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THE TADIĆ PHILOSOPHY: AN OVERVIEW OF THE GUIDING PRINCIPLES AND UNDERLYING IDEAS IN THE WORK OF MARKO TADIĆ

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To Marko Tadić for his 70th birthday

Abstract. This paper provides an overview of the guiding principles and underlying ideas in the work of Marko Tadić. His research is mostly concerned with the representation theory of reductive groups over local fields. From the authors' perspective, the most important guiding principles in his work are the essential simplicity of harmonic analysis, even in the non-commutative non-compact case, the Lefschetz principle saying that the representation theory over archimedean and non-archimedean fields should be studied in a unified way, and the principle of comparison of Jacquet modules. Besides these, the most prominent and fruitful ideas are the structural external approach to the unitary dual, the unitarizability along the lines, the use of topology of various duals to get information in harmonic analysis and arithmetic of the underlying group, and the interplay between unitarizability and Arthur packets. All these principles and ideas are the subject of this paper.

1. Introduction

Writing a complete overview of someone's mathematical work is an extremely difficult task. Any such attempt necessarily reflects the opinions, mathematical taste, subjective viewpoints, biased highlights, incomplete conclusions and illusions of the authors. Having that in mind, we feel that this paper should begin with a disclaimer.

Disclaimer. *This paper is* not *an attempt of writing a complete account of the work of Marko Tadić. It is* neither *a historical account of the topics studied by him. All the mistakes and wishful thinking conclusions are solely the responsibility of the authors.*

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In other words, this paper is nothing else than our personal subjective viewpoint of the most important guiding principles and underlying ideas present in the work of Tadić. We are pretty sure that any other author, including Tadić himself, would not make the same choices and not point out the same ideas. The ideas described in this paper are reoccurring themes in his work, not necessarily his inventions, but used as the fundamental tools and guiding principles in the lines of his research over the years. They could be described in one word as the *Tadić philosophy*, as indicated in the title.

The paper begins with three guiding principles present in the entire mathematical opus of Tadić. The first principle is the *simplicity of harmonic analysis*, which seeks, despite the complicated setting and techniques, for simple principles and answers to the main problems of non-commutative harmonic analysis. The motivation and inspiration for this principle is the beauty and simplicity of classical harmonic analysis, in particular, Fourier series and, more generally, Fourier analysis on locally compact Abelian groups. The main evidence for this principle in the work of Tadić is the classification of the unitary dual of the general linear groups, which is at the same time simple and beautiful, despite the complicated setting of the representation theory of general linear groups and the techniques required to establish the final results. Although it is difficult to expect such a nice result in the case of the unitary dual of classical groups, the most recent work of Tadić indicates, in form of certain conjectures, that this problem could also get a simple answer in the framework of local Arthur packets.

The second guiding principle in the work of Tadić is the *Lefschetz principle*, formulated by Harish-Chandra, which predicts that "whatever is true for real groups is also true for p-adic groups." However, Tadić mostly applied this principle in the opposite direction, by transferring the results and techniques from the p-adic setting to the real and complex groups. The prominent example of this is again the classification of the unitary dual of general linear groups. More evidence is provided by the determinantal formula of Tadić.

The third guiding principle in the work of Tadić, at least from our perspective, is related to the comparison of Jacquet modules. This principle of the *comparison of Jacquet modules* claims that the more detailed understanding of parabolically induced representations in representation theory of reductive groups can be achieved by comparing Jacquet modules with respect to as many as possible parabolic subgroups. In other words, the more parabolic subgroups at our disposal, the more possible sources of information on the representation theory. The first implications of this principle led Tadić to several reducibility criteria for induced representations and their composition series. However, it is also the underlying principle in many other ideas in his work. The best example is perhaps the idea to apply the external approach to study the unitary dual of general linear groups and classical groups.

The rest of the paper is devoted to the description of the four, at least in our opinion, crucial ideas underlying the work of Tadić. The first is the external approach to the representation theory of reductive groups, which is based on the Hopf ring and Hopf algebra structures on the Grothendieck group of the category of finite length representations. This idea led not only to the classification of the unitary dual of general linear groups, but also to many results in the representation theory of classical p-adic groups.

The second idea is to use the approach to unitarizability along the lines in the study of the unitary dual of classical p-adic groups. It is based on the division of representations of a classical p-adic group into certain lines of representations. The idea is that the unitarizability of an irreducible representation could perhaps be reduced to the unitarizability of corresponding representations in a certain special types of these lines. Tadić poses two questions regarding these unitarizabilites. The affirmative answer to these questions would considerably simplify the quest for the unitary dual of classical p-adic groups.

The third idea underlying the work of Tadić is to consider the implications of the topology on the unitary and non-unitary dual to the representation theory and harmonic analysis on reductive groups, and its arithmetic consequences. This thread of his work deals with different aspects of representation theory of classical p-adic groups. For instance, the reducibility question for parabolically induced representations is studied in terms of certain converging sequences in the topology of the non-unitary dual. A special attention is paid to the isolated representations in the topology of both unitary and non-unitary dual, as well as certain other types of duals related to arithmetic. The isolated representations are always special in some sense, and often have important relations to the arithmetic questions for the underlying group.

The final idea described here is the most recent one. It is the idea to compare and combine the unitarizability problem with the description of the local Arthur packets. Since local Arthur packets consist of possible local components of representations in the discrete spectrum, these are all unitary. On the other hand, Tadić has identified certain critical points in the study of the unitary dual of classical p-adic groups. He then conjectured that a given representation at the critical point is unitarizable if and only if it is in a local Arthur packet.

The structure of the paper follows closely the guiding principles and underlying ideas mentioned above. Each of the remaining sections is devoted to the explanation and description of one of the principles and ideas. The only exception is the preliminary section, that follows this introduction, which provides the necessary background and notation for the rest of the paper. The sections, other than the preliminary section, are almost independent, and could be read in any order.

The reader should be aware that parts of the text are accessible even without reading the preliminaries. Most of the sections in this paper consist of two parts divided by the sign $* \quad * \quad *$. The first part, above the sign, is meant to provide an overview of the covered topic in more general terms, hopefully accessible to a wider mathematical community, and free of notational subtleties as much as possible. The second part, below the sign, is just a bit more involved, provides examples, more detailed descriptions and results on the covered topic in each section.

At the end of the introduction, we would like to thank Goran Muić for the invitation to contribute to this volume. The paper is dedicated to Marko Tadić on the occasion of his 70th birthday. We would like to use the opportunity to express our gratitude to Marko for all of his beautiful mathematics, ideas and thoughts. We wish for even more Marko's mathematical pearls in many years to come...

2. Preliminaries and notation

Let F be a p-adic field, that is, a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. Although many of the results discussed below are valid in any characteristic different than two, or even for any local field, we stick in this paper to the case of characteristic zero.

Let $\mathbf{G}$ be a reductive algebraic group defined over F. The group of its F-rational points is denoted by $G = \mathbf{G}(F)$, and referred to as a p-adic reductive group. Throughout the paper we assume that $\mathbf{G}$ is F-split, and most of the time G is one of the following groups:

- the general linear group $GL(n, F)$,
- the special linear group $SL(n, F)$,
- the symplectic group $Sp(2n, F)$,
- the odd special orthogonal group $SO(2n + 1, F)$.

The F-rank of the former two groups is $n - 1$ and the F-rank of the latter two groups is n.

We now introduce definitions related to the structure theory of the groups G. We avoid the exact general definitions in the framework of reductive groups, but give makeshift definitions adjusted to the cases of the general linear groups and classical groups above. These groups have natural realizations as matrix groups, obtained by fixing a basis in F^n on which these groups naturally act. The symplectic, respectively odd special orthogonal group, is defined as the subgroup of the special linear group preserving a non-degenerate skew-symmetric, respectively symmetric, bilinear form. The choice of an appropriate basis with respect to this form leads to convenient matrix realizations.

Structure of the general linear group. The conjugacy classes of parabolic subgroups in $GL(n, F)$ are in one-to-one correspondence with the set of ordered partitions of n into positive integers. Given such a partition $(n_1, \ldots, n_k)$, where $\sum_{i=1}^{k} n_i = n$, a parabolic subgroup in the corresponding conjugacy class is denoted by $P = P_{(n_1,\ldots,n_k)}$. Its Levi decomposition is the semidirect product decomposition of the form $P = MN$, where the Levi subgroup M is isomorphic to

$$M \cong GL(n_1, F) \times \cdots \times GL(n_k, F),$$

and N is the unipotent radical. In the case of $k = 1$, so that the partition is a singleton $n_1 = n$, we obtain $P = G$. In the case of $k = n$, so that the partition consists of n ones, we obtain a Borel subgroup B, which is a minimal parabolic subgroup. Its Levi decomposition is denoted by $B = TU$, where T is a maximal split torus in $GL(n, F)$ isomorphic to the direct product of n copies of $GL(1, F)$, and U is the unipotent radical. Throughout the paper we denote by ν the character $\nu = |\det|$ of $GL(m, F)$, where the absolute value is the p-adic absolute value on F, and m is any positive integer.

Structure of classical groups. Similarly as in the case of the general linear group, the conjugacy classes of parabolic subgroups of the classical p-adic group G_n of rank n, as introduced above, are in one-to-one correspondence with the set of ordered partitions of non-negative integers $m \leq n$ into positive integers. Given an ordered partition $(n_1, \ldots, n_k)$ of some $m \leq n$, a parabolic subgroup in the corresponding conjugacy class is denoted by $P = P_{(n_1,\ldots,n_k;n')}$, where $n' = n - m$. It has a Levi decomposition $P = MN$, with the Levi subgroup M isomorphic to

$$M \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'},$$

where $G_{n'}$ is the group of the same type as G_n, but of rank $n' = n - m \leq n$, and N is the unipotent radical. We set G_0 to be the trivial group. Note that in the case of $m = 0$, there is no ordered partition of m into positive integers, but still this degenerate case corresponds to the parabolic subgroup $P = G_n$. In the case of $k = m = n$, so that the partition consists of n ones, we obtain a Borel subgroup B of G_n, which is a minimal parabolic subgroup. Its Levi decomposition is denoted by $B = TU$, where T is a maximal split torus in G_n isomorphic to the direct product of n copies of $GL(1, F)$, and U is the unipotent radical.

Representations. A *representation* π of a p-adic group G as above on a complex vector space V is a homomorphism

$$\pi : G \to GL(V).$$

We write (π, V) for the representation, if the underlying vector space of the representation π should be specified. Note that V can be, and usually is, infinite-dimensional.

Let (π, V) be a representation of G. Let W be a subspace of V invariant under the action of G through π. We say that W is G-invariant. Then the representation of G on W is called a *subrepresentation* of the representation (π, V). The trivial subspace $\{0\}$ and the full space V are obviously subrepresentations of (π, V).

The representation (π, V) is *irreducible* if the only G-invariant subspaces of V are $\{0\}$ and V itself. In other words, these are the only subrepresentations of (π, V). Definitions of a subrepresentation and irreducibility usually require some notion of closedness, but this definition is enough for our purposes, since we may, for the moment, ignore altogether the topology on V.

Let (π, V) be a representation of G. A vector $v \in V$ is called *smooth* if it is fixed, through π, by an open compact subgroup of G. The representation (π, V) is *smooth* if each vector in V is smooth. The open compact stabilizers of smooth vectors depend on the vector $v \in V$.

For any open compact subgroup K of G, denote by V^K the subspace of V which consists of all K-fixed vectors in V. The representation (π, V) is *admissible* if it is smooth and V^K is finite-dimensional for each open compact subgroup K of G.

In the rest of the paper, all representations are assumed to be smooth and admissible, unless otherwise specified.

A representation (π, V) is a *represention of finite length* if its Jordan–Hölder series is finite. In that case, its length is called the *length* of the representation π. Recall that such Jordan–Hölder series of (π, V) is obtained from an increasing filtration

$$\{0\} = V_0 \subsetneqq V_1 \subsetneqq V_2 \subsetneqq \cdots \subsetneqq V_{\ell-1} \subsetneqq V_\ell = V$$

of V by G-invariant subspaces V_j such that, for $j = 1, \ldots, \ell$, the quotients of the filtration V_j/V_{j-1}, with the natural action of G arising from π, are irreducible representations of G. These irreducible representations, obtained as quotients of the filtration, are referred to as the *subquotients* or *irreducible constituents* of π. They form the Jordan–Hölder series of the representation π of finite length, which is in this context also called the composition series of π. In this situation, the length of π is ℓ. If σ is a subquotient of π, we write $\sigma \leq \pi$. More generally, if σ is a representation of G such that all its subquotients appear in the composition series of π, we also write $\sigma \leq \pi$.

A representation (π, V) is *unitarizable* if there exists an inner product $(\,\cdot\,|\,\cdot\,)$ on V such that

$$(\pi(g)v_1|\pi(g)v_2) = (v_1, v_2) \text{ for all } v_1, v_2 \in V \text{ and } g \in G.$$

Unitarizability of representations of reductive groups is one of the main topics of Tadić's research.

Let (π, V) and (π', W) be two representations of the same group G. Any linear operator A from V to W such that

$$A(\pi(g)v) = \pi'(g)(Av) \text{ for all } v \in V, g \in G,$$

that is, which commutes with the actions of G, is called the *intertwining operator* between representations π and π'. If there exists an intertwining operator which is an isomorphism, we say that these two representations are *isomorphic* and write $\pi \cong \pi'$.

Let (π, V) be a representation of G. On the space $\hat{V}$ of linear functionals on V, one can introduce a G-action, denoted by $\hat{\pi}$, in the following way. Let $\langle \cdot , \cdot \rangle$ be the usual linear algebra pairing between V and $\hat{V}$. We define

$$\langle \hat{\pi}(g)\hat{v}, v\rangle = \langle \hat{v}, \pi(g^{-1})v\rangle,$$

for $g \in G$, $v \in V$ and $\hat{v} \in \hat{V}$. This representation is not smooth. The space of smooth vectors in the representation $(\hat{\pi}, \hat{V})$ is G-invariant. The representation of G on the subspace of smooth vectors is called the *contragredient* representation of π and denoted by $(\widetilde{\pi}, \widetilde{V})$. It is a smooth and admissible representation of G. A representation π is called selfcontragredient if it is isomorphic to its contragredient, i.e., if $\pi \cong \widetilde{\pi}$.

For the fixed $v \in V$ and $\widetilde{v} \in \widetilde{V}$, the *matrix coefficient* associated to v and $\widetilde{v}$ is the function on G defined by the assignment

$$g \mapsto \langle \widetilde{v}, \pi(g)v\rangle.$$

The matrix coefficients allow the study of analytic properties of representations. A representation is *square-integrable* if all of its matrix coefficients are square-integrable on G modulo the center of G, it is *supercuspidal* if all of its matrix coefficients are compactly supported modulo center, and it is *tempered* if all of its matrix coefficients are $L^{2+\varepsilon}$ functions on G modulo center for all $\varepsilon > 0$.

Note that the condition for the matrix coefficients of a representation of $GL(m, F)$ to be L^2 modulo center, implies that this representation is necessarily unitarizable. The representations of $GL(m, F)$ of the form $\nu^x \delta^u$, where x is a real number and δ^u a square-integrable representation of $GL(m, F)$ are called *essentially square-integrable representations.* Conversely, for any essentially square-integrable irreducible representation δ, the corresponding x as above is unique. To conclude, for each essentially square integrable representation δ, there exists a unique real number $e(\delta)$ and a square-integrable representation δ^u such that

$$\delta \cong \nu^{e(\delta)}\delta^u,$$

where $e(\delta)$ is called the *exponent* of the representation δ.

It is easy to see that every irreducible supercuspidal representation ρ of G is essentially square integrable. Hence, there exists a unique exponent $e(\rho) \in \mathbb{R}$ and a unitary supercuspidal representation ρ^u such that

$$\rho \cong \nu^{e(\rho)}\rho^u.$$

The irreducible supercuspidal representations of Levi subgroups of G are the building blocks of all the irreducible representations of G. We explain momentarily in which sense.

Parabolic induction. Let G be the p-adic classical or general linear group, as introduced above. Let P be a parabolic subgroup of G. Let $P = MN$ be a Levi decomposition of P, where M is the Levi subgroup and N the unipotent radical. Recall that the Levi subgroup M is a direct product of smaller general linear groups and a smaller classical group in the case of classical groups. Let (σ, W) be a representation of M. We extend this representation trivially over N, so that we obtain a representation of P, still denoted by σ. The representation of G *parabolically induced* from σ is denoted by

$$\mathrm{Ind}_P^G(\sigma),$$

and defined as follows. The underlying vector space is the space V of all functions $f : G \to W$ which satisfy

(i) $f(mug) = \delta_P(m)^{1/2}\sigma(m)f(g)$, for all $m \in M, u \in N, g \in G$,
(ii) there exists an open compact subgroup K of G, which depends on f, such that $f(gk) = f(g)$ for every $k \in K$, $g \in G$.

The action of G on the space V of the parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is by right translations. If we denote that action by π, it is defined as

$$(\pi(g_0)f)(g) = f(gg_0),$$

for $g_0, g \in G$ and $f \in V$.

Here δ_P denotes the modular function of P, which is used in (i) to obtain a convenient normalization of parabolic induction. With this normalization, the parabolic induction preserves unitarizability. The parabolic induction may be viewed as a functor from the category of representations of finite length of the Levi subgroup M to the category of such representations of G. This holds because it can be proved that, besides smoothness and admissibility, parabolic induction preserves the property of finite length.

Importance of parabolic subgroups. The parabolic subgroups are of the main interest for the representation theory of the groups G as above. Namely, they play a crucial role in the process of parabolic induction, which builds a representation of the group G starting from a representation of a Levi subgroup.

In the case of the general linear group and classical groups, the Levi subgroups are direct products of smaller general linear groups, and a smaller classical group in the case of classical groups. This means that one can guess, very optimistically, a possibility of certain inductive procedure in describing representations of general linear groups or classical groups using description of the set of representations of smaller groups of the same type.

This is exactly what is achieved by the process of parabolic induction, not only in general considerations, such as the Langlands quotient classification of irreducible representations of any reductive group which is based on parabolic induction. It also happens in specific cases, such as the case of the general linear groups, in which using the more structural approach of considering at the same time all unitarizable representations of all general linear groups, i.e., of all ranks, yields remarkable results of Tadić. This is explained in detail in Section 5 and Section 6.

Supercuspidal representations as "building blocks". Having the necessary technology in form of parabolic induction in place, we can now explain in what sense are irreducible supercuspidal representations the "building blocks" of all irreducible representations of the p-adic group G as above.

For every irreducible representation π of G, there exists a parabolic subgroup P and an irreducible supercuspidal representation σ of M such that π is a subrepresentation of the corresponding induced representation $\mathrm{Ind}_P^G(\sigma)$ (cf. [10]).

This yields another way to characterize irreducible supercuspidal representations of G. They are precisely those irreducible representations of G which cannot occur as a subquotient of any representation parabolically induced from the representation of the Levi subgroup of a *proper* parabolic subgroup.

Essentially square-integrable representations of $GL(m,F)$. There is a nice characterization of essentially square-integrable representations of the general linear groups. Each irreducible essentially square-integrable representation δ of $GL(m,F)$ is obtained as the unique irreducible subrepresentation of the induced representation

$$\mathrm{Ind}_Q^{GL(m,F)}\left(\nu^{\frac{l-1}{2}}\rho\otimes\nu^{\frac{l-3}{2}}\rho\otimes\cdots\otimes\nu^{-\frac{l-1}{2}}\rho\right),$$

for an irreducible supercuspidal representation ρ of $GL(d,F)$ and a positive integer l, where $m=dl$ and Q is the parabolic subgroup of $GL(m,F)$ with the Levi subgroup isomorphic to the direct product of l copies of $GL(d,F)$. We denote the essentially square-integrable representation δ which is the unique subrepresentation of the induced representation above by

$$\delta=\delta[\nu^{-\frac{l-1}{2}}\rho,\nu^{\frac{l-1}{2}}\rho].$$

We often say, if ρ is fixed, that it is the essentially square-integrable representation attached to the segment $[-\frac{l-1}{2},\frac{l-1}{2}]$ of length l.

Note that the twisting of the representation $\delta[\nu^{-\frac{l-1}{2}}\rho,\nu^{\frac{l-1}{2}}\rho]$ by the character ν^x is realized as the shift of the exponents in the segment by x, i.e.,

$$\nu^x\delta[\nu^{-\frac{l-1}{2}}\rho,\nu^{\frac{l-1}{2}}\rho]\cong\delta[\nu^{x-\frac{l-1}{2}}\rho,\nu^{x+\frac{l-1}{2}}\rho].$$

Hence, replacing ρ by ρ^u results with the essentially square-integrable representation

$$\delta = \delta[\nu^{-\frac{l-1}{2}}\rho, \nu^{\frac{l-1}{2}}\rho] \cong \nu^{e(\rho)}\delta[\nu^{-\frac{l-1}{2}}\rho^u, \nu^{\frac{l-1}{2}}\rho^u].$$

The representation associated to the segment $[\nu^{-\frac{l-1}{2}}\rho^u, \nu^{\frac{l-1}{2}}\rho^u]$ on the right-hand side is a square-integrable representation, so that we may write

$$\delta^u = \delta[\nu^{-\frac{l-1}{2}}\rho^u, \nu^{\frac{l-1}{2}}\rho^u].$$

Then, $e(\delta) = e(\rho)$, and the essentially square-integrable representation δ is isomorphic to

$$\delta \cong \nu^{e(\delta)}\delta^u,$$

where $e(\delta)$ is the exponent of δ, and δ^u is the square-integrable representation defined above.

Notation for the parabolic induction. Essentially from the work of Bernstein and Zelevinsky [7], there is a special notation for the parabolic induction in the case of the general linear groups. Let $P = P_{(n_1,\dots,n_k)}$ be a parabolic subgroup of $GL(n,F)$ corresponding to the partition $(n_1,\dots,n_k)$ of n into positive integers. The Levi subgroup M of P is isomorphic to the direct product

$$M \cong GL(n_1,F) \times \cdots \times GL(n_k,F).$$

Given a representation $\sigma \cong \sigma_1 \otimes \cdots \otimes \sigma_k$ of M, where σ_j is a representation of the factor $GL(n_j,F)$ of M, the induced representation

$$\mathrm{Ind}_P^{GL(n,F)}(\sigma) = \mathrm{Ind}_{P_{(n_1,n_2,\dots,n_k)}}^{GL(n,F)}(\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k)$$

is denoted by

$$\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k.$$

Originally introduced by Faddéev in [13], this notation not only shortens the usual notation and omits the tedious bookkeeping regarding parabolic subgroups, but also suggests certain very important properties of parabolic induction, which are seamlessly incorporated in the notation. These are, among others, induction in stages and independence of the Jordan–Hölder series of the induced representation on the order of σ_j. The latter refers to the parabolic induction from the so-called associate parabolic subgroups which corresponds to changing the order of the factors above. It gives rise to possibly different representations, but with the same Jordan–Hölder series. The product notation is also compatible with the the structure of the Hopf algebra, obtained when irreducible representations of general linear groups of all ranks are organized in the (sum of) Grothendieck groups. More about that can be found in Section 6, as Tadić used this notation extensively in his work.

Inspired by the benefits of the notation for the parabolic induction in the case of general linear groups, Tadić invented the analogous notation in the case of classical groups. It appeared for the first time in his work with

Sally [36], and, systematically, in [57]. Let $P = P_{(n_1,\ldots,n_k;n')}$ be a parabolic subgroup of the classical group G_n of rank n corresponding to the partition $(n_1,\ldots,n_k)$ of $n-n'$. Let $\sigma \cong \sigma_1 \otimes \cdots \otimes \sigma_k \otimes \sigma'$ be a representation of the Levi subgroup M of P, where σ_j is a representation of the factor $GL(n_j,F)$ of M, and σ' a representation of the smaller classical group $G_{n'}$ appearing as a factor in M. The induced representation

$$\mathrm{Ind}_P^{G_n}(\sigma) = \mathrm{Ind}_{P_{(n_1,n_2,\ldots,n_k;n')}}^{G_n}(\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_k \otimes \sigma')$$

is denoted in the notation of Tadić by

$$\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k \rtimes \sigma'.$$

His crucial improvement here is the sign $\rtimes$, which separates the representation of the only factor in the direct product of M which corresponds to the classical group from the remaining factors which are general linear groups. This notation in the case of classical groups remains suggestive for the important properties of the parabolic induction, as well as very well adjusted to the structure of the Hopf module, as explained in Section 6.

In order to gain better insight in the structural information on representations of the considered group G, the above product notation is often slightly modified to distinguish the parabolically induced representation from its semi-simplification. Given a representation π of G of finite length, its *semi-simplification*, denoted by $\mathrm{s.s.}(\pi)$, is isomorphic to the (finite) direct sum of its irreducible subquotients. Thus, passing to the semi-simplification, part of the information is lost, but the benefits of the structural approach to representation theory are huge.

In this spirit, we make the following convention in this paper. The product notation means

$$\sigma_1 \times \cdots \times \sigma_k \rtimes \sigma' = \mathrm{s.s.}\left(\mathrm{Ind}_P^{G_n}(\sigma_1 \otimes \cdots \otimes \sigma_k \otimes \sigma')\right),$$

that is, the semi-simplification of the induced representation. Thus, in this paper, we carefully distinguish between the product notation and the Ind notation. If we should be aware that certain subquotient is in fact a subrepresentation or a quotient, then the product notation will be avoided. On the other hand, the benefits of the product notation, even after taking the semi-simplification, will become clear in the paper.

3. Simplicity of harmonic analysis

The beauty and simplicity of classical harmonic analysis is overwhelming. At the same time, it is one of the most powerful mathematical theories with the widest range of diverse applications in physics, engineering, computer science, music, and different parts of mathematics from differential equations, numerical mathematics and mathematical physics to differential geometry and number theory.

Harmonic analysis was also the starting point of Tadić's mathematical journey. Some of the first papers of Tadić such as [40], and his dissertation [38], are in harmonic analysis, as well as some subsequent papers such as [32]. In his dissertation, he studies the theory of spherical functions for the so-called Gel′fand pairs (G, K). Roughly speaking, one can think of F-points of a split reductive group G over F and the hyperspecial maximal compact subgroup K. He introduces certain Fréchet algebras of K-biinvariant functions on G with some additional properties, and, among other things, defines spherical Fourier transform on these spaces and studies its properties.

The most fundamental part of the classical harmonic analysis is the theory of Fourier series. In its most basic form, it is the theory of expansions of periodic functions into trigonometric series, the so-called spectral expansions. The frequencies of the sine and cosine functions in the series are multiples of the frequency given by the period of the considered periodic function. These are the so-called harmonics, which give the harmonic analysis its name. Under certain technical assumptions, the amplitudes of these harmonics determine the original function. The spectral expansion of periodic functions into elementary functions such as sines and cosines is extremely useful tool in many applications.

However, the full strength of harmonic analysis arises from certain algebraic structure which is compatible with spectral expansions. This is not so obvious from the trigonometric series, and a different point of view on the spectral expansion is required. We explain this in detail in the second part of this section below. Briefly said, the main point is that the period of a periodic function admits the additive structure of a compact Abelian group, and the same group acts on the space of periodic functions. Hence, the sine and cosine functions should be replaced by certain periodic functions that preserve the group structure on the domain and are invariant up to scalar for the group action. The spectral expansion according to these new functions is the proper point of view on Fourier series and the launching pad for the harmonic analysis in general.

The first step towards more general instances of Fourier expansions is the harmonic analysis on locally compact Abelian groups. In this case, the space of square-integrable complex functions on such a group is considered, with an action of the underlying group. As in the theory of Fourier series, the main point is to find a system of functions on the group which preserves the group structure and is invariant up to scalar under the group action. Each of these functions spans a group action invariant one-dimensional subspace in the space of square-integrable functions on the group. It is these invariant subspaces that are the key to the next level of harmonic analysis. Note that we ignore here the fact that the spectral decomposition in the case of arbitrary locally compact Abelian group could involve direct integrals, not only sums, of basic functions.

The simplest non-commutative instance of harmonic analysis is the harmonic analysis on compact Lie groups. Its aim is to decompose the functions on a Lie group according to the structure on the space of square-integrable complex functions on this group arising from certain natural group action. However, in this case, it is not possible to find a system of functions and form the spectral expansion with respect to these functions. The reason is that, unlike in the commutative case, the invariant subspaces are no longer one-dimensional. Hence, the harmonic analysis on compact Lie groups means the spectral decomposition of the space of functions on the group into invariant pieces which cannot be further decomposed. Such pieces are irreducible unitary representations of the underlying group. In the case of compact Lie groups these are all finite-dimensional.

The same story can be transferred, almost word by word, to the more general case of non-commutative non-compact locally compact groups, such as p-adic reductive groups. The harmonic analysis on such groups means the spectral decomposition of the space of square-integrable complex functions on the underlying group into pieces that are irreducible and invariant under the natural group action. These are again irreducible unitary representations, but in this case they are not necessarily finite-dimensional. Moreover, the irreducible unitary representations in the spectral decomposition do not exhaust all irreducible unitarizable representations. Nevertheless, the problem of classification of irreducible unitary representations of the considered group, which is referred to the problem of the unitary dual, is closely related and important for the spectral decomposition. Recall that the set of isomorphism classes of irreducible unitary representations of a group is called the unitary dual.

After a lengthy opening of this section, we finally come to the point. The classical theory of Fourier series is essentially very simple. One of the main guiding principles in the work of Tadić is that the harmonic analysis, in the more general context, should resemble the simplicity of Fourier series, despite the more complicated setting and methods. He makes this point explicit in [76, p. 1], by saying:

"[...] the area of harmonic analysis on locally compact groups, a theory which has its roots in the classical Fourier analysis. The classical theory is one of the most applied parts of math, in math as well as outside of math. The reason for this fact is certainly the power of the theory. But it is also related to the simplicity of basic principles of the classical theory. It is hard to expect such simplicity in the setting which we shall consider, since the groups with which we shall deal are much more complicated than the one of the classical theory [...]. Nevertheless, at some directions we get remarkably simple answers."

To illustrate the principle of simplicity of harmonic analysis, we recall now the Tadić classification of the unitary dual of the general linear group

$GL(n,F)$ over a p-adic field F. The classification of the unitary dual of $GL(n,F)$ is a very important result as one of the cornerstones of the harmonic analysis on the p-adic groups, but at the same time its formulation is so beautiful due to its simplicity, that we decided to include the statement here. It is indeed a "remarkably simple answer" as mentioned in the quote above.

Given a square-integrable representation δ of $GL(m,F)$, and a positive integer k, the parabolically induced representation

$$\mathrm{Ind}_Q^{GL(n,F)}\left(\nu^{\frac{k-1}{2}}\delta\otimes\nu^{\frac{k-3}{2}}\delta\otimes\cdots\otimes\nu^{-\frac{k-1}{2}}\delta\right)$$

has a unique irreducible quotient, where Q is the parabolic subgroup of $GL(n,F)$ with the Levi subgroup isomorphic to the direct product of k copies of $GL(m,F)$, and $n=km$. The quotient, denoted by $u(\delta,k)$, is often referred to as the Speh representation. Observe that if $k=1$, then $u(\delta,1)=\delta$. With this notation, the unitary dual of the general linear group is described as follows.

Classification of the unitary dual for GL_n (Tadić, [46]; see also [55], [44], [77], [49], [43], [47]). *Let $\mathcal{D}^u$ denote the set of isomorphism classes of square-integrable representations of all $GL(m,F)$, where m ranges over positive integers. Let*

$$\mathcal{IRR}^u=\bigcup_{n=0}^{\infty}\widehat{GL(n,F)}$$

denote the set of isomorphism classes of unitary irreducible representations of all $GL(n,F)$, where n ranges over non-negative integers. Let

$$\mathcal{B}=\left\{u(\delta,k),\,\nu^{\alpha}u(\delta,k)\times\nu^{-\alpha}u(\delta,k)\,:\,\delta\in\mathcal{D}^u,k\in\mathbb{Z}_{>0},0<\alpha<1/2\right\}.$$

Then, the set $\mathcal{IRR}^u$ is completely described as follows:

(i) if $\pi_1,\ldots,\pi_l\in\mathcal{B}$, then $\pi_1\times\cdots\times\pi_l\in\mathcal{IRR}^u$,
(ii) conversely, if $\pi\in\mathcal{IRR}^u$, then there exist $\pi_1,\ldots,\pi_l\in\mathcal{B}$, unique up to permutation, such that $\pi\cong\pi_1\times\cdots\times\pi_l$.

We turn now to the problem of the unitary dual of classical p-adic groups. Despite a simple form of the solution in the case of general linear groups, it seems that the expectations in the case of classical groups are different. Tadić himself contemplates on that in [76, Sect. 9] as follows:

"In the moment, the unitarizability problem for classical groups seems quite hard, and the solution will be definitely not simple. Despite this, it seems that regarding this problem we are now in better position then we have been regarding GL-unitarizability just before that problem was solved (at the beginning of 1980-es), although the solution there turned out to be very simple. The biggest problem in that time was that a simple answer (as in Theorem A) was not expected. The expectations were quite opposite (and even a possibility of an explicit solution of the problem was in question)."

This quotation is taken from the lectures given by Tadić in 2014 at the 40th Anniversary Midwest Representation Theory Conference, University of Chicago. Since then, he has made substantial progress regarding the unitary dual of classical p-adic groups [82], [80], [26], [27], [74]. In particular, the connections between the unitary dual and local Arthur packets, predicted by a conjecture of Tadić in [81] and elaborated in Section 9, might lead in future to the discovery of a hidden structure in the unitary dual of classical p-adic groups. If this optimistic scenario is fulfilled, the simplicity and beauty of harmonic analysis would be reflected by the simple structural description of the unitary dual of classical groups, just as in the case of general linear groups.

* * *

We now provide more details of the story of algebraic structure underlying harmonic analysis. We begin with the classical theory of Fourier series. Given a measurable periodic function $f : \mathbb{R} \to \mathbb{C}$ of period one, which is also square-integrable over $[0, 1]$, its Fourier series S_f is defined as the trigonometric series

$$S_f(x) = a_0 + \sum_{m=1}^{\infty} \left(a_m \cos(2\pi m x) + b_m \sin(2\pi m x)\right),$$

where the Fourier coefficients a_m and b_m are given by the formulas

$$\begin{aligned} a_0 &= \int_0^1 f(\xi)\,\mathrm{d}\xi, \\ a_m &= 2\int_0^1 f(\xi)\cos(2\pi m\xi)\,\mathrm{d}\xi, \quad m \geq 1, \\ b_m &= 2\int_0^1 f(\xi)\sin(2\pi m\xi)\,\mathrm{d}\xi, \quad m \geq 1. \end{aligned}$$

The convergence of the Fourier series is a delicate issue, but if f is smooth, its Fourier series S_f converges to the original function f, and provides a powerful tool for the study of periodic functions in terms of the sine and cosine functions.

We identify periodic functions of period one with functions on the segment $[0, 1]$ with equal values at the end-points 0 and 1. The identification is achieved by the restriction of a periodic function to $[0, 1]$. The space of such functions on $[0, 1]$, which are also measurable and square-integrable over $[0, 1]$, is denoted by $L^2([0, 1])$. Given $f, g \in L^2([0, 1])$, their inner product is defined as

$$(f|g) = \int_0^1 f(\xi)\overline{g(\xi)}\,\mathrm{d}\xi,$$

where $\overline{g(\xi)}$ denotes the complex conjugation. The Fourier coefficients are in fact determined as inner products of f by trigonometric functions. More precisely,

$$\begin{aligned} a_0 &= (f|1)\,, \\ a_m &= 2\cdot\big(f|\cos(2\pi m\,\cdot\,)\big), \quad m\geq 1, \\ b_m &= 2\cdot\big(f|\sin(2\pi m\,\cdot\,)\big), \quad m\geq 1, \end{aligned}$$

where 1 stands for the constant function of value one. The reason for such simple explanation is that the trigonometric functions

$$1,\, \cos(2\pi m x),\, \sin(2\pi m x), \quad m\geq 1$$

which appear in the Fourier series, form an orthogonal system for the inner product. Namely, the inner product of different functions in the system is zero, while the inner product of a function in the system by itself gives the inverse of the corresponding multiplicative constant in the formulas. For instance,

$$\big(\cos(2\pi m\,\cdot\,)|\cos(2\pi m'\,\cdot\,)\big) = \begin{cases} 0, & \text{if } m\neq m', \\ \frac{1}{2}, & \text{if } m=m'. \end{cases}$$

Observe that the inner product of $\cos(2\pi m\,\cdot\,)$ by itself is $1/2$, which is the inverse of the multiplicative constant 2 in the formula for a_m. In conclusion, the inner product structure on $L^2([0,1])$ provides, at least formally, a convenient way to determine the Fourier coefficients of a given periodic function.

In order to simplify even more the theory of Fourier series, more algebraic structure is invoked. First of all, a periodic function of period one can be viewed as a function on the interval $[0,1\rangle$, which is the set of representatives for the quotient $\mathbb{R}/\mathbb{Z}$ of additive groups. Thus, we obtain a group structure on the domain of the considered functions. The group $\mathbb{R}/\mathbb{Z}$ is a compact Abelian group, which is isomorphic to the circle group S^1 of complex numbers of modulus one. The homeomorphic isomorphism is given by the assignment

$$x\mapsto \cos(2\pi x)+i\sin(2\pi x)$$

for $x\in[0,1\rangle=\mathbb{R}/\mathbb{Z}$, where $i=\sqrt{-1}$ is the imaginary unit.

The orthogonal system of trigonometric functions is now replaced by the system consisting of exponential functions

$$\chi_n(x)=e^{2\pi i n x}, \quad n\in\mathbb{Z}.$$

These functions are clearly of period one, but they also preserve the group structure. More precisely, χ_n is a unitary character of the group $\mathbb{R}/\mathbb{Z}$, because

$$\chi_n(x+y)=e^{2\pi i n(x+y)}=e^{2\pi i n x}\cdot e^{2\pi i n y}=\chi_n(x)\chi_n(y).$$

and $|\chi_n(x)|=1$. Moreover, all the unitary characters of $\mathbb{R}/\mathbb{Z}$ are obtained this way. The assignment $k\mapsto\chi_k$ defines a natural isomorphism between the additive group $\mathbb{Z}$ of integers and the multiplicative group of unitary characters of $\mathbb{R}/\mathbb{Z}$. This means that the Pontryagin dual, which plays the role of the unitary dual in this setting, of the group $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{Z}$.

The inner product of characters equals

$$\begin{aligned}(\chi_n|\chi_m) &= \int_0^1 \chi_n(\xi)\overline{\chi_m(\xi)}\,\mathrm{d}\xi \\ &= \int_0^1 \chi_{n-m}(\xi)\,\mathrm{d}\xi \\ &= \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m, \end{cases}\end{aligned}$$

where in the first line we used the facts that $\chi_k\chi_l = \chi_{k+l}$ and $\overline{\chi_k} = \chi_{-k}$, and in the last line we integrate a character of a group over the whole group, so that the integral is zero if the character is non-trivial, and equal to the volume of the group if the character is trivial. Thus, the system χ_n, $n \in \mathbb{Z}$, is orthonormal. The sine and cosine functions in the Fourier series can be replaced by χ_n, $n \in \mathbb{Z}$, because

$$\begin{aligned} 1 &= \chi_0(x), \\ \cos(2\pi m x) &= \frac{\chi_m(x) + \chi_{-m}(x)}{2}, \quad m \geq 1, \\ \sin(2\pi m x) &= \frac{\chi_m(x) - \chi_{-m}(x)}{2i}, \quad m \geq 1. \end{aligned}$$

The Fourier series of f in the orthonormal system χ_n, $n \in \mathbb{Z}$, reads

$$S_f(x) = \sum_{n\in\mathbb{Z}} c_n \chi_n(x),$$

where the coefficients c_n are given by the inner product as

$$\begin{aligned} c_n &= (f|\chi_n) \\ &= \int_0^1 f(\xi)\overline{\chi_n(\xi)}\,\mathrm{d}\xi \\ &= \int_0^1 f(\xi)e^{-2\pi i n\xi}\,\mathrm{d}\xi, \quad n \in \mathbb{Z}. \end{aligned}$$

So far, this is quite similar to the story of sines and cosines, but the first crucial difference is that χ_n are unitary characters, that is, preserve the group structure.

The space $L^2([0,1]) = L^2(\mathbb{R}/\mathbb{Z})$ exhibits an action of the underlying Abelian group $\mathbb{R}/\mathbb{Z}$ by translations. More precisely, given $x_0 \in \mathbb{R}$ and a function $f \in L^2(\mathbb{R}/\mathbb{Z})$, the action is defined as

$$\left(R(x_0)f\right)(x) = f(x + x_0),$$

which is well-defined because f is of period one. Thus, R defines a representation of $\mathbb{R}/\mathbb{Z}$ on the complex vector space $L^2(\mathbb{R}/\mathbb{Z})$, which is referred to as the right regular representation. This is the additional algebraic structure on the space $L^2(\mathbb{R}/\mathbb{Z})$ that explains why the characters χ_n are perfect for decomposing periodic functions of period one. Observe that the right regular representation is unitary, that is, preserves the inner product on $L^2([0,1]) = L^2(\mathbb{R}/\mathbb{Z})$. Given $f, g \in L^2(\mathbb{R}/\mathbb{Z})$ and $x_0 \in \mathbb{R}$,

$$\begin{aligned}\big(R(x_0)f|R(x_0)g\big) &= \int_0^1 (R(x_0)f)\,(\xi)\overline{(R(x_0)g)\,(\xi)}\,\mathrm{d}\xi \\ &= \int_0^1 f(\xi + x_0)\overline{g(\xi + x_0)}\,\mathrm{d}\xi \\ &= \int_0^1 f(\xi)\overline{g(\xi)}\,\mathrm{d}\xi \\ &= (f|g)\,,\end{aligned}$$

where we used the additive change of variables $\xi \mapsto \xi - x_0$.

It turns out that the one-dimensional subspaces $\mathbb{C}\chi_n$ spanned by χ_n are invariant for the right regular representation R. Indeed, given $x_0 \in \mathbb{R}$,

$$\begin{aligned}(R(x_0)\chi_n)\,(x) &= \chi_n(x + x_0) \\ &= e^{2\pi i n(x+x_0)} \\ &= e^{2\pi i n x_0}\chi_n(x),\end{aligned}$$

for all $x \in \mathbb{R}$, so that

$$R(x_0)\chi_n = e^{2\pi i n x_0}\chi_n \in \mathbb{C}\chi_n.$$

This fact shows that the Fourier series, written in terms of characters χ_n, is compatible with the structure of the right regular representation. This is the underlying reason for the beauty, but also the power, of the harmonic analysis.

Let us pause for the moment at this point to observe the rich algebraic structure present in the theory of Fourier series. The domain of the considered functions is the compact Abelian group $\mathbb{R}/\mathbb{Z}$ which acts on the space $L^2(\mathbb{R}/\mathbb{Z})$ of all these functions by unitary operators, that is, preserves the inner product. The unitary dual of the compact Abelian group $\mathbb{R}/\mathbb{Z}$, referred to as the Pontryagin dual in this context, consists of all unitary characters of $\mathbb{R}/\mathbb{Z}$, which form the Abelian group isomorphic to $\mathbb{Z}$. The unitary characters of $\mathbb{R}/\mathbb{Z}$ are compatible with the group action on $L^2(\mathbb{R}/\mathbb{Z})$ by the right regular representation, so that each of them spans an invariant one-dimensional subspace. This marvelous picture is the starting point for harmonic analysis in more complicated settings.

In a more general setting, the harmonic analysis is actually developed by considering the spaces invariant for the right regular representation of the appropriate L^2 space. These spaces should be irreducible unitary representations of the underlying group. The Fourier series in such setting is the decomposition according to these unitary representations, either as a direct sum or a direct integral. In any case, the harmonic analysis on locally compact topological groups crucially depends on the understanding and classification of the unitary representations of the underlying group.

Let G be a locally compact topological group with the Haar measure $\mathrm{d}g$. Let $L^2(G)$ be the space of square-integrable measurable complex functions on G, i.e., all measurable functions $f : G \to \mathbb{C}$ such that

$$\int_G |f(g)|^2 \, \mathrm{d}g < \infty.$$

The inner product on $L^2(G)$ is defined by

$$(f_1|f_2) = \int_G f_1(g)\overline{f_2(g)} \, \mathrm{d}g.$$

The right regular representation R of G on $L^2(G)$ is defined as follows. Given $g_0 \in G$ and $f \in L^2(G)$, the action is given by

$$(R(g_0)f)(g) = f(gg_0),$$

for all $g \in G$. It is unitary, because given $g_0 \in G$,

$$\begin{aligned}\big(R(g_0)f_1|R(g_0)f_2\big) &= \int_G \left(R(g_0)f_1\right)(g)\overline{\left(R(g_0)f_2\right)(g)} \, \mathrm{d}g \\ &= \int_G f_1(gg_0)\overline{f_2(gg_0)} \, \mathrm{d}g \\ &= \int_G f_1(g)\overline{f_2(g)} \, \mathrm{d}g \\ &= (f_1|f_2),\end{aligned}$$

where we made the change of variables $g \mapsto gg_0^{-1}$.

Observe how the basic objects and algebraic structure in the case of the group G are precisely parallel to those in the theory of classical Fourier series, i.e., the case of the group $G = \mathbb{R}/\mathbb{Z}$. The differences appear, because in the case of noncommutative and noncompact G, the G-invariant spaces of $L^2(G)$ are typically infinite-dimensional. Understanding of these spaces is an important step in the study of the unitary dual of G, although there are also other irreducible representations of G that are unitarizable. Thus, the classification of the unitary dual of G requires essentially more complicated techniques and approaches than the classical harmonic analysis of the Fourier series. However, despite the necessity for technically more advanced methods, the Tadić philosophy is that, at the end of the day, harmonic analysis must be simple, just as the classical theory of Fourier series. The evidence for such claim stems from the results in the cases in which the unitary dual is known, including the results of Tadić, as already pointed out above.

4. The Lefschetz principle

The Lefschetz principle for reductive groups was formulated by Harish-Chandra, inspired by the Lefschetz principle in algebraic geometry. The introduction to his lecture notes on harmonic analysis on reductive p-adic groups, given at the Institute for Advanced Study in 1969, begins with the following paragraph:

"The object of these lectures is to illustrate, what I like to call the Lefschetz principle, which, in the context of reductive groups, says that whatever is true for real groups is also true for p-adic groups" [19, p. 1].

And the rest of the introduction goes on to pinpoint examples and his expectations at the time of giving these lectures, how certain facts from the harmonic analysis on real groups should be transferred to the harmonic analysis on p-adic groups. The Lefschetz principle of Harish-Chandra should be understood as a philosophical principle, not a strict claim or a conjecture.

The Lefschetz principle is a reoccurring theme in the work of Tadić as well. However, in his work the flow of ideas and results is mostly in the opposite direction, that is, from p-adic groups to the real groups. This point is very nicely summarized in the joke by Paul Sally, mentioned at the end of Tadić's lecture notes [76]. Sally called the principle present in the work of Tadić the "Schefletz principle", because Tadić primarily proves the results in the case of p-adic groups, and then, the same ideas are extended to the case of real and complex groups. Perhaps the best way to combine the two principles is to refer to them as the unified approach to the representation theory of p-adic and real groups.

A prominent example of the Schefletz principle in the work of Tadić is the classification of the unitary dual of the general linear group [46]. In the p-adic case, this is recalled in Section 3. However, the theorem of exactly the same form holds in the case of the general linear group over the field of real and complex numbers [45], [69]. The only difference is in the amount of square-integrable representations from which the Speh representations $u(\delta, k)$, which serve as the building blocks of all unitary representations, are constructed. In the case of $GL(n, \mathbb{R})$, the possible δ are just the unitary characters of $GL(1, \mathbb{R}) \cong \mathbb{R}^\times$ and the square integrable representations of $GL(2, \mathbb{R})$ classified by their lowest $O(2)$-types. There are no square-integrable representations of $GL(n, \mathbb{R})$ for $n > 2$. In the case of $GL(n, \mathbb{C})$, the only possible δ are the unitary characters of $GL(1, \mathbb{C}) \cong \mathbb{C}^\times$, as there are no square-integrable representations of $GL(n, \mathbb{C})$ for $n > 1$. We also point out the paper [60] on metaplectic covers.

As explained in Section 6, there exist different approaches to the representation theory of the general linear group and classical groups. Over an archimedean local field, the representation theory can be studied using differentiation that leads to the representation theory of Lie algebras. However, in the p-adic case, passing to the p-adic Lie algebra does not yield any non-trivial results. The useful algebra in the p-adic case is the Hecke algebra of locally constant functions on the group. But in the archimedean case, the algebra of locally constant functions on a Lie group does not give any non-trivial results. Similar phenomenon happens with the restriction to a maximal compact subgroup, which is useful in the archimedean representation theory, but in the p-adic case must be replaced by the theory of types, which are restrictions, in some sense, to certain open compact subgroups.

The upshot of these considerations is that all these methods are extremely useful in one setting, but uninteresting in the other. In other words, they are far from the Lefschetz principle, and it seems as a miracle that the final outcome, for instance, the classification of the unitary dual of the general linear groups, can be formulated in the same way in both archimedean and p-adic case.

The Lefschetz principle in the case of the unitary dual of general linear groups is brought back to life by the external approach to the representation theory of general linear groups and classical groups. This approach of Tadić, described in Section 6, is uniform, independent of the underlying local field, and can be translated word-by-word from the p-adic setting to the archimedean setting. The only minor differences, which arise from the nature of the local field, are in the number of square-integrable representations, and certain technical facts.

The Lefschetz principle is closely related to the simplicity of harmonic analysis principle elaborated in Section 3. It seems that whenever the Lefschetz principle is well understood for certain problem in representation theory of reductive groups, the solution can be formulated in a simpler way, and even the proofs can be better understood and nicely structured. Somehow the understanding of the Lefschetz principle is a prerequisite for finding better points of view at the representation theory and non-commutative harmonic analysis in both archimedean and p-adic setting.

Another example of the Lefschetz principle is provided by the determinantal formula of Tadić [58]. It gives the characters of Speh representations in terms of characters of essentially square-integrable representations simultaneously in the p-adic and archimedean case. The characters of irreducible representations are one of the central problems in representation theory from its beginnings. Already in the classical book of Gel′fand and Naymark[1] [15], they point out the three main goals of representation theory of reductive groups. These are unitary duals, characters of irreducible representations and Plancherel measures. The influence of the work of Gel′fand and Naymark is present in the work of Tadić, in particular, in the early stages of his study of

[1]Although the official English transliteration would be Naymark, the name is often spelled as Naimark, Naĭmark or even Neumark. The last one is used in our list of references, in which we follow the transliteration found in *MathSciNet* and *zbMATH Open* databases.

unitary duals.

* * *

In the rest of this section, the determinantal formula is explained in more detail. In [58], Tadić expressed the characters of twists of Speh representations as linear combinations of the characters of essentially square-integrable representations, or in other words, in terms of the characters of standard representations. Note that the knowledge about the characters of Speh representations guarantees the knowledge about the characters of the whole unitary dual, according to the work of van Dyk [83]. This can be viewed as a special instance of the use of Kazhdan–Lusztig polynomials, but in general, these are very complicated, while the formula of Tadić, turned out to be very simple and elegant. He used a very interesting idea to formally transfer the problem from p–adic to complex groups, thus, providing more evidence for the Lefschetz, or more precisely, Schefletz principle. In both cases, he used information about the end of certain complementary series representations.

Determinantal formula (Tadić, [58, Thm. 5.4]). *Let ρ be an irreducible supercuspidal representation of $GL(d,F)$, where d is a positive integer. Let δ be the essentially square-integrable representation of $GL(m,F)$ defined above as*

$$\delta = \delta[\nu^{-\frac{l-1}{2}}\rho, \nu^{\frac{l-1}{2}}\rho],$$

where l is a positive integer such that $m = dl$. Let k be a positive integer and W_k the group of permutations of the set $\{1, 2, \dots, k\}$. Let

$$W_k^{(l)} = \{w \in W_k : w(i) + l \geq i \text{ for all } 1 \leq i \leq k\}.$$

Then, the following identity

$$\nu^{\frac{k+l}{2}} u(\delta, k) = \sum_{w \in W_k^{(l)}} (-1)^{sgn(w)} \prod_{i=1}^{k} \delta[\nu^i\rho, \nu^{w(i)+l-1}\rho].$$

holds in the Grothedieck group of the category of finite length representations of $GL(n,F)$, where $n = mk$ and $u(\delta,k)$ is the Speh representation defined as in Section 3, but δ is not necessarily unitarizable.

Note that the name of the formula stems from the alternating signs in the sum above, and the summation is over the set which is an obvious subset of the symmetric group W_k, just as in the definition of the determinant.

The argument of Tadić was later even simplified by Chenevier and Renard [11]. This was generalized by Lapid and Mínguez [23] to a wider class of representations, the so-called "ladder representations", encompassing the Speh representations, using Jacquet modules and completely avoiding the unitarizability issues.

5. Comparison of Jacquet modules

In the representation theory of p-adic groups one of the most fundamental objects is the parabolically induced representation introduced in Section 2. It is the most important construction of representations arising from representations of smaller groups. The complete understanding of reducibility of parabolically induced representations and their composition series is required in the classification of irreducible representations and irreducible unitary representations of the considered group.

One of the powerful tools in the study of parabolically induced representations are Jacquet modules. The reasons for that are the Frobenius reciprocity, which relates the intertwining operators on the level of the induced representation of the full group to the intertwining operators of Jacquet modules on the level of the Levi subgroups, and the geometric lemma and the related structural formula, recalled in Section 6, which determines the Jacquet module of parabolically induced representations.

Let G be a reductive p-adic group and $P = MN$ a parabolic subgroup with the Levi subgroup M and the unipotent radical N. Given a representation π of a reductive p-adic group G in a complex vector space V, the Jacquet module of π with respect to P is the representation $r_M^G(\pi)$ of the Levi subgroup M acting on the space

$$V_N = V/V(N), \text{ where } V(N) = \operatorname*{span}_{u\in N, v\in V} \{\pi(u)v - v\}$$

by the formula

$$\Big(\big(r_M^G(\pi)\big)(m)\Big)(v + V(N)) = \delta_P(m)^{-1/2}\pi(m)v + V(N),$$

where $m \in M$, $v \in V$ and δ_P is the modular function of P. The role of the modular function is a convenient normalization of Jacquet modules, as in the definition of parabolic induction in Section 2.

The fundamental relation between induced representations and Jacquet modules is the Frobenius reciprocity. It says that, given a finite length representation π of G and a finite length representation σ of M, there is an isomorphism

$$\operatorname{Hom}_G\left(\pi, \operatorname{Ind}_P^G(\sigma)\right) \cong \operatorname{Hom}_M\left(r_M^G(\pi), \sigma\right)$$

of spaces of intertwining operators. In other words, the functor of taking Jacquet modules is left adjoint to the functor of parabolic induction.

The Jacquet modules are not the invention of Tadić. They were used earlier by Jacquet [20], Casselman [10] and Bernstein–Zelevinsky [7]. However, his idea to apply Jacquet modules with respect to several different parabolic subgroups and compare the results turned out extremely fruitful. His main point is that the Jacquet module techniques yield better results for larger groups, simply because there are more parabolic subgroups at our disposal [61], [36]. This idea is referred to as the Tadić philosophy by Blondel in [8].

To put it in his own words, quoted from [61], "Surprisingly, Jacquet modules are much more powerful in more complicated situations. The reason for this is simple: there are more standard parabolic subgroups. Therefore, one can compare information on Jacquet modules coming from different parabolic subgroups. For this approach to be most effective, one needs to direct one's attention to Jacquet modules with respect to large parabolic subgroups (which was not usually done in the early applications of this use of Jacquet modules). We note that this approach is particularly convenient for classical groups because the Levi subgroups of their parabolic subgroups are direct products of general linear groups and smaller classical groups."

The limitations of the Jacquet module techniques in the case of smaller groups are best observed in the case of $G = SL(2,F)$, as pointed out in [61]. In that case, the only proper parabolic subgroup, up to conjugacy, is the Borel subgroup B of upper-triangular matrices in $SL(2,F)$. Its Levi subgroup is the maximal split torus $T \cong F^\times$ of diagonal matrices in $SL(2,F)$. Given a multiplicative character χ of $F^\times$, one may construct the parabolically induced representation

$$\mathrm{Ind}_B^{SL(2,F)}(\chi).$$

Since B is, up to conjugacy, the only proper parabolic subgroup of $SL(2,F)$, the only Jacquet module that can be considered in this setting is

$$r_T^{SL(2,F)}\left(\mathrm{Ind}_B^{SL(2,F)}(\chi)\right),$$

which is not sufficient to solve even the basic questions. For example, in the case of χ^2 is trivial, the above induced representation is irreducible if χ is trivial and reducible if χ is not trivial. In the case $\chi = |\,|^\alpha$, with $\alpha \in \mathbb{R}$, it is irreducible if and only if $\alpha \neq \pm 1$. These facts cannot be obtained by a simple application of Jacquet modules.

On the other hand, in case of larger classical group G, the Jacquet module techniques provide reducibility and irreducibility criteria arising from comparing Jacquet modules with respect to different parabolic subgroups. Such criteria proved to be extremely useful in the study of parabolically induced representations in a series of papers of Tadić, e.g. [62], [61], [63], [31], [59], [64], [72], [71], [53], [66], [65], [56], [17].

* * *

As an example, which highlights the flavor of these results, we recall now the first and simplest reducibility criterion of [62, Sect. 3].

Reducibility criterion (Tadić [62, Lemma 3.1]). *Let G be a reductive p-adic group. Let π, π' and Π be representations of G of finite length. Suppose that* *(i)* $\pi \leq \Pi$ *and* $\pi' \leq \Pi$,

(ii) *there exist parabolic subgroups* P_1 *and* P_2 *of* G*, with Levi subgroups* M_1 *and* M_2*, respectively, such that*

$$r^G_{M_1}(\pi) \not\leq r^G_{M_1}(\pi'),$$
$$r^G_{M_2}(\pi) + r^G_{M_2}(\pi') \not\leq r^G_{M_2}(\Pi).$$

Then, π *is reducible and has a common irreducible subquotient with* π'*.*

Even without going into details, it is now clear why larger groups are more suitable for Jacquet module techniques. There are more parabolic subgroups, and therefore more possible choices for P_1 and P_2 in condition (ii) of Lemma 5. The comparison of Jacquet modules with respect to carefully chosen P_1 and P_2 yields the reducibility results required in applications.

In practice, as observed in [62, Rmk. 3.2], the reducibility criterion is applied to the setting in which the representations π, π' and Π are parabolically induced representations. More precisely, let P, P' and Q be parabolic subgroups of G, with Levi subgroups M, M' and L, respectively. Let σ, σ' and Σ be irreducible representations of M, M' and L. Then, we take π, π' and Π to be the induced representations

$$\pi = \mathrm{Ind}^G_P(\sigma),$$
$$\pi' = \mathrm{Ind}^G_{P'}(\sigma'),$$
$$\Pi = \mathrm{Ind}^G_Q(\Sigma).$$

The conditions (i) and (ii) of Lemma 5 in this setting read

(i) $\mathrm{Ind}^G_P(\sigma) \leq \mathrm{Ind}^G_Q(\Sigma)$ and $\mathrm{Ind}^G_{P'}(\sigma') \leq \mathrm{Ind}^G_Q(\Sigma)$

(ii) there exist parabolic subgroups P_1 and P_2 of G, with Levi subgroups M_1 and M_2, respectively, such that

$$r^G_{M_1}\left(\mathrm{Ind}^G_P(\sigma)\right) \not\leq r^G_{M_1}\left(\mathrm{Ind}^G_{P'}(\sigma')\right),$$
$$r^G_{M_2}\left(\mathrm{Ind}^G_P(\sigma)\right) + r^G_{M_2}\left(\mathrm{Ind}^G_{P'}(\sigma')\right) \not\leq r^G_{M_2}\left(\mathrm{Ind}^G_Q(\Sigma)\right).$$

Observe that condition (ii) contains Jacquet modules of induced representations, which should be determined. But this boils down to the so-called geometric lemma and the closely related structural formula, which is a slightly different story, and will be revisited in the context of classical groups in Section 6.

6. External approach to unitary duals

In the representation theory of non-compact non-commutative reductive groups over archimedean local fields, that is, the fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers, respectively, the methods which can be referred to as "internal" appear to be very useful. These methods substantially exploit the internal structure of the considered representations. The linearization of the problem by differentiation leads to the study of the representations of the Lie algebra of the considered Lie group, and the restriction to a maximal compact subgroup allows the use of the well-known representation theory of compact Lie groups. The combination of these two ideas was pursued in the work of many people in the field of real reductive groups, among which we only mention the work of Vogan [84].

In the case of p-adic reductive groups, these internal methods of the archimedean theory are not appropriate. The study of the p-adic Lie algebra and its complex representations, for instance, does not yield any non-trivial results. One of the ways to circumvent this problem is to use a different approach, which can be referred to as "external", because it never considers the internal structure of representations. It instead considers a certain algebraic structure on the set of isomorphism classes of representations of the given p-adic group. The original idea goes back to the work of Bernstein and Zelevinsky [87] in the case of the general linear group over a p-adic field. However, Tadić used this approach extensively in the case of general linear groups, extended it to the case of classical p-adic groups, and showed that the same approach works for the general linear groups over archimedean local fields as well.

The external approach to unitary duals led Tadić to the complete classification of the unitary dual of the general linear group over any local field, except that in the case of an archimedean local field the proof relied on certain technical conjecture of Kirillov. The conjecture of Kirillov was announced in [22], but not proved at the time of Tadić's classification result. It was later proved by Baruch in [6].

The statement of the classification is already given in Section 3 in the case of the general linear group over a p-adic field as an example of the simple final answer to a complicated problem in the non-commutative harmonic analysis. However, the same statement is true for the archimedean case, as well, which provides a superb example of the Lefschetz principle, as already mentioned in Section 4.

Even the unitary dual of the general linear group over a p-adic division algebra D can be approached by the same external method [52]. However, in this case, the complete proof at the time depended on certain facts, denoted by (U0)–(U4) by Tadić. The facts (U3) and (U4) are already proved in [52], as well as the fact that (U0) and (U1) together imply (U2). When Badulescu and Renard proved (U1) in [5], it remained to prove (U0), which reads as follows:

(U0) Let σ_1 and σ_2 be irreducible unitary representations of the general linear groups $GL(n_1, D)$ and $GL(n_2, D)$, respectively. Then, the parabolically induced representation $\sigma_1 \times \sigma_2$ of $GL(n_1+n_2, D)$ is irreducible.

The proof of (U0) was finally provided by Sécherre [37], who generalized the theory of types of Bushnel–Kutzko [9] to the case of the general linear group over a p-adic division algebra. Observe that the theory of types can be seen as the internal approach to the representation theory of p-adic groups, as it considers the restrictions, in some sense, to certain open compact subgroups. However, the recent work of Lapid and Mínguez [24] provides the complete proof of the classification of the unitary dual of $GL(n, D)$ in the spirit of Tadić's external approach. Their main point is that the classification is obtained from a weaker version of (U0), and then the original (U0) is a consequence of the classification. We mention some other contributions of Tadić to further understanding of the representation theory of general linear groups over division algebras and their relation to the representation theory of general linear groups over fields [75], [67], [78]. Since special linear groups are closely related to general linear groups, as their derived subgroups, their representation theories are closely related, and the exact form of it was elaborated by Tadić in [54]. The external approach also led to several contributions of Tadić to the representation theory of classical p-adic groups [62], [61], [63], [31], [59], [64].

* * *

Let us now describe in more detail the external approach to the representation theory of the general linear group and classical groups over a p-adic field, which is one of the reoccurring themes in the work of Tadić.

Let $\mathcal{R}_n$ denote the category of representations of $GL(n, F)$ of finite length. Let R_n be its Grothendieck group, that is, the free Abelian group with the $\mathbb{Z}$-basis consisting of the set of isomorphism classes of irreducible representations of $GL(n, F)$. The canonical map from the category $\mathcal{R}_n$ to its Grothendieck group R_n is called semi-simplification and denoted simply by s. s., as in Section 2. Given an object π in $\mathcal{R}_n$, its semi-simplification is the sum

$$\mathrm{s.\,s.}(\pi) = \sigma_1 + \cdots + \sigma_\ell,$$

where $\sigma_1, \ldots, \sigma_\ell$ are the (possibly isomorphic) irreducible subquotients of π, and ℓ is the length of π. Let

$$R = \bigoplus_{n=0}^{\infty} R_n$$

be the sum of Grothendieck groups, which carries the structure of a $\mathbb{Z}_{\geq 0}$-graded Abelian group. Note that we set $GL(0, F)$ to be the trivial group, and the basis of R_0 is the trivial representation of the trivial group $GL(0, F)$.

The multiplication on R is defined using parabolic induction. Given two irreducible representations $\pi_1 \in R_{n_1}$ and $\pi_2 \in R_{n_2}$ of the general linear groups $GL(n_1, F)$ and $GL(n_2, F)$, respectively, their product $\pi_1 \times \pi_2$ is an element of the Grothedieck group $R_{n_1+n_2}$, i.e., a representation of $GL(n_1+n_2,F)$, defined using parabolic induction as

$$\pi_1 \times \pi_2 = \mathrm{s.\,s.}\left(\mathrm{Ind}_{P_{(n_1,n_2)}}^{GL(n,F)}(\pi_1 \otimes \pi_2)\right),$$

where $P_{(n_1,n_2)}$ is the parabolic subgroup of $GL(n_1+n_2,F)$ with the Levi subgroup isomorphic to the product $GL(n_1,F)\times GL(n_2,F)$. The multiplication is then extended to arbitrary elements of R by $\mathbb{Z}$-linearity. The resulting multiplication map

$$m : R \otimes R \to R$$

is a $\mathbb{Z}$-linear map of $\mathbb{Z}_{\geq 0}$-graded Abelian groups. According to the properties of parabolic induction, the multiplication m gives rise to the structure of $\mathbb{Z}_{\geq 0}$-graded $\mathbb{Z}$-algebra on R. The unit element is the trivial representation $\mathbf{1}$ of the trivial group $GL(0,F)$, which may be viewed as a map $e : \mathbb{Z} \to R$ given by the assignment $e(k) = k\mathbf{1}$ for every integer $k \in \mathbb{Z}$.

Similarly, the comultiplication on R is defined in terms of Jacquet modules. Given an irreducible representation $\pi \in R_n$ of the general linear group $GL(n,F)$, the coproduct of π is defined using Jacquet modules as

$$m^*(\pi) = \sum_{k=0}^{n} \mathrm{s.\,s.}\left(r_{M_{(k,n-k)}}^{GL(n,F)}(\pi)\right),$$

where $M_{(k,n-k)} \cong GL(k,F)\times GL(n-k,F)$ is the Levi subgroup of the parabolic subgroup $P_{(k,n-k)}$ of $GL(n,F)$, and $r_{M_{(k,n-k)}}^{GL(n,F)}$ stands for the Jacquet module with respect to $P_{(k,n-k)}$, as above. The summands on the right-hand side are elements of $R_k \otimes R_{n-k}$, so that their sum is in the tensor product $R\otimes R$. The definition of m^* is extended to all of R by $\mathbb{Z}$-linearity. In other words,

$$m^* : R \to R \otimes R$$

is a comultiplication on R. It is coassociative due to the transitivity of Jacquet modules. The counit element in R is the $\mathbb{Z}$-linear map $e^* : R \to \mathbb{Z}$, defined on the $\mathbb{Z}$-basis of R by the assignment

$$e^*(\pi) = \begin{cases} 1, & \text{if } \pi = \mathbf{1}, \\ 0, & \text{otherwise,} \end{cases}$$

where π is an irreducible representation of $GL(n,F)$, and $\mathbf{1}$ denotes the trivial representation of the trivial group $GL(0,F)$. The comultiplication and counit give rise to the structure of $\mathbb{Z}_{\geq 0}$-graded $\mathbb{Z}$-coalgebra on R.

The algebra and coalgebra structures on R are compatible in the sense that the comultiplication is an algebra homomorphism of R and $R\otimes R$, or equivalently, the multiplication is the coalgebra homomorphism of $R\otimes R$ and R. This fact means that R exhibits a bialgebra structure. The final ingredient for the Hopf algebra is an anti-involution ϑ on R such that

$$m \circ (id \otimes \vartheta) \circ m^* = e \circ e^*.$$

It turns out that there is a unique such ϑ.

The Hopf algebra structure on R is a strong tool for the study of representation theory of $GL(n,F)$. However, the theory of derivatives of Gel′fand and Kazhdan [16] was historically the main tool for the study of representations of the general linear group over a p-adic field. Nevertheless, the external approach, as pursued by Tadić, led to the classification of the unitary dual of the general linear group over all local fields, including archimedean local fields, as well as over division algebras over local fields. Moreover, the external approach provided a unified proof of the classification of the unitary dual of the general linear group over all local fields and division algebras over local fields.

We proceed now to the external approach to the representation theory of classical p-adic groups. For simplicity of exposition, we stick to the cases of the symplectic and odd special orthogonal p-adic groups. Let G_n denote either the symplectic group $Sp(2n,F)$, or the odd special orthogonal group $SO(2n+1,F)$, over a p-adic field F. We follow closely the exposition in [59].

Similarly as in the case of the general linear group, let $\mathcal{R}_n(G)$ be the category of representations of G_n of finite length. Let $R_n(G)$ denote the Grothendieck group of $\mathcal{R}_n(G)$. It is a free Abelian group with the $\mathbb{Z}$-basis consisting of isomorphism classes of irreducible representations of G_n. Let

$$R(G) = \bigoplus_{n=0}^{\infty} R_n(G).$$

It is a $\mathbb{Z}_{\geq 0}$-graded Abelian group. For $n = 0$, the group G_0 is the trivial group, and the trivial representation of G_0 is again denoted by $\mathbf{1}$.

The Abelian group $R(G)$ has no multiplicative structure of a $\mathbb{Z}$-algebra. However, since the Levi subgroups of G_n are products of the general linear groups and a smaller classical group of the same type, the parabolic induction gives rise to an R-module structure on $R(G)$. More precisely, given an irreducible representation $\pi \in R_m$ of $GL(m,F)$ and an irreducible representation $\sigma \in R_n(G)$ of G_n, the scalar multiplication $\pi \rtimes \sigma$ is defined using parabolic induction as

$$\pi \rtimes \sigma = \mathrm{s.\,s.} \left(\mathrm{Ind}_{P_{(m;n)}}^{G_{m+n}} (\pi \otimes \sigma) \right),$$

where $P_{(m;n)}$ is the parabolic subgroup of G_{m+n} with the Levi subgroup isomorphic to the product $GL(m,F) \times G_n$. The result is an element of $R_{m+n}(G)$. Extending the scalar multiplication $\mathbb{Z}$-bilinearly gives rise to the map

$$\mu : R \times R(G) \to R(G),$$

which is a scalar multiplication on $R(G)$ with scalars from R. Thus, the $\mathbb{Z}_{\geq 0}$-graded Abelian group $R(G)$ carries the structure of $\mathbb{Z}_{\geq 0}$-graded R-module.

The R-comodule structure on $R(G)$ is defined as follows. Given an irreducible representation σ of G_n, the scalar comultiplication $\mu^*(\sigma)$ is defined using Jacquet modules as

$$\mu^*(\sigma) = \sum_{k=0}^{n} \mathrm{s.\,s.}\left(r^{G_n}_{M_{(k;n-k)}}(\sigma) \right),$$

where $M_{(k;n-k)} \cong GL(k,F) \times G_{n-k}$ is the Levi subgroup of a parabolic subgroup $P_{(k;n-k)}$ of G_n, and $r^{G_n}_{M_{(k;n-k)}}$ is the Jacquet module with respect to $P_{(k;n-k)}$. Clearly, $\mu^*(\sigma)$ is an element of $R \otimes R(G)$. The $\mathbb{Z}$-linear extension of μ^* gives rise to the scalar comultiplication as a map

$$\mu^* : R(G) \to R \otimes R(G).$$

Since

$$(1 \otimes \mu^*) \circ \mu^* = (m^* \otimes 1) \circ \mu^*$$

as maps from $R(G)$ to $R \otimes R \otimes R(G)$, it follows that μ^* is coassociative. The R-comodule structure on $R(G)$ is also $\mathbb{Z}_{\geq 0}$-graded.

So far, we have the R-module and R-comodule structures on $R(G)$, given by μ and μ^*, respectively. In order to have the Hopf R-module structure on $R(G)$, it is required that μ and μ^* are compatible in the sense that $\mu^* : R(G) \to R \otimes R(G)$ is a homomorphism of R-modules with the R-module structure on $R \otimes R(G)$ given by the comultiplication m^* on R. More precisely, given $r_0 \in R$ and $r \otimes s$ in $R \otimes R(G)$, the scalar multiplication is defined as

$$r_0 \,.(r \otimes s) = m^*(r_0)(r \otimes s),$$

where the multiplication by $m^*(r_0)$ on the right-hand side is given by the $R \otimes R$-module structure on $R \otimes R(G)$ defined by m on the first and by μ on the second factor. However, it turns out that this requirement for the Hopf R-module structure on $R(G)$ is not satisfied. This fact can be observed by looking at the representation theoretic consequences, which the Hopf R-module structure on $R(G)$ would imply. A simple argument leading to contradiction is given in [59, Rmk. 7.3].

Hence, it is necessary to introduce certain twisted variant of the Hopf R-module structure on $R(G)$. This is achieved by twisting the R-module structure on $R \otimes R(G)$ in the definition of the Hopf R-module. More precisely, the m^* in the definition of the scalar multiplication on $R \otimes R(G)$ by scalars in R should be replaced with another ring homomorphism from R to $R \otimes R$. This homomorphism is denoted by M^* and defined as follows. Let $\sim$ stand for the contragredient operator on irreducible representations of general linear groups, extended $\mathbb{Z}$-linearly to all of R. It is an automorphism of $\mathbb{Z}$-algebra R. Let t be the ring endomorphism on $R \otimes R$ which interchanges the two factors, i.e., $t(r_1 \otimes r_2) = r_2 \otimes r_1$ for $r_1, r_2 \in R$. Then, $M^* : R \to R \otimes R$ is defined as

$$M^* = (m \otimes 1) \circ (\sim \otimes m^*) \circ t \circ m^*.$$

It is a ring homomorphism. The M^*-Hopf R-module structure is defined by the requirement that the comultiplication $\mu^* : R(G) \to R \otimes R(G)$ is a homomorphism of R-modules, where the scalar multiplication on $R \otimes R(G)$ is defined as

$$r_0 .(r \otimes s) = M^*(r_0)(r \otimes s),$$

and the scalar multiplication on the right-hand side is defined as above. One of the main points of the paper [59] is that $R(G)$ is indeed the M^*-Hopf R-module.

The twisted Hopf module structure on $R(G)$ leads to the full and systematic exploitation of the Jacquet module machinery. Together with the geometric lemma and the reducibility criteria mentioned in Section 5, it enabled Tadić to obtain many important results in representation theory and to continue the quest for the unitary dual of classical groups.

7. Unitarizability along the lines

In this section we consider the unitarizability problem for irreducible representations of classical p-adic groups. For our purposes, it is sufficient to have in mind the two classical groups introduced in Section 2, so that $G = Sp(2n, F)$ or $G = SO(2n+1, F)$. The problem is easily reduced to the case of so-called weakly real representations. This is a technical condition, omitted here, but recalled below in the second part of this section.

Given a weakly real irreducible representation π of G, there is a finite set of representations of classical groups of the same type and smaller (or equal) rank, associated to π by Jantzen [21]. These are denoted by $X_{\rho_i}(\pi)$, indexed by a certain finite set of self-contragredient supercuspidal representations ρ_i of general linear groups. Each $X_{\rho_i}(\pi)$ belongs to a family of representations of the classical groups which is referred to as the line of representations. This is related to the form of their supercuspidal support, but we leave the details for the second part of this section.

Jantzen has already proved in [21] that the passage from π to $X_{\rho_i}(\pi)$ preserves several fundamental properties, such as square-integrability, for instance. The idea of Tadić in this context can be summarized by two questions he posed in [68], see also [80].

Unitarizability along the lines questions (Tadić, [68]). *Let π be a weakly real irreducible representation of G. Let $X_{\rho_i}(\pi)$, for $i = 1, \dots, k$, be the representations associated to π in [21].*

1. ***Preservation of unitarizability:*** *Is it true that π is unitarizable if and only if all $X_{\rho_i}(\pi)$ are unitarizable, i.e., could unitarizability problem be reduced to unitarizability along the lines?*
2. ***Independence of unitarizability:*** *Is it possible to transfer unitarizability results between different lines of representations under certain natural conditions?*

These two questions are somehow the guideline for a possible approach to the unitarizability problem for classical p-adic groups. The affirmative answer to the first question would reduce the unitarizability problem to the unitarizability along the lines, that is, to the unitarizability problem for representations belonging to a line of representations. The affirmative answer to the second question would further reduce the study of unitarizability along the lines to certain special types of lines of representations.

The motivation for these two questions arises from the classification of the unitary dual of general linear groups. In Section 3, we recalled the classification of the unitary dual of the general linear group over a local field. However, the case of the general linear group over a local division algebra, takes a similar form, except that the so-called supercuspidal reducibility is at different points. That is, if ρ and ρ' are supercuspidal representations of general linear groups over a local field, then the induced representation $\rho \times \rho'$ reduces if and only if $\rho' \cong \nu^{\pm 1}\rho$. However, if ρ and ρ' are supercuspidal representations of general linear groups over a local division algebra, then the induced representation $\rho \times \rho'$ reduces if and only if $\rho' \cong \nu^{\pm s_0}\rho$, where s_0 depends on ρ. The latter situation is what inspires the unitarizability along the lines approach. Essentially, the independence of unitarizability would imply that the solution to the unitarizability problem depends only on the point s_0 at which the induced representation reduces.

Although it is still not clear whether the unitarizability along the lines will be the right approach to the unitary dual of classical p-adic groups, it is a great example of Tadić's way of thinking. He always seeks for unifying arguments between different settings which reduce the considered problem to certain fundamental facts. In a way, his approaches are always very structured and uniform.

* * *

Let π be an irreducible representation of the classical p-adic group G_n, as above, where n blue stands for its rank. As already mentioned, the problem of unitarizability of π can be reduced to the case of so-called weakly real representations. This means that there exists a parabolic subgroup Q, with the Levi subgroup isomorphic to the product $GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$ with $n_1 + \cdots + n_k + n' = n$, selfcontragredient supercuspidal representations ρ_i' of $GL(n_i, F)$, a supercuspidal representation σ of $G_{n'}$, and real numbers $x_i \in \mathbb{R}$, for $i = 1, \ldots, k$, such that π is a subrepresentation of the induced representation

$$\pi \hookrightarrow \mathrm{Ind}_Q^{G_n}\left(\nu^{x_1}\rho_1' \otimes \nu^{x_2}\rho_2' \otimes \cdots \otimes \nu^{x_k}\rho_k' \otimes \sigma\right).$$

In this situation, we say that σ is the partial supercuspidal support of π. For a given π, such σ is uniquely determined.

Let X be a set of irreducible supercuspidal representations of general linear groups. Assume that $X = \widetilde{X} = \{\widetilde{\rho} : \rho \in X\}$. For an irreducible representation π of a classical p-adic group, one says that it is supported by $X \cup \{\sigma\}$ if there exist $\rho_i \in X$ such that π is a subrepresentation of the induced representation

$$\pi \hookrightarrow \mathrm{Ind}_P^{G_n}(\rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma),$$

where P is an appropriate parabolic subgroup of G_n.

Let ρ be an irreducible selfcontragredient supercuspidal representation of a general linear group. We define the line X_ρ of supercuspidal representations associated to ρ as

$$X_\rho = \{\nu^x \rho : x \in \mathbb{R}\}.$$

We denote by $\mathrm{Irr}(X_\rho, \sigma)$ the set of all isomorphism classes of irreducible representations of the considered classical groups supported by $X_\rho \cup \{\sigma\}$.

For the given weakly real irreducible representation π of the classical group G_n and an irreducible selfcontragredient supercuspidal representation ρ of a general linear group, there exists a representation of the classical group $G_{m'}$ of rank $m' \leq n$, denoted by $X_\rho(\pi)$, supported by $X_\rho \cup \{\sigma\}$ and an irreducible representation of the general linear group $GL(m, F)$ with $m + m' = n$, denoted by $X_\rho^c(\pi)$, supported outside of X_ρ, such that π is a subrepresentation of the induced representation

$$\pi \hookrightarrow \mathrm{Ind}_{P_{(m;m')}}^{G_n}\left(X_\rho^c(\pi) \otimes X_\rho(\pi)\right),$$

where $P_{(m;m')}$ is a parabolic subgroup of G_n with the Levi subgroup isomorphic to the product $GL(m, F) \times G_{m'}$.

Obviously, for all but finitely many selfcontragredient ρ, we have that $X_\rho(\pi) \cong \sigma$. Those for which we have $X_\rho(\pi) \ncong \sigma$, we denote by $\rho_1, \ldots, \rho_k$. Jantzen has proved in [21] that the correspondence

$$\pi \leftrightarrow (X_{\rho_1}(\pi), X_{\rho_2}(\pi), \ldots, X_{\rho_k}(\pi))$$

is a bijection between the set of all irreducible representations supported by $X_{\rho_1} \cup \cdots \cup X_{\rho_k} \cup \{\sigma\}$ and the product $\prod_{i=1}^k \mathrm{Irr}(X_{\rho_i}, \sigma)$.

Having all the notation in place, we can now explain more thoroughly the two questions of Tadić raised at the beginning of this section. The question regarding the preservation of unitarizability can be rephrased as the question whether the bijection defined by Jantzen preserves unitarizability. Although Jantzen has shown that this bijection preserves some of the fundamental properties of representations, such as square-integrability, unitarizability is not among them. The affirmative answer to this question would reduce the unitarizability problem to the problem about unitarizability of the representations supported by just one line.

Let us now explain what the question regarding independence of unitarizability means. Assuming that the answer to the first question is positive, the unitarizability question reduces to the question about the unitarizability of the representations supported by just one line. The independence of unitarizability question asks whether it could be that the unitarizable representations supported in two different lines, but additionally satisfying a natural assumption, have totally analogous description. More precisely, given a self-contragredient supercuspidal representation ρ of a general linear group and a supercuspidal representation σ of a classical group, as above, there exists a unique $\alpha_{\rho,\sigma} \in \mathbb{R}_{\geq 0}$ such that

$$\nu^{\alpha_{\rho,\sigma}} \rho \rtimes \sigma$$

reduces. Moreover, since the characteristic of the p-adic field F is zero, it follows from the work of Arthur and Mœglin [86] that $\alpha_{\rho,\sigma} \in \frac{1}{2}\mathbb{Z}$. If there is another pair ρ', σ' such that $\alpha_{\rho,\sigma} = \alpha_{\rho',\sigma'}$, there is a canonical bijection

$$E : \mathrm{Irr}(X_\rho, \sigma) \to \mathrm{Irr}(X_{\rho'}, \sigma').$$

The second question now pertains to the following question: is it true that $\pi \in \mathrm{Irr}(X_\rho, \sigma)$ is unitarizable if and only if $E(\pi) \in \mathrm{Irr}(X_{\rho'}, \sigma')$ is unitarizable?

The evidence for the positive answer to both of these questions of Tadić is provided by the case of generic representations [25] and unramified representations [35].

8. Topology of the unitary dual

A keen interest in the topology and geometry of the unitary and non-unitary duals of reductive groups is a common thread in Tadić's work for decades [41], [50], [48], [42], [51], [39], [73]. Some observations and findings obtained from the topology of these duals of reductive algebraic groups, both p-adic and real, were the primary motivation for future studies in harmonic analysis on these groups. These findings, in turn, informed very interesting arithmetic interpretations.

Although the following considerations are valid for any reductive algebraic group over a local field, for simplicity of exposition we consider only the groups G introduced in Section 2 over a p-adic field F.

Let $\widehat{G}$ denote the unitary dual of G, that is, the set of isomorphism classes of unitarizable irreducible representations of G. Then, $\widehat{G}$ is naturally a topological space with the topology as defined below in the second part of this section.

Let $\widetilde{G}$ denote the non-unitary dual of G, that is, the set of isomorphism classes of all irreducible representations of G. Certain convenient characterization of the topology on $\widehat{G}$ in terms of convergence of matrix coefficients can be used to define a topology on $\widetilde{G}$. It turns out that $\widehat{G}$ is closed in $\widetilde{G}$. Tadić has also shown many different characterizations of this topology on $\widetilde{G}$. In particular, this is exactly the topology introduced by Fell in [14].

It turns out, as Tadić observed, that even if one is only interested in the unitary dual $\widehat{G}$, certain knowledge of $\widetilde{G}$ is indispensable. For example, the reducibility of the parabolically induced representations can be formulated in terms of number of limits of certain converging sequences in the non-unitary dual $\widetilde{G}$, see [50] for more details. Note that the topology of $\widetilde{G}$ is not Hausdorff. The irreducibility and reducibility of the parabolic induction is crucial in the determination of complementary series representation and parabolic reduction arguments, which are both very important in the determination of the unitary dual.

To support these claims, consider the general linear group $GL(n,F)$. Let $u(\delta,k)$ be the Speh representation, defined in Section 3, where δ is a square-integrable representation of $GL(m,F)$ and k a positive integer such that $km = n$. The Tadić classification of the unitary dual of $GL(n,F)$, recalled in Section 3, implies that the Speh representation $u(\delta,k)$ is unitarizable, as well as the induced representation $\nu^{\alpha}u(\delta,k)\times\nu^{-\alpha}u(\delta,k)$ for $\alpha\in\langle 0,\frac{1}{2}\rangle$. The latter is an example of a complementary series representation. The irreducibility of the induced representation for α in that range is crucial for its unitarizability.

On the other hand, one of the local ways to establish the unitarizability of the Speh representations $u(\delta,k)$ is by induction over k. For $k=1$, the Speh representation is just $u(\delta,1)=\delta$, which is unitary. Observing that $u(\delta,k-1)\times u(\delta,k+1)$ appears as a subquotient of $\nu^{\alpha}u(\delta,k)\times\nu^{-\alpha}u(\delta,k)$ at $\alpha=\frac{1}{2}$, i.e., at the end of the complementary series, together with the fact that $u(\delta,k-1)\times u(\delta,k+1)$ is irreducible, parabolic reduction gives that $u(\delta,k+1)$ must be unitarizable, as well. This illustrates the importance of reducibility and irreducibility information to gain knowledge about the unitary dual.

From the Tadić classification of the unitary dual of $GL(n,F)$, recalled in Section 3, it is transparent what a prominent role the Speh representations have. They appear to be the building blocks for the unitary dual, by using relatively simple procedures like parabolic induction or formation of the complementary series. On the other hand, they are isolated in the unitary dual of $GL(n,F)$, except in the special cases of $k=2$ and of δ attached to a segment of length two. The idea of Tadić that this topological property is what makes them very important in the construction of the whole unitary dual turned out to be very fruitful. They are indeed the building blocks for the unitary dual of the general linear groups, both in the archimedean and p-adic case, and, in both cases, they are topologically isolated. On the other hand, cohomological representations are very rare in the p-adic case, unlike in the real case, so that the cohomological argument does not, at least not in a straightforward manner, bring a unifying argument. However, as we have just seen, topology of the unitary dual does! For the record, this is another example of the Lefschetz principle, described in Section 4, present in the work of Tadić.

His philosophy that it is very important to find isolated representations in the appropriate duals gave important results in different contexts. For example, for a general reductive group G over a p-adic field, it is proved in [50] that the representation π is isolated modulo center in $\widetilde{G}$ if and only if it is supercuspidal. It is also known that the isolated representations in $L^2(G)$, in which the support in the corresponding direct integral decomposition consists of tempered representations, are precisely square-integrable representations [18]. In both cases, we see that all the representations isolated in the appropriate unitary duals have certain direct characterization in the harmonic analysis on that group.

Topological consideration of various other types of duals, especially those arising from the setting of adèlic groups, such as the automorphic dual and Ramanujan dual at the local place, yield striking arithmetic information. Tadić pursued these ideas and obtained interesting arithmetic consequences, which we elaborate below.

* * *

We begin the technical part of this section with a definition of topology on the unitary dual $\widehat{G}$, where G is one of the groups introduced in Section 2. Let $\mathcal{C}_0(G)$ denote the convolution algebra of locally constant compactly supported functions on G. One can introduce a norm on $\mathcal{C}_0(G)$ in the following way

$$||f|| = \sup\{||\pi(f)|| : \pi \in \widehat{G}\},$$

where $f \in \mathcal{C}_0(G)$, and the norm on the right-hand side is the operator norm. Recall that $\pi(f)$ is defined as

$$\pi(f)v = \int_G f(g)\pi(g)v\, dg,$$

where v is a vector in the space on which π acts, and dg is a Haar measure on G.

Let $C^*(G)$ be the completion of $\mathcal{C}_0(G)$ with respect to this norm. It is a C^* algebra. We denote by $\widehat{C^*(G)}$ the set of isomorphism classes of all non-zero representations of $C^*(G)$. This set is in bijection with $\widehat{G}$. We define topology on the set $\widehat{C^*(G)}$ using the canonical epimorphism

$$\widehat{C^*(G)} \to \mathrm{Prim}(C^*(G)),$$

given by the assignment $\pi \mapsto \ker\pi$, to the set $\mathrm{Prim}(C^*(G))$ of primitive ideals of $C^*(G)$ equipped with the Jacobson topology. The topology on $\widehat{C^*(G)}$ is then defined as the weakest topology in which this epimorphism is continuous. Because of the aforementioned bijection, this also defines topology on $\widehat{G}$.

It is important to observe that the same procedure could not be conducted for the non-unitary dual $\widetilde{G}$ of G. However, the topology on $\widehat{G}$ can be described in terms of convergence of matrix coefficients. This characterization of topology can be carried over to define topology on $\widetilde{G}$. As already mentioned, Tadić studied this topology extensively, and proved several characterizations, and in particular showed that it is precisely the Fell topology introduced in [14].

The idea of Tadić is that representations isolated in any type of dual of G are of the utmost importance, either for harmonic analysis on G or the arithmetic considerations related to G. The example of Speh representations, which are isolated in the topology of the unitary dual of $GL(n,F)$, is already discussed above. We turn now our attention to arithmetic consequences of topology on various duals. Hence, for the rest of this section, we move to the adèlic setting.

For the moment, up to the end of this section, let F be an algebraic number field, that is, a finite extension of $\mathbb{Q}$. For a given place v of F, let F_v denote the completion of F at v. Let $\mathbb{A}_F$ be the ring of adèles of F. Let G be the classical group or the special linear group defined over F.

The automorphic dual $\widehat{G}_{v,\mathrm{aut}}$ of G at the place v is defined as the support of the representation of $G(F_v)$ acting on $L^2(G(F)\backslash G(\mathbb{A}_F))$ by right translations. Here the support means the support of the corresponding measure in the spectral expansion of $L^2(G(F)\backslash G(\mathbb{A}_F))$ as a direct integral.

On the other hand, $\widehat{G}_v^1$ denotes the unramified dual with respect to the fixed maximal compact subgroup K_v of $G(F_v)$. It is an open subset of the unitary dual $\widehat{G}_v$ of $G(F_v)$. Recall that a representation of $G(F_v)$ is called unramified or spherical if its space contains a K_v-invariant vector. Let $\widehat{G}^1_{v,\mathrm{aut}}$ denote the unramified part of the automorphic dual of G at the place v. Sometimes $\widehat{G}^1_{v,\mathrm{aut}}$ is called the Ramanujan dual of G at the place v.

We remark that $\widehat{G}_{v,\mathrm{aut}}$ is largely unknown, even in the simplest of cases. For example, $\widehat{SL(2)}_{\infty,\mathrm{aut}}$ is largely unknown, but its description entails many important conjectures, e.g., the Selberg $\frac{1}{4}$-conjecture [70], [12]. Following the Tadić philosophy, it is very important to find isolated representations in the topology of $\widehat{G}_v^1$ and $\widehat{G}^1_{v,\mathrm{aut}}$. Note that the isolated representations in $\widehat{G}_v^1$ which are automorphic, are automatically isolated in $\widehat{G}^1_{v,\mathrm{aut}}$, but the converse does not hold in general.

The unramified unitary dual of the groups in question is completely known by [35]. Not only that the whole unramified unitary dual is obtained, but also its isolated points, and, among them, the points which are isolated in the whole unitary dual. On the other hand, motivated by his studies of unipotent automorphic representations, Muić introduces in [33] the negative and strongly negative (unramified) representations. Roughly, they are unramified representations which satisfy the conditions on Jacquet modules which are opposite to the ones that satisfy tempered and square-integrable representations, respectively, by the Casselman criterion [10]. In other words, they are unramified representations which are Aubert duals (cf. [3] for the definition of Aubert duality) of tempered and square-integrable representations, respectively. We denote by $\widehat{G}^1_{v,\mathrm{neg}}$ the negative representations in the unramified unitary dual. Muić proved that the strongly negative representations are local components of the automorphic representations [34]. Furthermore, it is proved in [70] that

$$\widehat{G}^1_{v,\mathrm{neg}} \subseteq \widehat{G}^1_{v,\mathrm{aut}},$$

that is, all the negative unramified representations of $G(F_v)$ are in the Ramanujan dual at v.

We recall now a very significant conjecture of Clozel, known under the name Arthur+ε conjecture [12, Conj. 2]. The unramified representations of $G(F_v)$ are parameterized, via the Satake isomorphism, by the conjugacy classes of semisimple elements in the complex dual group $G^\vee$ of G. Let t_π denote a semisimple representative of the Satake parameter of π. In our unramified setting, the local Arthur parameter is, briefly said, certain admissible homomorphism

$$\psi : W_{F_v} \times SL(2,\mathbb{C}) \to G^\vee,$$

where W_{F_v} is the Weil group of F_v and $G^\vee$ is the complex dual group of G. The Arthur parameter ψ is called unramified and isobaric of weight zero, if $\psi|_{SL(2,\mathbb{C})}$ is algebraic, $\psi|_{I_{F_v}}$ is trivial, where I_{F_v} is the inertia subgroup of W_{F_v}, and ψ maps the Frobenius element $Frob_v$ to a maximal compact subgroup of $G^\vee$.

The Arthur+ε conjecture states, roughly, that if π is an unramified local component of an irreducible automorphic representation, then there exists a parameter ψ, unramified and isobaric of weight zero as above, such that

$$t_\pi = \psi\left(Frob_v, \begin{bmatrix} q_v^{\frac{1}{2}} & 0 \\ 0 & q_v^{-\frac{1}{2}} \end{bmatrix}\right),$$

where q_v is the order of the residue field of F_v.

If one assumes that the Arthur+ε conjecture holds, it follows that

$$\widehat{G}^1_{v,\mathrm{neg}} = \widehat{G}^1_{v,\mathrm{aut}}.$$

This would also give us the complete description of isolated representations in the Ramanujan dual. Namely, [70, Prop. 2.5] then gives that the isolated representations in the Ramanujan dual are precisely all the strongly negative unramified representations.

It is also interesting to note that there is a huge discrepancy in the relationship of the Ramanujan dual and the whole unramified unitary dual between the split classical p-adic group $G(F_v)$, i.e., the symplectic and odd special orthogonal group, and the special linear group $SL(n, F_v)$. Indeed, if one assumes Clozel's Arthur+ε conjecture, there are many more isolated points in $\widehat{G}^1_{v,\mathrm{aut}}$ than in $\widehat{G}^1_v$ in the case of symplectic or odd special orthogonal group $G(F_v)$, because the isolated points in $\widehat{G}^1_{v,\mathrm{aut}}$ would be exactly the strongly negative representations. In that case, for example when $G = Sp(340, F_v)$ of rank 170, as Tadić points out in [70], the number of isolated unramified representations makes up just $1,93\%$ of the number of strongly negative unramified representations. In the case of $SL(n, F_v)$, these numbers coincide, and are both equal to one.

We mention now two more important instances of the explicit use of the topology of the unitary dual of p-adic reductive groups. The first is the convergence in the unitary dual [41], which is instrumental in the proof of the Tadić classification of the unitary dual of $GL(n, F)$ for a p-adic field F in [46]. More precisely, it is used in the Appendix to prove some crucial claims without referring to the properties of the Zelevinsky–Aubert involution [3], [4], which were still unknown at that time. The second use of topology is to obtain bounds for appearance of unitarizable subquotients in the parabolically induced representations [79].

The notion of a rigid representation is introduced in the setting of representations of general linear groups. The set of rigid unitarizable representations is defined as the set of isomorphism classes of all irreducible unitary representations which are induced from the Speh representations, but without complementary series. It turns out that one of the consequences of the generalized Ramanujan conjecture would be equality of the automorphic and rigid duals for the general linear group, with Speh representations as the only ones isolated in the topology of the automorphic dual. A certain partial generalization of this notion is possible for the classical p-adic groups. These topological consideration lead us to the final topic of the paper in the section that follows.

9. Interplay of Arthur packets and unitarizability

This idea is the most recent line of thought in the work Tadić. In the classification of the unitary dual of p-adic groups, the point of departure is the so-called supercuspidal reducibility as in Section 7. The final goal is to determine the unitary dual in terms of supercuspidal reducibilities as parameters. In that quest, the most difficult step is the study of parabolically induced representations at the so-called critical points.

On the other hand, local Arthur packets are defined as the sets of possible local components of the automorphic representations in the global Arthur packets [1]. The latter are candidates for representations in the spectral decomposition of the discrete part of the space of L^2 automorphic forms on the adèlic group, and thus, necessarily unitarizable. Briefly said, members of local Arthur packets form a family of unitarizable representations of G. The representations in this family are sometimes referred to as the representations of Arhtur's type.

According to Tadić, the interplay between the representations at critical points and those of Arthur's type, could be the crucial ingredient in the understanding of unitary duals of classical p-adic groups. These representations also seem to be distinguished in terms of the topology of the unitary dual introduced in Section 8. These expectations can be formulated by the following conjecture of Tadić [81].

Conjecture on unitarizability and Arthur packets (Tadić, [81]). *Let π be an irreducible representation of the classical p-adic group G.*

1. *Suppose that π is an irreducible subquotient at a critical point. Then the following two claims are equivalent:*
   (a) π is unitarizable.
   (b) π is a member of a local Arthur packet.
2. *If π is an isolated representation in the unitary dual, then π is a representation at a critical point.*
3. *Each isolated representation π in the unitary dual is a member of a local Arthur packet.*

The support for the validity of this conjecture Tadić has found in three cases: the corank three case [82], in the case of the generic unitary dual [25] and in the case of unramified unitary dual [35]. We end with a quote from [81] regarding the conjecture:

"[...] the above conjecture expects that in the case of the critical points, which is the most delicate part of the unitarizability, the mysterious line between unitarizability and non-unitarizability is drawn by Arthur packets. It may easily happen that the above conjecture is not true, but still we expect that this approach of thinking is in a good direction. This conjecture may be very hard to prove (if it is true)."

* * *

In the rest of this section, we introduce the technical notions required for better understanding of the Tadić conjecture above. It is still assumed, for simplicity, that the group G is one of the classical p-adic groups introduced in Section 2.

We begin with the definition of critical points. Given an irreducible supercuspidal representation ρ of the general linear group, there is a unique real number $e(\rho)$, referred to as the exponent of ρ, and a unitary irreducible supercuspidal representation ρ^u such that $\rho \cong \nu^{e(\rho)}\rho^u$. Recall that a segment $[a, b]$ is a sequence of real numbers of the form $a, a+1, a+2, \ldots, b$, where $b-a+1$ is a positive integer. Given a unitary irreducible supercuspidal representation ρ^u and an irreducible supercuspidal representation σ of the classical group, there exists a unique non-negative real number $\alpha_{\rho^u,\sigma} \geq 0$ such that the induced representation

$$\nu^{\pm\alpha_{\rho^u,\sigma}}\rho^u \rtimes \sigma$$

reduces. This was already used in Section 7.

Let $\rho_1, \dots, \rho_k$ be irreducible supercuspidal representations of general linear groups, and σ an irreducible supercuspidal representation of G. Assume that for each $i \in \{1, 2, \dots, k\}$ the following holds :

- ρ_i^u is selfcontragredient, i.e., $\rho_i^u \cong \widetilde{\rho_i^u}$,
- the multiset $\{e(\rho_j) : \rho_j^u \cong \rho_i^u\}$ of exponents, which can be assumed to be non-negative, is such that the underlying set forms a segment in $\frac{1}{2}\mathbb{Z}$,
- the set of exponents from the previous point contains the reducibility exponent $\alpha_{\rho_i^u,\sigma}$.

Then, the induced representation $\rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \sigma$ is referred to as the representation at the critical point, or of critical type. Any irreducible subquotient π of that induced representation is then called the irreducible subquotient at the critical point.

In order to proceed towards the conjecture of Tadić, we recall of the representations of Arthur's type. They are the local constituents of the global Arthur packets, which parameterize, according to Arthur's fundamental work [1], possible automorphic representations in the discrete part of the spectral decomposition of L^2 automorphic forms on the adèlic group. The explicit construction of the representations in the local Arthur packets, i.e., of Arthur's type, over a p-adic field is done mostly by Mœglin in a series of papers culminating with [30], [29], [85], [2].

In Section 8, we used a special form of Arthur packets which is appropriate to deal with unramified representations. We now recall a more general definition for the considered groups, which are all split. The local Arthur parameter for the split classical p–adic group G in question is a homomorphism

$$\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \to G^\vee$$

subject to certain technical requirements, where W_F is the Weil group of F and $G^\vee$ is the complex dual group of G. The local Arthur packet attached to this parameter is a finite multiset, which is actually a set in the considered cases according to [30], of irreducible representations of G subject to certain character relations.

We would like to point out that one of the main initial ingredients of Mœglin's construction of local Arthur packets is the Mœglin–Tadić classification of the square-integrable representations of classical p-adic groups [28], [31]. In their joint work, as a motivation, Tadić used the following idea to construct irreducible summands in the L-parameter attached to square-integrable representations. Assume that σ is an irreducible square-integrable representation of a classical group and ρ a selfcontragredient irreducible supercuspidal representation of a general linear group. Tadić examined, for a positive integer a, the following induced representation

$$\mathrm{Ind}_P^G(\delta[\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho] \otimes \sigma),$$

where $\delta[\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]$ is the square-integrable representation attached to the segment $[\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]$ as in Section 2. It turns out that for one parity of a, this representation is never reducible, and for almost all a of the other parity, it is reducible. The finite set of exceptions to reducibility will form the summands $\rho \otimes S_a$ in the Langlands parameter of σ, where S_a is the unique a-dimensional irreducible representation of $SL(2, \mathbb{C})$. This idea, that one can describe the summands in the L-parameter of a representation, thus, the object exhibiting both analytical and arithmetical properties, in terms of reducibility of parabolically induced representations, came from the case of the general linear group. Of course, one had to ensure that this candidate for L-parameter indeed satisfies all the required properties.

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Tadićeva filozofija: pregled temeljnih principa i ključnih ideja u radovima Marka Tadića

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Sažetak. Ovaj rad daje pregled temeljnih principa i ključnih ideja u radovima Marka Tadića. Njegovo istraživanje se najvećim dijelom bavi teorijom reprezentacija reduktivnih grupa nad lokalnim poljima. Iz perspektive autora, najvažniji temeljni principi u njegovom radu su suštinska jednostavnost harmonijske analize, čak i u nekomutativnom nekompaktnom slučaju, Lefschetzov princip koji kaže da bi se teorija reprezentacija nad arhimedskim i nearhimedskim poljima trebala proučavati na iste načine te princip usporedbe Jacquetovih modula. Osim tih principa, najvažnije i najplodonosnije ideje su strukturni vanjski pristup unitarnom dualu, unitarizabilnost duž pravaca, korištenje topologije raznih duala da se dobiju informacije o harmonijskoj analizi i aritmetici promatrane grupe te veza između unitarizabilnosti i Arthurovih paketa. Svi ti principi i ideje su tema ovog rada.

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