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WEIRD K-ACTIONS ON $\mathcal{U}(\mathfrak{g})$ FOR $\mathfrak{so}(n,1)$ AND $\mathfrak{su}(n,1)$

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For Marko Tadić, with our admiration and appreciation

Abstract. Let $\mathfrak{g}_0$ be either $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$, $\mathfrak{g}$ its complexification, K a maximal compact subgroup of the adjoint group of $\mathfrak{g}_0$, $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})^K$ its subalgebra of K-invariants. A consequence of our results in [2] is that besides the usual adjoint action of K on $\mathcal{U}(\mathfrak{g})$ there is another action of K commuting with the adjoint action and leaving $\mathcal{U}(\mathfrak{g})^K$ pointwise invariant. The case $\mathfrak{g}_0 = \mathfrak{so}(2,1) \simeq \mathfrak{su}(1,1)$ is trivial since K is commutative and the weird action of K coincides with the inverse of adjoint action. We investigate closely the weird action of K in the simplest nontrivial case $\mathfrak{g}_0 = \mathfrak{so}(3,1)$.

1. Notation

Our notation is usual: $\mathbb{C}$ are complex numbers, $\mathbb{R}$ real numbers, $\mathbb{Z}$ integers, $\mathbb{Z}_+$ nonnegative integers, $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$, $M_{n,m}(K)$ the space of $n \times m$ matrices with entries from a field K, $M_n(K) = M_{n,n}(K)$, $A_{j,k}$ is the (j,k)-entry of a matrix A, I_n is the unit $n \times n$ matrix, A^t denotes the transpose of a matrix A, A^* is the adjoint (= transpose and complex conjugate) of a matrix $A \in M_n(\mathbb{C})$. Furthermore, $\mathfrak{gl}(n,K)$ is the Lie algebra $M_n(K)$ with commutator $[A,B] = AB - BA$, $\mathfrak{sl}(n,K) = \{A \in \mathfrak{gl}(n,K);\ \mathrm{Tr}\,A = 0\}$, $\mathfrak{so}(n,1) = \{A \in M_{n+1}(\mathbb{R});\ A^t = -\Gamma A \Gamma\}$ with $\Gamma = \mathrm{diag}(1,\ldots,1,-1)$, $\mathfrak{so}(n) = \{B \in M_n(\mathbb{R});\ B^t = -B\}$, $\mathfrak{su}(n,1) = \{A \in \mathfrak{sl}(n+1,\mathbb{C});\ A^* = -\Gamma A \Gamma\}$, and $\mathfrak{u}(n) = \{B \in M_n(\mathbb{C});\ B^* = -B\}$; the complexifications of the real Lie algebras $\mathfrak{so}(n,1)$, $\mathfrak{so}(n)$, $\mathfrak{su}(n,1)$ and $\mathfrak{u}(n)$ are $\mathfrak{so}(n,1,\mathbb{C}) = \{A \in M_{n+1}(\mathbb{C});\ A^t = -\Gamma A \Gamma\}$, $\mathfrak{so}(n,\mathbb{C}) = \{B \in M_n(\mathbb{C});\ B^t = -B\}$, $\mathfrak{sl}(n+1,\mathbb{C})$ and $\mathfrak{gl}(n,\mathbb{C})$. Furthermore, $\mathrm{GL}(n,K)$ denotes the group of invertible matrices in $M_n(K)$ and $\mathrm{SL}(n,K) = \{A \in \mathrm{GL}(n,K); \det A = 1\}$. The matrix Lie groups of the introduced real Lie algebras are $\mathrm{SO}(n,1) = \{A \in \mathrm{SL}(n+1,\mathbb{R});\ A^t \Gamma A = \Gamma\}$ with the identity component $\mathrm{SO}_e(n,1) = \{A \in \mathrm{SO}(n,1);\ A_{n+1,n+1} \geq 1\}$, $\mathrm{SO}(n) =$

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$\{A \in \mathrm{SL}(n,\mathbb{R});\ A^tA = I_n\}$, $\mathrm{SU}(n,1) = \{A \in \mathrm{SL}(n+1,\mathbb{C});\ A^*\Gamma A = \Gamma\}$ and $\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C});\ A^*A = I_n\}$. We have

$$\mathfrak{so}(n,1) = \left\{ \begin{bmatrix} B & a \\ a^t & 0 \end{bmatrix};\ B \in \mathfrak{so}(n),\ a \in M_{n,1}(\mathbb{R}) \right\},$$

and

$$\mathfrak{su}(n,1) = \left\{ \begin{bmatrix} B & a \\ a^* & -\operatorname{Tr} B \end{bmatrix};\ B \in \mathfrak{u}(n),\ a \in M_{n,1}(\mathbb{C}) \right\}.$$

2. Preliminaries

Let $\mathfrak{g}_0$ be either $\mathfrak{so}(n,1)$ $(n \geq 2)$ or $\mathfrak{su}(n,1)$, $\mathfrak{g}$ its complexification, G the adjoint group of $\mathfrak{g}_0$, K its maximal compact subgroup, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ the corresponding Cartan decomposition and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its complexification.

In the case $\mathfrak{g}_0 = \mathfrak{so}(n,1)$ the adjoint group G can be identified with the group $\mathrm{SO}_e(n,1)$ and the adjoint action of $A \in \mathrm{SO}_e(n,1)$ on $\mathfrak{g}$ is given by $A.X = (\operatorname{Ad} A)X = AXA^{-1}$, $X \in \mathfrak{g}$. In this case we choose the maximal compact subgroup $K = \left\{ \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix};\ B \in \mathrm{SO}(n) \right\} \simeq \mathrm{SO}(n)$. Then

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(\mathbb{R});\ B \in \mathfrak{so}(n) \right\} \simeq \mathfrak{so}(n),$$

$$\mathfrak{p}_0 = \left\{ \begin{bmatrix} 0 & a \\ a^t & 0 \end{bmatrix};\ a \in M_{n,1}(\mathbb{R}) \right\},$$

and

$$\mathfrak{k} = \left\{ \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \in M_{n+1}(\mathbb{C});\ B \in \mathfrak{so}(n,\mathbb{C}) \right\} \simeq \mathfrak{so}(n,\mathbb{C}),$$

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & a \\ a^t & 0 \end{bmatrix};\ a \in M_{n,1}(\mathbb{C}) \right\}.$$

In the case $\mathfrak{g}_0 = \mathfrak{su}(n,1)$ we have $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$. We choose the Cartan decomposition of $\mathfrak{g}_0$

$$\mathfrak{k}_0 = \left\{ \begin{bmatrix} B & 0 \\ 0 & -\operatorname{Tr} B \end{bmatrix} \in M_{n+1}(\mathbb{C});\ B \in \mathfrak{u}(n) \right\} \simeq \mathfrak{u}(n),$$

and

$$\mathfrak{p}_0 = \left\{ \begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix};\ a \in M_{n,1}(\mathbb{C}) \right\}.$$

Then

$$\mathfrak{k} = \left\{ \begin{bmatrix} B & 0 \\ 0 & -\operatorname{Tr} B \end{bmatrix} \in M_{n+1}(\mathbb{C});\ B \in M_n(\mathbb{C}) \right\} \simeq \mathfrak{gl}(n,\mathbb{C}),$$

and

$$\mathfrak{p} = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix};\ a \in M_{n,1}(\mathbb{C}),\ b \in M_{1,n}(\mathbb{C}) \right\}.$$

Now G is identified with $\mathrm{SU}(n,1)/Z$, where Z is the center of $\mathrm{SU}(n,1)$:

$$Z = \{\alpha I_{n+1};\ \alpha \in \mathbb{C},\ \alpha^{n+1} = 1\} \simeq \mathbb{Z}_{n+1} := \mathbb{Z}/(n+1)\mathbb{Z}.$$

Then $K = \tilde{K}/Z$, where $\tilde{K}$ is a maximal compact subgroup of $\mathrm{SU}(n,1)$

$$\tilde{K} = \left\{ \begin{bmatrix} B & 0 \\ 0 & (\det B)^{-1} \end{bmatrix};\ B \in \mathrm{U}(n) \right\} \simeq \mathrm{U}(n).$$

Denote by $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k}) \subseteq \mathcal{U}(\mathfrak{g})$ the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{k}$. Furthermore, let $S(\mathfrak{g})$ and $S(\mathfrak{k}) \subseteq S(\mathfrak{g})$ be the symmetric algebras over $\mathfrak{g}$ and $\mathfrak{k}$; using the invariant non-degenerate trace bilinear form $(A,B) \mapsto \mathrm{Tr}\, AB$ one identifies $\mathfrak{g}$ and $\mathfrak{k}$ with its dual spaces $\mathfrak{g}^*$ and $\mathfrak{k}^*$, thus the symmetric algebras $S(\mathfrak{g})$ and $S(\mathfrak{k})$ with the polynomial algebras $\mathcal{P}(\mathfrak{g}) = S(\mathfrak{g}^*)$ and $\mathcal{P}(\mathfrak{k}) = S(\mathfrak{k}^*)$. The group G (and its subgroup K) acts by automorphisms on the algebras $\mathcal{U}(\mathfrak{g})$ and $\mathcal{P}(\mathfrak{g})$ and K acts by automorphisms on $\mathcal{U}(\mathfrak{k})$ and $\mathcal{P}(\mathfrak{k})$. Denote by $\mathcal{U}(\mathfrak{g})^G$, $\mathcal{P}(\mathfrak{g})^G$, $\mathcal{U}(\mathfrak{g})^K$, $\mathcal{P}(\mathfrak{g})^K$, $\mathcal{U}(\mathfrak{k})^K$ and $\mathcal{P}(\mathfrak{k})^K$ the subalgebras of invariants. Then $\mathcal{U}(\mathfrak{g})^G$ is the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{k})^K$ is the center $\mathcal{Z}(\mathfrak{k})$ of $\mathcal{U}(\mathfrak{k})$. Knop has proved in [1] the following theorem.

Theorem 2.1. *The multiplication induces an isomorphism of* $\mathcal{Z}(\mathfrak{g}) \otimes \mathcal{Z}(\mathfrak{k})$ *onto the algebra* $\mathcal{U}(\mathfrak{g})^K$ *and an isomorphism of* $\mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K$ *onto* $\mathcal{P}(\mathfrak{g})^K$.

Denote by $\mathcal{N}_K$ the set of all nilpotent elements of $\mathfrak{g}$ whose projection onto $\mathfrak{k}$ along $\mathfrak{p}$ is nilpotent in the reductive Lie algebra $\mathfrak{k}$, and let $\mathcal{H}$ be the subspace of $\mathcal{U}(\mathfrak{g})$ spanned by all powers (in $\mathcal{U}(\mathfrak{g})$) A^k, $A \in \mathcal{N}_K$, $k \in \mathbb{Z}_+$. The subspace $\mathcal{H}$ of $\mathcal{U}(\mathfrak{g})$ is invariant under the action of K. We have proved in [2] the following theorem.

Theorem 2.2. *Under above assumptions*

(i) *The multiplication induces an isomorphism of* $(\mathcal{U}(\mathfrak{g})^K, K)$*-modules* $\mathcal{U}(\mathfrak{g})^K \otimes \mathcal{H}$ *onto* $\mathcal{U}(\mathfrak{g})$.

(ii) *Let* $\hat{K}$ *be the set of equivalence classes of irreducible (finite-dimensional) representations of* K. *The multiplicity of any* $\delta \in \hat{K}$ *in the* K*-module* $\mathcal{H}$ *is equal to its degree* $d(\delta)$.

3. Weird action of K on $\mathcal{U}(\mathfrak{g})$

We recall briefly the proof of *(ii)* which leads to a weird action of K on $\mathcal{U}(\mathfrak{g})$. The inverse of the symmetrization $\mathcal{U}(\mathfrak{g}) \to S(\mathfrak{g}) = \mathcal{P}(\mathfrak{g})$ maps the K-submodule $\mathcal{H}$ onto the space $\mathcal{H}_K(\mathfrak{g})$ of K-harmonic polynomials on $\mathfrak{g}$:

$$\mathcal{H}_K(\mathfrak{g}) = \{f \in \mathcal{P}(\mathfrak{g});\ \partial(u)f = 0\ \forall u \in S_+(\mathfrak{g})^K\}.$$

Here $\partial : S(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})$ is the usual isomorphism of the symmetric algebra $S(\mathfrak{g})$ onto the algebra $\mathcal{D}(\mathfrak{g})$ of linear differential operators on $\mathcal{P}(\mathfrak{g})$ with constant coefficients: $\partial(X)$ is the derivation in the direction X for any $X \in \mathfrak{g}$. Furthermore, we denote by $S_+(\mathfrak{g})^K$ and $\mathcal{P}_+(\mathfrak{g})^K$ the maximal ideals (of codimension 1) of the algebras of K-invariants $S(\mathfrak{g})^K$ and $\mathcal{P}(\mathfrak{g})^K$:

$$S_+(\mathfrak{g})^K = \bigoplus_{k>0} S^k(\mathfrak{g})^K, \qquad \mathcal{P}_+(\mathfrak{g})^K = \bigoplus_{k>0} \mathcal{P}^k(\mathfrak{g})^K = \{P \in \mathcal{P}(\mathfrak{g})^k;\ P(0) = 0\}.$$

Then the set $\mathcal{N}_K$ of K-nilpotent elements of $\mathfrak{g}$ is the zero set of the ideal $\mathcal{P}(\mathfrak{g})\mathcal{P}_+(\mathfrak{g})^K$ generated by $\mathcal{P}_+(\mathfrak{g})^K$ in $\mathcal{P}(\mathfrak{g})$, i.e.

$$\mathcal{N}_K = \{X \in \mathfrak{g};\ P(X) = 0\ \forall P \in \mathcal{P}_+(\mathfrak{g})^K\}.$$

Now, by the Knop's theorem $\mathcal{P}(\mathfrak{g})^K \simeq \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{P}(\mathfrak{k})^K$. By Harish-Chandra isomorphism and by Chevalley's theorem on Weyl group invariants we know that the algebra $\mathcal{P}(\mathfrak{g})^G$ is generated by $\ell = \operatorname{rank}\mathfrak{g}$ homogeneous algebraically independent G-invariant polynomials $f_1, \ldots, f_\ell$ and the algebra $\mathcal{P}(\mathfrak{k})^K$ is generated by $k = \operatorname{rank}\mathfrak{k}$ homogeneous algebraically independent K-invariant polynomials $\varphi_1, \ldots, \varphi_k$. Thus, $\mathcal{P}(\mathfrak{g})^K$ is generated by $\ell + k$ homogeneous algebraically independent polynomials $f_1, \ldots, f_\ell, \varphi_1, \ldots, \varphi_k$ and so

$$\mathcal{N}_K = \{X \in \mathfrak{g};\ f_1(X) = \cdots = f_\ell(X) = \varphi_1(X) = \cdots = \varphi_k(X) = 0\}$$

is a Zariski closed subset of $\mathfrak{g}$ of dimension $\dim\mathfrak{g} - \ell - k$. More generally, for any $(\xi, \eta) = (\xi_1, \ldots, \xi_\ell, \eta_1, \ldots, \eta_k)$ in $\mathbb{C}^{\ell+k}$ we define a $K^{\mathbb{C}}$-stable Zariski closed set of the same dimension ($K^{\mathbb{C}}$ being the complexification of the group K)

$$\mathcal{N}_K(\xi, \eta) = \{X \in \mathfrak{g};\ f_j(X) = \xi_j,\ j = 1, \ldots, \ell,\ \varphi_i(X) = \eta_i,\ i = 1, \ldots, k\}.$$

For the Lie algebras $\mathfrak{so}(n,1)$ and $\mathfrak{su}(n,1)$ one finds that $\dim \mathcal{N}_K(\xi, \eta) = \dim K^{\mathbb{C}}$. We saw in [2] that for every $(\xi, \eta) \in \mathbb{C}^{\ell+k}$ the restriction of polynomials to $\mathcal{N}_K(\xi, \eta)$ induces an isomorphism of K-modules

$$\mathcal{H}_K(\mathfrak{g}) \approx \mathcal{P}(\mathcal{N}_K(\xi, \eta)) = \mathcal{R}(\mathcal{N}_K(\xi, \eta)).$$

Here $\mathcal{P}(S) = \{f|S;\ f \in \mathcal{P}(\mathfrak{g})\}$ for any subset $S \subseteq \mathfrak{g}$ and $\mathcal{R}(T)$ denotes the algebra of regular functions on an algebraic variety T. In [2] we have proved that there exists $X_0 \in \mathfrak{g}_0$ such that its stabilizer $K^{\mathbb{C}}_{X_0}$ in $K^{\mathbb{C}}$ is trivial. Then the dimension of the $K^{\mathbb{C}}$-orbit $\mathcal{O}_{X_0} = K^{\mathbb{C}}.X_0$ equals $\dim K^{\mathbb{C}}$. For $(\xi, \eta) = (f_1(X_0), \ldots, f_\ell(X_0), \varphi_1(X_0), \ldots, \varphi_k(X_0))$ we have $\mathcal{O}_{X_0} \subseteq \mathcal{N}_K(\xi, \eta)$ and the equality of dimensions implies that $\mathcal{O}_{X_0}$ is Zariski open in $\mathcal{N}_K(\xi, \eta)$, Thus the restriction to $\mathcal{O}_{X_0}$ is an isomorphism of $\mathcal{P}(\mathcal{N}_K(\xi, \eta)) = \mathcal{R}(\mathcal{N}_K(\xi, \eta))$ onto $\mathcal{P}(\mathcal{O}_{X_0})$. Using Peter-Weyl and Stone-Weierstrass theorems we have proved in [2] that in fact $\mathcal{P}(\mathcal{O}_{X_0}) = \mathcal{R}(\mathcal{O}_{X_0}) \approx \mathcal{R}(K^{\mathbb{C}})$.

Thus, as a K-module, $\mathcal{H} \approx \mathcal{H}_K(\mathfrak{g})$ is isomorphic to the left regular representation of K on $\mathcal{R}(K^{\mathbb{C}})$. Now $\mathcal{R}(K^{\mathbb{C}})$ carries also the right regular representation of K commuting with the left one. By the isomorphism $\mathcal{R}(K^{\mathbb{C}}) \approx \mathcal{H}$ we transfer this action of K to $\mathcal{H}$ and expand it to $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^K \otimes \mathcal{H}$ by leaving $\mathcal{U}(\mathfrak{g})^K$ pointwise invariant. The obtained representation of K on $\mathcal{U}(\mathfrak{g})$

we call **weird action** of K on $\mathcal{U}(\mathfrak{g})$. In the simplest case $\mathfrak{so}(2,1) \approx \mathfrak{su}(1,1)$ the compact group K is commutative and thus the weird action coincides with the adjoint action composed with the inverse map $x \mapsto x^{-1}$ in K.

In the cases $\mathfrak{so}(n,1)$, $n \geq 3$, and $\mathfrak{su}(n,1)$, $n \geq 2$, when K is not commutative, the weird action is not unique: it depends on the choice of $X_0 \in \mathfrak{g}_0$ such that its stabilizer $K^{\mathbb{C}}_{X_0}$ in $K^{\mathbb{C}}$ is trivial. Furthermore, in general the operators of the weird action are not automorphims of the algebra $\mathcal{U}(\mathfrak{g})$. One gets automorphisms if the weird action is trivially extended to the localization $\mathcal{U}(\mathfrak{g})_{\mathcal{U}(\mathfrak{g})^K\setminus\{0\}}$ and if we consider this localization as an algebra over the field of fractions $\mathcal{U}(\mathfrak{g})^K_{\mathcal{U}(\mathfrak{g})^K\setminus\{0\}}$ of the integral domain $\mathcal{U}(\mathfrak{g})^K$.

4. Weird action for $\mathfrak{g}_0 = \mathfrak{so}(3,1)$

We will compute the weird action in the simplest nontrivial case $\mathfrak{g}_0 = \mathfrak{so}(3,1)$. Computation will be on $\mathcal{P}(\mathfrak{g})$ instead of $\mathcal{U}(\mathfrak{g})$; one passes to $\mathcal{U}(\mathfrak{g})$ by symmetrization $\mathcal{P}(\mathfrak{g}) = S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$.

We choose a basis of $\mathfrak{g} = \mathfrak{so}(3,1,\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ as follows:

$$H = \begin{bmatrix} 0 & 2i & 0 & 0 \\ -2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ E = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & i & 0 \\ 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix},\ X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ -1 & -i & 0 & 0 \end{bmatrix}.$$

Then $\{H,E,F\}$ is a basis of $\mathfrak{k}$ and $\{Z,X,Y\}$ is a basis of $\mathfrak{p}$. The commutators are

$$\begin{array}{lllll} [H,E] = 2E, & [H,X] = 2X, & [E,X] = 0, & [F,X] = Z, & [Z,X] = -2E, \\ [H,F] = -2F, & [H,Z] = 0, & [E,Z] = 2X, & [F,Z] = 2Y, & [Z,Y] = -2F, \\ [E,F] = H, & [H,Y] = -2Y, & [E,Y] = Z, & [F,Y] = 0, & [X,Y] = -H. \end{array}$$

The algebra of G-invariants $S(\mathfrak{g})^G$ is generated by two algebraically independent homogeneous elements $D_1, D_2 \in S^2(\mathfrak{g})$ chosen as multiples of two Casimir elements corresponding to two simple factors $\mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$:

$$\begin{aligned} D_1 &= \tfrac{1}{4}H^2 + \tfrac{1}{4}Z^2 + \tfrac{1}{2}HZ + EF + EY - FX - XY, \\ D_2 &= \tfrac{1}{4}H^2 + \tfrac{1}{4}Z^2 - \tfrac{1}{2}HZ + EF - EY + FX - XY. \end{aligned}$$

The algebra of K-invariants $S(\mathfrak{k})^K$ is generated by a multiple $\Omega = H^2 + 4EF$ of the Casimir element in $S(\mathfrak{k})$. Instead of generators $\Omega, D_1, D_2 \in S^2(\mathfrak{g})$ of the algebra $S(\mathfrak{g})^K = S(\mathfrak{k})^K \otimes S(\mathfrak{g})^G$ we use $\Omega, \Delta, \Sigma \in S^2(\mathfrak{g})$, where

$$\begin{aligned} \Omega &= H^2 + 4EF, \\ \Delta &= Z^2 - 4XY = 2D_1 + 2D_2 - \Omega, \\ \Sigma &= HZ + 2EY - 2FX = D_1 - D_2. \end{aligned}$$

Thus, generators of the algebra $\mathcal{D}(\mathfrak{g})^K$ of K-invariant linear differential operators on $\mathcal{P}(\mathfrak{g})$ with constant coefficients are

$$\begin{aligned}\partial(\Omega) &= \tfrac{\partial^2}{\partial h^2} + 4\tfrac{\partial^2}{\partial e \partial f},\\ \partial(\Delta) &= \tfrac{\partial^2}{\partial z^2} - 4\tfrac{\partial^2}{\partial x \partial y},\\ \partial(\Sigma) &= \tfrac{\partial^2}{\partial h \partial z} + 2\tfrac{\partial^2}{\partial e \partial y} - 2\tfrac{\partial^2}{\partial f \partial x}.\end{aligned}$$

Here we have identified $\mathcal{P}(\mathfrak{g})$ with $\mathbb{C}[h,e,f,z,x,y]$, where $\{h,e,f,z,x,y\}$ is the basis of the dual space $\mathfrak{g}^*$ which is dual with respect to the chosen basis $\{H,E,F,Z,X,Y\}$ of $\mathfrak{g}$.

The adjoint representation of $\mathfrak{k}$ on $\mathfrak{g}$ extends to representation by derivations of the symmetric algebra $S(\mathfrak{g})$. Denote by π the represention of $\mathfrak{k}$ on $\mathcal{P}(\mathfrak{g})$ obtained by identification $\mathcal{P}(\mathfrak{g}) = S(\mathfrak{g})$ via the nondegenerate trace form $(A,B) \mapsto \operatorname{Tr} AB$ on $\mathfrak{g} = \mathfrak{so}(3,1,\mathbb{C})$. The operators of the representation π on $\mathcal{P}(\mathfrak{g})$ can be expressed as linear differential operators of first order:

$$\begin{aligned}\pi(H) &= -2e\tfrac{\partial}{\partial e} + 2f\tfrac{\partial}{\partial f} - 2x\tfrac{\partial}{\partial x} + 2y\tfrac{\partial}{\partial y},\\ \pi(E) &= -f\tfrac{\partial}{\partial h} + 2h\tfrac{\partial}{\partial e} - y\tfrac{\partial}{\partial z} - 2z\tfrac{\partial}{\partial x},\\ \pi(F) &= e\tfrac{\partial}{\partial h} - 2h\tfrac{\partial}{\partial f} - x\tfrac{\partial}{\partial z} - 2z\tfrac{\partial}{\partial y}.\end{aligned}$$

Let us now determine the K-harmonic polynomials on $\mathfrak{g}$:

$$\mathcal{H}_K(\mathfrak{g}) = \{P \in \mathcal{P}(\mathfrak{g});\ \partial(\Omega)P = \partial(\Delta)P = \partial(\Sigma)P = 0\} = \bigoplus_{n \in \mathbb{Z}_+} \mathcal{H}_K^n(\mathfrak{g}),$$

where $\mathcal{H}_K^n(\mathfrak{g}) = \mathcal{H}_K(\mathfrak{g}) \cap \mathcal{P}^n(\mathfrak{g})$.

By our results in [2] we have $\mathcal{P}(\mathfrak{g}) \approx \mathcal{P}(\mathfrak{g})^K \otimes \mathcal{H}_K(\mathfrak{g})$, thus

$$\dim\, \mathcal{P}^n(\mathfrak{g}) = \sum_{k=0}^{n} (\dim\, \mathcal{P}^k(\mathfrak{g})^K)(\dim\, \mathcal{H}_K^{n-k}(\mathfrak{g})).$$

Since $\mathfrak{g}$ is 6-dimensional, we have

$$\dim\, \mathcal{P}^n(\mathfrak{g}) = \dim\, S^n(\mathfrak{g}) = \binom{n+5}{5}.$$

Furthermore, we know that the subalgebra of K-invariants $S(\mathfrak{g})^K \approx \mathcal{P}(\mathfrak{g})^K$ is generated by three algebraically independent homogeneous elements $\Omega, \Delta, \Sigma \in S^2(\mathfrak{g})$. Thus, the dimensions of homogeneous spaces of K-invariants are

$$\dim\, \mathcal{P}^n(\mathfrak{g})^K = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{2}(k+1)(k+2) & n = 2k. \end{cases}$$

By induction on $n \in \mathbb{Z}_+$ one gets from these formulas:

PROPOSITION 4.1. *The dimensions of homogeneous spaces of K-harmonic polynomials on $\mathfrak{g}$ are*

$$\dim \mathcal{H}_K^n(\mathfrak{g}) = \begin{cases} 1 & n = 0 \\ 4n^2 + 2 & n \geq 1. \end{cases}$$

LEMMA 4.2. *For any $n \in \mathbb{Z}_+$ define $2n$ linearly independent homogeneous polynomials of degree n :*

$$A_j^n = f^{n-j} y^j, \quad 0 \leq j \leq n, \qquad B_j^{n-1} = f^{n-j-1} y^{j-1}(hy - fz), \quad 1 \leq j \leq n-1.$$

Then all these polynomials are in $\mathcal{H}_K^n(\mathfrak{g})$ and

$$\begin{array}{ll} \pi(H)A_j^n = 2nA_j^n, & \pi(E)A_j^n = 0,\ 0 \leq j \leq n, \\ \pi(H)B_j^{n-1} = (2n-2)B_j^{n-1}, & \pi(E)B_j^{n-1} = 0,\ 1 \leq j \leq n-1. \end{array}$$

The proof is by direct calculations with differential operators $\partial(\Omega)$, $\partial(\Delta)$, $\partial(\Sigma)$, $\pi(H)$ and $\pi(E)$.

Now, from the representation theory of $\mathfrak{k} \simeq \mathfrak{sl}(2,\mathbb{C})$ we see that A_j^n are highest weight vectors of $(2n+1)$-dimensional irreducible subrepresentations of π and bases of the corresponding invariant subspaces are $\{\pi(F)^k A_j^n;\ 0 \leq k \leq 2n\}$, $0 \leq j \leq n$. Furthermore, B_{j-1}^{n-1} are highest weight vectors of $(2n-1)$-dimensional irreducible subrepresentions of π and bases of the corresponding invariant subspaces are $\{\pi(F)^k B_{j-1}^{n-1};\ 0 \leq k \leq 2n-2\}$, $1 \leq j \leq n-1$. Since the homogeneous subspaces $\mathcal{H}_K^n(\mathfrak{g})$ are invariant under the representation π we conclude that all these subspaces are contained in $\mathcal{H}_K^n(\mathfrak{g})$. The sum of their dimensions (for $n \geq 1$) is

$$(n+1)(2n+1) + (n-1)(2n-1) = 4n^2 + 2 = \dim \mathcal{H}_K^n(\mathfrak{g}).$$

Thus, if we denote by π_n the equivalence class of $(2n+1)$-dimensional irreducible representations of K, we conclude:

PROPOSITION 4.3. *In the representation of K on $\mathcal{H}_K^n(\mathfrak{g})$ the multiplicity of the class π_n is $n+1$ and the multiplicity of the class π_{n-1} is $n-1$. Other classes do not appear in $\mathcal{H}_K^n(\mathfrak{g})$.*

Note that we have reproved (ii) of Theorem 2.2 in the case $\mathfrak{g}_0 = \mathfrak{so}(3,1)$: the multiplicity of π_n is $n+1$ in $\mathcal{H}_K^n(\mathfrak{g})$ and n in $\mathcal{H}_K^{n+1}(\mathfrak{g})$, so all together $2n+1 = d(\pi_n)$ in $\mathcal{H}_K(\mathfrak{g}) \approx \mathcal{H}$.

Now we calculate weird action ω of $\mathfrak{k}$ on $\mathcal{P}(\mathfrak{g})$. We choose the following $X_0 \in \mathfrak{g}_0 = \mathfrak{so}(3,1)$ whose stabilizer $K_{X_0}^{\mathbb{C}}$ in $K^{\mathbb{C}}$ is trivial:

$$X_0 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The group $K^{\mathbb{C}}$ consists of all complex matrices of the form

$$k = \begin{bmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

such that $kk^t = k^t k = I_4$ and $\det k = 1$. This means that

$$\begin{array}{ll} a_1^2 + a_2^2 + a_3^2 = 1, & a_1^2 + b_1^2 + c_1^2 = 1, \\ a_1b_1 + a_2b_2 + a_3b_3 = 0, & a_1a_2 + b_1b_2 + c_1c_2 = 0, \\ b_1^2 + b_2^2 + b_3^2 = 1, & a_2^2 + b_2^2 + c_2^2 = 1, \\ a_1c_1 + a_2c_2 + a_3c_3 = 0, & a_1a_3 + b_1b_3 + c_1c_3 = 0, \\ c_1^2 + c_2^2 + c_3^2 = 1, & a_3^2 + b_3^2 + c_3^2 = 1, \\ b_1c_1 + b_2c_2 + b_3c_3 = 0, & a_2a_3 + b_2b_3 + c_2c_3 = 0, \end{array}$$

and

$$\begin{array}{lll} a_1b_2 - a_2b_1 = c_3, & a_1c_2 - a_2c_1 = -b_3, & b_1c_2 - b_2c_1 = a_3, \\ a_1b_3 - a_3b_1 = -c_2, & a_1c_3 - a_3c_1 = b_2, & b_1c_3 - b_3c_1 = -a_2, \\ a_2b_3 - a_3b_2 = c_1, & a_2c_3 - a_3c_2 = -b_1, & b_2c_3 - b_3c_2 = a_1. \end{array}$$

Thus we find

$$(\mathrm{Ad}\, k)X_0 = kX_0k^{-1} = \begin{bmatrix} 0 & c_3 & -b_3 & a_1 \\ -c_3 & 0 & a_3 & b_1 \\ b_3 & -a_3 & 0 & c_1 \\ a_1 & b_1 & c_1 & 0 \end{bmatrix}.$$

We consider the restrictions of polynomials on $\mathfrak{g}$ to the $K^{\mathbb{C}}$-orbit of the element X_0. We get

$$\begin{array}{ll} h((\mathrm{Ad}\, k)X_0) = -\frac{i}{2}c_3, & e((\mathrm{Ad}\, k)X_0) = -\frac{i}{2}(a_3 + ib_3), \\ f((\mathrm{Ad}\, k)X_0) = -\frac{i}{2}(a_3 - ib_3), & z((\mathrm{Ad}\, k)X_0) = -\frac{1}{2}c_1, \\ x((\mathrm{Ad}\, k)X_0) = \frac{1}{2}(a_1 + ib_1), & y((\mathrm{Ad}\, k)X_0) = -\frac{1}{2}(a_1 - ib_1). \end{array}$$

For $C = hy - fz \in \mathcal{P}^2(\mathfrak{g})$ we get

$$C((\mathrm{Ad}\, k)X_0) = -\tfrac{1}{4}(a_2 - ib_2).$$

The restriction to the $K^{\mathbb{C}}$-orbit $K^{\mathbb{C}}.X_0$ is an isomorphism of the space $\mathcal{H}_K(\mathfrak{g})$ of K-harmonic polynomials onto the space $\mathcal{R}(K^{\mathbb{C}}.X_0)$ of regular functions on $K^{\mathbb{C}}.X_0$. As the stabilizer of X_0 in $K^{\mathbb{C}}$ is trivial, the action of $K^{\mathbb{C}}$ gives rise to the isomorphism $k \mapsto (\mathrm{Ad}\, k)X_0$ of algebraic varieties $K^{\mathbb{C}} \to K^{\mathbb{C}}.X_0$. Thus we can consider the restriction to the orbit $K^{\mathbb{C}}.X_0$ as an isomorphism of $\mathcal{H}_K(\mathfrak{g})$ onto $\mathcal{R}(K^{\mathbb{C}})$. This isomorphism transfers the adjoint representation of K to the left regular representation of K on $\mathcal{R}(K^{\mathbb{C}})$. We want to compute the representation ω of K on $\mathcal{H}_K(\mathfrak{g})$ obtained by the inverse isomorphism

$\mathcal{R}(K^{\mathbb{C}}) \to \mathcal{H}_K(\mathfrak{g})$ from the right regular representation of K on $\mathcal{R}(K^{\mathbb{C}})$. For $X \in \mathfrak{k}_0$ and for a K-harmonic polynomial $P \in \mathcal{H}_K(\mathfrak{g})$ we have

$$(\omega(X)P)((\mathrm{Ad}\, k)X_0) = \frac{\mathrm{d}}{\mathrm{d}t} P((\mathrm{Ad}\, k\mathrm{e}^{tX})X_0)\Big|_{t=0}, \qquad k \in K^{\mathbb{C}}.$$

To describe the action ω of $\mathfrak{k}$ on $\mathcal{H}_K(\mathfrak{g})$ it is enough to compute this action only on the highest weight vectors A_j^n and B_j^n for the adjoint representation π which are defined in Lemma 1. With the introduced notation $C = hy - fz \in \mathcal{P}^2(\mathfrak{g})$ we have

$$A_j^n = f^{n-j}y^j, \quad 0 \le j \le n, \qquad B_j^n = f^{n-j}y^{j-1}C, \quad 1 \le j \le n.$$

Explicit calculation from the definition of the representation ω on $\mathcal{H}_K(\mathfrak{g})$ leads to:

LEMMA 4.4. *The operators $\omega(H)$, $\omega(E)$ and $\omega(F)$ act on the polynomials f, y and $C = hy - fz$ as follows:*

$$\begin{array}{lll} \omega(H)f = 0, & \omega(E)f = -iy - 2C, & \omega(F)f = iy - 2C, \\ \omega(H)y = -4iC, & \omega(E)y = -if, & \omega(F)y = if, \\ \omega(H)C = iy, & \omega(E)C = -\frac{1}{2}f, & \omega(F)C = -\frac{1}{2}f. \end{array}$$

From the relations among the matrix elements of $k \in K^{\mathbb{C}}$ we have

$$(a_1 - ib_1)^2 + (a_2 - ib_2)^2 + (a_3 - ib_3)^2 =$$
$$(a_1^2 + a_2^2 + a_3^2) - (b_1^2 + b_2^2 + b_3^2) - 2i(a_1b_1 + a_2b_2 + a_3b_3) = 0.$$

Using the formulas for the restriction of the polynomials to the orbit $K^{\mathbb{C}}.X_0$ we find that on this orbit

$$4C^2 - f^2 + y^2 = \tfrac{1}{4}(a_2 - ib_2)^2 + \tfrac{1}{4}(a_3 - ib_3)^2 + \tfrac{1}{4}(a_1 - ib_1)^2 = 0.$$

Therefore, we conclude:

LEMMA 4.5. *Restricted to the orbit $K^{\mathbb{C}}.X_0$ one has the identity*

$$C^2 = \tfrac{1}{4}f^2 - \tfrac{1}{4}y^2.$$

From Lemmas 4.4 and 4.5 we compute the action ω on the π-highest weight vectors in $\mathcal{H}_K(\mathfrak{g})$:

THEOREM 4.6. *The weird representation ω of $\mathfrak{k}$ on $\mathcal{H}_K(\mathfrak{g})$ acts on the π-highest weight vectors A_j^n, B_j^n as follows:*

$$\begin{aligned}
&\omega(H)A_j^n = -4ijB_j^n,\\
&\omega(H)B_j^n = -i(j-1)A_{j-2}^n + ijA_j^n,\\
&\omega(E)A_j^n = -i(n-j)A_{j+1}^n - ijA_{j-1}^n - 2(n-j)B_{j+1}^n,\\
&\omega(E)B_j^n = -\tfrac{1}{2}(n-j+1)A_{j-1}^n + \tfrac{1}{2}(n-j)A_{j+1}^n - i(j-1)B_{j-1}^n - i(n-j)B_{j+1}^n,\\
&\omega(F)A_j^n = i(n-j)A_{j+1}^n + ijA_{j-1}^n - 2(n-j)B_{j+1}^n,\\
&\omega(F)B_j^n = -\tfrac{1}{2}(n-j+1)A_{j-1}^n + \tfrac{1}{2}(n-j)A_{j+1}^n + i(j-1)B_{j-1}^n + i(n-j)B_{j+1}^n.
\end{aligned}$$

As the weird action ω commutes with the adjoint action π, one obtains from Theorem 4.6 the action of the operators $\omega(H)$, $\omega(E)$ and $\omega(F)$ on the basis of $\mathcal{H}_K(\mathfrak{g})$:

$$\{\pi(F)^k A_j^n;\ n \in \mathbb{Z}_+,\ 0 \le j \le n,\ 0 \le k \le 2n\} \cup \\ \{\pi(F)^k B_j^n;\ n \in \mathbb{N},\ 1 \le j \le n,\ 0 \le k \le 2n\}.$$

The irreducible constituents of the representation ω of degree $(2n+1)$ are acting on the subspaces

$$\mathcal{H}_K(\mathfrak{g})^n = \operatorname{span}\{A_0^n, A_1^n, \ldots, A_n^n, B_1^n, \ldots, B_n^n\}, \quad \pi(F)^k \mathcal{H}_K(\mathfrak{g})^n, \quad 1 \le k \le 2n.$$

To find the highest vector for the action ω on $\mathcal{H}_K(\mathfrak{g})^n$ (and thus also for $\pi(F)^k \mathcal{H}_K(\mathfrak{g})^n$) one has to solve the equation

$$\omega(H)P = 2nP, \quad P = \sum_{j=0}^{n} \alpha_j A_j^n + \sum_{j=1}^{n} \beta_j B_j^n,$$

or, equivalently, $\omega(E)P = 0$. Using the formulas in Theorem 4.6 one obtains recursive equations for calculating the coefficients α_j and β_j. It turns out that in the case of even $n = 2m$ the coefficients α_j and β_j vanish for odd j and

$$\alpha_{2j} = (-1)^j 2^{2j} m \frac{(m+j-1)!}{(m-j)!(2j)!} \alpha_0,$$

$$\beta_{2j} = (-1)^{j+1} i 2^{2j} j \frac{(m+j-1)!}{(m-j)!(2j-1)!} \alpha_0,$$

$1 \le j \le m$. In the case of odd $n = 2m+1$ the coefficients α_j and β_j vanish for even j, and

$$\alpha_{2j+1} = (-1)^j 2^{2j} \frac{(m+j)!}{(m-j)!(2j+1)!} \alpha_1,$$

$$\beta_{2j+1} = (-1)^{j+1} i 2^{2j+1} \frac{(m+j)!}{(2m+1)(m-j)!(2j)!} \alpha_1,$$

$0 \le j \le m$. Thus

Proposition 4.7. *In the irreducible constituents* $\pi(F)^k \mathcal{H}_K(\mathfrak{g})^n$, $n \in \mathbb{Z}_+$, $0 \le k \le 2n$, *the highest weight vectors for the weird action* ω *are* $\pi(F)^k P_n$, *where*

$$P_{2m} = \sum_{j=0}^{m} (-1)^j 2^{2j} m \frac{(m+j-1)!}{(m-j)!(2j)!} A_{2j}^{2m} + \\ + \sum_{j=1}^{m} (-1)^{j+1} i 2^{2j} j \frac{(m+j-1)!}{(m-j)!(2j-1)!} B_{2j}^{2m},$$

and

$$P_{2m+1} = \sum_{j=0}^{m} (-1)^j 2^{2j} \frac{(m+j)!}{(m-j)!(2j+1)!} A_{2j+1}^{2m+1} + \\ + \sum_{j=0}^{m} (-1)^{j+1} i 2^{2j+1} \frac{(m+j)!}{(2m+1)(m-j)!(2j)!} B_{2j+1}^{2m+1}.$$

Thus, expressed through variables h, e, f, z, x, y and $C = hy - fz$ we have $P_0 = 1$, $P_1 = y - 2iC$, $P_2 = f^2 - 2y^2 + 4iyC$, $P_3 = f^2y - \frac{4}{3}y^3 - \frac{2i}{3}f^2C + \frac{8i}{3}y^2C$, $P_4 = f^4 - 8f^2y^2 + 8y^4 + 8if^2yC - 32iy^3C$, $P_5 = f^4y - 4f^2y^3 + \frac{16}{5}y^5 - \frac{2i}{5}f^4C + \frac{24i}{5}f^2y^2C - \frac{32i}{5}y^4C$ etc.

The weird action ω is extended to $\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{g})^K \otimes \mathcal{H}_K(\mathfrak{g})$ trivially on $\mathcal{P}(\mathfrak{g})^K$, i.e. $\omega(\mathfrak{k})\mathcal{P}(\mathfrak{g})^K = 0$. Our aim was to try to express this action using some K-invariant linear differential operators on $\mathcal{P}(\mathfrak{g})$ with polynomial coefficients. Unfortunately, it does not seem possible. Here is the action of the operators $\omega(H)$, $\omega(E)$ and $\omega(F)$ on the monomial bases of $\mathcal{P}^1(\mathfrak{g})$ and $\mathcal{P}^2(\mathfrak{g})$:

$\omega(H)h = 0,$	$\omega(E)h = -iz + ey + fx,$	$\omega(F)h = iz + ey + fx,$
$\omega(H)e = 0,$	$\omega(E)e = ix - 2hx - 2ez,$	$\omega(F)e = -ix - 2hx - 2ez,$
$\omega(H)f = 0,$	$\omega(E)f = -iy - 2hy + 2fz,$	$\omega(F)f = iy - 2hy + 2fz,$
$\omega(H)z = 2iey + 2ifx,$	$\omega(E)z = -ih,$	$\omega(F)z = ih,$
$\omega(H)x = 4ihx + 4iez,$	$\omega(E)x = ie,$	$\omega(F)x = -ie,$
$\omega(H)y = -4ihy + 4ifz,$	$\omega(E)y = -if,$	$\omega(F)y = if.$

$$\begin{aligned}
&\omega(H)h^2 = 0,\\
&\omega(H)f^2 = 0,\\
&\omega(H)he = 0,\\
&\omega(H)fz = -\tfrac{i}{2}y - 2ih^2y + 2ihfz + iefy + if^2x,\\
&\omega(H)hf = 0,\\
&\omega(H)fx = -iz + 2ihey + 2ihfx,\\
&\omega(H)hz = 12ihey + 12ihfx,\\
&\omega(H)fy = -4ihfy + 4if^2z,\\
&\omega(H)hx = -\tfrac{i}{2}x + 2ih^2x + 2ihez - ie^2y - ifx^2,\\
&\omega(H)z^2 = 4iezy + 4ifzx,\\
&\omega(H)hy = \tfrac{i}{2}y - 2ih^2y + 2ihfz + iefy + if^2x,\\
&\omega(H)zx = 4ihzx + 4iez^2 + 2iexy + 2ifx^2,\\
&\omega(H)e^2 = 0,\\
&\omega(H)zy = -4ihzy + 4ifz^2 + 2iey^2 + 2ifxy,\\
&\omega(H)ef = 0,\\
&\omega(H)x^2 = 8ihx^2 + 8iezx,\\
&\omega(H)ez = ix - 2ih^2x - 2ihez + ie^2y + iefx,\\
&\omega(H)xy = 4iezy + 4ifzx,\\
&\omega(H)ex = 4ihex + 4ie^2z,\\
&\omega(H)y^2 = -8ihy^2 + 8ifzy.\\
&\omega(H)ey = \tfrac{i}{2}y - 2ihey - 2ihfx,
\end{aligned}$$

$$\begin{aligned}
\omega(E)h^2 &= -\tfrac{4i}{3}hz + \tfrac{i}{3}ey - \tfrac{i}{3}fx + 2hey + 2hfx,\\
\omega(E)e^2 &= 2iex - 4hex - 4e^2z,\\
\omega(E)he &= ihx - iez - 2h^2x - 2hez + e^2y + efx,\\
\omega(E)ex &= ie^2 + ix^2 - 2hx^2 - 2ezx,\\
\omega(E)hf &= -ihy - ifz - 2h^2y + 2hfz + efy + f^2x,\\
\omega(E)f^2 &= -2ify - 4hfy + 4f^2z,\\
\omega(E)hz &= -\tfrac{2i}{3}h^2 + \tfrac{i}{3}ef - \tfrac{2i}{3}z^2 - \tfrac{i}{3}xy + ezy + fzx,\\
\omega(E)fy &= -if^2 - iy^2 - 2hy^2 + 2fzy,\\
\omega(E)hx &= -\tfrac{1}{4}e + ihe - izx + hzx + ez^2 + \tfrac{1}{2}exy + \tfrac{1}{2}fx^2,\\
\omega(E)z^2 &= -\tfrac{4i}{3}hz + \tfrac{i}{3}ey - \tfrac{i}{3}fx,\\
\omega(E)hy &= -\tfrac{1}{4}f - ihf + izy - hzy + \tfrac{1}{2}ey^2 + fz^2 + \tfrac{1}{2}fxy,\\
\omega(E)zx &= -ihx + iez,\\
\omega(E)ef &= \tfrac{4i}{3}hz - \tfrac{i}{3}ey + \tfrac{i}{3}fx - 2hey - 2hfx,\\
\omega(E)zy &= -ihy - ifz,\\
\omega(E)ez &= -\tfrac{1}{4}e - ihe + izx - hzx - ez^2 - \tfrac{1}{2}exy - \tfrac{1}{2}fx^2,\\
\omega(E)x^2 &= 2iex,\\
\omega(E)ey &= \tfrac{1}{2}h + \tfrac{2i}{3}h^2 - \tfrac{i}{3}ef + \tfrac{2i}{3}z^2 + \tfrac{i}{3}xy - ezy - fzx,\\
\omega(E)xy &= -\tfrac{4i}{3}hz + \tfrac{i}{3}ey - \tfrac{i}{3}fx,\\
\omega(E)fz &= \tfrac{1}{4}f - ihf - izy - hzy + \tfrac{1}{2}ey^2 + fz^2 + \tfrac{1}{2}fxy,\\
\omega(E)y^2 &= -2ify.\\
\omega(E)fx &= \tfrac{1}{2}h - \tfrac{2i}{3}h^2 + \tfrac{i}{3}ef - \tfrac{2i}{3}z^2 - \tfrac{i}{3}xy + ezy + fzx,
\end{aligned}$$

$$\begin{aligned}
\omega(F)h^2 &= \tfrac{4i}{3}hz - \tfrac{i}{3}ey + \tfrac{i}{3}fx + 2hey + 2hfx,\\
\omega(F)e^2 &= -2iex - 4hex - 4e^2z,\\
\omega(F)he &= -ihx + iez - 2h^2x - 2hez + e^2y + efx,\\
\omega(F)ex &= -ie^2 - ix^2 - 2hx^2 - 2ezx,\\
\omega(F)hf &= ihy + ifz - 2h^2y + 2hfz + efy + f^2x,\\
\omega(F)f^2 &= 2ify - 4hfy + 4f^2z,\\
\omega(F)hz &= \tfrac{2i}{3}h^2 - \tfrac{i}{3}ef + \tfrac{2i}{3}z^2 + \tfrac{i}{3}xy + ezy + fzx,\\
\omega(F)fy &= if^2 + iy^2 - 2hy^2 + 2fzy,\\
\omega(F)hx &= -\tfrac{1}{4}e - ihe + izx + hzx + ez^2 + \tfrac{1}{2}exy + \tfrac{1}{2}fx^2,\\
\omega(F)z^2 &= \tfrac{4i}{3}hz - \tfrac{i}{3}ey + \tfrac{i}{3}fx,\\
\omega(F)hy &= -\tfrac{1}{4}f + ihf - izy - hzy + \tfrac{1}{2}ey^2 + fz^2 + \tfrac{1}{2}fxy,\\
\omega(F)zx &= ihx - iez,\\
\omega(F)ef &= -\tfrac{4i}{3}hz + \tfrac{i}{3}ey - \tfrac{i}{3}fx - 2hey - 2hfx,\\
\omega(F)zy &= ihy + ifz,\\
\omega(F)ez &= -\tfrac{1}{4}e + ihe - izx - hzx - ez^2 - \tfrac{1}{2}exy - \tfrac{1}{2}fx^2,\\
\omega(F)x^2 &= -2iex,\\
\omega(F)ey &= \tfrac{1}{2}h - \tfrac{2i}{3}h^2 + \tfrac{i}{3}ef - \tfrac{2i}{3}z^2 - \tfrac{i}{3}xy - ezy - fzx,\\
\omega(F)xy &= \tfrac{4i}{3}hz - \tfrac{i}{3}ey + \tfrac{i}{3}fx,\\
\omega(F)fz &= \tfrac{1}{4}f + ihf + izy - hzy + \tfrac{1}{2}ey^2 + fz^2 + \tfrac{1}{2}fxy,\\
\omega(F)y^2 &= 2ify.\\
\omega(F)fx &= \tfrac{1}{2}h + \tfrac{2i}{3}h^2 + \tfrac{2i}{3}z^2 - \tfrac{i}{3}ef + \tfrac{i}{3}xy + ezy + fzx,
\end{aligned}$$

Finally, we note that when inspecting K-invariant linear differential operators on $\mathcal{P}(\mathfrak{g})$ with polynomial coefficients we found another representation κ of $\mathfrak{k}$ on $\mathcal{P}(\mathfrak{g})$ commuting with π. It is given by the following derivations of the algebra $\mathcal{P}(\mathfrak{g})$:

$$\begin{aligned}\kappa(H) &= -h\frac{\partial}{\partial h} - e\frac{\partial}{\partial e} - f\frac{\partial}{\partial f} + z\frac{\partial}{\partial z} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\\ \kappa(E) &= z\frac{\partial}{\partial h} - x\frac{\partial}{\partial e} + y\frac{\partial}{\partial f},\\ \kappa(F) &= h\frac{\partial}{\partial z} - e\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}.\end{aligned}$$

This representation does not commute with $\partial(\Omega)$, $\partial(\Delta)$ and $\partial(\Sigma)$, but the space of K-harmonic polynomials is nevertheless κ-invariant since the commutators are

$$\begin{array}{lll}[\kappa(H),\partial(\Omega)] = 2\partial(\Omega), & [\kappa(E),\partial(\Omega)] = 0, & [\kappa(F),\partial(\Omega)] = -2\partial(\Sigma),\\ [\kappa(H),\partial(\Delta)] = -2\partial(\Delta), & [\kappa(E),\partial(\Delta)] = -2\partial(\Sigma), & [\kappa(F),\partial(\Delta)] = 0,\\ [\kappa(H),\partial(\Sigma)] = 0, & [\kappa(E),\partial(\Sigma)] = -\partial(\Omega), & [\kappa(F),\partial(\Sigma)] = -\partial(\Delta).\end{array}$$

The homogeneous subspaces $\mathcal{P}^n(\mathfrak{g})$, thus also $\mathcal{H}^n_K(\mathfrak{g})$, are evidently κ-invariant. The action on the vectors A^n_j and B^n_j is:

$$\begin{array}{llll}\kappa(H)A^n_j = (2j-n)A^n_j, & \kappa(E)A^n_j = (n-j)A^n_{j+1}, & \kappa(F)A^n_j = jA^n_{j-1}, & 0 \le j \le n,\\ \kappa(H)B^n_j = (2j-n-1)B^n_j, & \kappa(E)B^n_j = (n-j)B^n_{j+1}, & \kappa(F)B^n_j = (j-1)B^n_{j-1}, & 1 \le j \le n.\end{array}$$

Thus, we see that the subspace span $\{A^n_j;\ 0 \le j \le n\}$ is κ-invariant and the corresponding subrepresentation is irreducible of degree $n+1$. The same holds for the subspaces span$\{\pi(F)^k A^n_j;\ 0 \le j \le n\}$, $1 \le k \le 2n$. Similarly, the subspace span $\{B^n_j;\ 1 \le j \le n\}$ (and also span $\{\pi(F)^k B^n_j;\ 1 \le j \le n\}$, $1 \le k \le 2n$) is κ-invariant and the corresponding subrepresentation is irreducible of degree n.

The subalgebra $\mathcal{P}(\mathfrak{g})^K$ of K-invariants is κ-invariant. We have $\mathcal{P}(\mathfrak{g})^K = \mathbb{C}[\omega,\delta,\sigma]$, where ω, δ and σ are quadratic polynomials:

$$\omega = h^2 + ef, \quad \delta = z^2 - xy, \quad \sigma = 2hz + ey - fx.$$

κ acts on them as follows

$$\begin{array}{lll}\kappa(H)\omega = -2\omega, & \kappa(E)\omega = \sigma, & \kappa(F)\omega = 0,\\ \kappa(H)\delta = 2\delta, & \kappa(E)\delta = 0, & \kappa(F)\delta = \sigma,\\ \kappa(H)\sigma = 0, & \kappa(E)\sigma = 2\delta, & \kappa(F)\sigma = 2\omega.\end{array}$$

Therefore, the subrepresention of κ on the 3-dimensional invariant subspace $\mathbb{C}^1[\omega,\delta,\sigma] = \text{span}\,\{\omega,\delta,\sigma\}$ is irreducible. Since the representation κ of $\mathfrak{k}$ acts by derivations, we conclude that all irreducible constituents of κ in $\mathcal{P}(\mathfrak{g})^K$ are of odd degree.

The representation κ on $\mathcal{P}(\mathfrak{g})$ is locally finite, thus the corresponding represention of $\mathfrak{k}_0$ integrates to a representation of a simply connected compact Lie group with the Lie algebra $\mathfrak{k}_0$. Since among the irreducible constituents of κ are not only those of odd degree but also those of even degree, this group is not $K \approx \mathrm{SO}(3)$ but its 2-fold covering group $\approx \mathrm{SU}(2)$. Finally, since $\mathfrak{k}_0$ acts by derivations, the action of the integrated representation on $\mathcal{P}(\mathfrak{g})$ is by automorphisms.

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Neobično K-djelovanje na $\mathcal{U}(\mathfrak{g})$ za $\mathfrak{so}(n,1)$ i $\mathfrak{su}(n,1)$

Hrvoje Kraljević

SAŽETAK. Neka je $\mathfrak{g}_0$ ili $\mathfrak{so}(n,1)$ ili $\mathfrak{su}(n,1)$, $\mathfrak{g}$ njezina kompleksifikacija, K maksimalna kompaktna podgrupa adjungirane grupe od $\mathfrak{g}_0$, $\mathcal{U}(\mathfrak{g})$ univerzalna omotačka algebra od $\mathfrak{g}$ i $\mathcal{U}(\mathfrak{g})^K$ njezina podalgebra K-invarijanata. Posljedica rezultata iz [2] je da osim uobičajenog adjungiranog djelovanja od K na $\mathcal{U}(\mathfrak{g})$ postoji i drugo djelovanje od K koje komutira s adjungiranim djelovanjem i ostavlja $\mathcal{U}(\mathfrak{g})^K$ po točkama invarijantnim. Slučaj $\mathfrak{g}_0 = \mathfrak{so}(2,1) \simeq \mathfrak{su}(1,1)$ je trivijalan jer je K komutativna i neobično djelovanje od K podudara se s inverzom adjungiranog djelovanja. U ovom članku detaljno smo proučili neobično djelovanje od K u najjednostavnijem netrivijalnom slučaju $\mathfrak{g}_0 = \mathfrak{so}(3,1)$.

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