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STEINER TRIPLE SYSTEMS OF ORDER 21 WITH SUBSYSTEMS

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Dedicated to the memory of Zvonimir Janko (1932–2022)

Abstract. The smallest open case for classifying Steiner triple systems is order 21. A Steiner triple system of order 21, an STS(21), can have subsystems of orders 7 and 9, and it is known that there are 12,661,527,336 isomorphism classes of STS(21)s with sub-STS(9)s. Here, the classification of STS(21)s with subsystems is completed by settling the case of STS(21)s with sub-STS(7)s. There are 116,635,963,205,551 isomorphism classes of such systems. An estimation of the number of isomorphism classes of STS(21)s is given.

1. Introduction

A *Steiner triple system* (STS) is a pair $(V, \mathcal{B})$, where V is a set of *points* and $\mathcal{B}$ is a set of 3-subsets of points, called *blocks*, such that every 2-subset of points occurs in exactly one block. The size of the point set is the *order* of the STS, and an STS of order v is denoted by STS(v). It is well known that an STS(v) exists iff

$$v \equiv 1 \text{ or } 3 \pmod 6. \tag{1.1}$$

For more information about Steiner triple systems, see [4, 5].

An STS(v) is said to be *isomorphic* to another STS(v) if there exists a bijection between the point sets that maps blocks onto blocks; such a bijection is called an *isomorphism*. An isomorphism of a Steiner triple system onto itself is an *automorphism* of the STS. The automorphisms of an STS form a group under composition, the *automorphism group* of the Steiner triple system.

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Classification of combinatorial designs is about finding a transversal of the isomorphism classes [11]. The Steiner triple systems have been classified up to order 19, and the numbers of isomorphism classes are 1, 1, 1, 2, 80, and 11,084,874,829 for orders 3, 7, 9, 13, 15, and 19, respectively. A classification of the STS(19)s was published in 2004 with a remark that the algorithm used would require hundreds of thousands of CPU years to classify the STS(21)s [9]. As this seems to be currently out of reach, it is reasonable to focus on subclasses of STS(21)s. Indeed, STS(21)s of various types have been considered in this context, including STS(21)s with a nontrivial automorphism group [8] (with earlier work in [3, 7, 18–20, 28, 29], also considering other properties), anti-Pasch STS(21)s [16], and resolutions of STS(21)s—that is, Kirkman triple systems—with subsystems [15].

A necessary condition for an STS(v) to have a nontrivial ($w > 3$) and proper ($w < v$) subsystem of order w, i.e., a sub-STS(w), is that $v \geq 2w + 1$; see [5, Lemma 6.1]. Classification of Steiner triple systems with sub-STS(7)s has been carried out for orders 15 and 19—see [21, Table 1.29] and [13], respectively—and for those with sub-STS(9)s for order 19—see [27].

The only possible nontrivial proper subsystems of STS(21)s are STS(7)s and STS(9)s. The STS(21)s with sub-STS(9)s are classified in [12]; there are 12,661,527,336 isomorphism classes of such designs. For STS(21)s with sub-STS(7)s, the special case of Wilson-type systems is handled in [13]. *Wilson-type* STS(21)s contain three sub-STS(7)s on disjoint point sets. In the current paper the classification problem for STS(21)s with subsystems is settled by completing the case of sub-STS(7)s.

THEOREM 1.1. *There are 116,635,963,205,551 isomorphism classes of* STS(21)*s containing at least one sub*-STS(7)*.*

The paper is organized as follows. An algorithm for classifying STS(21)s with sub-STS(7)s is described in Section 2, and the results are listed in Section 3. The number of isomorphism classes of STS(21)s with sub-STS(7)s is used in Section 4 to get an estimation of the total number of isomorphism classes of STS(21)s.

2. Classification

In this section, we present a classification algorithm for STS(21)s containing sub-STS(7)s. To facilitate reading, we give necessary definitions in Section 2.1. The general approach is outlined in Section 2.2, details about subtasks are given in Section 2.4, and some computational issues are considered in Section 2.5.

2.1. *Definitions.* A (v_r, b_k) configuration is an incidence structure with v points and b blocks, such that each block contains k points, each point occurs in r blocks, and two different blocks intersect in at most one point. If

$v = b$ and $k = r$, these are simply called v_k configurations. The definitions of isomorphism and automorphism of configurations are analogous to those for Steiner triple systems.

A *1-factor* in a graph, also called a perfect matching, is a 1-regular spanning subgraph and a *1-factorization* is a partition of the edges of the graph into 1-factors. A 1-factorization of a graph $G = (V, E)$ is isomorphic to a 1-factorization of a graph $G' = (V', E')$ if there is a bijection from V to V' that maps the 1-factors of the 1-factorization of G onto the 1-factors of the 1-factorization of G'.

2.2. *General approach.* On a general level, the current approach follows [12], in which all of Theorem 2.1 except the last statement already appeared.

THEOREM 2.1 ([12]). *Let* $(V, \mathcal{B})$ *be an* STS(v) *that has a sub*-STS(w) $(W, \mathcal{B}')$. *Then*

1. $\mathcal{B} = \mathcal{B}' \cup \mathcal{F} \cup \mathcal{D}$ *where* $\mathcal{F}$ *and* $\mathcal{D}$ *are the sets of blocks that intersect* W *in 1 and 0 points, respectively,*
2. $\mathcal{F} = \bigcup_{p \in W} \mathcal{B}_p$ *where* $\mathcal{B}_p$ *is the set of blocks in* $\mathcal{F}$ *that contain* $p \in W$,
3. $\mathcal{B}'_p = \{B \setminus \{p\} : B \in \mathcal{B}_p\}$ *is a 1-factor of a graph* G *with vertices* $V \setminus W$ *and edges* $\bigcup_{p \in W} \mathcal{B}'_p$,
4. $\{\mathcal{B}'_p : p \in W\}$ *is a 1-factorization of* G,
5. G *is* w*-regular and its complement* $\overline{G}$ *is* $(v - 2w - 1)$*-regular, and*
6. $\overline{G}$ *can be decomposed into a set of edge-disjoint 3-cycles—*$\mathcal{D}$ *being one possible set—which forms a*

$$((v - w)_{(v-2w-1)/2}, ((v - w)(v - 2w - 1)/6)_3)$$

configuration.

Using this theorem, any STS containing a sub-STS is decomposable into $\mathcal{B}' \cup \mathcal{F} \cup \mathcal{D}$. For the task of classifying all STS(v)s containing some sub-STS(w), one has now two starting points: either a classification of the 1-factorizations underlying $\mathcal{F}$ or a classification of the configurations corresponding to $\mathcal{D}$. Then, in both cases, one needs to combine this with a classification of $\mathcal{B}'$ to create an STS in all possible ways, taking symmetry into account.

The next sections illustrate the details for $v = 21$ and $w = 7$; the general setting is also described in [12].

2.3. *Application to STS(21)s containing sub-STS(7)s.* Let $(V, \mathcal{B})$ be an STS(21) that has a sub-STS(7) $(W, \mathcal{B}')$. Clearly $W \subseteq V$ and $\mathcal{B}' \subseteq \mathcal{B}$. Each block in $\mathcal{B} \setminus \mathcal{B}'$ intersects W in either 0 or 1 points, and those two sets of blocks are denoted by $\mathcal{D}$ and $\mathcal{F}$, respectively:

$$\mathcal{D} = \{B \in \mathcal{B} \setminus \mathcal{B}' : |B \cap W| = 0\} \text{ and}$$
$$\mathcal{F} = \{B \in \mathcal{B} \setminus \mathcal{B}' : |B \cap W| = 1\}.$$

Fix a point $p \in W$ and let $\mathcal{B}_p$ be the set of blocks in $\mathcal{F}$ that contain p. Further let

$$\mathcal{B}'_p = \{B \setminus \{p\} : B \in \mathcal{B}_p\}. \tag{2.1}$$

As a pair of points with one point in W and the other in $V \setminus W$ must occur in exactly one block of $\mathcal{F}$, the sets in $\mathcal{B}'_p$ partition $V \setminus W$. The sets in $\mathcal{B}'_p$ have size 2, and we may view them as edges in a graph with vertex set $V \setminus W$. The sets in $\mathcal{B}'_p$ form a 1-factor of that graph. With 7 possible values of p, we have 7 disjoint 1-factors of a 7-regular graph of order 14.

To complete the Steiner triple system, given a 7-regular graph G of order 14, one may find all 1-factorizations of G and in the complement $\overline{G}$ find all decompositions into 3-cycles (which is the graph analog of finding sets of triples that cover all unordered pairs) and combine these in all possible ways. Doing this for all possible choices of G gives all ways of extending the initial STS(7). Finally, isomorph rejection needs to be carried out during the process of combining parts. Specific details about using this approach in the current work—where the order $\mathcal{D} \to \mathcal{F} \to \mathcal{B}'$ for constructing the blocks $\mathcal{B}$ is actually used—are presented in Section 2.4. See also [12].

There are 21,609,301 isomorphism classes of 7-regular graphs of order 14 [24]; see also [26, Table 4.25]. Only a small number of graphs G have the property that the complement $\overline{G}$ can be decomposed into 3-cycles as described in the last statement in Theorem 2.1. Indeed, the required 14_3 configurations have already been classified.

There are 21,399 isomorphism classes of 14_3 configurations [2]. Checking the isomorphism classes of graphs underlying the 14_3 configurations shows that their number is 20,787. As this is about one thousandth of the number of regular graphs, the 14_3 configurations provide appropriate building blocks for our algorithm.

Example. There is a unique isomorphism class of an STS(21) that contains at least one sub-STS(7) and that admits an automorphism group of order 108, see Table 1. The incidence matrix of such a design visualizes the partitions of points and blocks in the general approach (note that the ordering of rows and columns within each subset does not necessarily coincide with the ordering given by the algorithm) and is presented in Figure 1.

Let $V = \{0, 1, \ldots, 20\}$. The design can be constructed by considering the group of order 108 generated by

$$(0, 9, 19)(2, 10, 16)(3, 4, 20, 8, 7, 18)(5, 6, 15, 14, 11, 13) \text{ and}$$
$$(0, 3, 4)(2, 5, 6)(7, 8, 9, 20, 18, 19)(10, 15, 13, 16, 11, 14)(12, 17)$$

and taking the 7 orbits under the action of this group with representatives

$$\{0, 1, 2\}, \{0, 3, 6\}, \{0, 9, 19\}, \{0, 10, 17\}, \{1, 12, 17\}, \{2, 5, 6\}, \{2, 10, 16\}.$$

```
     |111....|1111111..........................................|..............
     |1..11..|.......1111111...................................|..............
     |1....11|..............1111111............................|..............
W    |.1.1.1.|.....................1111111.....................|..............
     |..1.11.|............................1111111..............|..............
     |..11..1|...................................1111111.......|..............
     |.1..1.1|..........................................1111111|..............
-----+-------+-------------------------------------------------+--------------
     |.......|1......1......1......1......1......1......1......|111...........
     |.......|.1......1.....1.......1......1......1......1.....|...111........
     |.......|..1......1.....1.......1......1.....1.......1....|1.....11......
     |.......|...1....1.......1.....1........1.....1.......1...|.1......11....
     |.......|.1.....1.........1......1.......1.....1.......1..|......1.1.1...
     |.......|....1.....1....1.........1.......1.1............1.|...1.....11...
V\W  |.......|.....1.....1.......1.......1.......1....1.........1|1..1....1.....
     |.......|1.......1.........1.....1..........1....1.........1|...........111
     |.......|......1.....1......1.......1....1..........1.......1...|..1.1......1..
     |.......|....1.........1..1..........11..........1.......1..|.....1.1...1..
     |.......|...1......1.........1.....1..........1......1..1.....|..1....1....1.
     |.......|......1....1.........1......1.1...........1......1.|.1....6.....1.
     |.......|..1..........1.....1......1....1...........11......|....1....1...1
     |.......|.....1......1.........11.........1.........1....1....|.....1....1..1
-----+-------+-------------------------------------------------+--------------
     |       |  B_0  |  B_1  |  B_2  |  B_3  |  B_4  |  B_5  |  B_6  |
     |  B'   |                        F                        |      D
```

Figure 1. Incidence matrix for the STS(21) in the example

2.4. *Details of the approach.* We shall now give more specific details needed for implementing the general approach. Some of the computational subproblems will be considered separately in Section 2.5.

The point set. When building up an STS(21) $(V, \mathcal{B})$ containing a sub-STS(7), we let $V = \{0, 1, \ldots, 20\}$ such that $W = \{14, 15, \ldots, 20\}$ is the point set of the distinguished sub-STS(7) $(W, \mathcal{B}')$, called S'.

The 14_3 configuration. The distribution of the orders of the automorphism groups of the 21,399 14_3 configurations [2] is

$$1^{20,328}2^{916}3^{19}4^{91}6^{12}7^{1}8^{15}12^{7}14^{3}16^{3}24^{2}128^{1}56448^{1}.$$

The unique 14_3 configuration with automorphism group order $56448 = 2 \cdot 168^2$ consists of two disjoint STS(7)s and is the configuration of Wilson-type systems. Ignoring that configuration here, the groups to be considered have order at most 128, so there is no need for sophisticated group algorithms.

After fixing a configuration $(V \setminus W, \mathcal{D})$, we compute its automorphism group A, the underlying graph $\overline{G}$, and the complement G. Notice that the group A is trivial in most of the cases.

The 1-factorization. For a given graph G, we first determine the set F of 1-factors of G and then use the 1-factors in F to compute the set $\mathcal{F}'$ of all possible 1-factorizations of G. If the group A is nontrivial, isomorph rejection is further carried out by accepting precisely those 1-factorizations in $\mathcal{F}'$ that are lexicographically minimum under the action of A. For an accepted 1-factorization, the subgroup of A consisting of the elements that stabilize the 1-factorization is denoted by A'.

A 1-factor of G corresponds to a set $\mathcal{B}'_p$ defined in (2.1), and a 1-factorization of G gives a set of blocks $\mathcal{F} = \cup_{p=14}^{20} \mathcal{B}_p$ up to permutation of the points in $W = \{14, 15, \ldots, 20\}$ (we pick an arbitrary one). The group A' acts on $V \setminus W$. Blocks of $\mathcal{F}$ also have points in W, so we extend the action of A' to get a group A'' acting on V. The permutation of the points in W for an element in A'' is uniquely defined by how the original element in A' maps the 1-factors.

The sub-STS(7). There is a unique STS(7), the Fano plane, which has an automorphism group of order 168. Hence there are are $7!/168 = 30$ distinct labelled STS(7)s on 7 given points.

An isomorphism from one STS(21) with a sub-STS(7) to another maps the distinguished sub-STS(7) to a sub-STS(7). Hence there are two general situations: STS(21)s with exactly one sub-STS(7) and STS(21)s with more than one sub-STS(7). In the latter case, there are further several possibilities for how the point sets of two sub-STS(7)s may intersect. Such an intersection must form a (possibly trivial) sub-STS, so possible intersection sizes are 0, 1, and 3.

If the intersection size is 0, then there is necessarily a third sub-STS(7) whose point set is disjoint from the point sets of the first two sub-STS(7)s, that is, we have a Wilson-type system and the 14_3 configuration discussed earlier. Wilson-type STS(21)s have exactly three sub-STS(7)s [13, Lemma 1]. As the mentioned 14_3 configuration is not considered here, this case will not occur in the search.

Isomorph rejection when extending blocks $\mathcal{D} \cup \mathcal{F}$ with blocks $\mathcal{B}'$ is analogous to the situation when extending blocks $\mathcal{D}$ with blocks $\mathcal{F}$, considered earlier. Now, out of the 30 possibilities, those sub-STS(7)s that are lexicographically minimum under the action of A'' are accepted. The subgroup of A'' consisting of the elements that stabilize the accepted sub-STS(7) is denoted by A'''.

The blocks $\mathcal{B} = \mathcal{D} \cup \mathcal{F} \cup \mathcal{B}'$ now form an STS(21) with a *distinguished* sub-STS(7), and if those are the objects to classify we would be done. But in the classification of STS(21)s with *at least one* sub-STS(7), there is still one final step.

The final isomorph rejection. If there is exactly one sub-STS(7) in the constructed design $(V, \mathcal{B})$, then we accept the STS(21); its automorphism group is the group A''' computed earlier. Otherwise, we proceed by finding all sub-STS(7)s in V (as we have seen, these will intersect W in exactly 1 or 3 points; some precomputations for finding them can be done based on $\mathcal{D}$ and $\mathcal{F}$). We now determine whether the distinguished sub-STS(7) is a canonically minimum sub-STS(7), to be discussed in Section 2.5, and accept it if that is the case. The automorphism group of an accepted STS(21) is obtained as a by-product of the computations.

2.5. *Computational subproblems.* We shall here discuss some of the main computational subproblems that are encountered when implementing the presented approach and that are not standard problems related to data structures and algorithms.

Automorphism groups and canonical forms. Automorphism groups and canonical forms are conveniently computed with `nauty` [22] after an appropriate transformation of the combinatorial structure to a graph.

To order the sub-STS(7)s of an STS(21) one may use the standard graph encoding of the incidence matrix of the design, add one vertex for each sub-STS(7), and let the 7 vertices corresponding to the points of the sub-STS(7) form the neighborhood of an added vertex. Then the canonical order of vertices given by `nauty` imposes an order on the sub-STS(7)s. More precisely, `nauty` determines an order of the *orbits* of vertices under the action of the automorphism group of the graph. Therefore we get an induced ordering of the orbits of sub-STS(7)s under the action of the automorphism group of the STS(21).

For small group orders, the abstract type of the automorphism groups of the classified designs can be identified based on the multiset of orders of elements. The abstract type can further be computed using `AllSmallGroups` and `StructureDescription` in `GAP` [6]. In the current work, seven groups (of orders 27, 54, 108, 294, and 1008) had to be treated manually and separately.

1-factors and 1-factorizations. We use a backtrack algorithm to compute all 1-factors of general graphs. Given the set of 1-factors of a graph, the problem of finding all 1-factorizations can be phrased in the framework of exact cover [10], whereby the instances can be solved, for example, using the `libexact` [14] software.

3. Results

The total number of isomorphism classes of STS(21)s containing at least one sub-STS(7) is 116,635,963,205,551, which splits into 116,635,961,039,200 cases that are not of Wilson-type and 2,166,351 cases that are of Wilson-type [13].

More detailed information can be found in Table 1 and Table 2. The column headers in Table 1 are the order of the automorphism group (O), the number of contained sub-STS(7)s (U), the number of unordered pairs of sub-STS(7)s that intersect in 1 and 3 points (I_1 and I_3, respectively), the abstract type of the automorphism group (A), and finally the number of isomorphism classes of STS(21)s with these properties (#).

For completeness, Table 1b from [13] is included. For all Wilson-type STS(21)s, we have $U = 3$, $I_1 = 0$, and $I_3 = 0$ by [13, Lemma 1].

The notation for the abstract types of groups is as follows: C_n is the cyclic group of order n, S_n is the symmetric group of order $n!$, A_n is the alternating group of order $n!/2$, D_n is the dihedral group of order n, and $\mathrm{PSL}(v,q)$ is the projective special linear group in $\mathbb{F}_q^v$. For two groups G and H, $G \times H$ is the direct product of G and H, $G \rtimes H$ is a semidirect product of G and H, and G^n is $G \times G \times \cdots \times G$ (n times).

A central open problem for specific STS(21)s is whether systems exist that are doubly resolvable. The current work gives nothing new with respect to this problem, because Kirkman triple systems of order 21 with sub-STS(7)s have already been classified and tested [15].

The whole classification including the detection of the abstract group types took about 1300 CPU days on the equivalent of one core of an Intel Xeon E5-2665 @ 2.40GHz.

Verification. We perform two tests to validate results. Let $\mathcal{S}$ be a transversal of the isomorphism classes of the STS(21)s with sub-STS(7)s that are not of Wilson-type—this is the outcome of the current classification—and let $\mathcal{C}$ be a transversal of the isomorphism classes of the 14_3 configurations excluding the configuration leading to Wilson-type STS(21)s. Further, let $s_7(S)$ be the number of sub-STS(7)s in the system S, and let $f(C)$ be the number of 1-factorizations with labelled 1-factors of the complement of the graph underlying the configuration C.

In the first test, we count in two different ways all pairs of labelled STS(21)s that are not of Wilson-type and their contained sub-STS(7)s. By the Orbit–Stabilizer Theorem, we have

$$\sum_{S\in\mathcal{S}} \frac{21!}{|\operatorname{Aut}(S)|}\cdot s_7(S) = \sum_{C\in\mathcal{C}} \frac{21!}{|\operatorname{Aut}(C)|}\frac{7!}{168}\cdot f(C). \tag{3.1}$$

The fact that Wilson-type STS(21)s have exactly three sub-STS(7)s [13] and will not appear in the search is essential for the double counting to work.

In the second test, we extract the STS(21)s with sub-STS(7)s from the STS(21)s with nontrivial automorphisms classified in [8] and compare the numbers with those in Table 1a.

Both of these two tests were passed in the computations of the current work. In particular, both sides of (3.1) gave a count of

$$5{,}988{,}986{,}139{,}804{,}614{,}556{,}727{,}954{,}636{,}800{,}000.$$

4. Estimating the number of STS(21)s

The classification of the STS(21)s with sub-STS(7)s gives a lower bound on the number of isomorphism classes of STS(21)s but can also be used for an estimation of that number. The authors are not aware of any published estimations.

Quackenbush [25] conjectured that almost all Steiner triple systems have no nontrivial subsystems. Later, however, Kwan [17] used a random model

TABLE 1. Numbers of STS(21)s containing at least one sub-STS(7)

(A) non-Wilson-type

O	U	I_1	I_3	A	#
1	1	0	0	C_1	116,051,875,827,936
1	2	1	0	C_1	31,778,146,776
1	2	0	1	C_1	550,238,290,596
1	3	1	2	C_1	593,663,600
1	3	3	0	C_1	60,352,088
1	3	0	3	C_1	1,385,739,943
1	4	1	5	C_1	6,391,040
1	4	2	4	C_1	198,304
1	4	3	3	C_1	1,607,028
1	4	0	6	C_1	157,886
1	5	1	9	C_1	576
1	5	2	8	C_1	50,192
1	5	3	7	C_1	30,024
1	5	4	6	C_1	1,704
1	6	3	12	C_1	1,790
1	6	4	11	C_1	688
1	7	5	16	C_1	124
2	1	0	0	C_2	19,270,679
2	2	1	0	C_2	84,080
2	2	0	1	C_2	814,880
2	3	1	2	C_2	18,912
2	3	3	0	C_2	43,062
2	3	0	3	C_2	132,334
2	4	1	5	C_2	9,088
2	4	2	4	C_2	64
2	4	3	3	C_2	2,448
2	5	2	8	C_2	224
2	5	3	7	C_2	2,092
2	5	4	6	C_2	16
2	6	3	12	C_2	140
2	6	4	11	C_2	32
2	7	5	16	C_2	188
2	9	9	27	C_2	2
3	1	0	0	C_3	177,205
3	2	1	0	C_3	3,152
3	2	0	1	C_3	5,508

O	U	I_1	I_3	A	#
3	3	3	0	C_3	655
3	3	0	3	C_3	4,152
3	4	3	3	C_3	132
3	4	0	6	C_3	6
3	5	4	6	C_3	16
3	6	3	12	C_3	18
4	1	0	0	C_2^2	6,268
4	1	0	0	C_4	628
4	3	3	0	C_2^2	260
4	3	0	3	C_2^2	870
4	5	3	7	C_2^2	136
4	7	5	16	C_2^2	24
4	9	9	27	C_2^2	3
6	1	0	0	C_6	849
6	1	0	0	S_3	192
6	3	3	0	C_6	146
6	3	3	0	S_3	39
6	3	0	3	C_6	91
6	3	0	3	S_3	31
6	4	3	3	S_3	16
6	6	3	12	S_3	4
6	9	9	27	S_3	2
7	1	0	0	C_7	27
8	1	0	0	$C_4 \times C_2$	8
8	1	0	0	D_8	164
9	3	3	0	C_3^2	1
9	3	0	3	C_3^2	3
12	3	3	0	D_{12}	4
12	3	0	3	D_{12}	10
12	9	9	27	D_{12}	3
14	1	0	0	C_{14}	14
16	1	0	0	$C_2 \times D_8$	8
18	3	3	0	$C_3 \times S_3$	11
18	3	0	3	$C_3 \times S_3$	6
36	9	9	27	S_3^2	1
108	9	9	27	$(C_3^2 \rtimes C_6) \rtimes C_2$	1

(B) Wilson-type

O	A	#
1	C_1	2,156,186
2	C_2	8,914
3	C_3	685
4	C_2^2	253
4	C_4	18
6	C_6	94
6	S_3	103
8	C_2^3	22
8	D_8	19
9	C_3^2	3
12	A_4	2
12	D_{12}	4
16	$C_2 \times D_8$	4
18	$C_3^2 \rtimes C_2$	2
18	$C_3 \times S_3$	5
21	$C_7 \rtimes C_3$	2
24	$C_2 \times A_4$	9
24	$C_2^2 \times S_3$	1
24	S_4	7
42	$C_2 \times (C_7 \rtimes C_3)$	1
42	$C_7 \rtimes C_6$	5
48	$C_2 \times S_4$	2
72	$(C_3 \times A_4) \rtimes C_2$	1
72	$A_4 \times S_3$	4
126	$S_3 \times (C_7 \rtimes C_3)$	1
144	$S_3 \times S_4$	1
294	$C_7^2 \rtimes C_6$	1
882	$(C_7 \rtimes C_3)^2 \rtimes C_2$	1
1008	$\mathrm{PSL}(3,2) \times S_3$	1

to find evidence for the number of sub-STS(7)s in an STS(v) to have expectation $\Theta(1)$, referring to similar work in [23] on Latin squares. The models used in [17] and [23] are random 3-uniform hypergraphs and random integer matrices, respectively.

TABLE 2. Aggregated numbers of STS(21)s containing at least one sub-STS(7)

O	#
1	116,635,942,616,481
2	20,387,155
3	191,529
4	8,460
6	1,567
7	27

O	#
8	213
9	7
12	23
14	14
16	12
18	24

O	#
21	2
24	17
36	1
42	6
48	2
72	5

O	#
108	1
126	1
144	1
294	1
882	1
1008	1

An STS(v) has $v(v-1)/6$ blocks out of $v(v-1)(v-2)/6$ 3-subsets of a v-set, that is, a ratio of $p := 1/(v-2)$ of the 3-subsets are blocks. We may now form a random 3-uniform hypergraph on v vertices by including blocks with probability p (note that $p := 1/v$, which is used in [17], works when studying asymptotics). We denote the number of labelled STS(w)s on w points by $N(w)$. We have seen earlier that $N(7) = 30$. The number of labelled STS(w)s on v points, where $v \geq w$, is $M(v, w) := N(w)\binom{v}{w}$. The probability for a given STS(w) to occur in the random model is $p^{w(w-1)/6}$.

The linearity of the expected value allows now to compute the expected number of sub-STS(w)s

$$\mu(v, w) := \frac{N(w)\binom{v}{w}}{(v-2)^{w(w-1)/6}}$$

and, abbreviating $\mu(\infty, w) = \lim_{v\to\infty} \mu(v, w)$, we have $\mu(\infty, 7) = 1/168 \approx 0.00595$ and $\mu(\infty, w) = 0$ for $w > 7$.

Let S be the set of positive integers fulfilling (1.1). Analogously to the conjecture in [23, p. 346], see also [17], we state the following.

CONJECTURE 4.1. *The expectation of the number of sub-*STS(w)*s in an* STS(v) *tends to* 1/168 *for* $w = 7$ *and to* 0 *for* $w > 7$ *as* $v \in S$ *tends to infinity.*

Assuming Poisson distribution [17] for the number of sub-STS(7)s in an STS(v), the proportion of STS(v)s containing at least one sub-STS(7) is approximately $\alpha = 1 - e^{-1/168} \approx 0.00593$ for large v. Consequently, an estimation of the total number of STS(v)s can be obtained by dividing the number of STS(v)s with at least one sub-STS(7) by α.

As almost all Steiner triple systems have no nontrivial automorphisms [1], an estimation for the number of isomorphism classes of STS(v)s can be obtained by dividing the number of isomorphism classes of STS(v)s with at least one sub-STS(7) by α.

There are two orders for which we have data that can be compared with such an estimation. There are only 80 isomorphism classes of STS(15)s, and 7 is the maximum possible order of a proper subsystem. The earlier comment that almost all Steiner triple systems have no nontrivial automorphisms does not hold in this case, but we can use [21, Table 1.29] to arrive at a proportion of approximately 15α for labelled STS(15)s with sub-STS(7)s. There are 11,084,874,829 isomorphism classes of STS(19)s, and 86,701,547 of these have sub-STS(7)s [13] giving a proportion of about 1.3α.

For the number of isomorphism classes of STS(21)s, using the classification results of the current paper we calculate

$$116{,}635{,}963{,}205{,}551/\alpha \approx 1.965 \cdot 10^{16},$$

which indicates that the number could be somewhat greater than 10^{16}, perhaps between $1 \cdot 10^{16}$ and $2 \cdot 10^{16}$.

In the estimation one might consider utilizing $\mu(21,7) \approx 0.00389$ rather than $\mu(\infty,7)$, but notice that $\mu(19,7) \approx 0.00368$ underestimates the true value by a factor greater than 2, and the situation here might be analogous to that for sub-Latin squares considered in [23]. In that paper, it is conjectured that the expected number of 3×3 sub-Latin squares of a randomly chosen $n \times n$ Latin square tends to $1/18$ as n tends to infinity, and numerical data show that the value given by the random model for a fixed parameter, $f(n) = 12\binom{n}{3}^3 n^{-9}$, underestimates the computed value for small parameters. For example, for $n = 10$, the asymptotic value (≈ 0.0556) is closer to the exact value (≈ 0.0536) than $f(n)$ (≈ 0.0207).

It is not clear whether an STS with subsystems is more or less prone to have resolutions. If the correlation is weak, then the fact that there are 12,520,021 isomorphism classes of Kirkman triple systems of order 21 with sub-STS(7)s [15] could be used to calculate

$$12{,}520{,}021/\alpha \approx 2.111 \cdot 10^{9},$$

which would hint that there might be somewhat more than one billion isomorphism classes of Kirkman triple systems of order 21.

The estimations for the numbers of isomorphism classes of STS(21)s and KTS(21)s will hopefully be useful in the estimation of computational resources in future attempts to classify those structures and in other related studies.

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