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ON BASES OF $\mathfrak{g}$-INVARIANT ENDOMORPHISM ALGEBRAS

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To Marko Tadić for his 70th birthday.

Abstract. Let $\mathfrak{g}$ be a complex simple Lie algebra. Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$. Let V_λ be the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Our main result is a criterion of the existence of $Z(\mathfrak{g})$-bases for the $\mathfrak{g}$-invariant endomorphism algebra $R_\lambda =: \mathrm{Hom}_\mathfrak{g}(\mathrm{End}\, V_\lambda, U(\mathfrak{g}))$. Then we obtain a Clifford algebra analogue, namely a criterion of the existence $C(\mathfrak{g})^\mathfrak{g}$-bases for $R^C_\lambda =: \mathrm{Hom}_\mathfrak{g}(\mathrm{End}\, V_\lambda, C(\mathfrak{g}))$. We also describe a criterion of the existence of bases generated by powers of the Casimir element for $R_{\lambda,\nu} =: \mathrm{Hom}_\mathfrak{g}(\mathrm{End}\, V_\lambda, \mathrm{End}\, V_\nu)$.

1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra with a Cartan subalgebra $\mathfrak{h}$. Suppose that $\pi\colon \mathfrak{g} \to \mathrm{End}\, W$ is an irreducible finite-dimensional representation of $\mathfrak{g}$. Regarding $\mathrm{End}\, W$ as a $\mathfrak{g}$-module, the space of $\mathfrak{g}$-homomorphisms $\mathrm{Hom}_\mathfrak{g}(\mathfrak{g}, \mathrm{End}\, W)$ is called the space of adjoint operators in type $\mathrm{End}\, W$ by physicists [9] (the definition of adjoint operators is given in [9, Definition 1.1], but it will not be needed here). In case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, the Wigner-Eckart theorem states that [1, Theorem C. 4]:

$$\dim \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathfrak{sl}(2,\mathbb{C}), \mathrm{End}\, W) \leq 1.$$

This formula was generalized to any simple Lie algebra $\mathfrak{g}$ by Okubo and Myung [9], as they showed that

$$\dim \mathrm{Hom}_\mathfrak{g}(\mathfrak{g}, \mathrm{End}\, W) \leq r,$$

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where $r = \operatorname{rank}\mathfrak{g} = \dim\mathfrak{h}$. Suppose that the highest weight ν of a finite-dimensional simple $\mathfrak{g}$-module V_ν is expressed

$$\nu = m_1\omega_1 + \cdots + m_r\omega_r, \tag{1.1}$$

with fundamental weights $\omega_1, \ldots, \omega_r$ and non-negative integers $m_1, \ldots, m_r$. Then it is shown [9, Theorem 3.1]

$$\dim \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g}, \operatorname{End} V_\nu) = n(\nu),$$

where $n(\nu)$ is the number of nonzero m_i's in (1.1). In particular, it implies that the adjoint representation of a simple Lie algebra $\mathfrak{g}$ always occurs in $\operatorname{End} W$ for any nontrivial $\mathfrak{g}$-module W.

The above formula is better understood in the framework of $\mathfrak{g}$-invariant endomorphism algebras which we explain now. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. There is a surjective homomorphism of algebras

$$\pi_\nu \colon U(\mathfrak{g}) \to \operatorname{End} V_\nu.$$

Then π_ν induces a surjective linear map from the space of universal adjoint operators to the space of adjoint operators in type $\operatorname{End} V_\nu$:

$$A(\mathfrak{g}) = \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g}, U(\mathfrak{g})) \to \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g}, \operatorname{End} V_\nu).$$

Since there is an embedding of $\mathfrak{g} \hookrightarrow \operatorname{End} V_\lambda$ for any nontrivial simple $\mathfrak{g}$-module V_λ, we consider the following algebras of $\mathfrak{g}$-endomorphisms:

$$R_\lambda =\colon (\operatorname{End} V_\lambda \otimes U(\mathfrak{g}))^{\mathfrak{g}} \cong \operatorname{Hom}_{\mathfrak{g}}(\operatorname{End} V_\lambda, U(\mathfrak{g})),$$

and

$$R_{\lambda,\nu} =\colon (\operatorname{End} V_\lambda \otimes \operatorname{End} V_\nu)^{\mathfrak{g}} \cong \operatorname{Hom}_{\mathfrak{g}}(\operatorname{End} V_\lambda, \operatorname{End} V_\nu).$$

Let V be a $\mathfrak{g}$-module (possibly infinite-dimensional) with an infinitesimal character χ_ν. Kostant [7] proves that the tensor product of $V_\lambda \otimes V$ is of finite length, hence a direct sum of modules with generalized infinitesimal character. Moreover, the occurring characters are of form $\chi_{\nu+\mu_i}$ with μ_i being some weights of V_λ. In Kostant's proof, R_λ and $R_{\lambda,\nu}$ play pivotal roles.

The aim of this paper is to describe bases of R_λ and $R_{\lambda,\nu}$ generated by a Casimir element C, and equivalently by a certain matrix valued element $M_\lambda(C)$ to be defined in the following. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Kostant [6, Theorem 21] showed that there is a $\mathfrak{g}$-submodule E of $U(\mathfrak{g})$ such that the multiplication

$$Z(\mathfrak{g}) \otimes E \to U(\mathfrak{g})$$

is a $\mathfrak{g}$-module isomorphism. It follows that $U(\mathfrak{g})$ and R_λ are free $Z(\mathfrak{g})$-modules. Consider the map

$$\delta_\lambda : U(\mathfrak{g}) \to \operatorname{End} V_\lambda \otimes U(\mathfrak{g})$$

defined by

$$\delta_\lambda(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes x \text{ for } x \in \mathfrak{g},$$

which extends to a homomorphism of associative algebras. If $u \in Z(\mathfrak{g})$, then $\delta(u)$ is in R_λ.

Let B be the Killing form of $\mathfrak{g}$. Let x_i be a basis of $\mathfrak{g}$ and x_i^* be the dual basis with respect to B. The Casimir element C defined by

$$C = \sum_{i=1}^{m} x_i x_i^*$$

is in $Z(\mathfrak{g})$, and clearly it is independent of choice of the basis x_i. It follows that

$$\delta_\lambda(C) = \pi_\lambda(C) \otimes 1 + \sum_{i=1}^{m} \pi_\lambda(x_i) \otimes x_i^* + \sum_{i=1}^{m} \pi_\lambda(x_i^*) \otimes x_i + 1 \otimes C.$$

We set

$$M_\lambda(C) = \sum_{i=1}^{m} \pi_\lambda(x_i) \otimes x_i^*.$$

It is readily checked that $M_\lambda(C)$ is also independent of choice of the basis x_i, and thus it equals $\sum_{i=1}^{m} \pi_\lambda(x_i^*) \otimes x_i$. Then

$$\delta_\lambda(C) = \pi_\lambda(C) \otimes 1 + 2M_\lambda(C) + 1 \otimes C.$$

We write d_λ for $\dim V_\lambda$. Recall that a principal $\mathfrak{sl}_2$ in $\mathfrak{g}$ is a three-dimensional subalgebra spanned by $\{X, H, Y\}$ in $\mathfrak{g}$ such that

$$[H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = H$$

and the orbit of X under the adjoint group of $\mathfrak{g}$ is the principal nilpotent orbit. By a conjugation, we may and will assume that H is in the Cartan subalgebra $\mathfrak{h}$.

Theorem A (Theorem 3.1). *The following assertions are equivalent:*

(i) $1, \delta_\lambda(C), \ldots, \delta_\lambda(C)^{d_\lambda - 1}$ *form a basis of* $Z(\mathfrak{g})$*-module* R_λ*.*

(ii) $1, M_\lambda(C) \ldots, M_\lambda(C)^{d_\lambda - 1}$ *form a basis of* $Z(\mathfrak{g})$*-module* R_λ*.*

(iii) V_λ *is irreducible when restricted to a principal* $\mathfrak{sl}_2$ *in* $\mathfrak{g}$*.*

In Section 3 we obtain a complete list of V_λ's satisfying Condition (iii). In these cases, we get bases of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, U(\mathfrak{g}))$ consisting of m_i-th powers of either $\delta_\lambda(C)$ or $M_\lambda(C)$, where m_i's are the exponents of $\mathfrak{g}$ (cf. Section 3 for the definition of exponents).

By a theorem of Kostant [8, Theorem D], the Clifford algebra $C(\mathfrak{g})$ with respect to the Killing form of $\mathfrak{g}$ decomposes into the tensor product

$$C(\mathfrak{g}) = J \otimes E,$$

where $J = C(\mathfrak{g})^{\mathfrak{g}}$ and $E = \mathrm{End}\, V_\rho$. Here (π_ρ, E_ρ) is the irreducible representation of $\mathfrak{g}$ with highest weight ρ. We set the Clifford algebra analogue R_λ^C to be the invariant endomorphism algebra

$$R_\lambda^C :\ = \mathrm{Hom}_{\mathfrak{g}}(\mathrm{End}\, V_\lambda, C(\mathfrak{g})).$$

Then R_λ^C is a free J-module of rank equal to $\dim R_{\lambda,\rho}$. Note that

$$\rho = \omega_1 + \cdots + \omega_r.$$

For the irreducible representation (π_ρ, E_ρ) of the highest weight ρ, we define the map

$$\delta_{\lambda,\rho} : U(\mathfrak{g}) \to \mathrm{End}\, V_\lambda \otimes \mathrm{End}\, V_\rho$$

by

$$\delta_{\lambda,\rho}(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes \pi_\rho(x) \text{ for } x \in \mathfrak{g},$$

which extends to a homomorphism of associative algebras. Then

$$\delta_{\lambda,\rho}(C) = \pi_\lambda(C) \otimes 1 + 2M_{\lambda,\rho}(C) + 1 \otimes \pi_\rho(C),$$

where

$$M_{\lambda,\rho}(C) = \sum_{i=1}^{m} \pi_\lambda(x_i) \otimes \pi_\rho(x_i^*).$$

Recall that a simple $\mathfrak{g}$-module V_λ is said to be minuscule if $\langle \lambda, \alpha \rangle = 0, -1, 1$ for all roots α (cf. Section 4 for the list of the minuscule representations).

Theorem B (Theorem 4.3). *Assume that λ is minuscule. Then R_λ^C is a free J-module of rank d_λ. Moreover, the following assertions are equivalent:*

(i) $1, \delta_{\lambda,\rho}(C), \ldots, \delta_{\lambda,\rho}(C)^{d_\lambda - 1}$ *form a J-basis of R_λ^C.*

(ii) $1, M_{\lambda,\rho}(C), \ldots, M_{\lambda,\rho}(C)^{d_\lambda - 1}$ *form a J-basis of R_λ^C.*

(iii) $\delta_{\lambda,\rho}(C)$ *acts on the distinct irreducible summands of $V_\lambda \otimes V_\rho$ with distinct eigenvalues.*

Now we consider the map

$$\delta_{\lambda,\nu} : U(\mathfrak{g}) \to \mathrm{End}\, V_\lambda \otimes \mathrm{End}\, V_\nu$$

defined by

$$\delta_{\lambda,\nu}(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes \pi_\nu(x) \text{ for } x \in \mathfrak{g},$$

which extends to a homomorphism of associative algebras. Then

$$\delta_{\lambda,\nu}(C) = \pi_\lambda(C) \otimes 1 + 2M_{\lambda,\mu}(C) + 1 \otimes \pi_\nu(C),$$

where

$$M_{\lambda,\nu}(C) = \sum_{i=1}^{m} \pi_\lambda(x_i) \otimes \pi_\nu(x_i^*).$$

Theorem C (Theorem 4.4). *Let d be a positive integer. The following assertions are equivalent:*

(i) $1, \delta_{\lambda,\nu}(C), \ldots, \delta_{\lambda,\nu}(C)^{d-1}$ *form a basis of $R_{\lambda,\nu}$.*

(ii) $1, M_{\lambda,\nu}(C), \ldots, M_{\lambda,\nu}(C)^{d-1}$ *form a basis a basis of $R_{\lambda,\nu}$.*

(iii) $V_\lambda \otimes V_\nu = \bigoplus_{i=1}^{d} V_{\gamma_i}$ *decomposes into a direct sum of d non-isomorphic simple $\mathfrak{g}$-modules with distinct $\delta_{\lambda,\nu}(C)$-eigenvalues.*

We note that the algebra R_λ was investigated from a different perspective by Kirillov [4, 5] as 'quantum family algebra'. There was following up work on commutativity of R_λ and existence of certain M-type bases for R_λ by Rozhkovskaya [11]. Let $S(\mathfrak{g})$ denote the symmetric algebra of $\mathfrak{g}$. The related associated algebra $(\operatorname{End} V_\lambda \otimes S(\mathfrak{g}))^{\mathfrak{g}}$ is called 'classical family algebra' by Kirillov and it appeared in Panyushev's work on determination of the Dynkin polynomials and calculation of equivariant cohomology [10]. Their work inspired us to find the main result of this paper.

Our paper is organised as follows. In Section 2 we recall the basic properties of the algebras of $\mathfrak{g}$-endomorprhisms due to Kostant. In Section 3 we prove our main theorem on $Z(\mathfrak{g})$-bases for R_λ. In Section 4 we describe the bases for $R_{\lambda,\nu}$ and the Clifford analogue of our main theorem that is proved in Section 3.

2. Preliminaries on R_λ and $R_{\lambda,\nu}$

Fix a finite-dimensional simple $\mathfrak{g}$-module V_λ with highest weight λ. Let

$$\pi\colon U(\mathfrak{g}) \to \operatorname{End} V_\pi$$

be an arbitrary $\mathfrak{g}$-module having an infinitesimal character. In describing the infinitesimal characters of the tensor product $V_\lambda \otimes V_\pi$, Kostant [7] introduced the following algebras

$$R_\lambda = (\operatorname{End} V_\lambda \otimes U(\mathfrak{g}))^{\mathfrak{g}}$$

and

$$R_{\lambda,\pi} = (\operatorname{End} V_\lambda \otimes \pi[U(\mathfrak{g})])^{\mathfrak{g}}.$$

Kostant used the notation R and R_π for these two algebras [7]. Our notation indicates their dependence on λ. In particular, if π_ν is any finite-dimensional simple module with highest weight ν, then we use simpler notation $R_{\lambda,\nu}$ for R_{λ,π_ν}, namely

$$R_{\lambda,\nu} = (\operatorname{End} V_\lambda \otimes \operatorname{End} V_\nu)^{\mathfrak{g}} \cong \operatorname{End}_{\mathfrak{g}}(V_\lambda \otimes V_\nu).$$

Consider the map

$$\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

defined by

$$\delta(x) = x \otimes 1 + 1 \otimes x \text{ for } x \in \mathfrak{g},$$

which extends to a homomorphism of associated algebras.

By composing with π_λ on the first factor, we have the map

$$\delta_\lambda : U(\mathfrak{g}) \to \operatorname{End} V_\lambda \otimes U(\mathfrak{g})$$

defined by

$$\delta_\lambda(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes x \text{ for } x \in \mathfrak{g},$$

which extends to a homomorphism of associative algebras. Then R_λ is the commutant of $\delta(U(\mathfrak{g}))$ in $\operatorname{End} V_\lambda \otimes U(\mathfrak{g})$. For any $u \in Z(\mathfrak{g})$, $\delta(u)$ is in R_λ. Thus, $\delta(Z(\mathfrak{g}))$ is in the center of R_λ.

As shown in [6, Theorem 21] there is a $\mathfrak{g}$-submodule E of $U(\mathfrak{g})$ such that the multiplication

$$Z(\mathfrak{g}) \otimes E \to U(\mathfrak{g})$$

is a $\mathfrak{g}$-isomorphism. It follows that R_λ is a free $Z(\mathfrak{g})$-module.

Let $\triangle_\lambda = \{\mu_1, \cdots, \mu_k\}$ be the set of weights of V_λ and d_i be the multiplicity of μ_i. In other words, we have the weight space decomposition

$$V_\lambda|_{\mathfrak{h}} = \bigoplus_i \mathbb{C}_{\mu_i}^{\oplus d_i}.$$

Following Kostant we make the following definition.

Definition 2.1. *We say that λ is totally subordinate to ν if the number of irreducible constituents in $V_\lambda \otimes V_\nu$ is equal to $d_\lambda\colon = dim V_\lambda$.*

Proposition 2.2. *[7, Theorem 4.7] If λ is totally subordinate to ν, then there is an isomorphism of algebras*

$$R_{\lambda,\nu} \to \bigoplus_{i=1}^{k} \mathrm{Mat}_{d_i}(\mathbb{C}).$$

Proposition 2.3. *[7, Theorem 4.8] R_λ is a free $Z(\mathfrak{g})$-module of rank r, where $r = \sum_{i=1}^k d_i^2$.*

Proposition 2.4. *[7, Theorem 4.9] Suppose $u \in Z(\mathfrak{g})$ is not a constant. Then there exists a monic polynomial $P_{u,\lambda}(X)$ of degree k with coefficients in $Z(\mathfrak{g})$, such that $P_{u,\lambda}(X)$ is the minimal polynomial of $\delta_\lambda(u)$.*

Remark 2.5. The minimal polynomial of $\delta_\lambda(u)$ can be obtained from u by using the Harish-Chandra isomorphism [7, (4.9.4)- (4.9.6)].

Theorem 2.6. *The following statements are equivalent:*
(i) *R_λ is commutative.*
(ii) *V_λ has simple $\mathfrak{h}$-spectrum (every $d_i = 1$).*
(iii) *For any non-constant $u \in Z(\mathfrak{g})$, $1, \delta(u), \ldots, \delta(u)^{d_\lambda - 1}$ form a basis of R_λ over the fractional field $K(\mathfrak{g})$ of $Z(\mathfrak{g})$.*

Proof. $(i) \implies (ii)$: Commutativity of R_λ imples that $R_{\lambda,\nu}$ is commutative for any ν. Take a ν so that λ is totally subordinate to ν. By Proposition 2.1 there is an isomorphism of algebras

$$R_{\lambda,\nu} \to \bigoplus_{i=1}^{k} \mathrm{Mat}_{d_i}(\mathbb{C}).$$

Thus, (ii) follows from (i).

$(ii) \implies (iii)$: It follows from Proposition 2.2 that $1, \delta(u), \ldots, \delta(u)^{k-1}$ are linearly independent over $Z(\mathfrak{g})$, and thus they form a basis of R_λ over the $K(\mathfrak{g})$.

$(iii) \implies (i)$ is obvious. □

Remark 2.7. *The following is a complete list of irreducible representations of simple Lie algebras with simple $\mathfrak{h}$-spectrum. This list is well-known to experts. For instance, it appears in Howe's 1992 Schur Lecture Notes [2].*

$\mathfrak{g}$	λ the highest weight
$A_n(n \geq 1)$	$\omega_k, k = 1, \ldots, n$ $k\omega_1, k\omega_n, k = 1, 2, \ldots$
$B_n(n \geq 2)$	ω_1 ω_n (spin representation)
$C_n(n \geq 3)$	ω_1
C_3	ω_3
$D_n(n \geq 4)$	ω_1 ω_{n-1}, ω_n (spin representations)
G_2	ω_1 (dim $= 7$)
E_6	ω_1 (dim $= 27$) ω_6 (dim $= 27$)
E_7	ω_1 (dim $= 56$)

3. $Z(\mathfrak{g})$-Bases of R_λ

We see from Theorem 2.5 that any $\delta(u)$ ($u \in Z(\mathfrak{g})$ not a constant) generates R_λ over $K(\mathfrak{g})$. In this section we seek u so that $\delta(u)$ generates R_λ over $Z(\mathfrak{g})$. Naturally, it has to be the element of the smallest positive degree, namely the Casimir element

$$C = \sum_{i=1}^{m} x_i x_i^*.$$

We have

$$\delta_\lambda(C) = \pi_\lambda(C) \otimes 1 + 2M_\lambda(C) + 1 \otimes C, \tag{3.1}$$

where

$$M_\lambda(C) = \sum_{i=1}^{m} \pi_\lambda(x_i) \otimes x_i^*.$$

Clearly, as $Z(\mathfrak{g})$-module, R_λ is generated by powers of $\delta_\lambda(C)$ if and only if it is generated by powers of $M_\lambda(C)$.

To prove our main result Theorem 3.1 we first recall the concept of generalised exponents [6, Page 394] and a remarkable theorem of Kostant [6]. Let $I(\mathfrak{g}) = S(\mathfrak{g})^{\mathfrak{g}}$. We identify $I(\mathfrak{g})$ with the algebra $P(\mathfrak{g}^*)^{\mathfrak{g}}$ of $\mathfrak{g}$-invariant polynomials on $\mathfrak{g}^*$. Let $I^+(\mathfrak{g})$ be the augmentation ideal in $I(\mathfrak{g})$, namely the ideal of polynomials vanishing at the origin. Denote by $J(\mathfrak{g})$ the ideal in $S(\mathfrak{g})$ generated by $I^+(\mathfrak{g})$. The space $H(\mathfrak{g})$ of harmonic polynomials on $\mathfrak{g}^*$ is defined as the orthogonal complement to $J(\mathfrak{g})$ in $S(\mathfrak{g})$. Kostant showed that there is an isomorphism of graded $\mathfrak{g}$-modules:

$$S(\mathfrak{g}) \cong I(\mathfrak{g}) \otimes H(\mathfrak{g}).$$

Moreover, each irreducible representation π_λ has finite multiplicity in $H(\mathfrak{g})$. More precisely, if $s = m_\lambda(0)$ is the multiplicity of the zero weight in V_λ, then there exist numbers $e_1(\lambda), \ldots, e_s(\lambda)$ (not necessarily distinct) such that π_λ occurs in the homogeneous components $H^{e_1(\lambda)}(\mathfrak{g}), \ldots, H^{e_s(\lambda)}(\mathfrak{g})$. The numbers $e_1(\lambda), \ldots, e_s(\lambda)$ are called the generalised exponents related to the representation π_λ. Since $H(\mathfrak{g})$ is a self-dual $\mathfrak{g}-$module, the generalised exponents are the same for λ and λ^*. For the adjoint representation of a simple Lie algebra $\mathfrak{g}$, the generalised exponents coincide with the exponents of $\mathfrak{g}$.

We list of the exponents of simple Lie algebra $\mathfrak{g}$. This list will be used in the proof of Proposition 3.3.

$\mathfrak{g}$	exponents
$A_n(n \geq 1)$	$1, 2, \ldots, n$
$B_n(n \geq 2)$	$1, 3, 5, \ldots, 2n-1$
$C_n(n \geq 3)$	$1, 3, 5, \ldots, 2n-1$
$D_n(n \geq 4)$	$1, 3, 5, \ldots, 2n-3, n-1$
E_6	$1, 4, 5, 7.8.11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$

Recall that a principal $\mathfrak{sl}_2$ in $\mathfrak{g}$ is a three-dimensional subalgebra spanned by a triple $\{X, H, Y\}$ in $\mathfrak{g}$ such that

$$[H, X] = 2X, \ [H, Y] = -2Y, \ [X, Y] = H$$

and the orbit of X under the adjoint group of $\mathfrak{g}$ is the principal nilpotent orbit. It turns out that there is one conjugacy class of principal $\mathfrak{sl}_2$'s for which the semisimple element H is conjugate to

$$2\rho^\vee = \sum_{\alpha \in \phi^+} \alpha^\vee,$$

where Φ^+ is a fixed system of positive roots of $\mathfrak{g}$ and $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ is the dual root in $\mathfrak{h}$.

Let $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ be the set of simple roots. We choose the simple root vectors $X_1, \ldots, X_r$ and let $Y_1, \ldots, Y_r$ be the corresponding root vectors for the negatives of the simple roots normalized by the condition

$$[X_i, Y_i] = H_i := \alpha_i^\vee.$$

Since the difference of simple roots is never a root, we have $[X_i, Y_j] = 0$ for $i \neq j$. Set $c_i = \langle \omega_i, \rho^\vee \rangle$ where ω_i are the fundamental weights. Then $\rho^\vee = \sum c_i \alpha_i^\vee$,and

$$X = X_1 + \cdots + X_r,\ H = 2\rho^\vee = c_1 H_1 + \cdots + c_r H_r,\ Y = Y_1 + \cdots + Y_r$$

form a basis of a principal $\mathfrak{sl}_2$. The height of λ is defined by

$$\mathrm{ht}(\lambda) = \langle \lambda, \rho^\vee \rangle.$$

THEOREM 3.1. *The following statements are equivalent:*
(i) $1, \delta_\lambda(C), \ldots, \delta_\lambda(C)^{d_\lambda - 1}$ *form a basis of the* $Z(\mathfrak{g})$*-module* R_λ.
(ii) $1, M_\lambda(C), \ldots, M_\lambda(C)^{d_\lambda - 1}$ *form a basis of the* $Z(\mathfrak{g})$*-module* R_λ.
(iii) V_λ *is irreducible when restricted to a principal* $\mathfrak{sl}_2$ *in* $\mathfrak{g}$.

PROOF. Note that $\delta_\lambda(C) = |\lambda + \rho|^2 - |\rho|^2$ is a scalar. It follows from (3.1) that (i) and (ii) are equivalent. A necessary condition for R_λ having a basis generated by powers of one element is that R_λ is commutative. By Theorem 2.6, V_λ is multiplicity free. It follows that the rank of the free $Z(\mathfrak{g})$-module R_λ is $d_\lambda = \dim V_\lambda$. The condition (ii) is equivalent to that the generalised exponents of $\mathrm{End}\, V_\lambda$ are $0, 1, \ldots, d_\lambda - 1$. Note that the largest possible exponent of $\mathrm{End}\, V_\lambda$ is $2\mathrm{ht}(\lambda)$. Here $\mathrm{ht}(\lambda)$ is the height of λ, which is equal to the highest weight of the principal $\mathfrak{sl}_2$. Thus, condition (ii) is equivalent to that $2\mathrm{ht}(\lambda) = d_\lambda - 1$, which is equivalent to that V_λ is an irreducible module for the principal $\mathfrak{sl}_2$. □

We note that in Theorem 3.1 the set of integers $\{1, \ldots, d_\lambda\}$ that appeared as the powers of $\delta_\lambda(C)$ or $M_\lambda(C)$ is exactly the union of the sets of generalised exponents of all irreducible constituents V_{γ_i}'s in

$$\mathrm{End}\, V_\lambda \cong V_\lambda^* \otimes V_\lambda = \bigoplus V_{\gamma_i}.$$

Consequently, in Proposition 3.3 below the integers (i for type A_n and $2i - 1$ for others) appeared as the powers of $\delta_{\omega_1}(C)$ or $M_{\omega_1}(C)$ exactly the exponents for the corresponding simple Lie algebra $\mathfrak{g}$.

PROPOSITION 3.2. *Here is the list of simple* $\mathfrak{g}$*-modules that are irreducible when restricted to a principal* $\mathfrak{sl}_2$ *in* $\mathfrak{g}$.

$\mathfrak{g}$	λ *the highest weight*
A_n	ω_1, ω_n
A_1	$k\omega_1, k = 1, 2, \ldots$
B_n	ω_1
B_2	ω_2
C_n	ω_1
G_2	ω_1 $(dim = 7)$

PROOF. By Theorem 3.1, R_λ is commutative. It follows from Theorem 2.6 that V_λ is of simple $\mathfrak{h}$-spectrum. Then it is readily checked that the list in Remark 2.7 implies the conclusion. $\square$

Now we consider a special case when $\lambda = \omega_1$, the fundamental weight corresponding to the natural representation for a classical simple Lie algebra or the 7-dimensional irreducible representation for G_2. Denote by d the dimension of V_{ω_1}. Then

$$d = \begin{cases} n, & \mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}) \text{ or } \mathfrak{so}(n,\mathbb{C}) \\ 2n, & \mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C}) \\ 7, & \mathfrak{g} \text{ is of type } G_2. \end{cases}$$

By the natural embedding of

$$\mathfrak{g} \hookrightarrow \mathrm{Mat}_d(\mathbb{C}) \cong \mathrm{End} V_{\omega_1},$$

we have the embedding

$$A(\mathfrak{g}) = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, U(\mathfrak{g})) \hookrightarrow (\mathrm{Mat}_d(\mathbb{C}) \otimes U(\mathfrak{g}))^{\mathfrak{g}}.$$

As a consequence, we have the following proposition. Given a matrix M, let M^T denote the transpose of M.

PROPOSITION 3.3. *One has following $Z(\mathfrak{g})$-bases for $A(\mathfrak{g})$:*
A_n: $M_{\omega_1}(C)^i - \frac{tr M_{\omega_1}(C)^i}{n+1} I_{n+1}$ *with* $1 \le i \le n$;
B_n: $M_{\omega_1}(C)^{2i-1} - (M_{\omega_1}(C)^{2i-1})^T$ *with* $1 \le i \le n$;
C_n: $M_{\omega_1}(C)^{2i-1} - (M_{\omega_1}(C)^{2i-1})^T$ *with* $1 \le i \le n$;
G_2: $M_{\omega_1}(C)^{2i-1} - (M_{\omega_1}(C)^{2i-1})^T$ *with* $i = 1, 3$.

PROOF. This is verified case by case. First, we consider the case $\mathfrak{g}$ is of type A_n. Then we have $\mathrm{End}\, V_{\omega_1} \cong \mathfrak{g} \oplus \mathbb{C}$, and clearly a basis is given as above.

In the second case when $\mathfrak{g}$ is either of type B_n or of type C_n, we have the following decomposition into irreducible representations

$$\mathrm{End}\, V_{\omega_1} \cong \mathrm{Sym}^2 V_{\omega_1} \oplus \wedge^2 V_{\omega_1} \cong (\mathbb{C} \oplus V_{2\omega_1}) \oplus \mathfrak{g}.$$

Here the adjoint representation $\mathfrak{g}$ has highest weight ω_2 and is contained in $\wedge^2 V_{\omega_1}$. Then we readily verify by checking the exponents that the corresponding expressions of $M_{\omega_1}(C)$ with appropriate powers are in $(\mathfrak{g} \otimes U(\mathfrak{g}))^{\mathfrak{g}}$ inside $R_{\omega_1} = (\mathrm{End}\, V_{\omega_1} \otimes U(\mathfrak{g}))^{\mathfrak{g}}$.

In the last case when $\mathfrak{g}$ is of type G_2, we have

$$\mathrm{End}\, V_{\omega_1} \cong \mathrm{Sym}^2 V_{\omega_1} \oplus \wedge^2 V_{\omega_1} \cong (\mathbb{C} \oplus V_{2\omega_2}) \oplus (V_{\omega_1} \oplus \mathfrak{g}).$$

Here the adjoint representation $\mathfrak{g}$ has highest weight ω_2 and is contained in $\wedge^2 V_{\omega_1}$. Again we readily verify by checking the exponents that the corresponding expressions of $M_{\omega_1}(C)$ with appropriate powers are in $(\mathfrak{g} \otimes U(\mathfrak{g}))^{\mathfrak{g}}$ inside $R_{\omega_1} = (\mathrm{End}\, V_{\omega_1} \otimes U(\mathfrak{g}))^{\mathfrak{g}}$. $\square$

COROLLARY 3.4. *It follows from Theorem 3.1 that one gets the following $Z(\mathfrak{g})$-bases for $A(\mathfrak{g})$:*
A_n: $\delta_{\omega_1}(C)^i$ *with* $1 \leq i \leq n$;
B_n: $\delta_{\omega_1}(C)^{2i-1} - (\delta_{\omega_1}(C)^{2i-1})^T$ *with* $1 \leq i \leq n$;
C_n: $\delta_{\omega_1}(C)^{2i-1} - (\delta_{\omega_1}(C)^{2i-1})^T$ *with* $1 \leq i \leq n$;
G_2: $\delta_{\omega_1}(C)^{2i-1} - (\delta_{\omega_1}(C)^{2i-1})^T$ *with* $i = 1, 3$.

4. BASES FOR $R_{\lambda,\nu}$

Recall that the map

$$\delta_{\lambda,\nu} : U(\mathfrak{g}) \to \operatorname{End} V_\lambda \otimes \operatorname{End} V_\nu$$

is defined by

$$\delta_{\lambda,\nu}(x) = \pi_\lambda(x) \otimes 1 + 1 \otimes \pi_\nu(x) \text{ for } x \in \mathfrak{g},$$

which extends to a homomorphism of associative algebras. We set

$$M_{\lambda,\nu}(C) = \sum_{i=1}^{m} \pi_\lambda(x_i) \otimes \pi_\nu(x_i^*).$$

Then

$$\delta_{\lambda,\nu}(C) = \pi_\lambda(C) \otimes 1 + 2M_{\lambda,\mu}(C) + 1 \otimes \pi_\nu(C). \tag{4.1}$$

We also recall from Section 2 that $\triangle_\lambda = \{\mu_1, \cdots, \mu_k\}$ is the set of weights of V_λ and d_i the multiplicity of μ_i. If λ is totally subordinate to ν, then we have an isomorphism

$$R_{\lambda,\nu} \to \bigoplus_{i=1}^{k} \operatorname{Mat}_{d_i}(\mathbb{C}).$$

It follows from Theorem 2.6 that we have the following proposition.

PROPOSITION 4.1. *Assume that V_λ has simple $\mathfrak{h}$-spectrum and λ is totally subordinate to ν. Then the following statements are equivalent:*
(i) $1, \delta_{\lambda,\nu}(C), \ldots, \delta_{\lambda,\nu}(C)^{d_\lambda - 1}$ *form a basis of* $R_{\lambda,\nu}$.
(ii) $1, M_{\lambda,\nu(}C), \ldots, M_{\lambda,\nu}(C)^{d_\lambda - 1}$ *form a basis of* $R_{\lambda,\nu}$.
(iii) $\delta_{\lambda,\nu}(C)$ *acts on the distinct irreducible summands of* $V_\lambda \otimes V_\nu$ *with distinct eigenvalues.*

PROOF. It follows from (4.1) that (i) and (ii) are equivalent. Clearly, the equivalence (iii) and (i) is due to the expression of the determinant for the corresponding Vandermonde matrix. □

Now we deal with the minuscule representations V_λ. Recall that V_λ is said to be minuscule if $\langle \lambda, \alpha \rangle = 0, -1, 1$ for all roots α. Here is the list of the minuscule representations (cf. [3, Page 72, Exercise 13]).

$\mathfrak{g}$	λ the highest weight
$A_n (n \geq 1)$	$\omega_k, k = 1, \ldots, n$
$B_n (n \geq 2)$	ω_n (spin representation)
$C_n (n \geq 3)$	ω_1
$D_n (n \geq 4)$	ω_1 ω_{n-1}, ω_n (spin representations)
E_6	ω_1 (dim $= 27$) ω_6 (dim $= 27$)
E_7	ω_1 (dim $= 56$)

Proposition 4.2. *Suppose that λ is minuscule. Assume that $n(\nu) = r$ ($=$ rank $\mathfrak{g}$). Then λ is totally subordinate to ν. As a consequence of Proposition 4.1, one has the following equivalent statements:*

(i) $1, \delta_{\lambda,\nu}(C), \ldots, \delta_{\lambda,\nu}(C)^{d_\lambda - 1}$ *form a basis of $R_{\lambda,\nu}$.*

(ii) $1, M_{\lambda,\nu(}C), \ldots, M_{\lambda,\nu}(C)^{d_\lambda - 1}$ *form a basis of $R_{\lambda,\nu}$.*

(iii) $\delta_{\lambda,\nu}(C)$ *acts on the distinct irreducible summands of $V_\lambda \otimes V_\nu$ with distinct eigenvalues.*

Proof. Let α be a simple root of $\mathfrak{g}$ with respect to the fixed system of positive roots. Then $|\langle \lambda, \alpha \rangle| \leq 1$, since λ is minuscule, and $|\langle \mu, \alpha \rangle| \leq 1$ for all weights μ of V_λ. On the other hand, we have $\langle \nu, \alpha \rangle \geq 1$ due to $n(\nu) = r$. Thus, $V_\lambda \otimes V_\nu$ decomposes into d_λ (non-isomorphic) irreducible representations with highest weights $\nu + \mu_i$, where μ_i are the weights of V_λ. Therefore, λ is totally subordinate to ν. The rest of the conclusions follow from Proposition 4.1. $\square$

By a theorem of Kostant [8, Theorem D], the Clifford algebra $C(\mathfrak{g})$ with respect to the Killing form of $\mathfrak{g}$ decomposes into the tensor product

$$C(\mathfrak{g}) = J \otimes E,$$

where $J = C(\mathfrak{g})^{\mathfrak{g}}$ and $E = \operatorname{End} V_\rho$. We set the Clifford algebra analogue R_λ^C to be the invariant endomorphism algebra

$$R_\lambda^C : = \operatorname{Hom}_{\mathfrak{g}}(\operatorname{End} V_\lambda, C(\mathfrak{g})).$$

Then R_λ^C is a free J-module of rank equal to $\dim R_{\lambda,\rho}$. Note that

$$\rho = \omega_1 + \cdots + \omega_r.$$

The following theorem is an immediate consequence of Proposition 4.2.

Theorem 4.3. *Assume that λ is minuscule. Then R_λ^C is a free J-module of rank d_λ. Moreover, the following statement are equivalent:*

(i) $1, \delta_{\lambda,\rho}(C), \ldots, \delta_{\lambda,\rho}(C)^{d_\lambda - 1}$ *form a J-basis of R_λ^C.*

(ii) $1, M_{\lambda,\rho}(C), \ldots, M_{\lambda,\rho}(C)^{d_\lambda - 1}$ *form a J-basis of R_λ^C.*

(iii) $\delta_{\lambda,\rho}(C)$ *acts on the distinct irreducible summands of $V_\lambda \otimes V_\rho$ with distinct eigenvalues.*

In the remaining part of this section, we deal with the general situation for any λ, ν. Clearly, if $V_\lambda \otimes V_\nu$ decomposes into a direct sum of d non-isomorphic irreducible representations

$$V_\lambda \otimes V_\nu = \bigoplus_{i=1}^{d} V_{\gamma_i},$$

then $R_{\lambda,\nu}$ is a commutative $\mathbb{C}$-algebra and $\dim R_{\lambda,\nu} = d$.

THEOREM 4.4. *Let d be a positive integer. Then the following statements are equivalent:*

(i) $1, \delta_{\lambda,\nu}(C), \ldots, \delta_{\lambda,\nu}(C)^{d-1}$ *form a basis of* $R_{\lambda,\nu}$.

(ii) $1, M_{\lambda,\nu}(C), \ldots, M_{\lambda,\nu}(C)^{d-1}$ *form a basis of* $R_{\lambda,\nu}$.

(iii) $V_\lambda \otimes V_\nu = \bigoplus_{i=1}^{d} V_{\gamma_i}$ *decomposes into a direct sum of d non-isomorphic simple $\mathfrak{g}$-modules with distinct $\delta_{\lambda,\nu}(C)$-eigenvalues.*

PROOF. It follows from (4.1) that (i) and (ii) are equivalent. We now show that (i) and (iii) are equivalent. Either (i) or (iii) implies that $R_{\lambda,\nu}$ is commutative which is equivalent to $V_\lambda \otimes V_\nu$ decomposing into a direct sum of d distinct simple $\mathfrak{g}$-modules. Under the assumption that $V_\lambda \otimes V_\nu$ decomposes into a direct sum of d non-isomorphic simple $\mathfrak{g}$-modules

$$V_\lambda \otimes V_\nu = \bigoplus_{i=1}^{d} V_{\gamma_i},$$

we have $R_{\lambda,\nu}$ is commutative algebra with $\dim R_{\lambda,\nu} \leq d$. Thus, Condition (i) holds (namely $1, \delta_{\lambda,\nu}(C), \ldots, \delta_{\lambda,\nu}(C)^{d-1}$ form a basis of $R_{\lambda,\nu}$) if and only that $1, \delta_{\lambda,\nu}(C), \ldots, \delta_{\lambda,\nu}(C)^{d-1}$ are linear independent. This is in turn equivalent to that the determinant of the following Vandermonde matrix is nonzero:

$$\begin{pmatrix} 1 & (|\gamma_1+\rho|^2-|\rho|^2) & \cdots & (|\gamma_1+\rho|^2-|\rho|^2))^{d-1} \\ 1 & (|\gamma_2+\rho|^2-|\rho|^2) & \cdots & (|\gamma_2+\rho|^2-|\rho|^2)^{d-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (|\gamma_d+\rho|^2-|\rho|^2) & \cdots & (|\gamma_d+\rho|^2-|\rho|^2)^{d-1} \end{pmatrix},$$

which is equivalent to the condition that $\delta_{\lambda,\nu}(C)$-eigenvalues $|\gamma_2+\rho|^2-|\rho|^2$ on the irreducible constituents V_{γ_i} are distinct. $\square$

REMARK 4.5. Suppose that $V_\lambda \otimes V_\nu = \bigoplus_{i=1}^{d} V_{\gamma_i}$ decomposes into a direct sum of d non-isomorphic simple $\mathfrak{g}$-modules. Then the irreducible constituents V_{γ_i} have distinct infinitesimal characters $\chi_{\gamma_i+\rho}$. For almost all $u \in Z(\mathfrak{g})$, one has

$$\chi_{\gamma_i+\rho}(u) \neq \chi_{\gamma_j+\rho}(u), \text{ for } i \neq j. \tag{4.2}$$

Such $u \in Z(\mathfrak{g})$ satisfying the above Condition (4.2) are called generic with respect to λ and ν. It follows from Theorem 4.4 that $1, \delta_{\lambda,\nu}(u), \ldots, \delta_{\lambda,\nu}(u)^{d-1}$ form a basis of $R_{\lambda,\nu}$ provided $u \in Z(\mathfrak{g})$ is generic with respect to λ and ν.

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O bazama $\mathfrak{g}$-invarijantnih algebri endmorfizama

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Sažetak. Neka je $\mathfrak{g}$ kompleksna prosta Liejeva algebra. Neka je $Z(\mathfrak{g})$ centar univerzalne omotačke algebre $U(\mathfrak{g})$. Neka je V_λ konačno–dimenzionalni ireducibilan $\mathfrak{g}$-modul najveće visine λ. Glavni rezultat ovog rada je kriterij postojanja za $Z(\mathfrak{g})$–baze $\mathfrak{g}$-invarijantnih algebri endmorfizama $R_\lambda =: \mathrm{Hom}_{\mathfrak{g}}(\mathrm{End}\, V_\lambda, U(\mathfrak{g}))$. Nadalje, dokazujemo Clifford algebra analog tj. kriterij egzistencije $C(\mathfrak{g})^{\mathfrak{g}}$–baze za $R_\lambda^C =: \mathrm{Hom}_{\mathfrak{g}}(\mathrm{End}\, V_\lambda, C(\mathfrak{g}))$. Osim toga, opisujemo kriterij egzistencije baza generiranih potencijama Casimirovog elementa za $R_{\lambda,\nu} =: \mathrm{Hom}_{\mathfrak{g}}(\mathrm{End}\, V_\lambda, \mathrm{End}\, V_\nu)$.

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