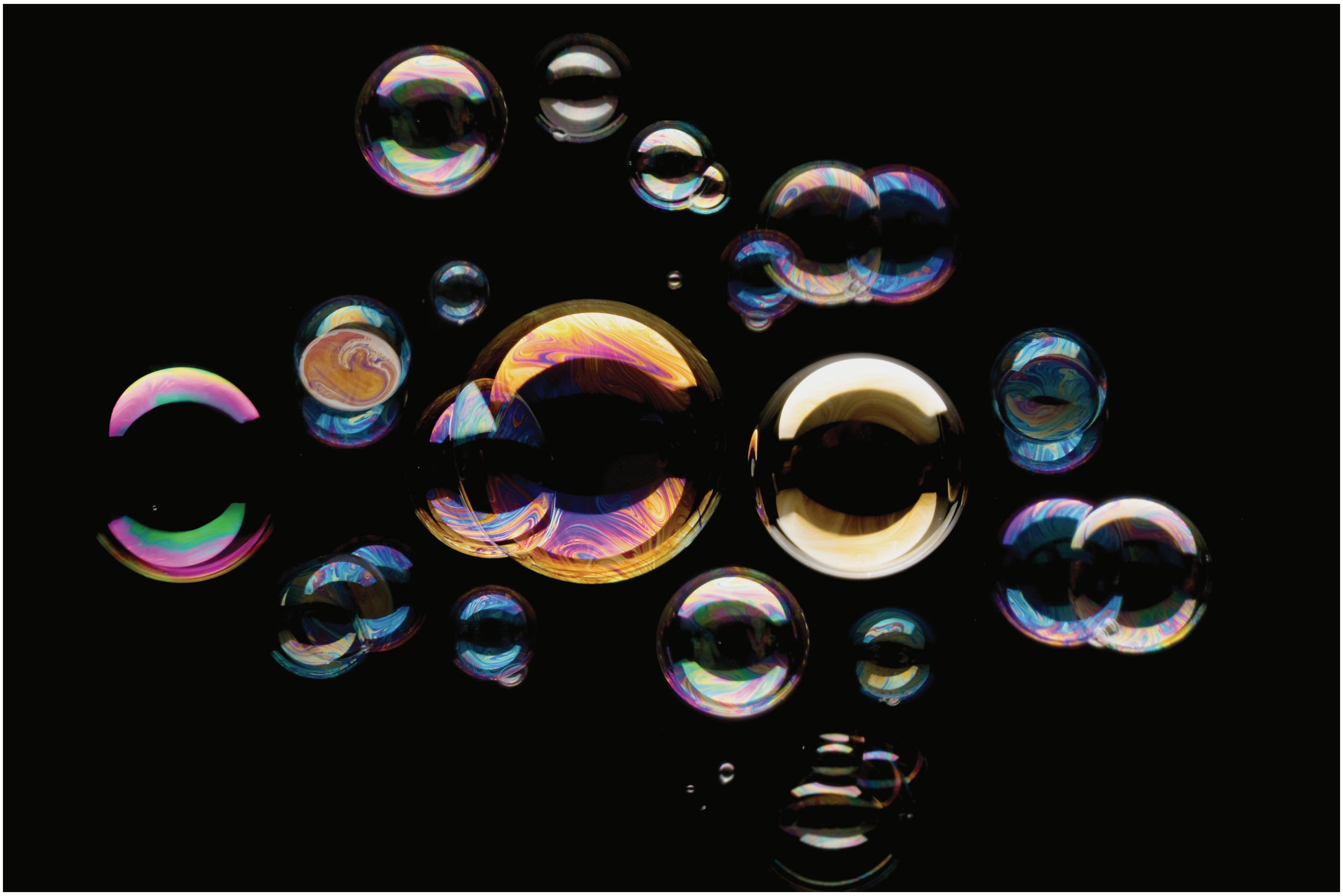
Bubbling Blow-Up in Critical Elliptic and Parabolic Problems



Monica Musso

Mathematical models are often expressed by nonlinear partial differential equations. Solutions of a given partial differential equation can be interpreted as attainable states for the underlying model. In steady as well as in time-dependent problems, a central issue is to determine the behaviour of solutions or the presence of blow-up.

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Blow-up takes place in regions or instants where solutions, or some quantities depending on them, become unbounded or exhibit irregular behaviour. This usually means that the original model loses validity near these regions and space-time scaling is required to make an accurate description. A particular type of blow-up are the ones that are triggered by *bubbling*. We will briefly discuss bubbling blow-up in two classical critical elliptic and parabolic problems.

Bubbling in critical elliptic problems. Many problems of physical and geometrical interest have a variational structure. For such problems, the failure of compactness at certain energy levels reflects highly interesting phenomena related to internal symmetries of the systems under study. In several of these situations, *bubbling* may occur. The term *bubbling* refers to the presence of families of solutions that at main order look like scalings of a single profile which in the limit become singular but at the same time have an approximately constant energy level. Such phenomena have been observed for the first time by Sacks-Uhlenbeck (1981) in the context of two-dimensional harmonic maps, and independently by Wente (1980) in the context of surfaces of prescribed constant mean curvature.

Classical models where bubbling occurs are semilinear boundary value problems near criticality in $\mathbb{R}^N$. Consider the problem of finding positive solutions to

$$\Delta u + u^q = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary $\partial\Omega$ and $q > 1$. This equation in (1) is sometimes called the Lane-Emden-Fowler equation. It was used first in the mid-19th century in the study of internal structure of stars, on the other hand it constitutes a basic model equation for steady states of reaction-diffusion systems, nonlinear Schrodinger equations, fast diffusion equations, and nonlinear dispersive equations. The case $q = \frac{N+2}{N-2}$ is especially meaningful. In geometry, it is related to the well-known problem of finding conformal metrics on a given manifold with prescribed scalar curvature, as in the Yamabe problem. In the study of nonlinear dispersive equations, it is related to the soliton resolution conjecture for nonlinear wave equations and nonlinear wave maps [10], [16].

The critical exponent $q = \frac{N+2}{N-2}$ sets a threshold where the structure of the solution set of (1) suffers a dramatic change. If $q < \frac{N+2}{N-2}$ a solution may always be found by minimizing the Rayleigh quotient

$$Q(u) \equiv \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{2}{q+1}}}, \quad u \in H_0^1(\Omega) \setminus \{0\}. \tag{2}$$

In fact, the quantity $S_q(\Omega) \equiv \inf_{u \in H_0^1(\Omega)\setminus\{0\}} Q(u)$ is achieved thanks to compactness of Sobolev embeddings $H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$ for $q < \frac{N+2}{N-2}$. A suitable scalar multiple of a minimizer turns out to be a solution of (1). The case $q \geq \frac{N+2}{N-2}$ is considerably more delicate: for $q = \frac{N+2}{N-2}$ compactness of the embedding is lost while for $q > \frac{N+2}{N-2}$ there is no such an embedding. This obstruction is not just technical for the solvability question, but essential. If Ω is strictly star-shaped around a point $x_0 \in \Omega$ and u solves (1) then Pohozaev's identity (1965) yields

$$\begin{aligned}\left(\frac{N-2}{2} - \frac{N}{q+1}\right)&\int_\Omega u^{q+1}\,dx \\ &= -\frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x - x_0)\cdot \nu d\sigma < 0,\end{aligned}$$

where ν is the unit outer normal to $\partial\Omega$. Hence necessarily $q < \frac{N+2}{N-2}$, and thus no solutions at all exist if $q \geq \frac{N+2}{N-2}$.

Pohozaev's result puts in evidence the central role of topology or geometry in the domain for solvability. Kazdan and Warner (1975) observed that Problem (1) is actually solvable for any $q > 1$ if Ω is a radial annulus, as compactness in the Rayleigh quotient Q is gained within the class of radially symmetric functions. On the other hand Coron (1984) found via a variational method that (1) is solvable at the critical exponent $q = \frac{N+2}{N-2}$ whenever Ω is a domain exhibiting a small hole. Substantial improvement of this result was found by Bahri and Coron [1], proving that if some homology group of Ω with coefficients in $\mathbf{Z}_2$ is not trivial, then (1) has at least one solution for q critical, in particular in any three-dimensional domain which is not contractible to a point. Examples showing that this condition is actually not necessary for solvability at the critical exponent were found by Dancer (1988), Ding (1989) and Passaseo (1989, 1998).

The change of structure of the solution set taking place at the critical exponent is strongly linked to the presence of unbounded sequences of solutions or *bubbling solutions*. By a *bubbling solution* for (1) near the critical exponent we mean an unbounded sequence of solutions u_n of (1) for $q = q_n \to \frac{N+2}{N-2}$. Setting

$$M_n \equiv \max_\Omega u_n = u_n(\xi_n) \to +\infty,$$

we see then that the scaled function

$$v_n(y) \equiv M_n^{-1} u_n(\xi_n + M_n^{-(q_n-1)/2} y),$$

satisfies

$$\Delta v_n + v_n^{q_n} = 0$$

in the expanding domain $\Omega_n = M_n^{(q_n-1)/2}(\Omega - \xi_n)$. Assuming for instance that ξ_n stays away from the boundary of Ω, elliptic regularity implies that locally over compacts around the origin, v_n converges up to subsequences to a positive solution of

$$\Delta U + U^{\frac{N+2}{N-2}} = 0$$

in entire space. Positive solutions to this equation are known from the classical works of Rodemich (1966), Aubin (1976), Obata (1972) and Talenti (1976) to be the functions

$$U_{\lambda,\xi}(y) = \lambda^{-\frac{N-2}{2}} U\left(\frac{y-\xi}{\lambda}\right),$$

where

$$U(y) = (N(N-2))^{\frac{N-2}{4}} \left(\frac{1}{1+|y|^2}\right)^{\frac{N-2}{2}}$$

for any scalar $\lambda > 0$ and any point $\xi \in \mathbb{R}^N$.

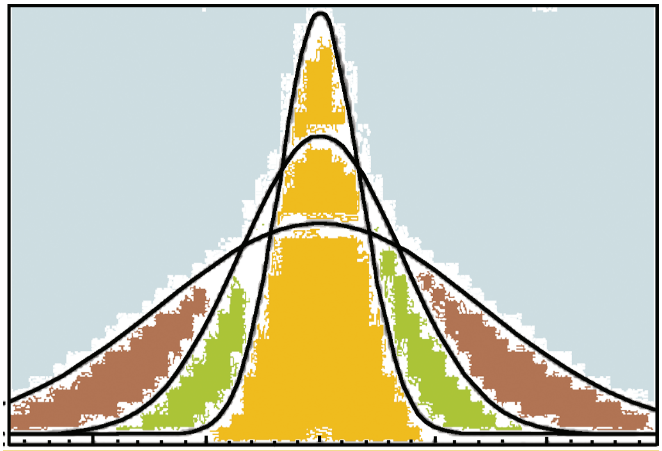

Figure 1. Three bubbles with different values for the concentration parameter λ, all centered at the same point.

These are the only positive solutions [4] and they are known as *the bubbles*. They corresponds precisely to an extremal of the critical Sobolev embedding

$$\mathcal{S}_N = \inf_{u\in C_0^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}}. \tag{3}$$

Coming back to the original variable, one expects then that "near ξ_n" the behavior of $u_n(x)$ can be approximated as

$$\begin{aligned} u_n(x) &\sim \lambda_n^{-\frac{N-2}{2}} U\left(\frac{x-\xi_n}{\lambda_n}\right)(1+o(1)) \\ \text{with}\quad \lambda_n &:= M_n^{-\frac{2}{N-2}}. \end{aligned} \tag{4}$$

A natural problem is that of constructing solutions exhibiting this property around one or several points of the domain when the exponent q approaches the critical value $\frac{N+2}{N-2}$.

For q slightly sub-critical, $q = \frac{N+2}{N-2} - \varepsilon$, $\varepsilon > 0$, a solution u_ε given by a minimizer of the Rayleigh quotient (2) clearly cannot remain bounded as $\varepsilon \downarrow 0$, since otherwise Sobolev's constant $\mathcal{S}_N$ in (3) would be achieved by a function supported in Ω. In this case, u_ε has asymptotically just a single maximum point ξ_ε and the asymptotic (4) holds globally in Ω with $M_\varepsilon \sim \varepsilon^{-\frac{1}{2}}$. Moreover, ξ_ε approaches a critical point of Robin's function $H(x,x)$. Here $H(x,y)$ is the regular part of Green's function $G(x,y)$ for the Laplace operator in Ω under Dirichlet boundary conditions.

This conclusion can be refined to the case of solutions u_ε exhibiting bubbling at multiple points, for both slightly sub-critical and super-critical exponents $q = \frac{N+2}{N-2} \mp \varepsilon$. The general result can be phrased in the following terms: Given a *nondegenerate critical point* or a *topologically nontrivial critical point* of the reduced functional of

$$\begin{aligned} (\xi,\lambda) &= (\xi_1,\dots,\xi_k,\lambda_1,\dots,\lambda_k) \in \Omega^k \times \mathbb{R}_+^k, \\ \Psi_k^{\mp\varepsilon}(\xi,\lambda) &= \sum_{j=1}^k H(\xi_j,\xi_j)\lambda_j^{N-2} \\ &\quad - 2\sum_{i<j} G(\xi_i,\xi_j)\lambda_i^{\frac{N-2}{2}}\lambda_j^{\frac{N-2}{2}} \\ &\quad \mp 2\varepsilon \log(\lambda_1\cdots\lambda_k), \end{aligned} \tag{5}$$

there exists a k-bubble solution

$$\begin{aligned} u_\varepsilon(x) &\sim \sum_{j=1}^k \lambda_{j\varepsilon}^{-\frac{N-2}{2}} U\left(\frac{x-\xi_j}{\lambda_{j\varepsilon}}\right), \\ \text{with}\quad \lambda_{j\varepsilon} &\sim \varepsilon^{\frac{1}{N-2}} \quad \text{as} \quad \varepsilon \to 0, \end{aligned}$$

to problem (1) with $q = \frac{N+2}{N-2} \mp \varepsilon$. Needless to say, it is a delicate task to find critical points for this reduced functional for a general domain Ω, and they may even not exist.

But what is the origin of the reduced functional $\Psi_k^{\mp\varepsilon}$? An alternative way to find a solution to (1) is as a critical point of the energy functional

$$E_q(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q+1}\int_\Omega u^{q+1}. \tag{6}$$

The scaled bubbles give a precise description of the solution near the blow up points. Far from these points, the bubbles need to be correct to match the zero Dirichlet boundary condition. An efficient way to do this is by using a proper multiple of the regular part $H(x,y)$ of the Green's function. Hence a better approximate solution is given by

$$v_\varepsilon(x) = \sum_{j=1}^k \lambda_{j\varepsilon}^{-\frac{N-2}{2}} U\left(\frac{x-\xi_j}{\lambda_{j\varepsilon}}\right) - \lambda_{j\varepsilon}^{\frac{N-2}{2}} H(x,\xi_j).$$

The energy evaluated at v_ε has the expansion, for $\varepsilon \to 0$,

$$E_{\frac{N+2}{N-2}\mp\varepsilon}(v_\varepsilon) \sim k\,\mathcal{S}_N + \Psi_k^{\mp\varepsilon}(\xi,\lambda).$$

Hence finding a critical point of the reduced functional suggests the existence of a solution u_ε close (in some topology) to the sum v_ε of k corrected bubbles. As mentioned before, in the sub-critical setting $q = \frac{N+2}{N-2} - \varepsilon$, the natural topology for this problem is the energy space $H_0^1(\Omega)$. In this regime, the reported results have been obtained by Brezis-Peletier [3], Han (1991), Rey (1990) and Bahri-Li-Rey [2]. In the super-critical regime $q = \frac{N+2}{N-2} + \varepsilon$ the embedding $H_0^1(\Omega) \hookrightarrow L^{q+1}(\Omega)$ is not available. New weighted L^∞ spaces were first introduced in [7] to treat the super-critical regime.

A main implication of the result in [7] states that in a domain with a small hole problem (1) with $q = \frac{N+2}{N-2} + \varepsilon$ has a two-bubble solution. More generally, if several spherical holes are drilled, a solution obtained by gluing together several two-bubbles can be found. Two-bubble solutions are the simplest to be obtained: single-bubble solutions for Problem (1) with $q = \frac{N+2}{N-2} + \varepsilon$ *do not exist*, as shown by M. Ben Ayed, K. El Mehdi, M. Grossi, O. Rey (2003). Solutions with different blow-up orders, known as *tower of bubbles* were found in [18].

Bubbling in critical parabolic problems. The parabolic version of problem (1)

$$\begin{aligned} \partial_t u &= \Delta u + u^q \quad \text{in } \Omega \times [0,T], \\ u &= 0 \quad \text{on } \partial\Omega \times [0,T], \end{aligned} \tag{7}$$

for $0 < T \le \infty$, is a widely studied classical problem, usually referred to as the Fujita problem, after his work in 1969. The heat operator $\partial_t u = \Delta u$ in (7) describes the diffusion of a density-function $u = u(x,t)$, where x is the space variable and t denotes time, and the term $f(u) = u^q$ represents a source. This is the simplest model of semilinear parabolic equations, which are ubiquitous as they can be found in numerous applications ranging from physics and biology to materials and social sciences. We refer the reader to reference [20] for a comprehensive survey on Problem (7) and more general versions of it.

Despite its simple look, Problem (7) encodes the fundamental features of a general semilinear parabolic problem. If the initial condition $u_0 = u_0(x)$ at time $t = 0$ is smooth and has value 0 on the boundary of Ω, Problem (7) has a unique (classical) solution $u = u(x,t)$ defined on some time interval $[0,T)$ with $0 < T \le \infty$. If we call $T = T(u_0)$ the maximal possible time of existence, the solution cannot be extended beyond T. If $T < \infty$, then necessarily the solution blows up at T, in the sense that $\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty$ as $t \nearrow T$. If $T = \infty$, we say that the solution is global. In this case, two possibilities can occur: either u remains bounded as $t \to \infty$, or

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

The latter is sometimes referred to as infinite time blow-up or grow-up of u. One of the fundamental problems concerning equation (7) is whether the infinite time blow-up can actually occur for some u_0 or not.

When q is the critical Sobolev exponent $q = \frac{N+2}{N-2}$, one expects that *blow-up by bubbling* for specific situations appears in the form

$$u(x,t) \sim \sum_{j=1}^{k} \lambda_j(t)^{-\frac{N-2}{2}} U\left(\frac{x - \xi_j(t)}{\lambda_j(t)}\right) \tag{8}$$

where now $\lambda_j(t)$ and $\xi_j(t)$ are functions of the time variable t, with $\lambda_j(t) \to 0$ as $t \to T$. Those solutions are usually asymptotically not self-similar and, while not generic, their presence is among the most important features of the full dynamics since they correspond to threshold solutions between different generic regimes.

Consider an initial condition of the form $u_0(x) = \alpha\varphi(x)$, where φ is a fixed positive smooth function in Ω with zero boundary value and α is a positive constant, and denote by $u_\alpha(x,t)$ the unique (local) solution to (7) with this initial condition. For all sufficiently small α, it is possible to prove that $u_\alpha(x,t)$ is globally defined and that $u_\alpha(x,t) \to 0$ uniformly for $x \in \Omega$ as $t \to \infty$. To see this, let λ_1 be the first eigenvalue of $-\Delta$ in Ω under Dirichlet boundary conditions and ϕ_1 a positive first eigenfunction:

$$-\Delta\phi_1(x) = \lambda_1\phi_1(x), \quad \text{in} \quad \Omega,\ \phi_1(x) = 0 \quad \text{on} \quad \partial\Omega.$$

Let $\delta > 0$ and consider the function $\bar{u}(x,t) = \delta e^{-\gamma t}\phi_1(x)$, where $0 < \gamma < \lambda_1$. Then a direct computation gives

$$\partial_t \bar{u} - \Delta\bar{u} - \bar{u}^q = \delta\phi_1 e^{-\gamma t}[(\lambda_1 - \gamma) - \delta^{q-1}\phi_1^{q-1}] > 0,$$

provided $\delta > 0$ is small. By the maximum principle, $\bar{u}(x,t)$ is a supersolution of (7). Hence any solution to (7), whose initial value at time $t = 0$ is bounded above by $\bar{u}(x,0)$, stays bounded by $\bar{u}(x,t)$ at all times. If we take $0 < \alpha$ small, then $u_\alpha(x,0) = \alpha\varphi(c) \le \bar{u}(x,0)$ and hence, for some positive constant $C > 0$, and any $x \in \Omega$

$$u_\alpha(x,t) \le Ce^{-\gamma t}, \quad \text{as} \quad t \to \infty.$$

On the other hand, if we now take α in the initial condition $u_0(x) = \alpha\phi(x)$ to be large, then $u_\alpha(x,t)$ blows-up in finite time. To see this, we assume that the solution $u_\alpha(x,t)$ is defined in $\Omega\times[0,T)$, we mutiply the equation against $\phi_1(x)$ and integrate on Ω. Using the divergence Theorem, we get

$$\begin{aligned} \frac{\partial}{\partial t}\int_\Omega u_\alpha(x,t)\phi_1(x)\,dx &= -\lambda_1\int_\Omega u_\alpha(x,t)\phi_1(x)\,dx \\ &+ \int_\Omega u_\alpha^q(x,t)\phi_1(x)\,dx. \end{aligned}$$

Let $g(t) = \int_\Omega u_\alpha(x,t)\phi_1(x)\,dx$. Then

$$g'(t) \ge -\lambda_1 g(t) + Cg^q(t)$$

for some positive constant C. Besides,

$$g(0) = \int_\Omega u_\alpha(x,0)\phi_1(x)\,dx = \alpha\int_\Omega \varphi(x)\phi_1(x)\,dx.$$

Then for α large, we have that $-\lambda_1 g(0) + Cg^q(0) > 0$. Then $g'(t) \ge 0$ for all $t \in [0,T)$ and, after integration

$$T \le \int_{g(0)}^{\infty} \frac{1}{-\lambda_1 g + Cg^q}\,dg.$$

This gives $T < \infty$, and blow-up in finite time occurs.

A consequence of the facts we just proves is that the number

$$\alpha_\star = \sup\{\alpha > 0 \ :\ \lim_{t\to\infty}\|u_\alpha(\cdot,t)\|_{L^\infty(\Omega)} = 0\}$$

is well defined and $0 < \alpha_\star < \infty$. The solution $u_{\alpha_\star}(x,t)$ somehow lies in the dynamic threshold between solutions globally defined in time and those that blow-up in finite time. Ni, Sacks, Tavantzis (1984) prove that this solution is a well-defined L^1-weak solution of the Fujita problem, but it is not clear whether it will be smooth for all times.

When $1 < q < \frac{N+2}{N-2}$, $u_{\alpha_\star}(x,t)$ is uniformly bounded and smooth, and up to subsequences it converges to a (positive) solution of the stationary problem (1). When $q > \frac{N+2}{N-2}$, Ω is a ball, and $u_{\alpha_\star}$ is radially symmetric then $u_{\alpha_\star}(x,t) \to 0$ as $t \to \infty$. The case $q = \frac{N+2}{N-2}$ is completely different: Galaktionov and Vázquez [13] proved that if $\Omega = B(0,1)$ and if the threshold solution $u_{\alpha*}$ is radially symmetric, then no finite time singularities for $u_{\alpha*}(r,t)$ occur and it must become unbounded as $t \to +\infty$, thus exhibiting infinite-time blow up

$$\lim_{t\to\infty} \|u_{\alpha_\star}(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Galaktionov and King [12] discovered that this radial blow-up solution to (7) at $q = \frac{N+2}{N-2}$ does have a bubbling asymptotic profile as $t \to +\infty$ of the form (8) with $k = 1$ and $\lambda_1(t) \sim t^{-\frac{1}{N-4}} \to 0$ for $N \geq 5$. This critical infinite-time bubbling occurs also in the nonradial setting, as shown in [6]: there are global solutions to (7) with $q = \frac{N+2}{N-2}$ which have infinite time blow-up at any collection of points $p = (p_1,\dots,p_k) \in \Omega^k$ if p lies in the open region of Ω^k where a certain $k \times k$ matrix $\mathcal{G}(q)$ is positive definite. The matrix $\mathcal{G}$ is explicitly defined in terms of the Robin's and the Green's functions in Ω, introduced in the previous section:

$$\begin{aligned}
\mathcal{G}(p) &= \left(\mathcal{G}_{ij}\right)_{i,j=1,\dots,k} \\
\mathcal{G}_{ii} &= H(p_i,p_i), \\
\mathcal{G}_{ij} &= -G(p_i,p_j) \quad i \neq j.
\end{aligned}$$

In other words, if $\mathcal{G}(p)$ is positive definite, there exist an initial datum u_0 and smooth functions $\xi_j(t) \to p_j$ and $0 < \lambda_j(t) \to 0$, as $t \to +\infty$, $j = 1,\dots,k$, such that the positive solution u_p of Problem (7) at $q = \frac{N+2}{N-2}$ has the form (8) with

$$\begin{aligned}
\lambda_j(t) &= O(t^{-\frac{1}{N-4}}), \\
\xi_j(t) &= p_j + O(t^{-\frac{2}{N-4}}) \quad \text{as } t \to +\infty.
\end{aligned} \tag{9}$$

A consequence of the construction in [6] is that this bubbling phenomena has codimension k-stability in the sense that there exists a codimension k manifold in $C^1(\bar{\Omega})$ that contains $u_p(x,0)$ such that if u_0 lies in that manifold and it is sufficiently close to $u_p(x,0)$, then the solution $u(x,t)$ of problem (7) has exactly k bubbling points $\tilde{p}_j$, $j = 1,\dots,k$ which lie close to the p_j, with the form (8).

Positive definiteness of $\mathcal{G}(q)$ trivially holds if $k = 1$. For $k = 2$ this condition holds if and only if

$$H(q_1,q_1)H(q_2,q_2) - G(q_1,q_2)^2 > 0,$$

in particular it does not hold if both points q_1 and q_2 are too close to a given point in Ω. Given $k > 1$ we can always find k points where $\mathcal{G}(q)$ is positive definite: it suffices to take points located at a uniformly positive distance one to each other, and then let them lie sufficiently close to the boundary.

The role of the matrix $\mathcal{G}(q)$ in elliptic bubbling phenomena of the stationary version of (1) at $q = \frac{N+2}{N-2}$ has been known for a long time. But what is its origin in the parabolic setting? The energy functional introduced in (6) is a Lyapunov functional for (7): for a solution $u(x,t)$ to (7) we compute

$$\frac{d}{dt}E_q(u(x,t)) = -\int_\Omega |\partial_t u|^2\, dx.$$

Thus along an approximate solution

$$v(x,t) = \sum_{j=1}^k \lambda_j^{-\frac{N-2}{2}} U\left(\frac{x-\xi_j}{\lambda_j}\right) - \lambda_j^{\frac{N-2}{2}} H(x,\xi_j)$$

the energy

$$t \to E_{\frac{N+2}{N-2}}(v(x,t)) \sim k\,\mathcal{S}_N + \Psi_k^0(\xi(t),\lambda(t))$$

is decreasing, and we may end up at the k-bubble energy $k\mathcal{S}_N$ as $t \to \infty$ only if its value is greater than $k\mathcal{S}_N$. If the matrix $\mathcal{G}(p)$ is positive definite that fact is guaranteed, as a simple look at the definition of Ψ_k^0 in (5) suggests. A formal consideration of balancing needed for the functions $\lambda_j(t)$ and $\xi_j(t)$ yields at main order to the following system of nonlinear ODEs

$$\dot{\lambda}_j + \nabla_{\lambda_j}\Psi_k^0(\xi,\lambda) = 0, \quad \dot{\xi}_j + \nabla_{\xi_j}\Psi_k^0(\xi,\lambda) = 0.$$

Under the positivity assumption on $\mathcal{G}$, the first equation has an admissible solution $\lambda_j(t) \to 0$, as $t \to \infty$, and the asymptotics (9) follow by a direct computation. This construction is valid on manifolds [15] but it is still open in dimensions 3 and 4, where one expects the paramenters λ and ξ to satisfy a system of *nonlocal* nonlinear differential equations, in analogy with a known result in dimension 3 on the whole space. Solutions with different blow-up orders at infinity, generating what is now known as *infinite-time tower bubbling*, have been obtained in [9].

When solutions blow-up in *finite-time*, the key issue is to understand how and where explosion can take place. The blow-up is said to be of type I if we have that

$$\limsup_{t\to T}(T-t)^{\frac{1}{q-1}}\|v(\cdot,t)\|_{L^\infty(\Omega)} < +\infty$$

and of type II if

$$\limsup_{t\to T}(T-t)^{\frac{1}{q-1}}\|v(\cdot,t)\|_{L^\infty(\Omega)} = +\infty.$$

Type I means that the blow-up takes place like that of the ODE $v_t = v^q$, so that in the explosion mechanism the nonlinearity plays the dominant role. The second alternative is rare and far less understood. The delicate interplay of diffusion, nonlinearity and geometry of the domain is responsible for that scenario.

The role of the Sobolev critical exponent $q = \frac{N+2}{N-2}$ is well-known to be central in the possible type of blow-up. When $1 < q < \frac{N+2}{N-2}$ solutions can only have type I blow-up, as it was first established by Giga and Kohn (1984) for the case of Ω convex. This is also the case for $q = \frac{N+2}{N-2}$ and radial solutions of (7); see Filippas, Herrero, Velázquez (2000).

Type II blow-up solutions are much harder to be detected. Herrero and Velázquez (1994) found a radial solution that blows-up with type II rate, for $N \geq 11$ and $q > q_{JL}(N)$ where $q_{JL}(N)$ is the *Joseph-Lundgren exponent* defined as

$$q_{JL}(d) = \begin{cases} \infty, & \text{if } 3 \leq N \leq 10, \\ 1 + \frac{4}{N-4-2\sqrt{N-1}}, & \text{if } d \geq 11. \end{cases}$$

The local profile locally resembles a time-dependent, asymptotically singular scaling of a positive radial solution of $\Delta w + w^q = 0 \quad \text{in } \mathbb{R}^N$ [5]. In this range for exponents q, these solutions are stable. Matano and Merle [17] prove that in the radially symmetric case no Type II blow-up can take place if $\frac{N+2}{N-2} < q \leq q_{JL}(N)$, a result that precisely complements that for the Herrero-Velázquez range.

A question that has remained conspicuously open for many years is whether or not type II blow-up solutions of (7) can exist in the Matano-Merle range $\frac{N+2}{N-2} < q < q_{JL}(N)$. The answer is positive [8]: in dimension $N \geq 7$ and $q = \frac{N+1}{N-3}$ (the critical Sobolev exponent in dimension $N-1$) and in a class of domains with axial symmetry there exists a solution to (7) which remains uniformly bounded outside any neighborhood of a certain curve $\Gamma \subset \partial\Omega$ while

$$\lim_{t\to T}(T-t)^{\gamma}\|u(\cdot,t)\|_{L^\infty(\Omega)}, > 0$$

$$\gamma = \frac{(N-3)(N-4)}{2(N-5)}.$$

Notice that for $q = \frac{N+1}{N-3}$ we have $\frac{1}{q-1} = \frac{N-3}{4} < \gamma$ so that u exhibits type II blow-up. This is again a blow-up by bubbling: at main order it is a bubble in dimension $N-1$ centered along a copy of the curve Γ, shifted inside Ω and at distance $d(t)$ from $\partial\Omega$, with scaled by $\lambda(t)$. The blow-up region is thus approaching the boundary, but the phenomena still describes an interior bubbling as $\lambda(t) \sim (T-t)^{\frac{N-4}{N-5}}$, whereas $d(t) \sim (T-t)$ as $t \to T^-$. In other words, the energy density $|\nabla u(x,t)|^2$ concentrates in the form of a Dirac mass for the curve Γ, generating bubbling blow-up along a curve. Bubbling blow-up along higher-dimensional sets, like surfaces, is still unknown.

Let me conclude this note with a brief description of the general strategy used in the proofs of the results that have been presented here. The procedure to construct solutions exhibiting the expected blow-up behaviour is to identify a first approximation with the anticipated features and then to get an actual solution, rather than an approximate one, by using a perturbation argument. Finding the *remainder* is a delicate and difficult step since the behaviour of a solution near the region of blow-up may depend in an intricate way on the entire dynamics, hence it is essential to have a precise control on the perturbation. The *inner-outer* scheme used in [6] consists in writing the solution as the approximation plus a remainder, and in expressing the remainder itself as sum of two parts, identified as the inner and the outer parts. The inner and the outer parts solve a coupled system of nonlinear partial differential equations, with the property that the main operator for the inner part catches the features of the problem near the singularity and it is expressed in the variable of the blowing-up limit profile, whilst the principal operator in the outer part sees the whole picture in the original scale. A key and delicate issue for the scheme to work is to ensure a fine control on the coupling between the inner and outer parts. Energy methods may fail to achieve this control, and sufficiently fast decay at infinity needs to be prescribed on the inner part. This general approach is quite flexible and has been successfully used in several other contexts, among which the construction of concentrated vorticities for the two-dimensional Euler's equations for incompressible fluids, in singularity formation for the two-dimensional harmonic map flow into S^2 as well as in the infinite-time blow-up for the Patlak-Keller-Segel model for chemotaxis.

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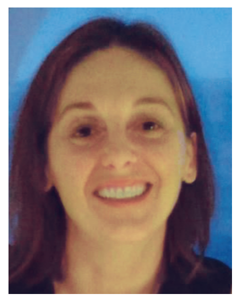

Monica Musso

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