used (3.21) as well as the asymptotic hyperbolicity.

Direct computation from the explicit form of $u_R$ gives that $|u_R''| + |u_R'| + |u_R| \leq Cu_R^{\frac{n+2}{n-2}}$ in $B_R$ where $C > 0$ is a constant depending only on $R$ and $n$. Therefore,

$$\begin{aligned}
&-c_n\Delta_{\mathring{g}}\bar{u} + S_g\bar{u} + n(n-1)\bar{u}^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}\bar{u} + b^i\mathring{\nabla}_i\bar{u} \\
&\qquad\qquad\qquad \geq n(n-1)\left(A^{\frac{n+2}{n-2}} - A\big(1 + C\varepsilon(r(x_0))\big)\right)u_R^{\frac{n+2}{n-2}}
\end{aligned}$$

and so we can now choose $A = 1 + \tilde{C}\varepsilon(r(x_0))$ with a sufficiently large constant $\tilde{C} > 0$ depending only on $R$ and $n$ so that $\bar{u}$ is a strict super–solution to (3.22).

Consequently, we may apply Lemma 3.2.7 to conclude that $u < \bar{u}$ on $\mathring{B}_R(x_0)$. In particular, at $x_0$ we have

$$u(x_0) < \left(1 + \tilde{C}\varepsilon(r(x_0))\right)\left(\frac{1}{\tanh R}\right)^{\frac{n-2}{2}}.$$

As $x_0$ was chosen arbitrarily other than the condition that $\mathring{B}_R(x_0) \subset M^+$, we may take $r(x_0)$ arbitrarily large and therefore obtain that

$$\limsup_{r(x)\to\infty} u(x) \leq \left(\frac{1}{\tanh R}\right)^{\frac{n-2}{2}}.$$

We send $R \to \infty$ and the proposition immediately follows. □

#### 3.2.2.2 The General Asymptotically Locally Hyperbolic Case

We now generalise the result of the previous sub-section where we now allow the manifold to be asymptotically locally hyperbolic.

**Proposition 3.2.9.** *Let $(M, g)$ be asymptotically locally hyperbolic in the sense of Definition 3.1.1 and suppose that, for some fixed $x_0 \in M$*

$$\liminf_{r(x)\to\infty} S_g(x) \geq -n(n-1). \tag{3.25}$$

*Then all solutions of the* (Ya) *on $(M, g)$ satisfy*

$$\limsup_{r(x)\to\infty} u(x) \leq 1. \tag{3.26}$$

We note that, in the argument made for the asymptotically hyperbolic case in the previous subsection, we used the fact that the solutions $u_R$ for the Yamabe equation (3.24) on $B_R(x_0)$ given by (3.23) are independent of $x_0$. If we attempt to directly apply this argument more generally, even in the case that $g$ is *exactly* equal to one of the locally hyperbolic reference metrics $\mathring{g}$, the obtained solution $u_R$ is necessarily dependent on $x_0$ since the underlying reference metric is no longer homogeneous.

To circumvent this issue, we switch our domains from balls to annuli and study a family of approximate solutions on the annuli in place of the explicit solutions on balls. The choice of switching to annuli has the advantage that the dependence on the centre $x_0$ is reduced to a dependence only on the radial coordinate $r(x_0)$, which can be tracked using certain conformal properties of the equation (see (3.32)). In the following Lemmas 3.2.10 and 3.2.11 below, we establish the existence of such a family of approximate solutions which we then use to provide a proof of Proposition 3.2.9 above.

**Lemma 3.2.10.** *There exists a positive solution $u_1$ of the equation*

$$-c_n\left(u_1'' + (n-1)u_1'\right) - n(n-1)u_1 + n(n-1)u_1^{\frac{n+2}{n-2}} = 0 \text{ on } (-1,1) \tag{3.27}$$

*satisfying $u_1(r) \to \infty$ as $r \to \pm 1$ and there exists a constant $C > 0$ depending on $n$ such that $u_1$ satisfies*

$$\begin{aligned} \frac{1}{C}(1-|r|)^{-\frac{n-2}{2}} \leq u_1(r) &\leq C(1-|r|)^{-\frac{n-2}{2}}, \\ u_1'(r) &\leq C(1-|r|)^{-\frac{n}{2}}, \\ u_1''(r) &\leq C(1-|r|)^{-\frac{n+2}{2}}. \end{aligned} \tag{3.28}$$

*Proof.* This result is essentially due to [ACF92]. Consider the model locally hyperbolic manifold $(\mathbb{R} \times \mathbb{T}^{n-1}, g_\delta)$ where $(\mathbb{T}^{n-1}, \mathring{h})$ is the flat Torus. Define the annulus $A_1 := \{|r| \leq 1\} \subset M$. We note here that any solution of (3.27) would automatically be a solution of the Yamabe equation

$$-c_n \Delta_{g_\delta} u_1 - n(n-1)u_1 + n(n-1)u_1^{\frac{n+2}{n-2}} = 0$$

on the annulus in this particular model space. From [ACF92, Theorem 1.2], we know that, on the annulus $A_1$, there exists a unique solution $u_1$ of the Yamabe equation satisfying $u_1 \to \infty$ at the boundary $\partial A_1$. Additionally, from Theorem 1.3 of [ACF92], we know the behaviour of $u_1$ at the boundary satisfies (3.28).

We note that the Yamabe equation is invariant under symmetries of the Torus and so, by uniqueness, $u_1$ must be radial and so the Yamabe equation reduces to the ODE

$$-c_n \left(u_1'' + (n-1)u_1'\right) - n(n-1)u_1 + n(n-1)u_1^{\frac{n+2}{n-2}} = 0$$

as desired. □

Equation (3.27) has the following important scaling property that we will make use of: if we define the family $\{u_R\}$ by

$$u_R(r) = \left(\frac{(e^2-1)e^R}{e^{2+R} + e^{2R+r} - e^{2+r} - e^R}\right)^{\frac{n-2}{2}} u_1\left(\log\left(\frac{(e^{2R}-1)e^{r+1}}{e^{2+R} + e^{2R+r} - e^{2+r} - e^R}\right)\right), \tag{3.29}$$

then each $u_R$ provides a solution of the ODE

$$-c_n \left(u_R'' + (n-1)u_R'\right) - n(n-1)u_R + n(n-1)u_R^{\frac{n+2}{n-2}} = 0 \text{ on } (-R, R). \tag{3.30}$$

Furthermore, by (3.28) of $u_1$ we have that each $u_R$ satisfies that $u_R(r) \to \infty$ as $r \to \pm R$ and, for some constant $C_R > 0$ depending on $R$ and $n$, that

$$\begin{aligned} \frac{1}{C_R}(R - |r|)^{-\frac{n-2}{2}} \leq u_R(r) &\leq C_R(R - |r|)^{-\frac{n-2}{2}}, \\ u_R'(r) &\leq C_R(R - |r|)^{-\frac{n}{2}}, \\ u_R''(r) &\leq C_R(R - |r|)^{-\frac{n+2}{2}}. \end{aligned} \tag{3.31}$$

We now establish locally uniform convergence of this family to 1 as $R \to \infty$.

**Lemma 3.2.11.** *The family* $\{u_R\}_{R>0}$ *of positive solutions of the equation* (3.30) *defined above is decreasing with respect to* $R$ *and satisfies* $u_R \searrow 1$ *uniformly on compact sets as* $R \to \infty$.

*Proof.* If $R_1 < R_2$, we may apply Lemma 3.2.7 on the annulus $A_{R_1}(0)$ to see that $u_{R_2} < u_{R_1}$. Consequently, $\{u_R\}$ is monotone decreasing.

Define the point-wise limit $u_\infty(r) := \lim_{R\to\infty} u_R(r)$. We now show that $\{u_R\}$ converges locally in $C^2$ to $u_\infty$. To this end, a locally uniform bound on the first derivative will suffice, as from this we immediately obtain local boundedness of $u_R''$ and $u_R'''$ (the latter after differentiating the ODE once). The Arzela-Ascoli theorem then provides the desired convergence. To obtain such a bound on the first derivative, note that on any compact domain $[-R_0, R_0]$ and for $R > R_0 + \varepsilon$, where $\varepsilon > 0$ is some small positive constant, we have that, due to the fact $\{u_R\}$ is monotone decreasing,

$$u_R'' + (n-1)u_R' = \frac{n(n-2)}{4}\left(u_R^{\frac{n+2}{n-2}} - u_R\right)$$

is bounded in $[-R_0, R_0]$ uniformly as $R \to \infty$. Integrating the above from some point $s \in [-R_0, R_0]$ to $r \in [-R_0, R_0]$ we obtain that $u_R'(r) - u_R'(s)$ is bounded in $[-R_0, R_0]$ uniformly as $R \to \infty$. Integrating the above, now in $s$, from $-R_0$ to $R_0$ we see that $u_R'$ is bounded in $[-R_0, R_0]$ uniformly as $R \to \infty$.

We proceed to show that $u_\infty$ is constant by using the scaling property used in defining the family $u_R$. In particular, for any $R, S > 0$ we can verify, by (3.29), that

$$u_R(r) = \left(\frac{(e^{2S}-1)e^R}{e^{2S+R} + e^{2R+r} - e^{2S+r} - e^R}\right)^{\frac{n-2}{2}} u_S\left(\log\left(\frac{(e^{2R}-1)e^{r+S}}{e^{2S+R} + e^{2R+r} - e^{2S+r} - e^R}\right)\right). \tag{3.32}$$

Consider the value of $u_R$ at $r = 0$. We then have

$$u_R(0) = \left(\frac{(e^{2S}-1)e^R}{e^{2S+R} + e^{2R} - e^{2S} - e^R}\right)^{\frac{n-2}{2}} u_S\left(\log\left(\frac{(e^{2R}-1)e^{S}}{e^{2S+R} + e^{2R} - e^{2S} - e^R}\right)\right)$$

then writing $R = S + \Lambda$

$$u_{S+\Lambda}(0) = \left(\frac{e^{3S+\Lambda} - e^{S+\Lambda}}{e^{3S+\Lambda} + e^{2(S+\Lambda)} - e^{2S} - e^{S+\Lambda}}\right)^{\frac{n-2}{2}} u_S\left(\log\left(\frac{e^{3S+2\Lambda} - e^{S}}{e^{3S+\Lambda} + e^{2(S+\Lambda)} - e^{2S} - e^{S+\Lambda}}\right)\right).$$

Taking the limit as $S \to \infty$ while keeping $\Lambda$ fixed we then obtain that for all $\Lambda$

$$u_\infty(0) = u_\infty(\log\left(e^\Lambda\right)) = u_\infty(\Lambda).$$

Consequently, $u_\infty \equiv C_\infty$ for some constant $C_\infty \geq 0$.

Recall that $u_R$ converges locally in $C^2$ to $u_\infty$ and so $u_\infty$ solves the Yamabe equation. Consequently, as $u_\infty$ is constant, either $u_\infty \equiv 0$ or $u_\infty \equiv 1$.

It then suffices to show that $u_\infty(0) > 0$ to conclude the proof. We take the limit in (3.29) as $R \to \infty$ at $r = 0$ to obtain

$$\lim_{R\to\infty} u_R(0) = \lim_{R\to\infty} \left((e^2-1)e^{-R}\right)^{\frac{n-2}{2}} u_1\left(1-(e^2-1)e^{-R} + \mathcal{O}(e^{-2R})\right)$$

and we note also that

$$1-(e^2-1)e^{-R} + \mathcal{O}(e^{-2R}) \to 1.$$

Then, from the asymptotic rates (3.28) for $u_1$, we conclude that

$$u_\infty(0) = \lim_{R\to\infty} u_R(0) \geq \lim_{R\to\infty} C\left((e^2-1)e^{-R}\right)^{\frac{n-2}{2}}\left((e^2-1)e^{-R}+\mathcal{O}(e^{-2R})\right)^{-\frac{n-2}{2}} = C > 0.$$

Consequently, we conclude that $u_\infty \equiv 1$. □

Having established the properties of the family $u_R$ above, we now prove Proposition 3.2.9 by showing that these functions can be used to generate super-solutions with the desired properties in the general asymptotically locally hyperbolic case.

*Proof of Proposition 3.2.9.* We will use the family of solutions from Lemma 3.2.11 to provide a bound on the limsup of a solution to the Yamabe equation on an arbitrary asymptotically locally hyperbolic manifold $(M, g)$ with reference metric $\mathring{g}$.

First, we rewrite the Yamabe equation for $g$ as a perturbation of an elliptic equation with respect to the reference metric $\mathring{g}$ as in (3.20). In particular, we have that any solution $u$ of the Yamabe equation for $g$ satisfies

$$-c_n\Delta_{\mathring{g}}u + S_g u + n(n-1)u^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}u + b^i\mathring{\nabla}_i u = 0. \tag{3.33}$$

where $|a^{ij}| = \mathcal{O}_1(e^{-(\alpha+2)r})$ and $|b^i| = \mathcal{O}(e^{-\alpha r})$.

Fix $R > 0$ and consider some point $x_0 \in M^+$ such that $r_0 := r(x_0) > R$. To bound the value of $u$ at $x_0$, we define the annulus $\Omega_R := \Omega(R, r_0) = \{x \in M^+ : |r(x) - r_0| < R\}$ and define a candidate super-solution to (3.33) on $\Omega_R$,

$$\bar{u}(x) := Au_R(r(x) - r_0)$$

where $u_R$ is defined in (3.29) and $A > 0$ is some constant to be determined. In the argument below, all implicit constants in various $\mathcal{O}$ terms are independent of both $A$ and $R$.

We first note that

$$-c_n\Delta_{\mathring{g}}\bar{u} = -c_n\left(\partial_{rr}\bar{u} + (n-1)q_k(r)\partial_r\bar{u}\right) = A\left(-c_n\left(u_R'' + (n-1)u_R'\right) + \mathcal{O}(e^{-2r}|u_R'|)\right)$$

where

$$q_k(r) = \begin{cases} \coth(r) & k = 1, \\ 1 & k = 0, \\ \tanh(r) & k = -1, \end{cases}$$

where $k$ corresponds to that in the definition of the reference metrics in (3.1) and we use only that $q_k = 1 + \mathcal{O}(e^{-2r})$.

Consequently, from (3.30), we have that $\bar{u}$ satisfies

$$-c_n\Delta_{\mathring{g}}\bar{u} - n(n-1)\bar{u} + n(n-1)\bar{u}^{\frac{n+2}{n-2}} = n(n-1)\left(A^{\frac{n+2}{n-2}} - A\right)u_R^{\frac{n+2}{n-2}} + A\mathcal{O}\left(e^{-2r}|u_R'|\right)$$

for the reference metric $\mathring{g}$ on $\Omega_R \subset M^+$ and $\bar{u}$ blows up as $x \to \partial\Omega_R$. If follows that

$$\begin{aligned}
&-c_n\Delta_{\mathring{g}}\bar{u} + S_g\bar{u} + n(n-1)\bar{u}^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}\bar{u} + b^i\mathring{\nabla}_i\bar{u} \\
&\quad = -c_n\Delta_{\mathring{g}}\bar{u} - n(n-1)\bar{u} + n(n-1)\bar{u}^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}\bar{u} + b^i\mathring{\nabla}_i\bar{u} + (S_g + n(n-1))\bar{u} \\
&\quad = n(n-1)\left(A^{\frac{n+2}{n-2}} - A\right)u_R^{\frac{n+2}{n-2}} + A\left(a^{ij}\mathring{\nabla}_{ij}u_R + b^i\mathring{\nabla}_iu_R + (S_g + n(n-1))u_R\right) \\
&\quad \geq n(n-1)\left(A^{\frac{n+2}{n-2}} - A\right)u_R^{\frac{n+2}{n-2}} - A\varepsilon(r(x_0))\left(|u_R''| + |u_R'| + |u_R|\right)
\end{aligned}$$

where $\varepsilon(r) \geq 0$ and $\lim_{r\to\infty}\varepsilon(r) = 0$; here we have used (3.25) as well as the asymptotic local hyperbolicity.

From the asymptotic rates in (3.31), we see that there exists a constant $C_R > 0$ depending on $R$ such that $|u_R''| + |u_R'| + |u_R| \le C_R u_R^{\frac{n+2}{n-2}}$ near the boundary. Therefore,

$$-c_n \Delta_{\mathring{g}} \bar{u} + S_g \bar{u} + n(n-1)\bar{u}^{\frac{n+2}{n-2}} + a^{ij}\mathring{\nabla}_{ij}\bar{u} + b^i \mathring{\nabla}_i \bar{u}$$
$$\ge n(n-1)\left(A^{\frac{n+2}{n-2}} - A\left(1 + C_R\varepsilon(r_0)\right)\right) u_R^{\frac{n+2}{n-2}}.$$

We can now choose $A = 1 + \tilde{C}\varepsilon(r_0)$ with a sufficiently large constant $\tilde{C}$ independent of $x_0$ so that the RHS above is positive and so $\bar{u}$ is a strict super-solution to (3.33).

We may now apply Lemma 3.2.7 to conclude that $u < \bar{u}$ on $\Omega_R$ and so we have shown that any solution to the Yamabe equation $u$ satisfies

$$u(x_0) < (1 + \tilde{C}\varepsilon(r_0))u_R(0).$$

Consequently, as $x_0$ was arbitrary and $\varepsilon(r) \to 0$ as $r \to \infty$, we may take $r_0$ arbitrarily large to obtain

$$\limsup_{r(x)\to\infty} u(x) \le u_R(0).$$

As $R > 0$ was arbitrary, we may now take the limit as $R \to \infty$ to conclude, from Lemma 3.2.11, that

$$\limsup_{r(x)\to\infty} u(x) \le 1$$

as desired. □

#### 3.2.2.3 Discussion of Uniqueness and an A Priori Lower Bound

In this sub-section, we briefly discuss ideas which may lead towards a proof of a lower bound for solutions to (Ya) on asymptotically locally hyperbolic manifolds. Though some progress was made, the author was unable to complete a proof of a lower bound within the time constraints of this DPhil project.

We again provide a motivating argument for a lower bound in the case that our manifold is the hyperbolic space, in a similar way to the motivating discussion in sub-section 3.2.2.1. Again, the proof strategy summarised below is present in the literature, for example in [LN74] and [ACF92]. Once more, we conveniently express the hyperbolic space as the disk model $(B_1, g_{\text{hyp}})$ where $B_1$ is the Euclidean unit ball and suppose $u$ is a solution to the Yamabe equation for $(B_1, g_{\text{hyp}})$ i.e.

$$-c_n \Delta_{g_{\text{hyp}}} u - n(n-1)u = -n(n-1)u^{\frac{n+2}{n-2}} \text{ on } B_1.$$

We may construct a sub-solution to (Ya) by taking some larger ball $B_R \supset B_1$ and letting $g_R$ denote the hyperbolic metric on $B_R$ such that the restriction of $g_R$ to $B_1$ can be written $g_R = u_R^{\frac{4}{n-2}} g_{\text{hyp}}$, the hyperbolic metric on $B_R$. We now consider the prospective solution $\tilde{g} = u^{\frac{4}{n-2}} g_{\text{hyp}} = (\frac{u}{u_R})^{\frac{4}{n-2}} g_R$ on $B_1$. We deduce from completeness that $\frac{u}{u_R} \to \infty$ as $\tilde{g}$ is assumed to be complete and so we may apply our previous argument to see that $\frac{u}{u_R} \geq 1$ and so $u \geq u_R$. Letting $R \searrow 1$ gives $u_R \to 1$ and so $u \geq 1$.

The author believes the approach used to treat the model case outlined above will also yield a lower bound on the value of a solution $u$ to (Ya) for an asymptotically locally hyperbolic manifolds in the sense of Definition 3.1.1 with the requirement that the corresponding conformal metric is complete. This has been the case for similar settings presented in [ACF92] and [AILA18], albeit under stronger conditions on the curvature decay.

We finally note that an a priori lower bound of 1 on solutions of (Ya) satisfying the completeness requirement would allow us, via a maximum principle argument, to improve the maximality of the solution $u$ in Theorem A (established in the next section) to uniqueness of solutions for the Yamabe problem on asymptotically locally hyperbolic manfiolds.

### 3.2.3 Part III: Asymptotic Behaviour of the Conformal Factor at Infinity

We now turn to the second part of Theorem A, namely to establish the particular asymptotic behaviour of the solution obtained in the Section 3.2.1 when additionally given a full decay of the scalar curvature.

We begin by creating a particular super-solution on $M$. The following super-solution will provide an asymptotic upper bound on our solution obtained in Proposition 3.2.5 and will allow us to conclude that the conformal metric remains locally asymptotically hyperbolic to the same order (up to an additional factor of $r$ in the critical case that $\alpha = n$). We first address the case that $\alpha < n$.

**Lemma 3.2.12.** *Let $(M, g)$ be asymptotically locally hyperbolic of order $\alpha < n$ and $S_g \geq -n(n-1) - Ce^{-\alpha r}$ for some constant $C > 0$. There exist constants $A > 0$ and $R > 0$ such that the function $u_+ \in H^1_{loc}(M)$ defined by*

$$u_+(r) := \begin{cases} 1 + Ae^{-\alpha r} & \text{on } \{r \geq R\} \\ 1 + Ae^{-\alpha R} & \text{on } M_0 \cup \{r < R\} \end{cases},$$

*is a super-solution to* (Ya) *on $M$.*

In the same way as in Lemma 3.2.3, we must establish the inequality

$$\begin{aligned} -c_n\left(1 + \mathcal{O}(e^{-2\alpha r})\right) u_+'' + c_n\left((n-1)\frac{f_k'}{f_k}(r + r_0) + \mathcal{O}(e^{-\alpha r})\right) u_+' \\ - \left(n(n-1) + \mathcal{O}(e^{-\alpha r})\right) u_+ \geq -n(n-1)u_+^{\frac{n+2}{n-2}}. \end{aligned} \tag{3.34}$$

*Proof.* We need to find large $A$ and $R$ such that (3.34) holds in the two regions $r \leq R$ and $r > R$ and establish transmission conditions.

We first note that the transmission condition

$$0 = \lim_{r \nearrow R} u_+'(r) \geq \lim_{r \searrow R} u_+'(r) = -\alpha A e^{-\alpha R}$$

certainly holds.

We proceed similarly as in the proof of Lemma 3.2.3 with the difference that (as $u_+ > 1$) we can estimate directly on the RHS that

$$-n(n-1)u_+^{\frac{n+2}{n-2}} \leq -n(n-1)\left(1 + \frac{n+2}{n-2}(u_+ - 1)\right).$$

In the following the value of $C$ may change from line to line but always depends only on $g$ and we choose $R$ larger if necessary so that $R > 1$. We compute for $r > R$, using that $\frac{f_k'}{f_k}(r+r_0) > 1 - Ce^{-2r}$ (see Section 3.1.1) and $u_+' < 0$,

$$-c_n\Delta_g u_+ + S_g u_+ + n(n-1)u_+^{\frac{n+2}{n-2}} \geq -c_n A e^{-\alpha r}\Bigg[\alpha^2 - (n-1)\alpha - n$$
$$+ Ce^{-\alpha r} + Ce^{-2r} + \frac{C}{A}\Bigg].$$

Thus, we must choose $A$ and $R$ such that

$$\left[\alpha^2 - (n-1)\alpha - n\right] + Ce^{-\alpha R} + Ce^{-2R} + \frac{C}{A} \leq 0. \tag{3.35}$$

As $\alpha^2 - (n-1)\alpha - n < 0$ we can select an $A_0$ and $R$ such that the RHS of the above is positive for all $A > A_0$. Fix $R$ from hereon. It remains to find an $A > A_0$ such that (3.35) holds on $r \leq R$.

To establish that $u_+$ is a super-solution on $M_0 \cup \{r < R\}$, note that there exists an $A > A_0$ sufficiently large such that

$$C_1\left(1 + Ae^{-\alpha R}\right) \leq n(n-1)\left(1 + Ae^{-\alpha R}\right)^{\frac{n+2}{n-2}},$$

where $C_1 := \sup_{M_0 \cup \{r<R\}} |S_g|$. Consequently we have that, on $M_0 \cup \{r < R\}$,

$$-c_n\Delta_g u_+ + S_g u_+ \geq -C_1(1 + Ae^{-\alpha R}) \geq -n(n-1)(1 + Ae^{-\alpha R})^{\frac{n+2}{n-2}}$$

and so $u_+$ is also a super-solution on $M_0 \cup \{r < R\}$. □

Once again, as in the proof of Lemma 3.2.3, we find that the leading term $\alpha^2 - (n-1)\alpha - n$ in (3.35) vanishes and so we are unable to complete the proof. We address this deficiency with the adjustment in the lemma below.

**Lemma 3.2.12′.** *Let $(M, g)$ be asymptotically locally hyperbolic of order $\alpha = n$ and $S_g \geq -n(n-1) - Ce^{-nr}$ for some constant $C$. There exist constants $A > 0$ and $R > 0$ such that the function $u_+ \in H^1_{loc}(M)$ defined by*

$$u_+(r) := \begin{cases} 1 + Are^{-nr} & \text{on } \{r \geq R\} \\ 1 + ARe^{-nR} & \text{on } M_0 \cup \{r < R\} \end{cases},$$

*is a super-solution to* (Ya) *on $M$.*

*Proof.* The proof is the same as in Lemma 3.2.12 with the only difference being in choosing $R$ so that the transmission conditions hold and that $u_+$ satisfies (3.34) for $r > R$.

We first note that the transmission condition

$$0 = \lim_{r \nearrow R} u'_+(r) \geq \lim_{r \searrow R} u'_+(r) = A(1 - nR)e^{-nR}$$

certainly holds for $R > \frac{1}{n}$.

We proceed in the same way as in the proof of Lemma 3.2.12. We compute for $r > R$, using that $\frac{f'_k(r+r_0)}{f_k(r+r_0)} > 1 - Ce^{-2r}$ (see Section 3.1.1),

$$-c_n \Delta_g u_+ + S_g u_+ + n(n-1)u_+^{\frac{n+2}{n-2}} \geq -c_n A e^{-nr}\left[-(n+1) + Cre^{-2r} + \frac{C}{A}\right].$$

Thus, we must choose $A$ and $R$ such that

$$-(n+1) + CRe^{-2R} + \frac{C}{A} \leq 0. \tag{3.36}$$

It is then clear that the RHS of the above is positive for $A$ and $R$ large enough. We can then complete the proof that $u^+$ is a super-solution on $M_0 \cup \{r < R\}$ in the same way as in the proof of Lemma 3.2.12, thus completing the proof. □

Having established the super- solutions above, we use them to gain control on the solution $u$ obtained in the previous section.

**Lemma 3.2.13.** *Let $(M, g)$ be asymptotically locally hyperbolic of order $\alpha \leq n$ with $|S_g + n(n-1)| \leq Ce^{-\alpha r}$ for some constant $C$. Then the smooth solution $u$ of* (Ya) *on $M$ obtained in Proposition 3.2.5 is of the form $u = 1 + \hat{u}$ where*

$$\hat{u} = \begin{cases} \mathcal{O}(e^{-\alpha r}) \text{ if } \alpha < n, \\ \mathcal{O}(re^{-nr}) \text{ if } \alpha = n. \end{cases}$$

*Furthermore, $u$ is maximal in the sense that any solution $\tilde{u}$ of* (Ya) *satisfies $\tilde{u} \leq u$.*

*Proof.* From Proposition 3.2.5, we have that $u \geq u_- \geq 1 - Ce^{-\alpha r}$ where $u_-$ is the sub-solution constructed in Lemma 3.2.3 (or correspondingly $u \geq u_- \geq 1 - Cre^{-nr}$ with $u_-$ from Lemma 3.2.3′ if $\alpha = n$). Additionally, Proposition 3.2.9 gives that $\limsup_{r(x)\to\infty} u \leq 1$. Consequently, we have that $\lim_{r(x)\to\infty} u(x) = 1$ and it remains only to control the decay from above.

To this end, we will use the super-solution $u_+ \leq 1 + Ce^{-\alpha r}$ constructed in Lemma 3.2.12 (or correspondingly $u_+ \leq 1 + Cre^{-nr}$ from 3.2.12′ if $\alpha = n$). We will show that $u \leq u_+$.

For the sake of contradiction, we suppose that $\inf \frac{u_+}{u} = \frac{1}{C} < 1$ for some constant $C > 1$. As $\lim_{r(x)\to\infty} \frac{u_+(x)}{u(x)} = 1$, the infimum would have to be achieved and so we could define the function $v := Cu_+ - u$ which would satisfy $v \geq 0$ and would achieve 0 at some minimum. We could then apply the same maximum principle argument as in Lemma 3.2.7. In particular, as $C > 1$, $v$ satisfies

$$\begin{aligned} -c_n \Delta_g v - n(n-1)v &= -n(n-1)\left(Cu_+^{\frac{n+2}{n-2}} - u^{\frac{n+2}{n-2}}\right) \\ &> -n(n-1)\left((Cu_+)^{\frac{n+2}{n-2}} - u^{\frac{n+2}{n-2}}\right) = c(x)v, \end{aligned}$$

where

$$c(x) = \begin{cases} \frac{(Cu_+)^{\frac{n+2}{n-2}} - u^{\frac{n+2}{n-2}}}{Cu_+ - u} & \text{for } v \neq 0, \\ \frac{n+2}{n-2}(Cu_+(x))^{\frac{4}{n-2}} & \text{if } v = 0. \end{cases}$$

We could then apply the strong maximum principle to the linear differential inequality above, noting that the sign of the zero order term $(n(n-1) + c(x))$ does not matter as the value of the minimum in question is 0, see [GT01, Section 3.2]. We would then have that $v \equiv 0$, a contradiction as $v \to C - 1 > 0$ as $r(x) \to \infty$. We conclude that $\inf \frac{u_+}{u} \geq 1$ and so $u \leq u_+$ as desired.

Consequently, we can certainly write $u$ in the form $u = 1 + \hat{u}$ with $\hat{u} = \mathcal{O}(e^{-\alpha r})$ ($\hat{u} = \mathcal{O}(re^{-nr})$ if $\alpha = n$) as an immediate consequence of the asymptotic decays of $u_-$ and $u_+$.

Given another solution $\tilde{u} > 0$ of (Ya), we know that $\limsup_{r(x)\to\infty} \tilde{u}(x) \leq 1$ again from Proposition 3.2.9, and so we have that $\liminf_{r(x)\to\infty} \frac{u}{\tilde{u}} \geq 1$ as $\lim_{r(x)\to\infty} u(x) = 1$.

Consequently, if $\inf \frac{u}{\tilde{u}} < 1$ then the infimum must be attained and so we may apply the same maximum principle argument above to conclude that $\tilde{u} \leq u$. □

**Remark 3.2.14.** *One may also use the method of sub- and super-solutions directly using Lemmas 3.2.3, 3.2.3′, 3.2.12 and 3.2.12′ in order to prove the existence of a solution with the asymptotic behaviour above. The question of whether this solution is the same as the solution obtained in Proposition 3.2.5 has not been resolved within the time constraints of this thesis (see the discussion on a priori lower bounds and uniqueness in Section 3.2.2.3).*

We are almost ready to complete the proof of Theorem A. However, in order to ensure that the corresponding metric to our conformal factor $u$ $(= 1 + \hat{u})$ is asymptotically locally hyperbolic, we must first establish stronger $C^2$ decay of $\hat{u}$.

**Lemma 3.2.15.** *Let $(M, g)$ be asymptotically locally hyperbolic of order $\alpha < n$ with $|S_g + n(n-1)| \leq Ce^{-\alpha r}$ for some constant $C > 0$. Suppose $\varphi = \mathcal{O}(e^{-\alpha r})$ satisfies*

$$- c_n \Delta_g \varphi + S_g(1 + \varphi) = -n(n-1)(1 + \varphi)^{\frac{n+2}{n-2}} \tag{3.37}$$

*on $M$. Then $\varphi$ satisfies*

$$\left| \mathring{\nabla} \varphi + \mathring{\nabla}^2 \varphi \right|_{\mathring{g}} = \mathcal{O}\left(e^{-\alpha r}\right)$$

*or equivalently*

$$\partial_r \varphi = \mathcal{O}\left(e^{-\alpha r}\right), \qquad |\nabla_{\mathring{h}} \varphi|_{\mathring{h}} = \mathcal{O}\left(e^{-(\alpha-1)r}\right),$$

$$\partial_{rr} \varphi = \mathcal{O}\left(e^{-\alpha r}\right), \qquad |\nabla_{\mathring{h}} \left(\partial_r \varphi\right)|_{\mathring{h}} = \mathcal{O}\left(e^{-(\alpha-1)r}\right), \qquad |\nabla^2_{\mathring{h}} \varphi|_{\mathring{h}} = \mathcal{O}\left(e^{-(\alpha-2)r}\right).$$

In the following we use the notation of [GT01] to denote certain interior norms of which those needed in the following we outline here for the convenience of the reader. In particular, we will use the following weighted $C^k$ and $C^{k,\alpha}$ norms on a bounded Euclidean domain $W$:

$$|f|^{(\sigma)}_{k;W} = \sum_{j=0}^{k} \sup_{\substack{x \in W \\ |\beta| = j}} d_x^{k+\sigma} |\partial^\beta f(x)|,$$

$$|f|^{(\sigma)}_{k,\alpha;W} = |f|^{(\sigma)}_{k;W} + \sup_{\substack{x,y \in W \\ |\beta| = k}} d_{x,y}^{k+\alpha+\sigma} \frac{|\partial^\beta f(y) - \partial^\beta f(x)|}{|x - y|^\beta},$$

where $\beta$ is some multi-index and $d_x$ and $d_{x,y}$ denote the distance from $x$ to the boundary of $W$ and the minimum distance from the boundary of $x$ and $y$, respectively. When $\sigma = 0$ we use the notation $|\cdot|^*_{k,\beta;W} := |\cdot|^{(0)}_{k,\beta;W}$ and if additionally $k = 0$ also we simply write $|\cdot|_{0;W}$ for the sup-norm on $W$. When the domain $W$ in question is clear, we will omit it in this notation.

Our approach will be to apply Schauder estimates to (3.37) on neighbourhoods further and further along the asymptotic end. This is related to treatments in [ACF92] and [AILA18]. More precisely, we will rewrite equation (3.37) as a perturbation of a fixed elliptic operator on a Euclidean domain via a "rescaling" of the cross-sectional coordinates. This allows us to apply standard Schauder estimates to this fixed operator and then track the dependence on the distance along the asymptotic end in the perturbation coefficients and the rescaling of the cross-sectional coordinates.

*Proof.* Take an arbitrary point $x_0 = (r_0, \theta_0)$ in the exterior region $M^+ = \mathbb{R}_{\geq 0} \times N$ with $r_0 := r(x_0) \gg 1$. Fix a finite set of coordinate patches $\{U_i \subset N\}$ which cover $N$. Given $x_0$, we choose one such chart containing $\theta_0$ which maximises the distance $d^h_{\theta_0}$ with respect to the metric $\mathring{h}$ from $\theta_0$ to the boundary of the coordinate chart $U_i$. Consequently, as $\{U_i\}$ is an open covering and $N$ is compact, there must exist a $\delta > 0$ independent of $\theta_0$ so that $d^h_{\theta_0} > \delta$; if this were not the case, a sequence minimising $\max_{U_i \ni \theta_0} d^h_{\theta_0}$ would have an accumulation point in $N$ which would have zero distance to the boundary of any chart $U_i$ containing it, a contradiction. From here on write $U = U_i$ and define $U_\delta \subset U$ to be the set of all points in $U$ that are at least a distance $\delta$, with respect to $h$, away from the boundary.

We fix a coordinate system $\{\theta^a\}$ on $U$ where $\theta^a : U \to \mathbb{R}^{n-1}$; without loss of generality, by shrinking the original choice of charts slightly, we assume that the coordinates are defined in an open neighbourhood of $\bar{U}$ so that $\mathring{h}_{ab}$ is positive definite on $\bar{U}$. For simplicity, we identify $U$ with a subset of $\mathbb{R}^{n-1}$ via the coordinates $\theta^a$. Consider the neighbourhood $\Omega$ of $x_0$ by $\Omega := (r_0 - 1, r_0 + 1) \times U$ and define a mapping

$$\Psi(r, \theta) = (\tilde{r}, \tilde{\theta}) := (r - r_0, e^{r_0}\theta).$$

We define $\tilde{U} := \{\tilde{\theta} : \theta \in U\}$, $\tilde{U}_\delta := \{\tilde{\theta} : \theta \in U_\delta\}$ and $\tilde{V} := \Psi(\Omega) = (-1,1) \times \tilde{U}$. We consider $\tilde{V}$ as a subset of $\mathbb{R}^n$ and $(\tilde{r}, \tilde{\theta})$ as coordinates on $\Omega$. We denote partial derivatives in these coordinates by $\tilde{\partial}$, that is we write $\frac{\partial}{\partial \tilde{r}} = \tilde{\partial}_r$ and $\frac{\partial}{\partial \tilde{\theta}^a} = \tilde{\partial}_a$. We note here that $\tilde{\partial}_r = \partial_r$ and $\tilde{\partial}_a = e^{-r_0}\partial_a$.

We first note that, in the $(\tilde{r}, \tilde{\theta})$ coordinates, $\Delta_{\mathring{g}}$ can be expressed in non-divergence form as

$$\Delta_{\mathring{g}} = \tilde{\partial}_{rr} + \frac{e^{2r_0}}{f_k^2}\mathring{h}^{ab}\tilde{\partial}_{ab} + (n-1)q_k\tilde{\partial}_r + \frac{e^{2r_0}}{f_k^2}\frac{\tilde{\partial}_a\left(\mathring{h}^{ab}\sqrt{\mathring{h}}\right)}{\sqrt{\mathring{h}}}\tilde{\partial}_b =: \tilde{L},$$

where $f_k = \mathcal{O}(e^r)$ is as in the definition of the reference locally hyperbolic metric (3.1) and

$$q_k(r) = \frac{f_k'(r)}{f_k(r)} = \begin{cases} \coth(r) & k = 1, \\ 1 & k = 0, \\ \tanh(r) & k = -1. \end{cases}$$

As the coordinates $\theta^a$ are defined on an open neighbourhood of $\bar{U}$, $\tilde{L}$ defined above has a uniform (in $r_0$) lower bound on the ellipticity constant and uniform upper bound on the Hölder norms of the coefficients with respect to the Euclidean metric.

We now write $\Delta_g$ as a perturbation of $\Delta_{\mathring{g}}$ as in (3.20) to obtain

$$\Delta_g\varphi - \Delta_{\mathring{g}}\varphi = a^{ij}\mathring{\nabla}_{ij}\varphi + b^i\mathring{\nabla}_i\varphi = \mathcal{O}\left(e^{-\alpha r_0}\left(|\tilde{\partial}^2\varphi| + |\tilde{\partial}\varphi|\right)\right)$$

where $a^{ij} := (g^{ij} - \mathring{g}^{ij})$, $b^i := g^{jk}(\Gamma^i_{jk} - \mathring{\Gamma}^i_{jk})$, and the modulus is taken with respect to the Euclidean metric on $\tilde{V}$ and the final equality is a consequence of the decay established in Lemma 3.1.2. Consequently, we may rewrite (3.37) as

$$\begin{aligned} -c_n\tilde{L}\varphi - n\varphi &= R(\varphi) + c\varphi + \mathcal{O}\left(e^{-\alpha r_0}\left(|\tilde{\partial}^2\varphi| + |\tilde{\partial}\varphi|\right)\right) \\ &= \underbrace{R(\varphi) + \mathcal{O}\left(e^{-\alpha r_0}\left(|\tilde{\partial}^2\varphi| + |\tilde{\partial}\varphi| + |\varphi|\right)\right)}_{=f} \end{aligned}$$

where $c = S_g + n(n-1)$ and we define

$$R(x) := (1+x)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2}x.$$

We now apply the Schauder estimates of [GT01, Theorem 6.2] to the equation above on $\tilde{V}$. We obtain that, fixing some arbitrary $\beta > 0$,

$$|\varphi|^*_{2,\beta} \leq C_1(|\varphi|_0 + |f|^{(2)}_{0,\beta}) \tag{3.38}$$

where all norms are taken on $\tilde{V}$ and $C_1$ depends only on the ellipticity constant of $\tilde{L}$, the Hölder norms of the coefficients of $\tilde{L}$, $n$ and $\beta$. Consequently, as established earlier, this means that $C_1$ is independent of $r_0$.

We estimate

$$\begin{aligned} |f|^{(2)}_{0,\beta} &\leq |R(\varphi)|^{(2)}_{0,\beta} + Ce^{-\alpha r_0} \left| |\tilde{\partial}^2\varphi| + |\tilde{\partial}\varphi| + |\varphi| \right|^{(2)}_{0,\beta} \\ &\leq |R(\varphi)|^{(2)}_{0,\beta} + C_2 e^{-\alpha r_0} |\varphi|^*_{2,\beta} \end{aligned}$$

where $C_2$ depends only on $g$ and the coordinate choice but again not on $r_0$. For the first term, we write the error in the Taylor expansion as

$$R(\varphi) = C \int_0^{\varphi} (1+t)^{\frac{6-n}{n-2}} (\varphi - t)\ dt = \mathcal{O}(\varphi^2).$$

Differentiating the above yields

$$\begin{aligned} |\tilde{\partial} R(\varphi)| &\leq C \int_0^{\varphi} (1+t)^{\frac{6-n}{n-2}} |\tilde{\partial}\varphi - t|\ dt \\ &\leq C|\tilde{\partial}\varphi| \int_0^{\varphi} (1+t)^{\frac{6-n}{n-2}}\ dt + C\int_0^{\varphi} |t|(1+t)^{\frac{6-n}{n-2}}\ dt \\ &\leq \mathcal{O}(\varphi)|\tilde{\partial}\varphi| + \mathcal{O}(\varphi^2) \end{aligned}$$

so that

$$\begin{aligned} |R(\varphi)|^{(2)}_{0,\beta} &\leq |R(\varphi)|_0 + [\tilde{\partial} R(\varphi)]^{(2)}_0 \\ &\leq \mathcal{O}(e^{-2\alpha r_0}) + \mathcal{O}\left(e^{-\alpha r_0}\right) |\varphi|^*_{2,\beta}. \end{aligned}$$

In summary, we now have

$$|\varphi|^*_{2,\beta} \leq C_1(|\varphi|_0 + |f|^{(2)}_{0,\beta}) \leq Ce^{-\alpha r_0}|\varphi|^*_{2,\beta} + Ce^{-\alpha r_0}$$

where $C$ is independent of $r_0$. We then absorb the $|\varphi|^*_{2,\beta}$ term on the RHS of the above by taking $x_0$ far into the asymptotic region so that $r_0$ is very large. We obtain

$$|\varphi|^*_{2,\beta} \leq Ce^{-\alpha r_0}$$

on $\tilde{V}$ with $C$ independent of $r_0$.

It remains to convert the bound on $|\varphi|^*_{2,\beta}$ into a bound at $x_0$ on the partial derivatives $\partial_i \varphi(x_0)$ in our original coordinates on $\Omega$. We first establish a bound on the partial derivatives $\tilde{\partial}_i \varphi(x_0)$ in the coordinates on $\tilde{V}$ at $x_0$. To this end, we fix a neighbourhood

$$\tilde{W} := \left(-\frac{1}{2}, \frac{1}{2}\right) \times \tilde{U}_\delta \subset \tilde{V}$$

which certainly contains $x_0$ by the definition of $\delta$. From the definition of the norm $|\cdot|^*_{2,\beta}$ and defining $\tilde{d}_x$ to be the Euclidean distance to the boundary of $x$ on $\tilde{V}$, we see that

$$\|\varphi\|_{C^2(\tilde{W})} \leq \left(\inf_{x\in\tilde{W}} \tilde{d}_x\right)^{-2} |\varphi|^*_{2,\beta} \leq Ce^{-\alpha r_0} \left(\min\left(\frac{1}{2}, \delta f_k(r)\right)\right)^{-2} \leq Ce^{-\alpha r_0}$$

where $C$ remains independent of $x_0$ for $r_0$ sufficiently large as $f_k(r) = \mathcal{O}(e^r)$.

Recalling that $\tilde{\partial}_r = \partial_r$ and $\tilde{\partial}_a = f_k^{-1}(r)\partial_a$ we then finally obtain that, on $\Omega$ we have that

$$\partial_r \varphi(x_0) = \mathcal{O}(e^{-\alpha r_0}), \qquad \partial_a \varphi(x_0) = \mathcal{O}(e^{-(\alpha-1)r_0}),$$

$$\partial_{rr}\varphi(x_0) = \mathcal{O}(e^{-\alpha r_0}), \qquad \partial_{ra}\varphi(x_0) = \mathcal{O}(e^{-(\alpha-1)r_0}), \qquad \partial_{ab}\varphi(x_0) = \mathcal{O}(e^{-(\alpha-2)r_0}).$$

The conclusion follows directly from the above. □

The lemma above allows us to improve Lemma 3.2.13 to have $\hat{u}$ decay to zero in the full $C^2$ norm. We are now ready to prove Theorem A.

**Lemma 3.2.16.** *Suppose $(M, g)$ is an asymptotically locally hyperbolic manifold of order $\alpha \in (0, n)$. In addition, suppose that the scalar curvature satisfies the stronger condition*

$$|S_g + n(n-1)| \leq Ce^{-\alpha r},$$

*for some constant $C > 0$, then there exists a smooth positive solution $u = 1 + \mathcal{O}(e^{-\alpha r})$ of* (Ya) *and the corresponding conformal metric is also asymptotically locally hyperbolic of order $\alpha$.*

*Proof.* As a consequence of the Proposition 3.2.5 and Lemma 3.2.13, we have the existence of a conformal factor $u = 1 + \hat{u}$ where $\hat{u} = \mathcal{O}(e^{-\alpha r})$ such that the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ satisfies $S_{\tilde{g}} \equiv -n(n-1)$ on all of $M$. It remains to show that $\tilde{g}$ is asymptotically locally hyperbolic of order $\alpha$. We note that we may apply Lemma 3.2.15 to $\hat{u}$ to obtain that

$$\partial_r \hat{u} = \mathcal{O}(e^{-\alpha r}), \qquad \partial_a \hat{u} = \mathcal{O}(e^{-(\alpha-1)r}),$$

$$\partial_{rr} \hat{u} = \mathcal{O}(e^{-\alpha r}), \qquad \partial_{ra} \hat{u} = \mathcal{O}(e^{-(\alpha-1)r}), \qquad \partial_{ab} \hat{u} = \mathcal{O}(e^{-(\alpha-2)r}).$$

We write explicitly $\tilde{g} = u^{\frac{4}{n-2}} \left( dr^2 + \sinh^2(r + r_0) \mathring{h} + \varepsilon_{ra} d\theta^a dr + \varepsilon_{ab} d\theta^a d\theta^b \right)$ where $\varepsilon_{ab} = \mathcal{O}_1(e^{-(\alpha-2)r})$ and $\varepsilon_{ra} = \mathcal{O}_1(e^{-(\alpha-1)r})$. Consider the new coordinate

$$z := r + \int_r^\infty \left( 1 - u^{\frac{2}{n-2}} \right) \, ds \, .$$

First note that

$$\int_r^\infty \left( 1 - u^{\frac{2}{n-2}} \right) \, ds = \mathcal{O}(e^{-\alpha r})$$

and so $\mathcal{O}(e^{-\alpha r}) = \mathcal{O}(e^{-\alpha z})$. We compute that

$$dz = u^{\frac{2}{n-2}} dr - \left( \int_r^\infty \frac{2}{n-2} u^{\frac{4-n}{n-2}} \partial_a \hat{u} \, ds \right) d\theta^a$$

and so we have

$$\begin{aligned} u^{\frac{4}{n-2}} dr^2 = dz^2 &+ 2 \left( \int_r^\infty \frac{2}{n-2} u^{\frac{4-n}{n-2}} \partial_a \hat{u} \, ds \right) d\theta^a dz \\ &+ \left( \frac{2}{n-2} \right)^2 \left( \int_r^\infty u^{\frac{4-n}{n-2}} \partial_a \hat{u} \, ds \right) \left( \int_r^\infty u^{\frac{4-n}{n-2}} \partial_b \hat{u} \, ds \right) d\theta^a d\theta^b. \end{aligned}$$

Substituting this into the expression for $\tilde{g}$ allows us to write

$$\tilde{g} = dz^2 + \sinh^2(z + r_0) \mathring{h} + \tilde{\varepsilon}_{za} d\theta^a dz + \tilde{\varepsilon}_{ab} d\theta^a d\theta^b$$

where

$$\tilde{\varepsilon}_{za} = 2\left(\frac{2}{n-2}\right)\left(\int_r^\infty u^{\frac{4-n}{n-2}}\partial_a \hat{u}\; ds\right) + u^{\frac{2}{n-2}}\varepsilon_{ra}$$

and

$$\begin{aligned}\tilde{\varepsilon}_{ab} =& \left(\frac{2}{n-2}\right)^2\left(\int_r^\infty u^{\frac{4-n}{n-2}}\partial_a \hat{u}\; ds\right)\left(\int_r^\infty u^{\frac{4-n}{n-2}}\partial_b \hat{u}\; ds\right)\\ &+ \left(\frac{2}{n-2}\right)u^{\frac{2}{n-2}}\left(\int_r^\infty u^{\frac{4-n}{n-2}}\partial_b \hat{u}\; ds\right)\varepsilon_{ra}\\ &+ \left(u^{\frac{4}{n-2}}\sinh^2(r+r_0) - \sinh^2(z+r_0)\right)\mathring{h}_{ab} + u^{\frac{4}{n-2}}\varepsilon_{ab}.\end{aligned}$$

To conclude that $\tilde{g}$ is asymptotically locally hyperbolic in the sense of Definition 3.1.1, it remains to show the appropriate decay properties of $\varepsilon_{za}$ and $\varepsilon_{ab}$. Using the fact that $\partial_z = (1 + \mathcal{O}(e^{-\alpha z}))\,\partial_r$, the $C^2$ decay of $\hat{u}$ from Lemma 3.2.15 and the fact that

$$\sinh^2(r+r_0) = \sinh^2(z+r_0) + \mathcal{O}_1(e^{-(\alpha-2)z})$$

yields, via direct computation, that $\tilde{\varepsilon}_{za} = \mathcal{O}_1(e^{-(\alpha-1)z})$ and $\tilde{\varepsilon}_{ab} = \mathcal{O}_1(e^{-(\alpha-2)z})$ as desired. □

In summary, in Section 3.2.1 we established the existence part of Theorem A and we have now completed the proof of the second part of Theorem A as a combination of Lemma 3.2.13 and Lemma 3.2.16.

# 3.3

# The Yamabe Problem for Asymptotically Warped Product Manifolds

We now return to our original motivating goal of understanding when a solution to the Yamabe problem for manifolds with $S_g \leq -n(n-1)$ outside of some compact set exists. Having proved Theorem A, we extend our class of manifolds by considering asymptotically warped product manifolds.

Once again, we suppose that we can write $(M, g)$ in two parts, $M = M_0 \cup M^+$, where $M_0$ is some compact manifold with boundary, $M^+ = \mathbb{R}^+ \times N$ with $(N, \mathring{h})$ some compact manifold. We suppose that the metric $g$ on $M^+$ is asymptotic (in a sense to be made precise in the subsequent sections) to the reference warped product

$$\mathring{g}_f = dz^2 + f^2(z)\mathring{h} \tag{3.39}$$

where $f$ is some positive function, usually referred to as the warping function. We note again the useful formula for the scalar curvature of such a warped product metric

$$S_{\mathring{g}_f} = -2(n-1)\frac{f''}{f} - (n-1)(n-2)\left(\frac{f'}{f}\right)^2 + \frac{S_{\mathring{h}}}{f^2}. \tag{3.40}$$

We suppose that $\partial M_0$ corresponds directly to the set $\{0\} \times N \subset M^+$. We refer to such a manifold with the exact reference metric $\mathring{g}$ as having a warped product end and to $g$ as having an asymptotically warped product end. This notion generalises the class of asymptotically locally hyperbolic metrics discussed in earlier sections.

We first make a remark regarding the approach used in the proof of Theorem A and how it may be applied to a broader class of asymptotically warped product manifolds. A key element of the proof of Theorem A is the construction of the sub-solution in Lemma 3.2.3. In the proof of Lemma 3.2.3, we note that the only properties of the underlying reference metrics (see Definition 3.1.1 and preceding discussion for details) that were used were their particular warped product form $\mathring{g} = dr^2 + f_k^2(r)\mathring{h}$ and that their warping functions $f_k$ satisfied

$$\liminf_{r\to\infty} \frac{f_k'}{f_k} \geq 1. \tag{3.41}$$

While the approach of Theorem A could likely be adapted to asymptotically warped product manifolds satisfying the above condition, we find that we are able to encompass a significantly larger class by first constructing a conformal change to an asymptotically locally hyperbolic metric and then applying our Theorem A to this conformal metric.

In particular, in Section 3.3.1, we prove that a necessary and sufficient criterion for a metric $\mathring{g}$ of a warped product end to be conformally locally hyperbolic is that $f$ satisfies

$$\int_0^\infty \frac{1}{f} < \infty. \tag{3.42}$$

We highlight that, while we will go on to construct radially symmetric conformal factors to prove the sufficiency of the above, *we do not assume that the conformal factors must be radially symmetric in proving the necessity of the above condition.*

We note here that condition (3.41) implies the criterion (3.42) and, furthermore, the latter is significantly weaker than condition (3.41) (for example, including warping functions of the form $f(z) = z^\alpha$ for $\alpha > 1$).

Furthermore, in Section 3.3.2, we address the yet broader class of those metrics $g$ which asymptote to a warped product satisfying the criterion above; we provide the asymptotic properties of the constructed coordinate change and conformal factor as well as rates for the asymptotic decay corresponding to those required in our main Definition 3.1.1 of asymptotic local hyperbolicity.

### 3.3.1 A Necessary and Sufficient Criterion for Conformal Equivalence to a Locally Hyperbolic End for Warped Product Ended Manifolds

We consider the metric $\mathring{g}_f$ as in (3.39). In this subsection we prove a necessary and sufficient condition on $f$ for $\mathring{g}_f$ to be conformal to the corresponding locally hyperbolic metric (3.1). In particular, we will show that

**Theorem 3.3.1.** *A metric $\mathring{g}_f$ with a warped product end as in* (3.39) *is conformal to a metric with a locally hyperbolic end if and only if*

$$\int_0^\infty \frac{1}{f(s)}\, ds < \infty. \tag{3.43}$$

We will establish the theorem above as a consequence of Lemmas 3.3.2, 3.3.4 and 3.3.5 proved in this subsection.

We begin our work to prove Theorem 3.3.1 by establishing that (3.43) is sufficient to show that $\mathring{g}_f$ is conformal to a metric with a locally hyperbolic end.

Consider a change of coordinates $r = K(z)$ for some strictly increasing $K$, potentially only on the tail $[z_0, \infty) \times N$ of the warped product end for some $z_0 > 0$. We would like to choose $K$ so that, in the new coordinate system, the metric (3.39) is manifestly conformal to the reference locally hyperbolic metric (3.1). For convenience, we choose $K(z_0) = 0$. In order to preserve the non-compact radial factor for the desired locally hyperbolic end $\mathbb{R}_{\geq 0} \times N$, we first need that $K$ satisfies

$$K(\infty) := \lim_{z\to\infty} K(z) = \infty.$$

In this coordinate system $\mathring{g}_f$ is written as

$$\mathring{g}_f = (K'(z))^{-2}\left(dr^2 + (K'(z)f(z))^2\mathring{h}\right).$$

It is then readily seen that, for the metric to then be conformal to the reference metric, we would require that $K$ solves

$$f(z)K'(z) = \begin{cases} \sinh(K(z)+r_0) & k = 1, \\ e^{K(z)+r_0} & k = 0, \\ \cosh(K(z)+r_0) & k = -1, \end{cases}$$

for $z \geq z_0$.

We establish the existence of a solution to the above with the desired properties under condition (3.43) on the warping factor $f$ in the following lemma.

**Lemma 3.3.2.** *Let $(M, \mathring{g}_f)$ be a manifold with a warped product end as in* (3.39)*. Suppose additionally that*

$$\int_0^\infty \frac{1}{f(s)}\, ds < \infty. \tag{3.44}$$

*Then there exists a conformal metric $\tilde{g}$ of $\mathring{g}_f$ such that $(M, \tilde{g})$ has a locally hyperbolic end.*

*Proof.* We first solve the ODEs

$$f(z)K'(z) = \begin{cases} \sinh(K(z) + r_0) & k = 1, \\ e^{K(z)+r_0} & k = 0, \\ \cosh(K(z) + r_0) & k = -1. \end{cases}$$

for $z \geq z_0$ with the initial condition $K(z_0) = 0$. As the equations above are separable, we arrive immediately at the following solutions

$$K(z) = \begin{cases} 2\,\mathrm{arctanh}\left(\tanh\left(\frac{r_0}{2}\right)\exp\left(\int_{z_0}^{z}\frac{1}{f(s)}\,ds\right)\right) - r_0 & k = 1, \\ \log\left(\frac{1}{e^{r_0} - \int_{z_0}^{z}\frac{1}{f(s)}\,ds}\right) + r_0 & k = 0, \\ 2\,\mathrm{arctanh}\left(\tan\left(\frac{1}{2}\int_{z_0}^{z}\frac{1}{f(s)}\,ds + \arctan\left(\tanh\left(\frac{r_0}{2}\right)\right)\right)\right) - r_0 & k = -1. \end{cases}$$

To conclude, we need to show that, for each $k = 1, 0, -1$, there exist $r_0$ and $z_0$ so that the solutions above are defined for all $z \geq z_0$ and the remaining requirements that $K(z)$ is increasing and $K(\infty) = \infty$ hold.

For $k = 1$, in order to see that $K(z)$ is well defined for all $z \geq z_0$ we require that

$$\tanh\left(\frac{r_0}{2}\right)\exp\left(\int_{z_0}^{\infty}\frac{1}{f(s)}\,ds\right) = 1.$$

As $\int_0^\infty \frac{1}{f} < \infty$, we may set $z_0 = 0$ and choose a corresponding $r_0$ so that the above is true. That $K$ is increasing is clear as $r_0 > 0$.

For $k = 0$ we have that $K(z)$ is increasing for any choice of $r_0$ and $z_0$. Again using that $\int_0^\infty \frac{1}{f} < \infty$, it is clear that, by setting $z_0 = 0$ and taking a corresponding $r_0$ so that

$$e^{r_0} - \int_0^\infty \frac{1}{f(s)}\,ds = 0$$

we obtain that $K(z)$ is well defined for all $z \geq z_0$ and $K(\infty) = \infty$ as desired.

Finally, for $k = -1$, we can see that $K(z)$ is well defined for all $z \geq z_0$ and $K(\infty) = \infty$ provided that

$$\frac{1}{2}\int_{z_0}^{\infty}\frac{1}{f(s)}\,ds + \arctan\left(\tanh\left(\frac{r_0}{2}\right)\right) = \frac{\pi}{4}.$$

As $\int_0^\infty \frac{1}{f} < \infty$, we may choose $z_0$ sufficiently large that $\int_{z_0}^\infty \frac{1}{f(s)}\,ds < \frac{\pi}{2}$ and a corresponding $r_0$ such that the above is true. It is evident that $K(z)$ is increasing for any choice of $r_0$ and $z_0$. □

**Remark 3.3.3.** *During the course of this project, we originally proved the above via an alternative, non-explicit method. Though we chose the above, more concise proof over this alternative proof, the latter does have some benefits in that it makes the similarity between the asymptotic behaviour of the conformal factor in the three cases* $k = -1, 0, +1$ *(which will be used later) more obvious than in the explicit solutions above. Furthermore, the alternative argument addresses a more general set of ODE problems and so it's conceivable that it could be of independent interest in this regard. For these reasons, we choose to include the argument in Appendix B, although this is non-essential content for the reader.*

Having established the sufficiency of condition (3.43) for $\mathring{g}_f$ to be conformal to a metric with a locally hyperbolic end, we now provide two lemmas that will allow us to also show its necessity, and so prove Theorem 3.3.1.

To this end, we will establish a connection between the quantity $\int_0^\infty \frac{1}{f}$ and the completeness and volume of the conformal metrics to $\mathring{g}_f$. First, in the case that $\int_0^\infty \frac{1}{f} = \infty$, we proceed in a similar manner to the previous lemma and explicitly construct a conformal metric to $\mathring{g}_f$ which is complete and has finite volume.

**Lemma 3.3.4.** *Let* $(M, \mathring{g}_f)$ *be a manifold with a warped product end as in* (3.39). *Suppose additionally that*

$$\int_0^\infty \frac{1}{f(s)}\,ds = \infty. \tag{3.45}$$

*Then there exists a conformal metric* $\breve{g}$ *of* $\mathring{g}_f$ *such that* $(M, \breve{g})$ *is complete and of finite volume.*

*Proof.* Using that $\int_0^\infty \frac{1}{f(s)}\,ds = \infty$, we will construct an explicit conformal factor between $g$ and a particular complete, finite volume metric

$$\breve{g} = d\breve{z}^2 + e^{-2\breve{z}}\mathring{h}.$$

In particular, define

$$K(z) := \log\left(1 + \int_0^z \frac{1}{f(s)} ds\right)$$

so that

$$K'(z) = \frac{1}{f(z)\left(1 + \int_0^z \frac{1}{f(s)} ds\right)}.$$

We then define a new coordinate $\breve{z} = K(z)$ so that

$$\mathring{g}_f = dz^2 + f^2(z)\mathring{h} = (K'(z))^{-2}\left(d\breve{z}^2 + (K'(z)f(z))^2\mathring{h}\right) = (K'(z))^{-2}\left(d\breve{z}^2 + e^{-2\breve{z}}\mathring{h}\right)$$

and so $\mathring{g}_f$ is conformal to $d\breve{z}^2 + e^{-2\breve{z}}\mathring{h}$ which can readily be seen to be both complete and of finite volume on $M^+$. $\square$

Having shown the above lemma, we now show that the integral $\int_0^\infty \frac{1}{f}$ diverging is in fact necessary for a metric $\mathring{g}_f$ with a warped product end to be conformal to a complete metric of finite volume. In particular, any metric $\breve{g}$ conformal to $\mathring{g}_f$ with finite volume will fail to be complete unless $\int_0^\infty \frac{1}{f} = \infty$.

**Lemma 3.3.5.** *Let $(M, \mathring{g}_f)$ be a manifold with a warped product end. Suppose that the metric $\mathring{g}_f$ is conformal to a complete, finite volume metric $\breve{g}$. Then, necessarily, $f$ must satisfy*

$$\int_0^\infty \frac{1}{f(s)}\, ds = \infty. \tag{3.46}$$

*Proof.* We recall that, on the exterior region $M^+$, we can write $\mathring{g}_f = dz^2 + f^2(z)\mathring{h}$, where we write $z : [0, \infty) \to M^+$ as a coordinate on the $\mathbb{R}_{\geq 0}$ fibre of $M^+$. By assumption, there exists some conformal factor $u(z, \theta)$ such that

$$\breve{g} = u^{\frac{4}{n-2}}(z, \theta)\mathring{g}_f.$$

We will show that the finite volume metric $\breve{g}$ can only be complete if $\int_0^\infty \frac{1}{f(s)}\, ds = \infty$. Consider the divergent curves $\gamma_\theta : [0, \infty) \to M^+$ defined by

$$\gamma_\theta(t) = (z(t), \theta)\,.$$

Note that $\dot{\gamma}_\theta(t) = \partial_z$. We compute directly the length

$$\begin{aligned}\mathcal{L}_{\breve{g}}(\gamma_\theta) &= \int_0^\infty \sqrt{\breve{g}\left(\dot{\gamma}_\theta(t), \dot{\gamma}_\theta(t)\right)}\ dt \\ &= \int_0^\infty u^{\frac{2}{n-2}}(z(t),\theta)\sqrt{\mathring{g}_f\left(\partial_z,\partial_z\right)}\ dt = \int_0^\infty u^{\frac{2}{n-2}}(z,\theta)\ dz.\end{aligned}$$

Consequently, we may write

$$\begin{aligned}\int_N \mathcal{L}_{\breve{g}}(\gamma_\theta)\ d\theta &= \int_N\int_0^\infty u^{\frac{2}{n-2}}(z,\theta)\ dz\ d\theta \\ &= \int_N\int_0^\infty \left(u^{\frac{2}{n-2}}(z,\theta)f^{\frac{n-1}{n}}(z)\right) f^{-\frac{n-1}{n}}(z)\ dz\ d\theta \\ &\leq \left(\int_N\int_0^\infty u^{\frac{2n}{n-2}}(z,\theta)f^{n-1}(z)\ dz\ d\theta\right)^{\frac{1}{n}}\left(\int_N\int_0^\infty \frac{1}{f(z)}\ dz\ d\theta\right)^{\frac{n-1}{n}} \\ &= Vol_{\breve{g}}(M^+)^{\frac{1}{n}} Vol_h(N)^{\frac{n-1}{n}}\left(\int_0^\infty \frac{1}{f(z)}dz\right)^{\frac{n-1}{n}}.\end{aligned}$$

By the assumptions on $\breve{g}$ we have both that $Vol_{\breve{g}}(M^+) < \infty$ and $\mathcal{L}_{\breve{g}}(\gamma_\theta) = \infty$ and so, as the LHS of the above is infinite, we must have that $\int_0^\infty \frac{1}{f(s)}ds = \infty$. □

The combination of the two lemmas above is sufficient to prove our main theorem, which we now show.

*Proof of Theorem 3.3.1.* The sufficiency of the condition $\int_0^\infty \frac{1}{f} < \infty$ is established in Lemma 3.3.2.

The only remaining step is to apply Lemma 3.3.4 and Lemma 3.3.5 to establish the necessity of the condition $\int_0^\infty \frac{1}{f} < \infty$. A consequence of Lemma 3.3.4 and Lemma 3.3.5 is that any two warped product metrics $\mathring{g}_{f_1}$ and $\mathring{g}_{f_2}$ with $\int_0^\infty \frac{1}{f_1} < \infty$ and $\int_0^\infty \frac{1}{f_2} = \infty$ cannot be conformal, as one is conformal to a complete metric of finite volume, by Lemma 3.3.4, and the other cannot be, by Lemma 3.3.5. We can compute for the reference locally hyperbolic metrics in (3.1), that

$$\int_0^\infty \frac{1}{f_k(s)}ds < \infty$$

for each $k = -1, 0, 1$ and so any metric conformal to one of the reference locally hyperbolic metrics must have $\int_0^\infty \frac{1}{f} < \infty$ as required. □

### 3.3.2 A Corresponding Existence Result for the Yamabe Problem on Manifolds with an Asymptotically Warped Product End

Given the necessary and sufficient criterion for conformal equivalence to the locally hyperbolic end established for manifolds with a warped product end in the previous sub-section, we now look to the implications of this equivalence for existence to the Yamabe problem for asmyptotically warped product manifolds.

**Remark 3.3.6.** *As established early on in the goals of this thesis, we are interested at all times in proving existence for the Yamabe problem for manifolds that have a representative in their conformal class that satisfies some kind of asymptotic negativity condition on the scalar curvature in the nature of* $\limsup_{r(x)\to\infty} S_g \le -n(n-1)$. *We note here that Theorem 3.3.1 of the previous section provides a criteria for the existence of such a representative within the conformal class of a warped product end, potentially showing a connection between negative scalar curvature conditions and the condition* $\int_0^\infty \frac{1}{f(s)} ds < \infty$.

We first make precise our definition of an asymptotically warped product end and establish a corresponding condition on the scalar curvature to ensure the required negativity condition of the scalar curvature on the corresponding conformally asymptotically locally hyperbolic end. In particular, we consider a Riemannian manifold $(M, g_f)$ which can be written $M = M_0 \cup (\mathbb{R}_{\ge 0} \times N)$ and we can write the metric $g_f$ on $\mathbb{R}_{\ge 0} \times N$ as

$$g_f = \mathring{g}_f + \varepsilon_{za} dz d\theta^a + \varepsilon_{ab} d\theta^a d\theta^b \tag{3.47}$$

where $\mathring{g}_f$ again is a reference warped product metric of the form $dz^2 + f(z)^2 \mathring{h}$ and $\varepsilon_{ij}$ is a perturbation satisfying, for some $\alpha \in (0, n)$,

$$\begin{aligned} \varepsilon_{ab} &= \mathcal{O}\left(f^2 H^\alpha\right), & \varepsilon_{za} &= \mathcal{O}\left(f H^\alpha\right), \\ \partial_z \varepsilon_{ab} &= \mathcal{O}\left(\left(f^3 H + f' f + f H^{-1}\right) H^\alpha\right), & \partial_z \varepsilon_{za} &= \mathcal{O}\left(\left(f^2 H + f' + H^{-1}\right) H^\alpha\right), \\ \partial_c \varepsilon_{ab} &= \mathcal{O}\left(f^2 H^\alpha\right), & \partial_c \varepsilon_{za} &= \mathcal{O}\left(f H^\alpha\right), \end{aligned} \tag{3.48}$$

where we have assumed $\int \frac{1}{f} < \infty$ and defined $H(z) := \int_z^\infty \frac{1}{f(s)} ds$. These are chosen so that $g_f$ is conformal to an asymptotically locally hyperbolic metric in the sense of Definition 3.1.1. In particular, we provide the following proposition:

**Proposition 3.3.7.** *Let $(M, g_f)$ be a manifold with an asymptotically warped product end $M^+$ on which $g_f$ is defined as in* (3.47) *with perturbation coefficients satisfying* (3.48) *for any $\alpha \in (0, n)$ and whose warping function satisfies*

$$\int_0^\infty \frac{1}{f} < \infty. \tag{3.49}$$

*Then $(M, g_f)$ is conformal to an asymptotically locally hyperbolic manifold $(M, \tilde{g})$ in the sense of Definition 3.1.1.*

*Proof.* We first define the coordinate change $r = K(z)$ and take the conformal factor $\kappa(z) := K'(z)$ where $K$ corresponds to the coordinate change obtained in the proof of Lemma 3.3.2 so that

$$\begin{aligned}\tilde{g} := \kappa^2 g &= dr^2 + f_k^2(r + r_0)\mathring{h} + \kappa^2 \varepsilon_{ab}\, d\theta^a d\theta^b + \kappa \varepsilon_{za}\, dr d\theta^a \\ &= dr^2 + f_k^2(r + r_0)\mathring{h} + \tilde{\varepsilon}_{ab}\, d\theta^a d\theta^b + \tilde{\varepsilon}_{ra}\, dr d\theta^a\end{aligned}$$

where, in the first line, we used that $dz = \kappa^{-1} dr$ and we define $\tilde{\varepsilon}_{ab} := \kappa^2 \varepsilon_{ab}$ and $\tilde{\varepsilon}_{ra} := \kappa \varepsilon_{za}$. For $\tilde{g}$ to satisfy Definition 3.1.1, it remains to show that the perturbation terms in the above satisfy the corresponding bounds in Definition 3.1.1, that is $\tilde{\varepsilon}_{ab} = \mathcal{O}_1\left(e^{-(\alpha-2)r}\right)$ and $\tilde{\varepsilon}_{ra} = \mathcal{O}_1\left(e^{-(\alpha-1)r}\right)$. We note the fact that

$$r(z) = \mathcal{O}\left(\log\left(\frac{1}{H}\right)\right) \text{ and } \kappa = \mathcal{O}\left(\frac{1}{fH}\right)$$

(they are exactly equal in the $S_{\mathring{h}} \equiv 0$ case) and so we compute directly that

$$\mathcal{O}(e^{-\alpha r}) = \mathcal{O}\left(H^\alpha\right).$$

Additionally, we have that $\partial_r = \kappa \partial_z$ and so, computing the required derivatives of the terms $\tilde{\varepsilon}_{ab}$ and $\tilde{\varepsilon}_{ra}$, we obtain the required bounds directly from the bounds in (3.48) on $\varepsilon_{ab}$, $\varepsilon_{za}$ and their derivatives. □

Having established the existence of a conformal metric $\tilde{g}$ of $g_f$ which is asymptotically locally hyperbolic, we require only an equivalent to the negative scalar curvature requirement

$$S_{\tilde{g}} \leq -n(n-1) + Ce^{-\alpha r}$$

in order to apply Theorem A and provide a solution to the Yamabe problem for $(M, g)$. We provide here a criterion for $\tilde{g}$ to satisfy the desired scalar curvature decay. In particular, we obtain

**Lemma 3.3.8.** *Under the hypotheses of Proposition 3.3.7, suppose further that there exists a constant $C > 0$ such that the warping function satisfies*

$$\left|\frac{1}{fH}\right| + \left|\frac{f'}{f}\right| \leq C \text{ and } \left|\frac{f''}{f}\right| \leq CH^{-\alpha}. \tag{3.50}$$

*Then, the scalar curvature of the conformal asymptotically locally hyperbolic metric $\tilde{g}$ obtained in Proposition 3.3.7 satisfies*

$$S_{\tilde{g}} \leq -n(n-1) + Ce^{-\alpha r} \text{ provided } S_{g_f} \leq S_{\mathring{g}_f} + C\frac{H^{\alpha-2}}{f^2} \text{ on } M^+, \tag{3.51}$$

*and likewise*

$$S_{\tilde{g}} \geq -n(n-1) + Ce^{-\alpha r} \text{ provided } S_{g_f} \geq S_{\mathring{g}_f} + C\frac{H^{\alpha-2}}{f^2} \text{ on } M^+. \tag{3.52}$$

*Proof.* Define $\tilde{g} := \kappa^2 g_f$ to be the asymptotically locally hyperbolic metric of order $\alpha$ obtained in Proposition 3.3.7 where again we define the coordinate change $r = K(z)$ and the conformal factor $\kappa = K'(z)$ where $K$ is the coordinate change obtained in the proof of Lemma 3.3.2 in the previous section. We note here that, as a consequence of this lemma, we have that $\mathring{g}_f = \kappa^{-2}\mathring{g}$.

Defining $v := \kappa^{\frac{n-2}{2}}$, we have the following formula

$$\begin{aligned} S_{\tilde{g}} &= v^{-\frac{n+2}{n-2}}\left(-c_n\Delta_{g_f}v + S_{g_f}v\right) \\ &= -n(n-1) + v^{-\frac{n+2}{n-2}}\left(-c_n\left(\Delta_{g_f}v - \Delta_{\mathring{g}_f}v\right) + \left(S_{g_f} - S_{\mathring{g}_f}\right)v\right). \end{aligned} \tag{3.53}$$

where the second equality comes from the fact that $v^{\frac{4}{n-2}}\mathring{g}_f$ has scalar curvature equal to $-n(n-1)$.

To conclude, we need only estimate the difference $\Delta_{g_f}v - \Delta_{\mathring{g}_f}v$. As $g_f = \kappa^{-2}\tilde{g}$ and $\mathring{g}_f = \kappa^{-2}\mathring{g}$, we have that

$$\Delta_{g_f}v = \Delta_{\kappa^{-2}\tilde{g}}v = \kappa^2(z(r))\left(\Delta_{\tilde{g}}v + (n-2)\tilde{g}^{rr}\partial_r\left(\log\left(\frac{1}{\kappa(z(r))}\right)\right)\partial_r v\right)$$

and

$$\Delta_{\mathring{g}_f}v = \Delta_{\kappa^{-2}\mathring{g}}v = \kappa^2(z(r))\left(\Delta_{\mathring{g}}v + (n-2)\mathring{g}^{rr}\partial_r\left(\log\left(\frac{1}{\kappa(z(r))}\right)\right)\partial_r v\right).$$

Consequently, we obtain

$$\begin{aligned}\Delta_{g_f}v - \Delta_{\mathring{g}_f}v &= \kappa^2(z(r))\left(\Delta_{\tilde{g}}v - \Delta_{\mathring{g}}v + (n-2)\left(\tilde{g}^{rr} - \mathring{g}^{rr}\right)\partial_r\left(\log\left(\frac{1}{\kappa(z(r))}\right)\right)\partial_r v\right)\\ &= \kappa^2(z(r))\Bigg(\left(\tilde{g}^{rr} - \mathring{g}^{rr}\right)\partial_{rr}v + \left(\tilde{g}^{ab} - \mathring{g}^{ab}\right)\mathring{\Gamma}^r_{ab}\partial_r v + \mathring{g}^{jk}\left(\Gamma^r_{jk} - \mathring{\Gamma}^r_{jk}\right)\partial_r v\\ &\qquad + (n-2)\left(\tilde{g}^{rr} - \mathring{g}^{rr}\right)\partial_r\left(\log\left(\frac{1}{\kappa(z(r))}\right)\right)\partial_r v\Bigg)\end{aligned}$$

where we use that $\tilde{g}$ is asymptotically locally hyperbolic to use (3.20) to compare the Laplacian of $\tilde{g}$ to that of the reference locally hyperbolic $\mathring{g}$ and we use $\Gamma^i_{jk}$ and $\mathring{\Gamma}^i_{jk}$ to refer to their corresponding Christofell symbols as in earlier sections. From here we use that $\partial_r = \kappa\partial_z$ and the decay of quantities for the corresponding asymptotically locally hyperbolic metric $\tilde{g}$ established in Lemma 3.1.2 to obtain

$$\Delta_{g_f}v - \Delta_{\mathring{g}_f}v = \mathcal{O}\left(H^{2\alpha}\right)\partial_{zz}v + \mathcal{O}\left(H^{\alpha}\right)\partial_z v.$$

The conditions (3.50) give us that $|\kappa| \leq C$, $|\kappa'| \leq C$ and $\kappa'' \leq CH^{-\alpha}$; consequently, $|\partial_z v| \leq C$ and $\partial_{zz}v \leq CH^{-\alpha}$ and so

$$\Delta_{g_f}v - \Delta_{\mathring{g}_f}v \leq CH^{-\alpha}.$$

Substituting the above bound for $\Delta_{g_f}v - \Delta_{\mathring{g}_f}v$ into (3.53) we obtain

$$S_{\tilde{g}} = -n(n-1) + \mathcal{O}(H^\alpha) + \left(S_{g_f} - S_{\mathring{g}_f}\right)\mathcal{O}\left(f^2H^2\right)$$

from which the conclusions of the lemma are apparent. $\square$

Having established conditions which allow us to compare a wide class of asymptotically warped product manifolds to conformal asymptotically locally hyperbolic manifolds, we are now able to apply our Theorem A in this more general setting to obtain:

**Theorem 3.3.9.** *Let $(M, g_f)$ be a manifold with an asymptotically warped product end with perturbation coefficients satisfying* (3.48) *for some $\alpha \in (0, n)$. Suppose additionally that the warping function satisfies* (3.49) *and* (3.50)*.*

*If the scalar curvature satisfies*

$$S_{g_f} \leq S_{\mathring{g}_f} + C\frac{H^{\alpha-2}}{f^2} \text{ on } M^+ \tag{3.54}$$

*for some constant $C > 0$, then there exists a positive smooth solution $u_f$ of the Yamabe equation for $g_f$ on $M$ satisfying*

$$\liminf_{r\to\infty}\left(u_f - \frac{1}{fH}\right) \geq 0$$

*and the corresponding conformal metric $\tilde{g}$ is complete and has constant scalar curvature $S_{\tilde{g}} \equiv -n(n-1)$ on $M$.*

*In addition, if the scalar curvature satisfies the stronger condition*

$$|S_{g_f} - S_{\mathring{g}_f}| \leq C\frac{H^{\alpha-2}}{f^2} \text{ on } M^+, \tag{3.55}$$

*then*

$$\left|u_f - \frac{1}{fH}\right| \longrightarrow 0 \text{ as } r \to \infty$$

*and $u_f$ is maximal in that any solution $\tilde{u}_f$ of the Yamabe equation for $g_f$ on $M$ satisfies $\tilde{u}_f \leq u_f$. Furthermore, the corresponding conformal metric $(M, \tilde{g})$ is asymptotically locally hyperbolic of order $\alpha$.*

*Proof.* As an immediate consequence of Proposition 3.3.7, $g_f$ is conformal to an asymptotically locally hyperbolic metric $g$ satisfying Definition 3.1.1. We will write the conformal factor from the proof of Proposition 3.3.7 again as $\kappa$ so that $\kappa^2 g_f = g$ and note again that $\kappa = \mathcal{O}\left(\frac{1}{fH}\right)$.

We may then apply Theorem A, using Lemma 3.3.8 to convert conditions (3.54) and (3.55) to their corresponding conditions for the scalar curvature of $g$, to obtain a solution $u$ to the Yamabe equation for $g$ with corresponding conformal metric $\tilde{g}$. We then have that $u_f^{\frac{4}{n-2}} g_f = \tilde{g}$, where $u_f := \kappa^{\frac{n-2}{2}} u$, and so $u_f$ must solve the Yamabe equation for $g_f$ and the remaining conclusions of Theorem A translate directly. $\square$

# 4

# Volume Ratios, Eigenvalue Estimates and the Yamabe Problem

In this chapter, we present a new perspective on the Yamabe problem via a study of volume ratio conditions of particular compact domains; we present a new result regarding eigenvalue estimates for the conformal Laplacian and explore its implications for existence of solutions to the Yamabe problem. From this new perspective, we begin to motivate further study through a series of examples and questions.

In Section 4.1, we provide an introduction of the concepts discussed in this chapter and an overview of our main results and their place in the literature. In Section 4.2, we prove our main results and a number of useful corollaries. Section 4.3 provides a number of motivating examples providing context for further study, as well as demonstrating the sharpness of the condition for our main eigenvalue estimate. Additionally, Section 4.3 also provides a comparison of the scope of the existence results in this chapter with that of the previous chapter on asymptotically locally hyperbolic manifolds.

# 4.1

# Introduction

Recall, again, that we would like to find, on a given non-compact Riemannian manifold $(M, g)$, a complete metric $\tilde{g} = u^{\frac{4}{n-2}} g$ conformal to $g$ with constant negative scalar curvature via a solution of the Yamabe equation

$$-c_n \Delta_g u + S_g u = -n(n-1) u^{\frac{n+2}{n-2}} \quad \text{on } M. \tag{Ya}$$

One may also desire an understanding of the asymptotic behaviour of such a conformal factor $u$; however, given the constraints of this DPhil project, in this chapter our priority will be to establish conditions for the existence of such a conformal factor for which the corresponding constant scalar curvature metric is complete, without regard for the particular asymptotic behaviour of the corresponding conformal factor.

As discussed in Chapter 1, our goal is to understand what additional requirements are needed on manifolds satisfying the natural negativity condition that $S_g \leq -\varepsilon < 0$ on some exterior region, in order to provide a solution to the Yamabe problem. In this chapter, our work is guided by an interest in the role of the first eigenvalue of the conformal Laplacian $-c_n \Delta_g + S_g$ on bounded domains, with respect to the homogeneous Dirichlet boundary conditions, in the existence of solutions to the Yamabe problem.

It has been shown by Aviles and McOwen in [AM88], that we can establish the existence of a solution to (Ya) if we can find some compact domain in $M$ on which the first eigenvalue of conformal Laplacian is negative. Our aim will be to find a geometric condition, in addition to the negativity of the scalar curvature, under which we can find a compact domain $\Omega \subset M$ which exhibits such a negative first eigenvalue.

We first establish definitions and terminology in Section 4.1.1, after which we briefly motivate our geometric condition of interest in Section 4.1.2. Finally, we discuss our main result and its implications in Section 4.1.3.

### 4.1.1 Definitions and Terminology

We recall that, for any given Riemannian manifold $(M, g)$, we refer to the elliptic operator $-c_n\Delta_g + S_g$ as the *conformal Laplacian.* Central to our discussion will be the first eigenvalue $\lambda$ of the conformal Laplacian with Dirichlet boundary data on some bounded domain $\Omega$. We recall the variational formulation for $\lambda$; in particular, we may define

$$\lambda := \inf_{\substack{\varphi \in H_0^1, \\ \|\varphi\|_{L^2}=1}} \int_\Omega c_n|\nabla_g\varphi|^2 + S_g\varphi^2 dV_g. \tag{4.1}$$

We note here, as this will be relevant in our main proof of this chapter, that a consequence of the above variational formulation is that we may show that $\lambda < 0$ for some domain $\Omega$ if we can exhibit any test function $\varphi \in H_0^1(\Omega)$ such that

$$\int_\Omega c_n|\nabla_g\varphi|^2 + S_g\varphi^2 dV_g < 0. \tag{4.2}$$

In the following, a central quantity will be the ratio of the volumes of two sets $\Omega_1$ and $\Omega_2$ which are separated by a fixed distance $R > 0$, that is

$$d_g(x, \partial\Omega_2) = R \text{ for each } x \in \partial\Omega_1.$$

More specifically, we will be interested in the size of the ratio between the inner region $\Omega_1$ and the outer region between $\Omega_1$ and $\Omega_2$, that is the quantity

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)}. \tag{4.3}$$

We will refer to conditions on the above quantity simply as *volume ratio conditions.*

### 4.1.2 Motivation

We consider here, why we might be interested in these volume ratio conditions discussed above as a natural geometric condition that one can impose to establish the existence of a negative first eigenvalue for the conformal Laplacian. A natural connection between the first eigenvalue and the geometry of the underlying manifold occurs in the equivalence between the isoperimetric inequality and the Sobolev inequality (see [Tal76], [Oss78], [Oss79], [BNT10]). In particular, we see that the isoperimetric inequality can be recovered from the Sobolev inequality via a series of approximations to the indicator function on a given domain and the converse can be seen via the coarea formula. In turn, the Sobolev inequality provides a direct link to the first eigenvalue of the Laplacian. For example, an application of the Poincaré inequality leads us to a first straight-forward but important lemma prohibiting the existence of a negative first eigenvalue for the Laplacian at small scales.

**Lemma 4.1.1.** *For any Riemannian manifold* $(M, g)$ *there exists, for each* $x \in M$*, an* $R > 0$ *such that for any* $r < R$ *the least eigenvalue of the conformal Laplacian* $-c_n \Delta_g + S_g$ *on* $B_r(x)$ *is strictly positive.*

In particular, we remark that the above lemma leads us naturally to consider only "large" domains within the manifold.

*Proof.* Bearing in mind formulation (4.1) for the eigenvalue problem we note that, by the Poincaré inequality for $H_0^1$ functions on a ball,

$$\int_{B_r} \left[c_n |\nabla_g \varphi|^2 + S_g \varphi^2\right] dV_g \geq \int_{B_r} \left(\frac{C}{r} + S_g\right) \varphi^2 dV_g > 0$$

with the last inequality certainly being true for $r < R$ where $R > 0$ is some constant depending on $n$ and $S_g$. □

Our general philosophy will be to consider similar volume ratio type conditions in the spirit of the isoperimetric inequality in establishing our desired negative first eigenvalue. In particular, our goals of this chapter will be focused around a few central questions:

- Can we establish the existence of a negative first eigenvalue for the conformal Laplacian on a bounded domain (e.g. on a ball or annulus) with negative scalar curvature in an arbitrary manifold $(M, g)$ by imposing a condition only of the volume ratio type as in (4.3)?

- If such a condition exists, can we find a sharp bound on the size of the volume ratio past which a negative first eigenvalue in general does not exist on the domain of interest?

- What is the optimal volume ratio bound required to deduce existence of solutions to the Yamabe problem?

We are able to address the first two questions in this chapter, in particular we will be able to find a sharp condition on the volume ratio for existence of a negative first eigenvalue for the conformal Laplacian. The third question is unresolved in this DPhil project, see Section 5.2.2 for a discussion of this question.

### 4.1.3 Overview and Discussion of Main Results

We now provide an overview of the new results obtained in this section, discuss the implications of these results for the Yamabe problem and compare the scope of our new results with existing literature and our results obtained in Chapter 3.

Our main result provides a condition on the volume ratio between two sets $\Omega_1$ and $\Omega_2$ as in (4.3) under which we may deduce the existence of a negative first eigenvalue of the conformal Laplacian on $\Omega_2$. However, in line with the overarching goals of this DPhil project, we first state the corresponding existence theorem for the Yamabe problem which will follow as a consequence of our main result. In particular, we prove:

**Theorem.** *Let $(M, g)$ be a Riemannian manifold and suppose there exist two open sets $\Omega_1 \subset \Omega_2$ with $C^1$ boundary which satisfy, for some $R > 0$, that*

$$d_g(x, \partial\Omega_2) = R \text{ for each } x \in \partial\Omega_1,$$

*that*

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)} \leq \sinh^2\left(\frac{\sqrt{n(n-2)}}{2}R\right)$$

*and that the scalar curvature satisfies* $S_g \leq -n(n-1)$ *on* $\Omega_2$*. Suppose, furthermore, that* $S_g \leq -\varepsilon < 0$ *everywhere outside of some compact set for some constant* $\varepsilon > 0$*. Then there exists a complete metric* $\tilde{g}$ *conformal to* $g$ *on* $M$ *with constant scalar curvature* $-n(n-1)$*.*

We highlight a characteristic of the existence theorem above. In particular, the existence result does not make *any* restrictions on the asymptotic behaviour of the manifold aside from on the scalar curvature. We contrast this to our existence result in Chapter 3, and to other existence results in the literature reviewed in Section 3.1.2, where significant additional asymptotic restrictions are required. In Section 4.3.3, we will provide examples of manifolds for which the existence theorem above holds but which fall outside the class of manifolds addressed in Chapter 3. It should also be noted, in line with our goal of the thesis as discussed previously, we do not make a global non-positivity condition on the scalar curvature.

On the other hand, there are many natural examples which do not exhibit domains satisfying the volume ratio condition required to apply our existence theorem above, most strikingly the hyperbolic space which is the natural model in the asymptotically hyperbolic setting in which most progress has been achieved on the non-compact Yamabe problem. As an example demonstrating the failure to meet the volume ratio condition in this case, we consider concentric geodesic balls in hyperbolic space which we may estimate directly that the volume ratio (for large $R$) behaves like

$$\frac{Vol_{\mathbb{H}^n}(B_{2R})}{Vol_{\mathbb{H}^n}(B_R)} \approx e^{(n-1)R}.$$

This is in contrast to the exponential rate of $\sqrt{n(n-2)}$ required in our existence result, which is less than $n-1$. To demonstrate that the theorem is not vacuous, we provide examples of multiply warped product manifolds in Section 4.3 to which our theorem applies, as well as in Section 4.3.3 where we provide examples which fall outside the scope of Chapter 3 as already mentioned. We discuss speculation on the nature of this gap between the results of Chapter 3 and this chapter in Section 5.2.2.

Our existence theorem discussed above is a direct consequence our second main theorem which provides an eigenvalue estimate based on a volume ratio condition in combination with the work of Aviles and McOwen in [AM88]. In particular our main theorem of this chapter is the following:

**Theorem B.** *Let $(M, g)$ be a Riemannian manifold and suppose there exist two open sets $\Omega_1 \subset \Omega_2$ with $C^1$ boundary which satisfy, for some $R > 0$,*

$$d_g(x, \partial\Omega_2) = R \text{ for each } x \in \partial\Omega_1,$$

*and*

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)} \leq \sinh^2\left(\frac{\sqrt{n(n-2)}}{2}R\right)$$

*and that the scalar curvature satisfies $S_g \leq -n(n-1)$ on $\Omega_2$. Then, the conformal Laplacian $-c_n\Delta_g + S_g$ for $(M, g)$ has a negative first eigenvalue on $\Omega_2$.*

We highlight that, in the important example of the hyperbolic space discussed above, there exists no bounded domain which has negative first eigenvalue for the conformal Laplacian with Dirichlet boundary conditions.

**Remark 4.1.2.** *In light of Lemma 4.1.1, we know that our theorem is necessarily vacuous as $R \to 0$ and this is reflected in the degeneration of the bound on the LHS of* (4.6)*.*

As will be discussed in Section 4.2.3, the existence of such a negative first eigenvalue for the conformal Laplacian as established in our theorem above can be used, in conjunction with a negativity condition on the scalar curvature on an exterior region, to establish the existence of a solution to the Yamabe problem via the work of Aviles and McOwen in [AM88].

# 4.2

# Proof of the Main Results

In this section we prove the main results which motivate this chapter which provide an estimate for the first eigenvalue of the conformal Laplacian and corresponding existence result for the Yamabe problem. In particular, we provide a condition on the volume ratios discussed in Section 4.1.1 such that a negative first eigenvalue exists on the corresponding compact domain. Later examples in this chapter will demonstrate that the condition on the volume ratio obtained is in fact sharp, at least up to the exponential rate, for the existence of a negative first eigenvalue. We then discuss the implications of this result for existence of solutions to the Yamabe problem.

Before we can prove our eigenvalue estimate, we first review some tools from the literature which will play an important role in the proof.

## 4.2.1 Sup-Norm Minimization Problems

In this section, we review tools from the literature on sup-norm minimisation problems. The section summarises certain excerpts from the paper [Aro65] which will be needed in our work.

By a sup-norm minimisation problem we mean the following. Fix some interval $[0, R] \subset \mathbb{R}$ and, for some function $\varphi : [0, R] \to \mathbb{R}$ which is absolutely continuous and satisfies the boundary conditions $\varphi(0) = 0$ and $\varphi(R) = 1$, define the functional

$$H(\varphi) := \sup_{0 \leq r \leq R} F\left(\varphi(r), \varphi'(r)\right).$$

We will refer to the set of all $\varphi$ satisfying the requirements above by $\mathcal{A}$. We are interested in finding

$$\inf_{\varphi \in \mathcal{A}} H(\varphi).$$

Problems such as the above were first addressed in a series of papers by Gunnar Aronnson, although for our needs the first of them ([Aro65]) shall suffice. To state the result which we will use in our work, we must first define some additional quantities and make some restrictions on $F$.

In particular, we suppose that $F(y,z)$ satisfies the following conditions:

1. $F$ is well defined and continuous for all $y, z \in \mathbb{R}$.
2. $\dfrac{\partial F}{\partial z}$ exists for all $y, z \in \mathbb{R}$ and $\dfrac{\partial F}{\partial z} \begin{cases} > 0 \text{ if } z > 0, \\ = 0 \text{ if } z = 0, \\ < 0 \text{ if } z < 0. \end{cases}$
3. $\lim\limits_{|z|\to\infty} F(y,z) = \infty$ for all $y \in \mathbb{R}$.

Given the above, we then define the following quantities (defined for $M > F(y,0)$)

$$\Phi_M(y) := \inf\{z : z > 0, F(y,z) = M\} \tag{4.4}$$

and

$$\mathcal{L}(M) := \int_0^1 \frac{1}{\Phi_M(t)}\, dt. \tag{4.5}$$

We may now state the result of Aronsson which will be required in our proof. For the sake of brevity, we state only the existence part of Theorem 4 of [Aro65] as the uniqueness of the minimiser will not be required in our work.

**Theorem** ([Aro65], Theorem 4)**.** *Suppose that $F$ satisfies conditions 1–3 above. Then the solution of the minimisation problem $\inf_{\varphi\in\mathcal{A}} H(\varphi)$ is summarised in the following:*

*a) Given $M$, a necessary and sufficient condition for the existence of a function $\varphi \in \mathcal{A}$ such that $H(\varphi) = M$ is that $\mathcal{L}(M) \leq R$.*

*b) There is a number $M_0$ such that $\mathcal{L}(M) \leq R$ if and only if $M \geq M_0$.*

*c) $M_0 = \inf_{\varphi\in\mathcal{A}} H(\varphi)$.*

*d) There exists a $C^1$ function $\tilde{\varphi} \in \mathcal{A}$ which achieves the infimum.*

### 4.2.2 An Eigenvalue Estimate via a Volume Ratio Condition

For the convenience of the reader, we restate the main result of this section on the existence of a negative first eigenvalue for the conformal Laplacian:

**Theorem B.** *Let $(M, g)$ be a Riemannian manifold and suppose there exist two open sets $\Omega_1 \subset \Omega_2$ with $C^1$ boundary which are separated by a fixed distance $R > 0$, that is*

$$d_g(x, \partial\Omega_2) = R \text{ for each } x \in \partial\Omega_1,$$

*which satisfy*

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)} \leq \sinh^2\left(\frac{\sqrt{n(n-2)}}{2}R\right) \tag{4.6}$$

*and that the scalar curvature satisfies $S_g \leq -n(n-1)$ on $\Omega_2$. Then, the conformal Laplacian $-c_n\Delta_g + S_g$ for $(M, g)$ has a negative first eigenvalue on $\Omega_2$.*

To aid in our discussion regarding warped product and multiply warped product type manifolds, we also provide the following corollaries of Theorem B for geodesic balls and for annuli. In particular, for geodesic balls we have

**Corollary 4.2.1.** *Let $(M, g)$ be a Riemannian manifold and suppose there exist constants $\alpha, R > 0$ and some geodesic ball $B_{(1+\alpha)R}$ which satisfies*

$$\frac{Vol_g(B_{(1+\alpha)R} \setminus B_R)}{Vol_g(B_R)} \leq \sinh^2\left(\frac{\alpha R\sqrt{n(n-2)}}{2}\right) \tag{4.7}$$

*and on which the scalar curvature satisfies $S_g \leq -n(n-1)$. Then, the conformal Laplacian $-c_n\Delta_g + S_g$ for $(M, g)$ has a negative first eigenvalue on $B_{(1+\alpha)R}$.*

In the case of a multiply warped product metrics (defined later, see (4.11)) which have a radial fibre whose coordinate we denote by $r$, we fix some value $r_0 \in \mathbb{R}$ and define the annular region

$$A_R(r_0) = \{x \in M : |r(x) - r_0| \leq R\}.$$

We may then obtain, again as a corollary of Theorem B,

**Corollary 4.2.2.** *Let $(M, g)$ be a Riemannian manifold and suppose there exist constants $\alpha, R > 0$ and some annulus $A_{(1+\alpha)R}(r_0)$ which satisfies*

$$\frac{Vol_g(A_{(1+\alpha)R}(r_0) \setminus A_R(r_0)}{Vol_g(A_R(r_0))} \leq \sinh^2\left(\frac{\alpha R\sqrt{n(n-2)}}{2}\right) \tag{4.8}$$

*and on which the scalar curvature satisfies $S_g \leq -n(n-1)$. Then, the conformal Laplacian $-c_n\Delta_g + S_g$ for $(M, g)$ has a negative first eigenvalue on $A_{(1+\alpha)R}(r_0)$.*

The proofs of the two corollaries follow directly from Theorem B with the appropriate choices made for $\Omega_1$, $\Omega_2$ and $R$. We now prove Theorem B.

*Proof of Theorem B.* We would like to find a suitable test function $\varphi \in H_0^1(\Omega_2)$ which realises the inequality

$$\int_{\Omega_2} c_n|\nabla_g\varphi|^2 + S_g\varphi^2 dV_g < 0. \tag{4.9}$$

We define the distance function $r : \overline{\Omega_2 \setminus \Omega_1} \to [0, R]$ by

$$r(x) := d_g(x, \partial\Omega_2).$$

We will construct a test function $\varphi(r)$ for (4.9). We presuppose that $\varphi$ has the form

$$\varphi(x) = \begin{cases} 1 & x \in \Omega_1 \\ \tilde{\varphi}(r(x)) & x \in \overline{\Omega_2 \setminus \Omega_1} \end{cases}$$

where $\tilde{\varphi}$ is a $C^1$ function satisfying $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}(R) = 1$ to be determined. We note that, as $r$ is Lipschitz, these conditions on $\tilde{\varphi}$ ensure that $\varphi \in H_0^1(\Omega_2)$ and so is a valid test function.

Assuming this form, we can bound the LHS of (4.9) by

$$\begin{aligned}\int_{\Omega_2} c_n|\nabla_g\varphi|^2 + S_g\varphi^2 dV_g &\leq \int_{\Omega_2} c_n(\varphi')^2 - n(n-1)\varphi^2 dV_g \\ &\leq -n(n-1)Vol_g(\Omega_1) + \int_{\Omega_2\setminus\Omega_1} c_n(\tilde{\varphi}')^2 - n(n-1)\tilde{\varphi}^2 dV_g \\ &\leq -n(n-1)Vol_g(\Omega_1) \\ &\quad + Vol_g(\Omega_2 \setminus \Omega_1) \sup_{0\leq r\leq R} \left(c_n(\tilde{\varphi}'(r))^2 - n(n-1)\tilde{\varphi}^2(r)\right).\end{aligned}$$

Consequently, we will have established (4.9), and so the existence of a negative first eigenvalue for the conformal Laplacian, provided that we have

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)} \leq n(n-1)\left(\sup_{0\leq r\leq R} F(\tilde{\varphi}, \tilde{\varphi}')\right)^{-1}$$

where we define $F(y,z) := c_n z^2 - n(n-1)y^2$.

We have thus reposed our problem as a minimization problem as addressed in the work of Aronsson in [Aro65] discussed in Section 4.2.1. In particular, we would like to find a $C^1$ interpolating function $\tilde{\varphi}$ which satisfies $\tilde{\varphi}(0) = 0$, $\tilde{\varphi}(R) = 1$ and minimizes the functional

$$\sup_{0\leq r\leq R} F(\tilde{\varphi}, \tilde{\varphi}').$$

We will appeal to Theorem 4 of [Aro65] (see Section 4.2.1) and first note that our $F$ clearly satisfies conditions 1, 2 and 3 from Section 4.2.1 required to apply the result. In applying [Aro65, Theorem 4], we must compute the quantities (4.4) and (4.5) for our particular $F$. We obtain

$$\Phi_M(y) := \inf\{z : z > 0, F(y,z) = M\} = \sqrt{\frac{M + n(n-1)y^2}{c_n}}$$

and

$$\mathcal{L}(M) := \int_0^1 \frac{dt}{\Phi_M(t)}\, dt = \frac{2}{\sqrt{n(n-2)}} \operatorname{arctanh}\left(\sqrt{\frac{n(n-1)}{M + n(n-1)}}\right).$$

Theorem 4 of [Aro65] then states that there exists a number $M_0$ such that $\mathcal{L}(M) \leq R$ if and only if $M \geq M_0$ and furthermore, that this $M_0$ satisfies

$$M_0 = \inf_{\substack{\tilde{\varphi}(0)=0,\\ \tilde{\varphi}(R)=1}} \left(\sup_{0\leq r\leq R} F(\tilde{\varphi}, \tilde{\varphi}')\right)$$

and that there exists a $\tilde{\varphi} \in C^1$ with $\tilde{\varphi}(0) = 0$, $\tilde{\varphi}(R) = 1$ achieving the infimum above.

We use the first of the two statements above to compute $M_0$. In particular, we compute that $\mathcal{L}(M) \leq R$ if and only if

$$n(n-1)\sinh^{-2}\left(\frac{R\sqrt{n(n-2)}}{2}\right) \leq M$$

and so we have our value for $M_0$ and the bound we are trying to show follows immediately from the existence of a minimiser $\tilde{\varphi}$. □

**Remark 4.2.3.** *Comparing the model barrier case of the hyperbolic space to our result we see that, as the dimension grows, the two exponential rates converge (that is,* $\sqrt{n(n-2)} \to (n-1)$*). As no negative first eigenvalue can exist on a compact domain in the hyperbolic space, this perhaps gives reason to think that, if we presuppose only a bound on the volume ratio, there is already not much room for improvement on the exponent in* (4.6)*.*

### 4.2.3 Consequences for the Yamabe Problem

We now discuss the implications of our eigenvalue result of Theorem B for the Yamabe problem. In particular, we will prove the following existence result:

**Theorem C.** *Suppose* $(M, g)$ *is a Riemannian manifold satisfying the assumptions of Theorem B and, furthermore, that* $S_g \leq -\varepsilon < 0$ *everywhere outside of some compact set for some* $\varepsilon > 0$*. Then there exists a complete metric* $\tilde{g}$ *conformal to* $g$ *on* $M$ *with constant scalar curvature* $-n(n-1)$*.*

We briefly provide an overview of the process of obtaining a solution to the Yamabe problem from the existence of a negative first eigenvalue for the conformal Laplacian on some compact domain. In particular, the proof of Theorem 4.2.3 above is a direct consequence of our Theorem B and the following theorem of Aviles and McOwen:

**Theorem** ([AM88])**.** *If* $(M, g)$ *is a complete Riemannian manifold with some compact domain* $\Omega$ *and function* $\varphi \in H_0^1(\Omega)$ *satisfying*

$$\int_\Omega c_n |\nabla_g \varphi|^2 + S_g \varphi^2 dV_g < 0 \tag{4.10}$$

*on some compact set* $\Omega$*, then there exists a conformal metric* $\tilde{g}$ *with constant negative scalar curvature.*

*Furthermore, if the scalar curvature satisfies, outside of some compact set,* $S_g \leq -\varepsilon$ *for some* $\varepsilon < 0$ *then the conformal metric* $\tilde{g}$ *is also complete.*

We highlight here that, in the proof of the theorem of Aviles and McOwen above, the requirements split into the criteria (4.10) for the existence of a positive solution to (Ya) and then the additional curvature requirements which allow us to conclude that the resulting conformal metric is complete. We also note that the role of condition (4.10) is to allow the construction of a sub-solution with which to initiate the sub- and super-solution argument of Aviles and McOwen discussed in Chapter 2.

**Remark 4.2.4.** *We point out a consequence of the local nature of the volume ratio condition on the exterior region in our existence Theorem C. In particular, if we consider non-compact manifolds with multiple ends, then, as long as all of the ends have asymptotically negative scalar curvature, only one of the ends need behave well enough to exhibit domains satisfying the volume ratio condition in order to solve the Yamabe problem on the entire manifold.*

# 4.3 Multiply Warped Product Spaces and Sharpness of the Volume Ratio Condition

In this section, we provide examples with two main goals in mind. Firstly, we would like to show that there is a large family of model spaces which can be treated by our existence result of Theorem C. These model spaces play a similar role to the model locally hyperbolic spaces in that they provide a reference metric for the asymptotic behaviour of manifolds on which we can aim to solve the Yamabe problem; we prove such a result in Corollary 4.3.3. Secondly, having found a sufficient condition on the volume ratio for the negativity of the first eigenvalue in Theorem B, we aim to determine whether the condition is sharp given the assumption that $S_g \leq -n(n-1)$. A useful class of explicit examples will be of the following multiply warped product form which are a generalisation of the warped product manifolds studied in Section 3.3 and provide a richer class necessary to achieve the two goals of this section.

We consider the multiply warped product manifolds $M = \mathbb{R} \times N_1 \times \ldots \times N_m$ where each $N_i$ is a compact manifold of dimension $n_i$ with $\sum_i n_i = n - 1$. We endow $M$ with a multiply warped metric $g$ written

$$g = dr^2 + \sum_i p_i^2(r) h_i \tag{4.11}$$

where $p_i : \mathbb{R} \to (0, \infty)$ and $h_i$ are warping functions and metrics, respectively, on each of the $m$ manifolds $N_i$. A computation shows that $(M, g)$ has scalar curvature

$$S_g = -2 \sum_i n_i \frac{p_i''}{p_i} - \sum_i n_i(n_i - 1) \left(\frac{p_i'}{p_i}\right)^2 - 2 \sum_{i<j} n_i n_j \frac{p_i' p_j'}{p_i p_j} + \sum_i \frac{S_{h_i}}{p_i^2}\ , \tag{4.12}$$

volume form

$$dV_g = \prod_i p_i^{n_i}\ dr\ dV_{h_1} \ldots dV_{h_m}, \tag{4.13}$$

and Laplacian

$$\Delta_g = \partial_{rr} + \sum_i n_i \frac{p_i'}{p_i} \partial_r + \sum_i \frac{1}{p_i^2} \Delta_{h_i}. \tag{4.14}$$

In this section, our goal will be to study a number of examples of the above type in order to provide model examples of manifolds exhibiting different volume ratios. Furthermore, we will use some of this set of examples to demonstrate that, at least up to the exponential rate $\sqrt{n(n-2)}$, the volume ratio condition in Theorem B is sharp. That is, we will show that for any exponential rate $\beta > \sqrt{n(n-2)}$, there exist examples of manifolds which exhibit equally distanced sets $\Omega_1$ and $\Omega_2$ satisfying

$$\frac{Vol_g(\Omega_2)}{Vol_g(\Omega_1)} \leq Ce^{\beta R}$$

for some $C$ independent of $R$ as well as the negative scalar curvature condition, but which do not have a negative first eigenvalue for the conformal Laplacian on $\Omega_2$. From this, we observe that Theorem B demonstrates that the exponential rate $\sqrt{n(n-2)}$ is somehow critical for the existence of a negative first eigenvalue for the conformal Laplacian.

We first give some specific classes of multiply warped product and compute their volume ratios in Section 4.3.1. We then provide an analysis of the first eigenvalue of the conformal Laplacian in these examples and demonstrate the sharpness of Theorem B in Section 4.3.2. Lastly, in Section 4.3.3, we compare the scope of our main existence result of this section with our existence results of Chapter 3.

### 4.3.1 Volume Ratios in Manifolds of Negative Scalar Curvature

We now provide a number of examples of the multiply warped product form in (4.11) which have negative scalar curvature and estimate the volume ratio of large balls in these examples.

Our first example directly generalises the locally hyperbolic metric with a flat torus cross-section (see (3.1) from Chapter 3) by choosing each warping function $p_i$ to be exponentials of independent rates. In the terminology of the multiply warped products (4.11) set out at the start of this chapter, we take each $N_i = \mathbb{S}^1$ (and so $M = \mathbb{R} \times N_1 \times \ldots \times N_{n-1} = \mathbb{R} \times \mathbb{T}^{n-1}$).

**Proposition 4.3.1.** *Let* $(M, g)$ *be a multiply warped product as in* (4.11) *with each* $N_i = \mathbb{S}^1$ *and* $p_i = e^{\alpha_i r}$ *for some* $\alpha_i \in \mathbb{R}$*. Defining* $\beta := \sum_i \alpha_i$*, we have that if* $|\beta| < \sqrt{n(n-2)}$ *and the scalar curvature, which may be written*

$$S_g = -2\beta^2 + 2\sum_{i<j} \alpha_i \alpha_j,$$

*satisfies* $S_g \leq -n(n-1)$ *then there exists a solution to the Yamabe problem on* $(M, g)$*.*

We will use Corollary 4.2.2 of our existence theorem to prove the above, which will require us to provide an estimate on the volume ratio of large balls in $(M, g)$. Before doing this, we first demonstrate that the above conditions on $\beta$ and the scalar curvature are not vacuous, i.e. that there exists a multiply warped product satisfying the requirements of the proposition.

To see this, we observe that under the constraint $\sum_i \alpha_i = \beta$, the quantity $-2\beta^2 + 2\sum_{i<j} \alpha_i \alpha_j$ is unbounded below for $\alpha_i \in \mathbb{R}$ and so choices of $\alpha_i$ such that $S_g \leq -n(n-1)$ certainly exist. Of additional interest is the fact that the only critical point

(which, therefore, must be a maximum) of $-2\beta^2 + 2\sum_{i<j} \alpha_i\alpha_j$ under the constraint is at $\alpha_i = \frac{\beta}{n-1}$ where $S_g = -\frac{n}{n-1}\beta^2$. Consequently, there exist choices of $\alpha_i$ such that $S_g \equiv -C \leq -n(n-1)$ exactly provided that the maximum $-\frac{n}{n-1}\beta^2 \geq -n(n-1)$, i.e. if $|\beta| \leq n-1$ which is certainly true under the requirements of Proposition 4.3.1.

We now estimate the volume ratio on large balls for manifolds satisfying the requirements of Proposition 4.3.1. Fixing some large $R$ and taking some ball $B_R(p_0)$, writing $p_0 = (r_0, x_0)$, we have immediately that

$$B_R(p_0) \subset \{p : |r(p) - r_0| \leq R\}.$$

On the other hand, defining $S_0 := \{r_0\} \times \mathbb{T}^{n-1}$ we have that

$$d_g(p, p_0) \leq d_g(p, S_0) + \mathrm{diam}(S_0) = r(p) - r_0 + e^{r_0}\mathrm{diam}(\mathbb{T}^{n-1}).$$

Consequently, writing $A_0 = e^{r_0}\mathrm{diam}(\mathbb{T}^{n-1})$, we have that

$$\{|r - r_0| \leq R - A_0\} \subset B_R(p_0) \subset \{|r - r_0| \leq R\}$$

from which we obtain

$$Vol_g(B_{2R}(p_0)) \leq \int_{\mathbb{T}^{n-1}} \int_{r_0-2R}^{r_0+2R} e^{\beta r} dr\ dx = \frac{1}{\beta} Vol(\mathbb{T}^{n-1}) \left(e^{\beta(r_0+2R)} - e^{\beta(r_0-2R)}\right)$$

and

$$Vol_g(B_R(p_0)) \geq \int_{\mathbb{T}^{n-1}} \int_{r_0-(R-A_0)}^{r_0+(R-A_0)} e^{\beta r} dr\ dx = \frac{1}{\beta} Vol(\mathbb{T}^{n-1}) e^{-\beta A_0} \left(e^{\beta(r_0+R)} - e^{\beta(r_0-R+2A_0)}\right)$$

from which we may estimate

$$\frac{Vol_g(B_{2R} \setminus B_R)}{Vol_g(B_R)} \leq e^{\beta A_0} \frac{e^{\beta(r_0+2R)} - e^{\beta(r_0-2R)}}{e^{\beta(r_0+R)} - e^{\beta(r_0-R+2A_0)}} - 1.$$

Consequently, manifolds satisfying the requirements of Proposition 4.3.1 satisfy the following bound on the volume ratio for large balls

$$\frac{Vol_g(B_{2R} \setminus B_R)}{Vol_g(B_R)} \leq Ce^{\beta R}\left(1 + \mathcal{O}(e^{-2(n-1)R})\right) - 1 \tag{4.15}$$

and so, taking $R$ large, we obtain

$$\frac{Vol_g(B_{2R} \setminus B_R)}{Vol_g(B_R)} \leq Ce^{\beta R}.$$

for some large constant $C > 0$.

We note that, in the proposition discussed above, one of its two ends must have finite volume (apart form in the case that $\beta = 0$). We now provide a version of Proposition 4.3.1 where the warping functions are replaced by $p_i = \cosh(\alpha_i r)$ which correspond to manifolds with infinite volume on both ends. We note that this choice of $p_i$ means that, without loss of generality, we can set $\alpha_i \geq 0$.

**Proposition 4.3.2.** *Let $(M, g)$ be a multiply warped product as in (4.11) with each $N_i = \mathbb{S}^1$ and $p_i = \cosh(\alpha_i r)$ for some $\alpha_i \geq 0$. We then have that, again defining $\beta := \sum_i \alpha_i$, if $\beta < \sqrt{n(n-2)}$ and $S_g \leq -n(n-1)$ for $r$ sufficiently large, then there exists a solution to the Yamabe problem on $(M, g)$.*

To see that there exist multiply warped product manifolds satisfying the conditions above, we need the additional lower bound requirement that $\beta > \sqrt{\frac{n(n-1)}{2}}$. This arises as a consequence of the fact that each $\alpha_i \geq 0$ and so, as the scalar curvature in the above may be written

$$S_g = -2\beta^2 + 2\sum_{i<j} (2 - \tanh(\alpha_i r)\tanh(\alpha_j r))\,\alpha_i\alpha_j,$$

we have that $S_g \geq -2\beta^2$. By taking large balls far along one of the ends of the example above, the volume form will be comparable to that of Proposition 4.3.1 and so the volume ratio computations carry over into this case. We therefore note that Proposition 4.3.2 contains examples of multiply warped product manifolds with volume ratios satisfying (4.8) provided $n > 3$ (in the case that $n = 3$ we are in a critical situation in that $\sqrt{\frac{n(n-1)}{2}} = \sqrt{n(n-2)}$).

Lastly, we demonstrate an example application of our theorem to provide a result for existence of solutions to the Yamabe problem for the manifolds which are asymptotic to the multiply warped products in the two propositions above.

**Proposition 4.3.3.** *Consider $(M, g)$ to have an asymptotically warped product end, that is*

$$g = \mathring{g} + \varepsilon \text{ where } \varepsilon = \varepsilon_{rr}dr^2 + \varepsilon_{ra}drd\theta^a + \varepsilon_{ab}d\theta^a d\theta^b$$

*and $\mathring{g}$ is as in either Proposition 4.3.1 or Proposition 4.3.2. If $S_g \leq -n(n-1)$ for large $r$ and $\det(g) \to \det(\mathring{g})$ as $r \to \infty$, then there exists a solution of the Yamabe problem on $(M, g)$.*

*Proof.* As the volume form asymptotes to that of the volume form in Proposition 4.3.1 or Proposition 4.3.2, we may choose a large annuli far along the asymptotic end and use the same argument to apply Corollary 4.2.2 to conclude that the desired solution exists. □

As we can see, as our approach only requires a condition on the volume ratios, we are able to allow a large family of perturbations to the metric, only requiring that the scalar curvature and volume form decay.

### 4.3.2 Eigenvalues for the Conformal Laplacian and Sharpness of Theorem B

In this section, we explore for which of the examples of the previous section the conformal Laplacian exhibits a negative first eigenvalue. In doing this, we will be able to test the strength of Theorem B. Notably, we will be able to demonstrate examples which show that the exponential rate in Theorem B is sharp.

We first discuss the family in Example 4.3.1 which fits more naturally into the annuli setting. We note that, as

$$\frac{Vol_g(A_{2R} \setminus A_R)}{Vol_g(A_R)} \leq Ce^{\beta R}$$

for large $R$, we certainly satisfy the requirements of Corollary 4.2.2 provided $\beta < \sqrt{n(n-2)}$ and $R$ is sufficiently large and so we have that a negative first eigenvalue for the conformal Laplacian must exist on $A_{2R}$. We now use the explicit form for the

Laplacian for a multiply warped product (4.14) to directly analyse the sign of the first eigenvalue for different values of $\beta$.

We first show that, without loss of generality, we may assume that our candidate eigenfunction $\varphi$ is radial on $A_{2R}$. In particular, suppose that $\varphi$ is an eigenfunction for the first eigenvalue $\lambda$ of the conformal Laplacian, that is

$$-c_n\Delta_g\varphi + S_g\varphi = \lambda\varphi.$$

As the eigenspace corresponding to the first eigenvalue $\lambda$ is necessarily 1 dimensional, we must have that $\varphi$ is radial by virtue of the symmetry of the Torus. Consequently, using formula (4.14) for the Laplacian, the PDE for $\varphi$ reduces to the constant coefficient ODE

$$\varphi'' + \beta\varphi' + \frac{\lambda + n(n-1)}{c_n}\varphi = 0.$$

In order for the ODE for $\varphi$ to have a positive solution which is 0 on the boundary of some interval, we must have a complex root for the corresponding characteristic equation. In particular, we require that

$$\beta^2 - \frac{(n-2)(\lambda + n(n-1))}{(n-1)} < 0$$

and so, if we impose that $\lambda < 0$, we must have $\beta < \sqrt{n(n-2)}$. Consequently, we conclude that, at least up to the exponential rate, the bound (4.6) required in Theorem B is in fact sharp in general.

We note that, again taking our annuli far along one of the ends, we can also provide a similar argument as for Proposition 4.3.1, up to some perturbation terms, for Proposition 4.3.2.

**Remark 4.3.4.** *We remark that the above sharpness of the estimate was quite unexpected to the author. In particular, the method of sup-norm minimisation employed in the proof of Theorem B, what one normally would consider as an $L^2$-type inequality, appeared as if it would be insufficient to obtain a good bound for the volume ratio condition* (4.6)*. The author's intent was to try and improve* (4.6) *but, in attempting to do so, the above sharpness was observed.*

### 4.3.3 Comparison with the Results of Chapter 3

We now compare our results to those obtained in Chapter 3 and elsewhere in the literature for asymptotically locally hyperbolic manifolds; as discussed previously, this setting is the area in which most progress on the non-compact Yamabe problem for negative curvature type manifolds has been made in the last 20–30 years and so any way that our new existence result of this chapter can address examples which fall outside of this class would be of interest.

We first summarise the comparison between these two chapters. Firstly, as previously observed, the volume ratios of sets in the reference locally hyperbolic space do not satisfy condition (4.6) and so our main existence result of this chapter does not apply there. However, we are able to provide examples demonstrating a number of interesting classes that fall outside the scope of the work in Chapter 3. Firstly, we give a family of examples of warped product ended manifolds of the type studied in Section 3.3.1 which satisfy $\int \frac{1}{f} = \infty$ and so are not conformal to a locally hyperbolic metric (by Theorem 3.3.1) but for which we can solve the Yamabe problem.

For example, a very straightforward demonstration of a class of manifolds covered by our new existence Theorem C is the following

**Example 4.3.5.** *Consider a manifold $(M, \mathring{g}_f)$ with a warped product end in the sense of* (3.39) *with warping function*

$$f(r) \leq Ce^{-\beta r} \text{ for some } 0 \leq \beta < \frac{\sqrt{n(n-2)}}{n-1}$$

*and with cross-section $(N, h)$ of constant negative scalar curvature. Then $S_{\mathring{g}_f} \to -\infty$ as $r \to \infty$ and large annuli far along the warped product end satisfy the requirements of Corollary 4.2.2. Consequently, we may solve the Yamabe problem on $(M, \mathring{g}_f)$.*

As discussed, this above example is not conformal to a locally hyperbolic ended manifold by Theorem 3.3.1 as any $f$ satisfying the conditions above satisfies $\int \frac{1}{f} = \infty$. Consequently, the results of Chapter 3 do not apply in this case.

Furthermore, given a model manifold of a type that exhibits the desired volume ratios, we are able to show that our theorem encompasses a large class of perturbations of this manifold, for example as in Proposition 4.3.3. In contrast to the perturbations seen in the asymptotically locally hyperbolic case, these perturbations needn't necessarily even satisfy decay in $C^0$ provided they preserve a decay condition on the volume form and the negativity of the scalar curvature and certainly do not make any direct restriction of higher derivative decay of the metric components.

# 5

# Directions for Further Study

Having established our main results in Chapter 3, extending results known in the well studied asymptotically hyperbolic case, and provided some first exploratory results for a new direction in Chapter 4, in this final chapter we overview some questions for further exploration beyond the scope of this DPhil project.

As mentioned, my perspective on the work of Chapter 4 is that it forms an early attempt to find a different perspective on the non-compact Yamabe problem and so our goal at this time is to ask further questions with the aim of understanding what implications the volume ratio type conditions might have on existence of solutions for the problem. Additionally, one initial motivation for the work in Chapter 3 was to try and find a condition on the restricted class of asymptotically warped product metrics which would give existence or non-existence of a complete solution to the Yamabe problem; we have gained some partial insight in this goal and also hope it may provide direction for potential further work in the existence question for non-compact problem in general.

This chapter is split into two sections, one on questions regarding Chapter 3 and the other on Chapter 4.

# 5.1

# Asymptotically Locally Hyperbolic and Warped Product Ended Manifolds

## 5.1.1 A Generalisation of the Quantity $\int \frac{1}{f}\, dr$ in the Warped Product Setting

As shown in Section 3.3, a key quantity in the asymptotically warped product setting is

$$\int_{M^+} \frac{1}{f(r)}\, dr, \tag{5.1}$$

the boundedness of which represents the case that the reference warped product end is conformally locally hyperbolic. Furthermore, we are able to use this condition to find a condition on all manifolds which are of asymptotically warped product type given which we can show that the conformal class admits an asymptotically locally hyperbolic representative. In the case that the integral (5.1) diverges, we are in the case that the warped product end is conformally complete and of finite volume of a "hyperbolic cusp" type (see the proof of Lemma 3.3.4).

Two questions that follow our work above are the following. Firstly, can we extend these ideas to address the broader question (mentioned in the introduction of this thesis) of when a given conformal class of a non-compact manifold admits a representative with negative scalar curvature outside of a compact set? An understanding of these types of conformal classes could provide insight, or a useful simplifying step, in solving the Yamabe problem on non-compact manifolds of negative curvature type. To the knowledge of the author, this question hasn't received any treatment in the literature.

The second question is concerning the quantity (5.1). In particular, we ask if this quantity can be generalised, or if an equivalent quantity can be found for a more general class of manifolds which are not of warped product type. As seen in the proof of Lemma 3.3.5 the condition is related to the connection between the volume and the completeness of the manifold (in particular, this quantity must be infinite if the manifold is both complete and of finite volume).

### 5.1.2 A Generalisation of our Approach to Multiply Warped Product Manifolds

It would be interesting to consider whether the sub-solution approach used to prove the existence part of Theorem A could be generalised to a multiply warped product ended manifold. Not only would this be of independent interest, but it would also provide a common ground on which to compare the results of Chapter 4 using local eigenvalue conditions to the asymptotically hyperbolic type approach which, as mentioned, does not satisfy the requirements of Chapter 4.

## 5.2

## Volume Ratio Conditions

### 5.2.1 Understanding the Volume Ratios of Domains in Manifolds with Negative Scalar Curvature

An important question is what types of manifolds with negative scalar curvature have domains exhibiting different values for the volume ratios as defined in Chapter 4. In particular we ask two specific questions of this nature:

1. Is there a restriction on the possible volume ratios of two equally spaced sets $\Omega_1$ and $\Omega_2$ given the condition that the manifold has negative scalar curvature?
2. Can we characterise the class of manifolds which satisfy the assumptions of Theorem B?

We can already provide some examples in this direction. For example, we can immediately see that there exist manifolds with negative scalar curvature which do not contain concentric geodesic balls with a volume ratio satisfying condition (4.6) of Theorem B. In particular, we analyse an example of Aviles and McOwen [AM88, Example 6.2] of a negative scalar curvature manifold on which the Yamabe problem cannot be solved.

**Example 5.2.1** ([AM88, Example 6.1])**.** *Let $M = \mathbb{R} \times \mathbb{T}^{n-1}$ where $\mathbb{T}^{n-1}$ is the flat $(n-1)$-dimensional torus. Consider the metric*

$$g = dr^2 + e^{-2r^2} d\Theta^2 \tag{5.2}$$

*where $d\Theta^2$ is the metric on $\mathbb{T}^{n-1}$ and whose scalar curvature is*

$$S_g = 4(n-1)(1 - nr^2) \ . \tag{5.3}$$

*In this case, there does not exist a conformal metric $\tilde{g}$ of constant scalar curvature* $-1$ *(even if we relax the condition that the conformal metric be complete).*

One can readily compute that any ball far away from $r = 0$ (where the scalar curvature is negative as required to apply Theorem B) exhibits volume ratios of size $\approx e^{R^2}$ for large, equally spaced sets and so we do not satisfy condition (4.6).

Other relevant examples have already been the subject of discussion in earlier sections; in particular, the hyperbolic space, which also does not have domains satisfying the volume ratio condition (4.6), and Example 4.3.1, which includes a whole class of manifolds with negative scalar curvature and which have domains satisfying (4.6).

### 5.2.2 A Sharp Volume Ratio Bound for Existence to the Yamabe Equation

We ask the question: Given our assumption that $S_g \leq -n(n-1)$ outside of some compact set, what is the optimal bound $\Theta(R)$ such that, given two equally spaced sets $\Omega_1$ and $\Omega_2$, the bound

$$\frac{Vol_g(\Omega_2 \setminus \Omega_1)}{Vol_g(\Omega_1)} \leq \Theta(R)$$

implies that a complete solution to the Yamabe problem exists?

Our existing work establishes that $\Theta(R) \geq \sinh^2\left(\frac{\sqrt{n(n-2)}}{2}R\right)$ via Theorem C and, as mentioned earlier, Example 5.2.1 shows that $\Theta(R) < e^{R^2}$. We have seen that the lower bound $\sinh^2\left(\frac{\sqrt{n(n-2)}}{2}R\right)$ is sharp as far as the existence of a negative first eigenvalue is concerned; therefore, it's natural to explore more carefully those manifolds which exhibit a solution for the Yamabe equation but which do not have a negative first eigenvalue for the conformal Laplacian on any compact set, the hyperbolic space being a clear example.

This ties interestingly into the question of whether the negativity of the first eigenvalue of the conformal Laplacian can be relaxed to a weaker eigenvalue condition, perhaps that the first eigenvalue is less than that of the hyperbolic space.

### 5.2.3 Exploring the Relation of the Volume Ratio Condition with the Isoperimetric Inequality

As mentioned in the motivation section, volume ratio type conditions similar to those we consider as an application to the first eigenvalue can be loosely motivated by comparison to the implications of the isoperimetric inequality to the first eigenvalue. It may be enlightening to try and obtain a better geometric understanding of these volume ratio conditions and, in particular, explore its relation (if any) to the isoperimetric inequality and then, more broadly, the implications of these to the Yamabe problem. We note an early computation in this direction although it is unclear to the author whether any insight can be derived from what follows.

In particular, if we write

$$F(r) := \frac{Vol_g(B_{2r} \setminus B_r)}{Vol_g(B_r)}$$

and

$$G(r) := \frac{Vol_g(\partial B_r)}{Vol_g(B_r)}.$$

We note the similarity of $G$ with the isoperimetric quotient $\frac{Vol_g(\partial B_r)^{\frac{n}{n-1}}}{Vol_g(B_r)}$ and also the Cheeger constant $\inf_A\{\frac{Vol_g(\partial A)}{Vol_g(A)} : \ Vol_g(A) \leq \frac{1}{2}Vol_g(M)\}$ for finite volume manifolds, although we highlight that any connection with the two is highly speculative. $F$ satisfies the ODE

$$F'(r) = (F(r) + 1)(G(2r) - G(r)).$$

In particular, we may write $F$ in terms of $G$ as

$$F(r) = A \exp \left( \int_r^{2r} G(s)ds - \int_0^r G(s)ds \right) - 1 \tag{5.4}$$

with boundary condition $F(0) = 2^n - 1$ (computed from the limiting Euclidean case)

$$F(0) = \lim_{r \to 0} A \exp \left( \int_r^{2r} G(s)ds \right) - 1 = \lim_{r \to 0} A \exp \left( \int_r^{2r} \frac{n}{s} \, ds \right) - 1 = 2^n A - 1, \tag{5.5}$$

so

$$F(r) = \exp \left( \int_r^{2r} G(s)ds \right) - 1.$$

As pointed out, the above computation offers some apparently loose connection to some quantities in the literature that may be worth further exploration.

# A

# Transmission Conditions

We review a criterion that allows us to "glue" a subsolution inside some domain with a subsolution outside the domain together to create a subsolution on an entire manifold. Specifically, we have:

**Proposition A.0.1.** *Suppose $(M,g)$ is a Riemannian manifold and $\Omega \subset M$ is open, bounded and has $C^2$ boundary. Suppose that $u_1 \in C^1(\bar{\Omega})$, $u_2 \in C^1(\overline{M \setminus \Omega})$ subsolutions of* (Ya) *in $\Omega$ and $M \setminus \Omega$ respectively. Define*

$$u = \begin{cases} u_1 & \text{in } \Omega \ , \\ u_2 & \text{in } M \setminus \Omega \ . \end{cases} \tag{A.1}$$

*If $u_1|_{\partial\Omega} \equiv u_2|_{\partial\Omega}$ and $\nabla u_1 \cdot \hat{n} \leq \nabla u_2 \cdot \hat{n}$ on $\partial\Omega$ (where $\hat{n}$ is the unit outward pointing normal on $\partial\Omega$) then $u \in H^1_{loc}(M)$ and $u$ is a subsolution of* (Ya) *on all of $M$.*

*Proof.* We would like to show that

$$\int_M c_n \nabla u \cdot \nabla\varphi + S_g u\varphi + n(n-1)u^{\frac{n+2}{n-2}}\varphi \, dV_g \leq 0 \tag{A.2}$$

for all $\varphi \geq 0$, $\varphi \in C_c^\infty(M)$.

The above is immediately true for any $\varphi$ supported away from the boundary of $\Omega$ by the properties of $u_1$ and $u_2$. For functions with support including the boundary, we can assume (via a partition of unity and the regularity of $\partial\Omega$) that the support $U$ of $\varphi$ is compactly contained in an open bounded $\tilde{U}$ with smooth boundary that admits local coordinates $\{x_1, ..., x_n\}$ such that $\{x_n = 0\} \cap \tilde{U} = \partial\Omega \cap \tilde{U}$, $\{x_n < 0\} \cap \tilde{U} = \Omega \cap \tilde{U}$ and $\partial_{x_n}$ coincides with the normal vector $\hat{n}$ of $\partial\Omega$ on $\{x_n = 0\}$.

Consider a sequence of $\psi_k \in C_c^{0,1}(\tilde{U})$ satisfying $\psi_k \equiv 1$ on $\{x_n < -2/k\} \cap U$, $\psi_k \equiv 0$ on $\{x_n > -1/k\} \cap U$ and $\nabla\psi_k = k\partial_{x_n}$ on $\{-2/k < x_n < -1/k\} \cap U$. We compute

$$\begin{aligned}
&\int_M c_n \nabla u \cdot \nabla(\varphi\psi_k) + S_g u(\varphi\psi_k) + n(n-1)u^{\frac{n+2}{n-2}}(\varphi\psi_k)\, dV_g \\
&\quad = \int_M c_n \psi_k \nabla u \cdot \nabla\varphi + S_g u(\varphi\psi_k) + n(n-1)u^{\frac{n+2}{n-2}}(\varphi\psi_k)\, dV_g + \int_M c_n \varphi \nabla u \cdot \nabla\psi_k\, dV_g \\
&\quad =: (I) + (II)\ .
\end{aligned}$$

It's clear that $(I) \to \int_\Omega c_n \nabla u \cdot \nabla\varphi + S_g u\varphi + n(n-1)u^{\frac{n+2}{n-2}}\varphi\, dV_g$. As for $(II)$, by the choice of $\psi_k$ we observe that, in the local coordinates

$$\begin{aligned}
(II) &= \int_U c_n \varphi \nabla u_1 \cdot \nabla\psi_k\ \sqrt{|g(x)|}\, dx = c_n k \int_{\{-2/k < x_n < -1/k\} \cap U} \varphi \partial_{x_n} u_1\ \sqrt{|g(x)|}\, dx \\
&= c_n k \int_{-2/k}^{-1/k} \left( \int \varphi \partial_{x_n} u_1\ \sqrt{|g(x)|}\, dx_1 ... dx_{n-1} \right) dx_n \\
&\overset{k\to\infty}{\longrightarrow} c_n \int_{\{x_n = 0\}} \varphi \partial_{x_n} u_1\ \sqrt{|g(x_1, ..., x_{n-1}, 0)|}\, dx_1 ... dx_{n-1} = c_n \int_{\partial\Omega} \varphi \nabla u_1 \cdot \hat{n}\, d\hat{V}_g\ ,
\end{aligned}$$

where $d\hat{V}_g$ is the volume form induced on $\partial\Omega$ (in the third identity we have used Fubini's theorem and in the convergence statement we have used the $C^1$ regularity of $u_1$). Thus, we conclude that

$$\begin{aligned}
&\int_M c_n \nabla u \cdot \nabla(\varphi\psi_k) + S_g u(\varphi\psi_k) + n(n-1)u^{\frac{n+2}{n-2}}(\varphi\psi_k)\, dV_g \\
&\qquad \longrightarrow \int_\Omega c_n \nabla u \cdot \nabla\varphi + S_g u\varphi + n(n-1)u^{\frac{n+2}{n-2}}\varphi\, dV_g + c_n \int_{\partial\Omega} \varphi \nabla u_1 \cdot \hat{n}\, d\hat{V}_g\ .
\end{aligned} \tag{A.3}$$

On the other hand, as $\psi_k\varphi$ is compactly supported in $\Omega$, the fact that $u_1$ is a subsolution gives that the LHS of (A.3) is $\leq 0$ and so

$$\int_\Omega c_n \nabla u \cdot \nabla\varphi + S_g u\varphi + n(n-1)u^{\frac{n+2}{n-2}}\varphi\, dV_g + c_n \int_{\partial\Omega} \varphi \nabla u_1 \cdot \hat{n}\, d\hat{V}_g \leq 0\ .$$

Applying a similar argument to $u_2$ and $\Omega^c$ we obtain that

$$\int_{\Omega^c} c_n \nabla u \cdot \nabla \varphi + S_g u \varphi + n(n-1) u^{\frac{n+2}{n-2}} \varphi \; dV_g - c_n \int_{\partial\Omega} \varphi \nabla u_1 \cdot \hat{n} \; d\hat{V}_g \leq 0 \; .$$

Summing the previous two inequalities gives

$$\int_M c_n \nabla u \cdot \nabla \varphi + S_g u \varphi + n(n-1) u^{\frac{n+2}{n-2}} \varphi \; dV_g + c_n \int_{\partial\Omega} \varphi (\nabla u_1 - \nabla u_2) \cdot \hat{n} \; d\hat{V}_g \leq 0 \; ,$$

and so if $\nabla u_1 \cdot \hat{n} \leq \nabla u_2 \cdot \hat{n}$ on $\partial\Omega$ we obtain that $u$ is a subsolution on all of $M$.

□

# B

# An ODE Result and Alternative Proof for Lemma 3.3.2

**Lemma B.0.1.** *Suppose $F : [0, \infty) \to (0, \infty)$ is smooth and either monotonically non-increasing or monotonically non-decreasing, $F(s) \to 1$ as $s \to \infty$ and $p : [0, \infty) \to \mathbb{R}$ is smooth and satisfies*

$$\int_0^\infty e^{P(s)} ds < \infty \text{ where } P(z) := \int_0^z p(s) ds.$$

*Then, there exists a smooth positive strictly increasing function $K : [0, \infty) \to [0, \infty)$ solving the following boundary value problem*

$$\begin{cases} K''(z) = F(K(z))(K'(z))^2 + p(z)K'(z), \\ K(0) = 0, \ K(\infty) = \infty. \end{cases} \tag{B.1}$$

*Proof.* We rewrite the second order ODE in (B.1) as a Ricatti-type equation in $\kappa := K'(z)$,

$$\begin{cases} \kappa'(z) = F(K(z))\kappa^2(z) + p(z)\kappa(z) \\ \int_0^\infty \kappa(z) \, dz = \infty. \end{cases} \tag{B.2}$$

We note that for any given function $q(z)$ the general solution to the Ricatti equation $\bar{\kappa}' = q\bar{\kappa}^2 - p\bar{\kappa}$ is given by

$$\bar{\kappa}(z) = \frac{e^{P(z)}}{C - \int_a^z q(s) e^{P(s)} ds}$$

where $C$ and $a$ are arbitrary constants.

Case 1: $F$ is monotone non-increasing.

We will solve the problem by an iterative scheme. Let

$$\kappa_0(z) = \frac{e^{P(z)}}{\int_z^\infty F(0)e^{P(s)}ds}$$

which is a solution of the ODE

$$\kappa_0' = F(0)\kappa_0^2 + p\kappa_0.$$

Now define iteratively, for $n \geq 0$

$$\kappa_{n+1} := \frac{e^{P(z)}}{\int_z^\infty F(K_n(s))e^{P(s)}ds} \tag{B.3}$$

where $K_n(z) := \int_0^z \kappa_n(s)ds$ and note here that $\kappa_{n+1}$ solves

$$\kappa_{n+1}' = F(K_n)\kappa_{n+1}^2 + p\kappa_{n+1}. \tag{B.4}$$

We first establish that $\kappa_n \leq \kappa_{n+1}$. Note that, as $\kappa_0 > 0$, necessarily $K_0 > 0$ and so $F(K_0) \leq F(0)$ and so $\kappa_0 \leq \kappa_1$. In general, suppose $\kappa_{n-1} \leq \kappa_n$, then $K_{n-1} \leq K_n$ and so $F(K_{n-1}) \geq F(K_n)$ and therefore the formula (B.3) gives $\kappa_n \leq \kappa_{n+1}$.

We now observe that, as $F(x) \geq 1$, we have the upper bound

$$\kappa_n \leq \frac{e^{P(z)}}{\int_z^\infty e^{P(s)}ds} =: \kappa^*(z).$$

As a consequence, there exists a $\kappa_\infty$ such that $\kappa_n(z) \nearrow \kappa_\infty(z)$ for all $z \geq 0$. We also note that the uniform upper bound $\kappa^*$ ensures that, via repeated differentiation of the ODE (B.4), on any closed interval $[0, R]$ the sequence $\{\kappa_n\}$ is uniformly bounded in $C^m$ for any $m$. Consequently, the sequence converges in $C^m_{loc}$ for any $m$ to $\kappa_\infty$, which belongs to $C^\infty$ and satisfies

$$\kappa_\infty' = F(K_\infty)\kappa_\infty^2 - p\kappa_\infty$$

as desired.

It remains only to check that $K_\infty(z) = \int_0^z \kappa_\infty(s)ds \to \infty$. To see this note that we have

$$\kappa_\infty \geq \kappa_0 = -\frac{d}{dz}\left(\frac{1}{F(0)}\log\left(\int_z^\infty e^{P(t)}dt\right)\right)$$

and so

$$K_\infty(z) \geq \frac{1}{F(0)}\log\left(\frac{\int_0^\infty e^{P(s)}ds}{\int_z^\infty e^{P(s)}ds}\right) \to \infty.$$

This completes the proof in the case that $F$ is monotone non-increasing.

Case 2: $F$ is monotone non-decreasing.

The argument is the same as above but now the sequence $\kappa_n$ is non-increasing and $\kappa^*$ provides a lower bound i.e. $\kappa_n \geq \kappa^*$. The convergence of $\kappa_n$ to $\kappa_\infty$ is obtained in the same way, the only difference is in obtaining that $K_\infty \to \infty$ which we now obtain from the lower bound $\kappa_\infty \geq \kappa^*$ as

$$K_\infty \geq \int_0^z \kappa^*(s)ds = \int_0^z \frac{e^{P(s)}}{\int_s^\infty e^{P(t)}dt}ds = \log\left(\frac{\int_0^\infty e^{P(s)}ds}{\int_z^\infty e^{P(s)}ds}\right) \to \infty.$$

□

We may now apply this lemma to prove our main result of this section.

**Proposition B.0.2.** *Let $(M, g)$ be a manifold with a warped product end. Then there exists a conformal metric $\tilde{g}$ of $g$ such that $(M, \tilde{g})$ has a locally hyperbolic end if and only if*

$$\int_0^\infty \frac{1}{f(s)}\, ds < \infty. \tag{B.5}$$

*Proof.* Consider a conformal factor $\kappa > 0$ which is radial on the warped product end, that is $\kappa(z, \theta) = \kappa(z)$ on $M^+$ and define $\tilde{g} := \kappa^2 g$. Then the metric $\tilde{g}$ on $M^+$ may be written

$$\tilde{g} = \kappa^2 dz^2 + (\kappa f)^2 \mathring{h}.$$

Define a corresponding coordinate change in $z$ by

$$r := \int_0^z \kappa(s)ds = K(z)$$

so that $dr^2 = \kappa^2 dz^2$ and so we can equivalently write

$$\tilde{g} = dr^2 + (\kappa(z)f(z))^2 \mathring{h}.$$

We note here that the metric $\tilde{g}$ is complete provided

$$\int_0^\infty \kappa(s)\, ds = \infty. \tag{B.6}$$

For $\tilde{g}$ to be locally hyperbolic, we require that

$$f(z)\kappa(z) = \begin{cases} \sinh(r + r_0) = \sinh(K(z) + r_0) & k = 1, \\ e^r = e^{K(z)+r_0} & k = 0, \\ \cosh(r + r_0) = \cosh(K(z) + r_0) & k = -1, \end{cases} \tag{B.7}$$

where $k$ corresponds to the sign of $S_{\mathring{h}}$ as in the definition of the reference locally hyperbolic metrics (3.1). Taking the logarithm of both sides and differentiating, we obtain corresponding ODEs for $\kappa(z)$,

$$\kappa'(z) = \begin{cases} \coth(K(z) + r_0)\kappa^2(z) + p(z)\kappa(z) \\ \kappa^2(z) + p(z)\kappa(z) \\ \tanh(K(z) + r_0)\kappa^2(z) + p(z)\kappa(z) \end{cases} \tag{B.8}$$

where $p = -\frac{f'}{f}$.

First suppose that $\int_0^\infty \frac{1}{f(s)}\, ds = \infty$. For any choice of $C > 0$ and $A > \coth(r_0)$, the function

$$\underline{\kappa} = \frac{1}{f(z)(C - A\int_0^z \frac{1}{f(s)} ds)}$$

provides a strict sub-solution to (B.8) for any $k \in \{1, 0, -1\}$ and blows up as $z \nearrow Z$ for some finite $Z > 0$. Consequently, for any solution of (B.8), we may choose $C$ sufficiently large that $\underline{\kappa}(0) \le \kappa(0)$ and therefore $\underline{\kappa}(z) \le \kappa(z)$ for all $z \ge 0$ and so $\kappa$ must blow up as $z \nearrow Z$ for some finite $Z > 0$.

Now suppose that $\int_0^\infty \frac{1}{f(s)}\, ds < \infty$. We would like to apply Lemma B.0.1 to the ODEs above to conclude the proof. The conditions on $F$ are certainly met and so we need only check that $\int_0^\infty e^{P(s)} ds < \infty$; however, this is clear as

$$P(s) = \int_0^s p(t)dt = \int_0^s -\frac{f'(t)}{f(t)} dt = \log\left(\frac{f(0)}{f(s)}\right)$$

and so

$$\int_0^\infty e^{P(s)} ds = \int_0^\infty \frac{f(0)}{f(s)} ds < \infty$$

by assumption. Therefore we may apply Lemma B.0.1 to obtain a solution $\kappa$ with the desired properties. □

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