DOI:10.22337/2587-9618-2023-19-4-14-26

NONLINEAR ANALYSIS OF SOLID REINFORCED CONCRETE STRUCTURES WITH CRACKS

Vladimir P. Agapov [1], Alexey S. Markovich [1,2]

[1] RUDN University, Moscow, RUSSIA

[2] Moscow State University of Civil Engineering, Moscow, RUSSIA

Abstract: A finite element method, as well as the algorithm and the program for solid reinforced concrete structures analysis have been developed, taking into account plastic deformations of concrete. A modified Willam & Warnke failure criterion was used, supplemented by a flow criterion. Two models of volumetric deformation of concrete have been developed: an elastic model under brittle fracture and an ideal elastic-plastic model. An eight-node solid finite element with linear approximation of displacement functions, which implements the deformation models above mentioned, is constructed. This finite element is adapted to the PRINS computational software, and as part of this program it can be used for physically nonlinear analysis of building structures containing three-dimensional reinforced concrete elements. Modern building codes prescribe to carry out calculations of concrete and reinforced concrete structures in a nonlinear formulation, taking into account the real properties of concrete and reinforcement. To verify the developed finite element, a series of test calculations of a beam in the condition of pure bending was carried out. Comparison of the calculation results with experimental data confirmed the high accuracy and reliability of the results obtained.

Keywords: finite element method, PRINS computational program, building structures, solid reinforced concrete structures, physical nonlinearity, plasticity, flow theory, structural mechanics

НЕЛИНЕЙНЫЙ АНАЛИЗ МАССИВНЫХ ЖЕЛЕЗОБЕТОННЫХ КОНСТРУКЦИЙ С УЧЕТОМ ТРЕЩИНООБРАЗОВАНИЯ

В.П. Агапов [1], А.С. Маркович [1,2]

[1] Российский университет дружбы народов им. Патриса Лумумбы, г. Москва, РОССИЯ

[2] Национальный исследовательский Московский государственный строительный университет, г. Москва, РОССИЯ

Аннотация: Разработан объемный восьмиузловой конечный элемент для расчета массивных железобетонных конструкций с учетом трещинообразования. При построении элемента в области напряженного состояния «сжатие – сжатие – сжатие» использован модифицированный критерий прочности Виллама – Варнке. Напряженное состояние бетона при возникновении трещины в режиме «сжатие – сжатие» рассматривалось как плосконапряженное и использовался модифицированный критерий прочности Мизеса – Губера. Поведение бетона при растяжении принималось линейным вплоть до возникновения трещины. Разработанный конечный элемент адаптирован к вычислительному комплексу ПРИНС и в составе этого комплекса может быть использован инженерами проектных и научных организаций для практических расчетов массивных железобетонных конструкций. Для верификации разработанного конечного элемента проведена серия тестовых расчетов балки, находящейся в условии чистого изгиба. Сравнение результатов расчета с экспериментальными данными подтвердило высокую точность и достоверность полученных результатов.

Ключевые слова: метод конечных элементов, вычислительный комплекс ПРИНС, строительные конструкции, массивные железобетонные сооружения, физическая нелинейность, пластичность, теория течения, механика деформируемых тел

INTRODUCTION

Modern building codes and regulations adopted in our country and abroad prescribe to carry out calculations of reinforced concrete structures in a nonlinear setting, taking into account the real properties of concrete and reinforcement. At the same time, the nonlinear deformation of concrete, taking into account real elastoplastic properties under the conditions of a volumetric stress state, was studied throughout the 20th century by both domestic and foreign scientists.
The prerequisites for the successful implementation of such calculations were created by the development of computer technology, on the one hand, and the development of numerical methods of structural mechanics, primarily the finite element method, on the other [1,2,3,4,5].
Nonlinear methods for calculating structures are implemented in a number of computer programs, such as NASTRAN [6], ANSYS [7], ABAQUS [8], ADINA [9], DIANA [10] and others. Common to all these programs is the use of algorithms based on the execution of step procedures.
However, it should be noted that the calculations of physically nonlinear structures in the above programs are performed using physical relationships based on certain experimental data. In this case, the obtained nonlinear equations for the structure as a whole are solved by approximate methods. To increase the reliability of the results, such calculations should be carried out using several programs. Therefore, engineers should have several available calculation tools in their arsenal. In this regard, the development of alternative computational methods and corresponding programs is still an urgent task.

MATERIALS AND METHODS

As is known, the system of nonlinear algebraic equations is solved by the Newton-Raphson method in full or modified form. The equilibrium equations at the loading step are written as:

$$\mathbf{K}_j^i \Delta \mathbf{u}_j^i = \mathbf{P}_j - \mathbf{F}_j^{i-1}, \quad (1)$$

where $\mathbf{K}_j^i$ is the tangential stiffness matrix, $\Delta \mathbf{u}_j^i$ is the vector of nodal displacements, $\mathbf{P}_j$ is the vector of nodal loads, $\mathbf{F}_j^{i-1}$ is the vector of nodal concentrated forces equivalent to element stresses, j is the step number, i is the iteration number.
A feature of the Newton-Raphson method for solving equation (1) is the calculation and factorization of the tangent stiffness matrix at each iteration. In the case of large-order systems, such calculations can be quite expensive.
When using the modified Newton-Raphson method, the stiffness matrix is calculated and factorized only once at the beginning of the step [4,5]. This simplifies the calculations, but requires more iterations to achieve the specified accuracy. Therefore, to accelerate convergence, different approaches are used based on the correction of the displacement vector at the current iteration [11,12,13].
In the PRINS computer application, the calculation of physically nonlinear structures by the finite element method is carried out in increments according to the equation [1]:

$$\left(\mathbf{K}_0 + \Delta \mathbf{K}\right) \Delta \mathbf{u} = \Delta \mathbf{P}, \quad (2)$$

where $\Delta \mathbf{K} = \frac{1}{2}\left(\mathbf{K}_1 - \mathbf{K}_0\right)$.
Stiffness matrices $\mathbf{K}_0$ and $\mathbf{K}_1$ are calculated at the beginning and at the end of the loading step.
Equation (2) is solved by the iterative method:

$$\mathbf{K}_0 \Delta \mathbf{u}_i = \Delta \mathbf{P} - \Delta \mathbf{K}_{i-1} \Delta \mathbf{u}_{i-1}, \quad (3)$$

where i is the iteration number.
Upon reaching the convergence of the iterative process, the full values of displacements and stresses are found using the formulas:

$$\mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \Delta\boldsymbol{\sigma}. \tag{4}$$

In general, stress increments are determined by the formula

$$\Delta\boldsymbol{\sigma} = \mathbf{C}_{ep}\Delta\boldsymbol{\varepsilon}, \tag{5}$$

where $\mathbf{C}_{ep}$ is the elastoplastic matrix of material characteristics.

At each loading step, the stress state is analyzed and, in the event of plastic deformations and cracks, the stresses are corrected taking into account the accepted material deformation diagrams. This requires a structural balancing process. Equilibrium iterations are performed according to formula (1), which is modified to the form:

$$\mathbf{K}_j \Delta\mathbf{u}_j^i = \mathbf{P}_j - \mathbf{F}_j^{i-1}. \tag{6}$$

The difference between formulas (1) and (6) is that in formula (1) the stiffness matrix changes from iteration to iteration (meaning equilibrium iteration), and in formula (6) it does not change, and is taken equal to the stiffness matrix, found at the end of the step during iterations according to formula (3).

Thus, in formula (3) the stiffness matrix is iterated, and according to formula (6) is iterated the vector of nodal forces, which is equivalent to internal stresses.

The stiffness matrix for a single finite element is found by the formula [2]:

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{C}_e \mathbf{B} d\mathbf{V}, \tag{7}$$

where $\mathbf{B}$ is the matrix relating the strain components of the element to the nodal displacement components (geometric matrix), $\mathbf{C}_e$ is the matrix relating the stress components to the strain components (physical matrix).

The technique for calculating a geometric matrix is well known (see, for example, [3]).

THE MAIN PREREQUISITES FOR THE ANALYSIS OF THE VOLUMETRIC STRESS STATE

Four types of stress state are considered: compression – compression – compression, compression – compression – tension, compression – tension – tension, tension – tension – tension.

The following types of material behavior are taken into account: loading, unloading and reloading. The deformation paths of the material corresponding to the indicated types of behavior are shown in figs. 1.

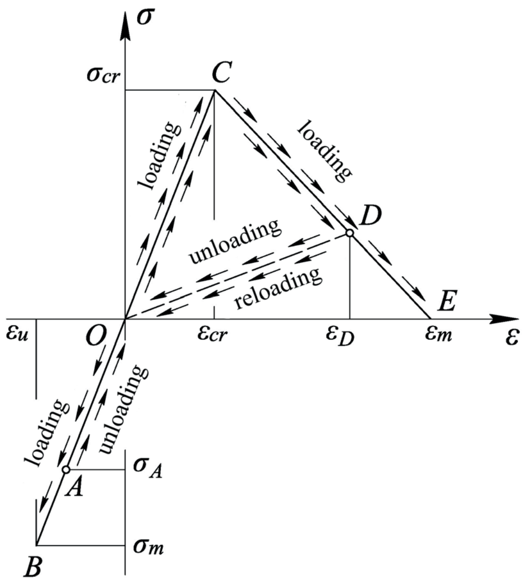


Figure 1. Material Deformation Path

Loading of tensile concrete after the formation of a crack (path *C – D – E*) occurs with the modulus of elasticity

$$E_i = \frac{\sigma_{cr}}{\varepsilon_m - \varepsilon_{cr}}, \tag{8}$$

where ε_m is the ultimate strain of concrete with cracks in tension and ε_{cr} is the cracking strain, respectively (Fig. 1).

Unloading and reloading of concrete with cracks (path *D – O*) occur according to a linear law with a fictitious modulus of elasticity

$$E_i' = \frac{1}{\varepsilon_D}\frac{\sigma_{cr}\left(\varepsilon_m - \varepsilon_D\right)}{\varepsilon_m - \varepsilon_{cr}},\ \varepsilon_{cr} < \varepsilon_D < \varepsilon_m \quad (9)$$

The stress in tensile concrete after the occurrence of a crack is found by the formula

$$\sigma = \sigma_{cr}\frac{\varepsilon_m - \varepsilon_D}{\varepsilon_m - \varepsilon_{cr}}, \quad (10)$$

which is easily obtained from the similarity of the triangles in fig. 1.

Deformation of concrete in the «compression – compression – compression» mode

When constructing a physical matrix, an elastic model was adopted for brittle fracture of concrete in the «compression – compression – compression» mode. The behavior of concrete is considered to be linearly elastic until reaching the fracture surface.

The five-parameter model proposed by Willam & Warnke [14] was adopted as the fracture surface. On fig. 2 shows the deviatoric section of this surface, and in fig. 3 shows the main meridians of tension and compression.

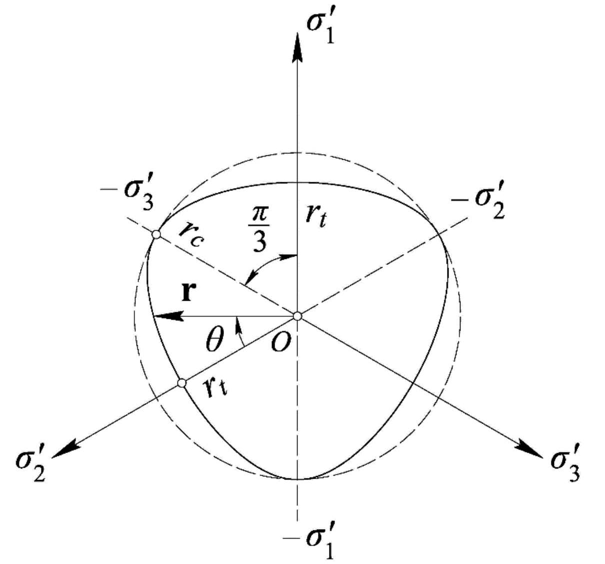


Figure 2. Deviatoric section of the fracture surface (r_t and r_c – the meridians of tension and compression, θ – the angle of the type of the stress state)

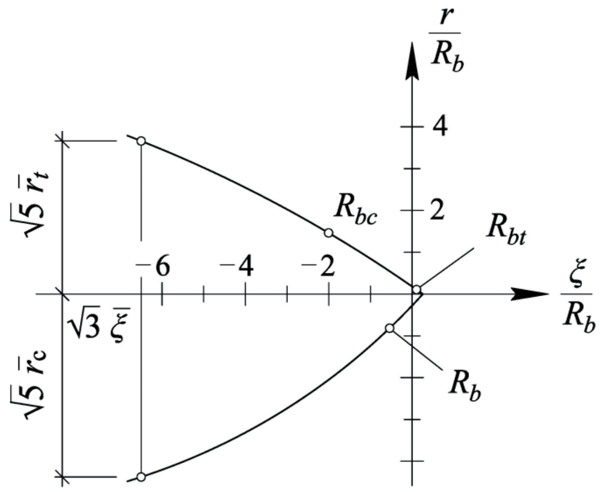


Figure 3. Tension and compression meridians of fracture surface (R_b and R_{bt} – ultimate strength of concrete for axial compression and tension, R_{bc} – ultimate strength of concrete for uniform biaxial compression, $r = \sqrt{5}\tau_m$, $\xi = \sqrt{3}\sigma_m$)

The failure criterion is determined by the formula [14]:

$$f(\boldsymbol{\sigma}) = f(\sigma_m, \tau_m, \theta) = \frac{1}{\bar{r}(\sigma_m, \theta)}\frac{\tau_m}{R_b} - 1 = 0, \quad (11)$$

where σ_m and τ_m are the average values of normal and shear stresses in the vicinity of the point, θ is the angle of the stress state, and $\bar{r}(\sigma_m, \theta) = \frac{r}{\sqrt{5}R_b}$ is the radius-vector of the point on the fracture surface in the deviator section (fig. 2).

The radius-vector $\bar{r}(\sigma_m, \theta)$ is determined by the formula [14]:

$$\bar{r}(\sigma_m, \theta) = \frac{2r_c\left(r_c^2 - r_t^2\right)\cos\theta + r_c\left(2r_t - r_c\right)\Psi}{4\left(r_c^2 - r_t^2\right)\cos^2\theta + \left(r_c - 2r_t\right)^2}, \quad (12)$$

where $\Psi = \sqrt{4\left(r_c^2 - r_t^2\right)\cos^2\theta + 5r_t^2 - 4r_t r_c}$.

The radius-vectors r_t and r_c (fig. 2) define the meridians of tensile $(\theta = 0)$ and compression $(\theta = \pi/3)$. In the Willam & Warnke model these meridians are square parabolas:

$$\frac{\tau_{m,t}}{R_b} = r_t = a_0 + a_1 \frac{\sigma_m}{R_b} + a_2 \left(\frac{\sigma_m}{R_b} \right)^2 ,$$

$$\frac{\tau_{m,c}}{R_b} = r_c = b_0 + b_1 \frac{\sigma_m}{R_b} + b_2 \left(\frac{\sigma_m}{R_b} \right)^2 . \qquad (13)$$

The coefficients a_0, a_1, a_2 and b_0, b_1, b_2 are obtained on the basis of experimental data for specific concrete grades (table 1).

$$a_0 = \frac{2}{3}\bar{R}_{bc} a_1 - \frac{4}{9}\bar{R}_{bc}^2 a_2 + \sqrt{\frac{2}{15}}\bar{R}_{bc} ,$$

$$a_1 = \frac{1}{3}\left(2\bar{R}_{bc} - \bar{R}_{bt}\right) a_2 + \sqrt{\frac{6}{5}} \frac{\bar{R}_{bt} - \bar{R}_{bc}}{2\bar{R}_{bc} + \bar{R}_{bt}} ,$$

$$a_2 = \frac{\sqrt{\frac{6}{5}}\bar{\xi}_t \left(\bar{R}_{bt} - \bar{R}_{bc}\right) - \sqrt{\frac{6}{5}}\bar{R}_{bt}\bar{R}_{bc} + \bar{r}_t\left(2\bar{R}_{bc} + \bar{R}_{bt}\right)}{\left(2\bar{R}_{bc} + \bar{R}_{bt}\right)\lambda}$$

$$\lambda = \bar{\xi}_t^2 - \frac{2}{3}\bar{R}_{bc}\bar{\xi}_t + \frac{1}{3}\bar{R}_{bt}\bar{\xi}_t - \frac{2}{9}\bar{R}_{bt}\bar{R}_{bc} , \qquad (14)$$

and

$$b_0 = -\rho b_1 - \rho^2 b_2 , \; b_1 = \left(\bar{\xi}_c + \frac{1}{3}\right) b_2 + \frac{\sqrt{\frac{6}{5}} - 3 r_c}{3\bar{\xi}_c - 1} ,$$

$$b_2 = \frac{r_c\left(\rho + \frac{1}{3}\right) - \sqrt{\frac{2}{15}}\left(\bar{\xi}_c + \rho\right)}{\left(\bar{\xi}_c + \rho\right)\left(\bar{\xi}_c - \frac{1}{3}\right)\left(\rho + \frac{1}{3}\right)} . \qquad (15)$$

<u>Table 1.</u> To the determination of the parameters of the Willam & Warnke model

	Type of stress state	Criterion	Stresses	Angle θ
1.	Uniaxial compression	R_b	$\sigma_3 = -R_b$, $\sigma_1 = \sigma_2 = 0$	0
2.	Uniaxial tension	R_{bt}	$\sigma_1 = R_{bt}$, $\sigma_2 = \sigma_3 = 0$, $\bar{R}_{bt} = \frac{R_{bt}}{R_b}$	0
3.	Uniform biaxial compression	R_{bc}	$\sigma_1 = \sigma_2 = -R_{bc}$, $\sigma_3 = 0$, $\bar{R}_{bc} = \frac{R_{bc}}{Rb} = 1,2$	0
4.	Triaxial compression in high compression mode $(\sigma_3 < \sigma_1 = \sigma_2,\ \theta = 0)$	$\frac{\sigma_m}{R_b} = -\bar{\xi}_t$, $\bar{\xi}_1 > 0$, $\frac{\tau_m}{R_b} = \bar{r}_t$	$\sigma_1 \neq 0$, $\sigma_2 \neq 0$, $\sigma_3 \neq 0$, $\bar{\xi}_1 = 3,67$*, $\bar{r}_t = 1,59$*	0
5.	Triaxial compression in low compression mode $(\sigma_3 > \sigma_1 = \sigma_2,\ \theta = \pi/3)$	$\frac{\sigma_m}{R_b} = -\bar{\xi}_c$, $\bar{\xi}_2 > 0$, $\frac{\tau_m}{R_b} = \bar{r}_c$	$\sigma_1 \neq 0$, $\sigma_2 \neq 0$, $\sigma_3 \neq 0$, $\bar{\xi}_c = 3,67$*, $\bar{r}_c = 1,94$*	$\pi/3$

Deformation of concrete in the modes «compression – compression – tension», «compression – tension – tension» and «tension - tension – tension» (taking into account cracking)

The physical matrix for concrete with one, two and three cracks is formed as follows.

When one crack is formed the stress state of concrete in two other directions is considered as plane stressed and, in this case, the modified von Mises – Huber strength criterion is used [29].

Initially, the stresses in the finite element are calculated in the global axes of the structure $x-y-z$. At the moment the first crack occurs the position of the main axes $1-2-3$ is fixed (fig. 4).

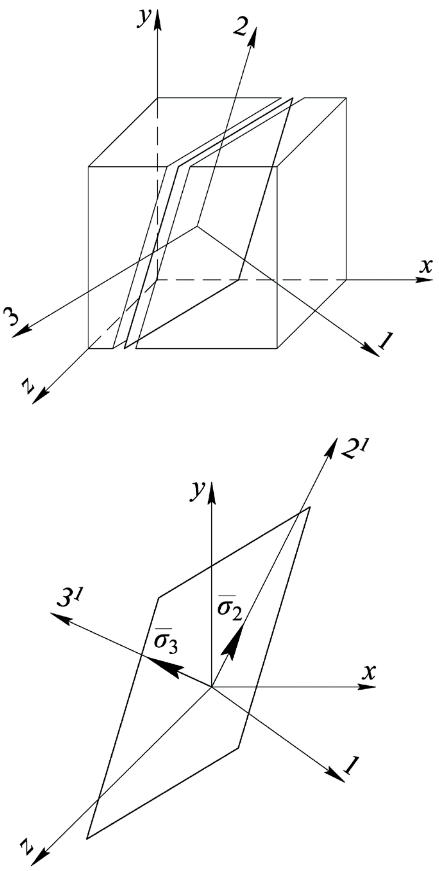


Figure 4. Analysis of the stress state of concrete during the formation of one crack

When the first crack occurs the material in the volume of the element breaks up, according to the accepted hypothesis, into a number of plane stressed plates. In each such plate there are such areas with normals 2^1 and 3^1, on which the normal stresses have extreme values. Further analysis of cracking is carried out according to the principal stresses and acting along axes 2^1 and 3^1 (fig. 4). These stresses, as well as the angle of rotation of axes 2^1 and 3^1 relative to axes 2 and 3 are found according to the general rules for the strength of materials. Further calculation of the structure is performed in axes $1-2^1-3^1$, for which the matrix of direction cosines of these axes is preliminarily calculated in the global axes $x-y-z$.

In the presence of two cracks the module of elasticity of concrete in the third direction is determined by the «stress-strain» diagram for a uniaxial stress state, and the shear modules are taken according to the recommendations given in [1].

In this case, the physical matrix is calculated as follows:

$$\mathbf{C}_e = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12}^c & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{23}^c & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{31}^c \end{bmatrix}, \qquad (16)$$

In the presence of three cracks the modulus of elasticity is zero.

The modulus E in matrix (16) is taken equal to either the initial modulus if the concrete is tensiled in the third main direction, or the tangent modulus of the stress-strain curve if the concrete is compressed in the third main direction.

Experiments show that cracked concrete with reinforcement transmits significant shear stresses. In this case, the magnitude of shear stiffness is influenced by such factors as the width of the crack opening, the coefficient of reinforcement, the diameter of the reinforcement, the structure of concrete, etc. [28]. Following the recommendations of [28], we take the shear modulus as a function of the current tensile strain.

For concrete with a crack in the first principal direction

$$G_{12}^c = 0,25G\left(1-\frac{\varepsilon_1}{0,004}\right),$$
$$G_{12}^c = 0 \text{ if } \varepsilon_1 > 0,004, \qquad (17)$$
$$G_{31}^c = G_{12}^c,$$

where G is the shear modulus of concrete without cracks, ε_1 is the tensile strain in the main direction 1.

For concrete with a crack in two directions:

$$G_{13}^c = 0,25G\left(1-\frac{\varepsilon_1}{0,004}\right),$$
$$G_{13}^c = 0 \text{ if } \varepsilon_1 > 0,004,$$
$$G_{23}^c = 0,25G\left(1-\frac{\varepsilon_2}{0,004}\right), \qquad (18)$$
$$G_{23}^c = 0 \text{ if } \varepsilon_2 > 0,004,$$
$$G_{12}^c = \frac{1}{2}\min\left[G_{13}^c, G_{23}^c\right].$$

For concrete with a crack in three directions:

$$G_1 = 0,25G\left(1-\frac{\varepsilon_1}{0,004}\right),$$
$$G_1 = 0 \text{ if } \varepsilon_1 > 0,004,$$
$$G_2 = 0,25G\left(1-\frac{\varepsilon_2}{0,004}\right), \qquad (19)$$
$$G_2 = 0 \text{ if } \varepsilon_2 > 0,004,$$
$$G_3 = 0,25G\left(1-\frac{\varepsilon_3}{0,004}\right),$$
$$G_3 = 0 \text{ if } \varepsilon_3 > 0,004,$$
$$G_{12}^c = G_{23}^c = G_{31}^c = \frac{1}{2}\min\left[G_1, G_2, G_3\right].$$

Based on the above algorithm, a program adapted to the PRINS computer application was developed.

RESEARCH RESULTS

To test the developed finite element, the results of experiments by Obernikhin D.V. and Nikulin A.I. were used from the article [26], in which the strength and crack resistance of reinforced concrete beams were studied.

Beam material – class B22.5 concrete. The beam had double longitudinal reinforcement with A500 Ø12 and A240 Ø8 mm rebar, respectively. The transverse reinforcement is reinforcement class Bp500 Ø5 mm (Fig. 5).

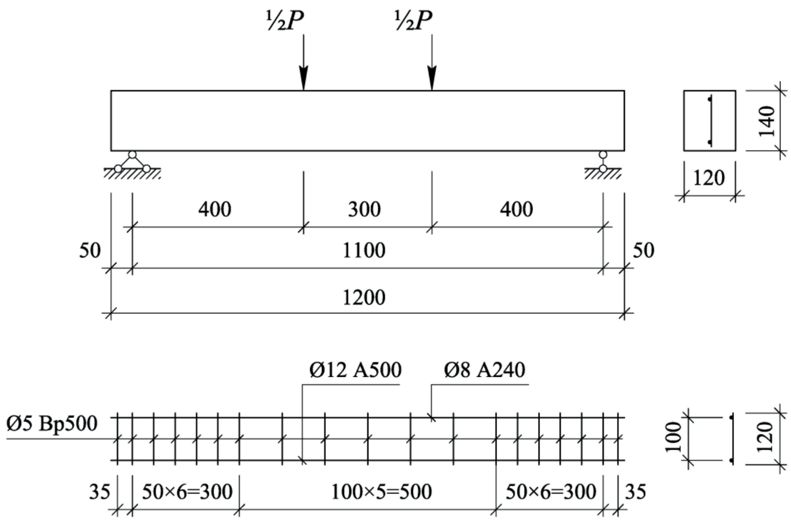
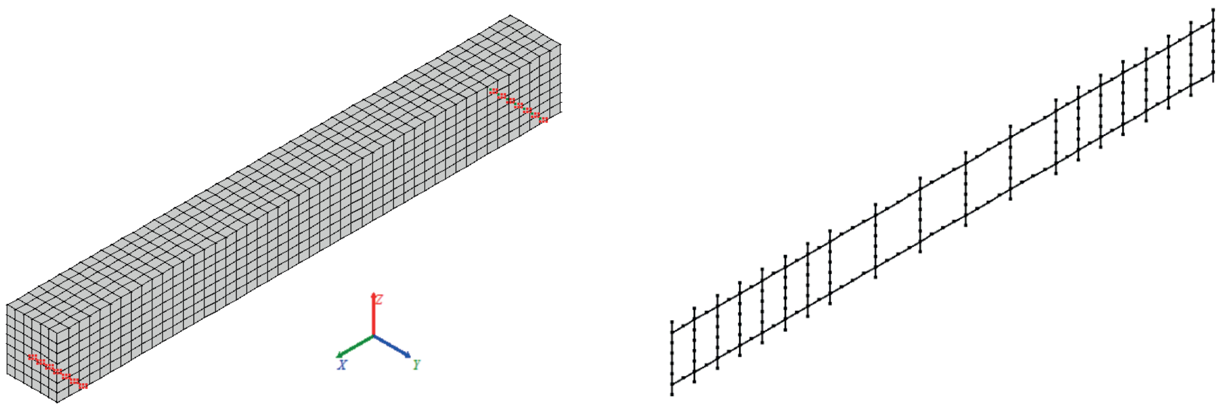


Figure 5. Geometric dimensions and reinforcement scheme of beams

The finite element model of the experimental beam consisted of 2016 three-dimensional eight-node elements (fig. 6). Beam reinforcement was modeled by one-dimensional bar elements with three degrees of freedom at the node. For longitudinal reinforcement the hypothesis of ideal elastoplastic behavior was accepted and the influence of the squares of rotation angles on longitudinal forces was taken into account. The transverse reinforcement rods were under conditions of linear elastic deformation.

The loading and fixing conditions of the beam corresponded to the methodology carried out by Obernikhin D.V. and Nikulin A.I. experiments [26].

It should be noted that earlier similar problems were considered by Rimshin V.I. and Amelin P.A. in [27] using the foreign complex ABAQUS.



a) *b)*

Figure 6. FE scheme of the experimental beam: a – FE model of the «body» of the beam, b – bar FE scheme of reinforcement

The total load on the beam was taken equal to *P* = 48 kN. The calculations were performed by the step-iterative method. 41 loading steps were set, the coefficient to the load at the first step was taken equal to 0.01, and at the remaining steps – 0.03.

According to the calculation results, the process of crack formation in concrete, beam displacements at fixed load values, as well as the ultimate (failure) load were researched.

The calculation results are presented in table 2.

Table 2. Beam calculation results

Analyzed value	Average displacements of beams (mm) under load, kN			Average cracking load, kN	Average failure load, kN
	16	24	32		
Experimental results [26]	1,76	3,23	4,90	6,41	40,218
Calculation results in the PRINS computer application	1,63	2,98	4,82	6,24	39,36
Divergence, %	7,98	8,39	1,66	2,72	2,18

As can be seen from table. 2, the divergences in the values of the ultimate (failure) load, obtained from the results of calculations in the PRINS computer program, in comparison with the experimental data [26] does not exceed 2.18%, which indicates the high accuracy of the developed finite element, as well as reliability and stability used nonlinear calculation algorithms.

The process of cracking of the considered beam is shown in Fig. 7.

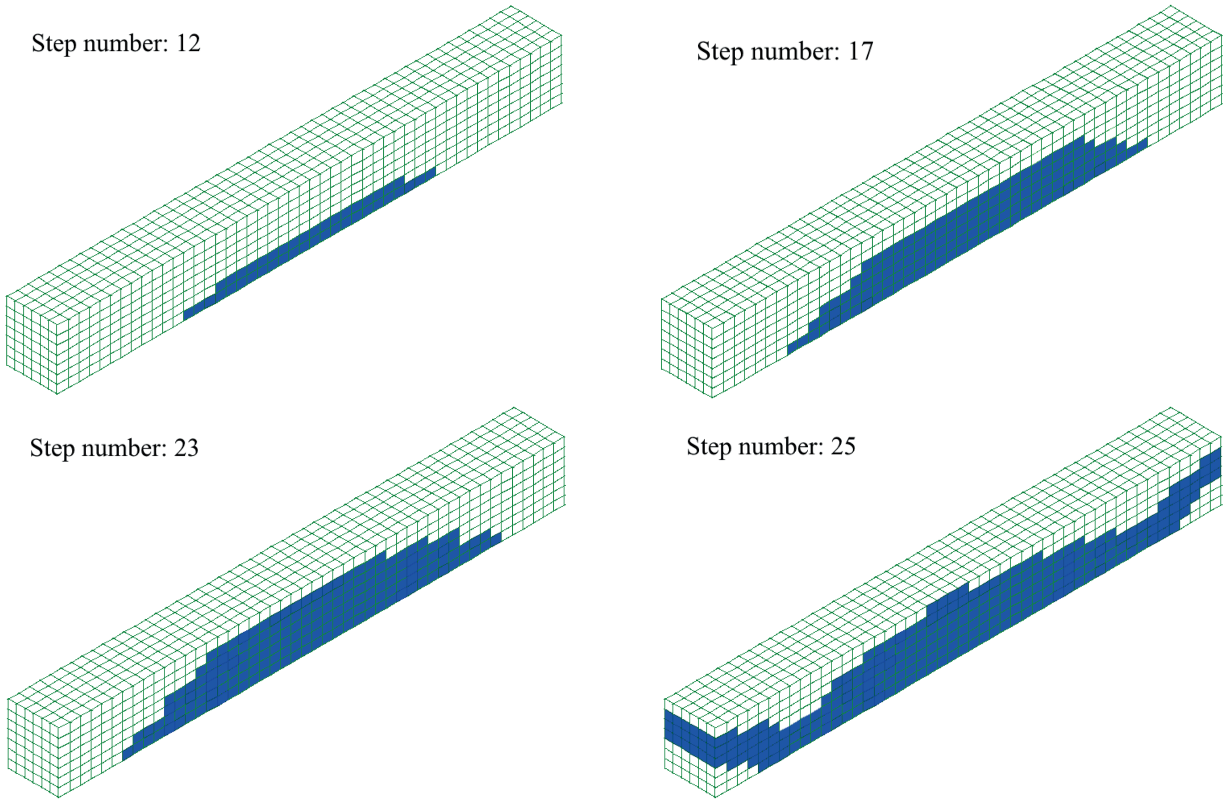
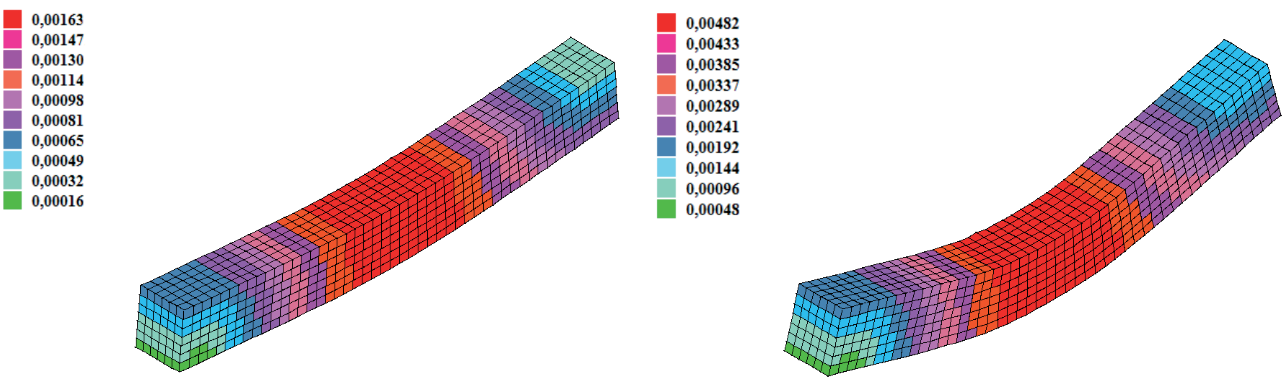


Figure 7. Cracking of the considered beam

The beam displacements corresponding to the total load of 16 kN and 32 kN are shown in fig. 8.



a) *b)*

Figure 8. Beam displacements (m): a – at a load of 16 kN, b – at a load of 32 kN

CONCLUSION

The authors have developed and implemented a physically nonlinear volumetric finite element for calculating massive reinforced concrete structures, which makes it possible to take into account the brittle fracture of concrete in the compression mode and the process of cracking in tensile concrete. When creating an element to take into account the work of concrete in the «compression – compression – compression» mode, the Willam & Warnke failure criterion was used.

When a crack occurs in concrete in the first main direction, the stress state of concrete in the other two directions (in the «tension – compression – compression» mode) is considered as plane-stressed and the modified von Mises – Huber failure criterion is used.

In the presence of two cracks, the modulus of elasticity of concrete in the third direction is determined by a simple stress-strain diagram, and, at the same time, the influence of shear stresses on the work of concrete with cracks is taken into account.

Since the reinforcement of the structure can be performed arbitrarily, its features can be fully taken into account by including one-dimensional bar elements characterizing the reinforcement in the finite element diagram of the structure. In this case, for reinforcing steel, it is possible to set both elastic and elastic-plastic deformation models.

To verify the created finite element, a series of test calculations of the beam, which is in the condition of a four-point bending (pure bending), was carried out. The subsequent comparison of the calculation results with the experimental data confirmed the high accuracy and reliability of the results obtained.

This finite element is adapted to the PRINS computer application and, as part of this software application, can be used by engineers of design and scientific organizations for practical calculations of reinforced concrete structures.

Thus, the PRINS software application can be successfully used to solve a large class of engineering problems [23,24,25].

REFERENCES

1. **Agapov V.P.** Metod konechnyh elementov v statike, dinamike i ustojchivosti konstrukcij [Finite element method in statics, dynamics and stability of structures]. Izdatel'stvo ASV, 2005. 245 p. (In Russ.)
2. **Zienkiewicz O.C., Taylor R.L.** The Finite Element for Solid and Structural Mechanics. Sixth edition. McGraw-Hill, 2005, 631 p.
3. **Bathe K.J., Wilson E.L.** Numerical methods in finite element analysis. – N.J.: Prentice-Hall, 1976, 528 p.
4. **Crisfield M.A.** Non-linear finite element analysis of solids and structures. John Wiley & Sons Ltd, 1977, 488 p.
5. **Oden J.T.** Finite elements in nonlinear continua. McGraw, Hill Book Company, New York, 1972, 464 p.
6. MSC NASTRAN 2016. Nonlinear User’s Guide SOL 400. – MSC Software, 2016, 790 p.
7. ANSYS Theory Reference. Release 5.6. ANSYS Inc. Canonsburg, PA, 1999, 1286 p.
8. ABAQUS 6.11. Theory manual. DS Simulia, 2011
9. ADINA. Theory and Modelling Guide. ADINA R & D, Inc. 71 Elton Avenue Watertown, MA 02472, USA, 705 p.
10. DIANA FEA User’s Manual. Release 10. – DIANA FEA BV, 2017
11. **Shanno D.F.** Conditioning of Quasi-Newton methods for function minimization, Math. Comp., 24, 1970, pp. 647-656
12. **Dennis J.E., Jr. and Jorge J. More**. Quasi-Newton Methods, Motivation and Theory. SIAM Review, Vol. 19, No. 1, January 1977. pp. 46-89
13. **H. Matthies and G. Strang.** The Solution of Nonlinear Finite Element Equations. International Journal for Numerical Methods in Engineering, Vol. 14, 1979, p. 1613-1626
14. **Willam K.J., Warnke E.P.** Constitutive Model for the Triaxial Behavior of Concrete. Proceedings of IABSE. Structural Engineering Report 19, Section III, 1975, pp. 1-30

15. SP 63.13330.2018 «SNiP 52-01-2003 Betonnye i zhelezobetonnye konstrukcii. Osnovnye polozheniya» [Concrete and reinforced concrete structures. Key points]. 2018. 148 p.
16. CEB, CEB-FIP Model Code 1990 / CEB Bulletin d'Information № 213/214, Comite Euro-International du Beton, Lausanne, Switzerland, 1993. 437 p.
17. Rekomendacii po opredeleniyu prochnostnyh i deformacionnyh harakteristik betona pri neodnoosnyh napryazhennyh sostoyaniyah [Recommendations for determining the strength and deformation characteristics of concrete in non-uniaxial stress states]. M. NIIZhB Gosstroya USSR, 1985, 72 p.
18. **Kupfer, H., Hilsdorf, H., Rusch, H.** Behavior of Concrete under Biaxial Stresses, ACI Journal, Proceedings Vol. 66, No. 8, August, 1969, pp. 656-666
19. **Launay P., Gachon H.** Strain and Ultimate Strength of Concrete under Triaxial Stress, Am. Concrete Inst. Spec. Publ. 34, pap. 13, 1972
20. **Mills L.L., Zimmerman R.M.** Compressive Strength of Plain Concrete under Multiaxial Loading Conditions, ACI Journal, Vol. 67, No. 10, 1970, October, pp. 802-807
21. **Korsun V.I.** Sopostavitel'nyj analiz kriteriev prochnosti dlya betonov [Comparative analysis of strength criteria for concrete] / V.I. Korsun, A.V. Nedorezov, S.Yu. Makarenko // Sovremennoe promyshlennoe i grazhdanskoe stroitel'stvo [Modern industrial and civil construction]. – 2014. – T. 10. – № 1. – P. 65-78. – EDN THXXCZ (In Russ.)
22. **Hansen T.C.** Triaxial test with concrete and cement paste: Report № 319 / T. C. Hansen. – Lyngby : Technical University of Denmark, 1995. – 54 p
23. **Agapov V.P. and Markovich A.S.** The family of multilayered finite elements for the analysis of plates and shells of variable thickness: La familia de elementos finitos multicapa para el análisis de placas y cascos de espesor variable. South Florida Journal of Development. 2021;2(4):034-5048. https://doi.org/10.46932/sfjdv2n4-007
24. **Agapov V.P., Markovich A.S.** Dynamic method for determining critical loads in the PRINS computer program. Structural Mechanics of Engineering Constructions and Buildings. 2020;16(5):380-389. (In Russ.) DOI: 10.22363/1815-5235-2020-16-5-380-389
25. **Agapov V.P., Markovich A.S.** Investigation of the accuracy and convergence of the results of thin shells analysis using the PRINS program. Structural Mechanics of Engineering Constructions and Buildings. 2021; 17(6):617-627. (In Russ.) DOI: 10.22363/1815-5235-2021-17-6-617-627
26. **Obernikhin D.V., Nikulin A.I.** Experimental studies of strength, crack resistance and deformability of reinforced concrete beams of trapezoidal and rectangular cross sections. Innovative science. 2016; 8(2):73-77.
27. **Rimshin V.I., Amelin P.A.** Numerical calculation of bent reinforced concrete elements of rectangular section in the Abaqus software. Structural Mechanics of Engineering Constructions and Buildings. 2022;18(6):552-563. (In Russ.) http://doi.org/10.22363/1815-5235-2022-18-6-552-563
28. **Cedolin L., S. Deipoli.** Finite element studies of shear-critical R/C beams. ASCE Journal of the Engineering Mechanics Division. June 1977. Vol. 103. No. EM3, Pp. 395-410.
29. **von Mises R.** Mechanik der festen Körper im plastisch-deformablen Zustand. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse. 1913; 1:582-592.

СПИСОК ЛИТЕРАТУРЫ

1. **Агапов В.П.** Метод конечных элементов в статике, динамике и

устойчивости конструкций. Изд-во АСВ, М.: 2005. 245 с.
2. **Zienkiewicz O.C., Taylor R.L.** The Finite Element for Solid and Structural Mechanics. Sixth edition. McGraw-Hill, 2005, 631 p.
3. **Bathe K.J., Wilson E.L.** Numerical methods in finite element analysis. – N.J.: Prentice-Hall, 1976, 528 p.
4. **Crisfield M.A.** Non-linear finite element analysis of solids and structures. John Wiley & Sons Ltd, 1977, 488 p.
5. **Oden J.T.** Finite elements in nonlinear continua. McGraw, Hill Book Company, New York, 1972, 464 p.
6. MSC NASTRAN 2016. Nonlinear User's Guide SOL 400. – MSC Software, 2016, 790 p.
7. ANSYS Theory Reference. Release 5.6. ANSYS Inc. Canonsburg, PA, 1999, 1286 p.
8. ABAQUS 6.11. Theory manual. DS Simulia, 2011
9. ADINA. Theory and Modelling Guide. ADINA R & D, Inc. 71 Elton Avenue Watertown, MA 02472, USA, 705 p.
10. DIANA FEA User's Manual. Release 10. – DIANA FEA BV, 2017
11. **Shanno D.F.** Conditioning of Quasi-Newton methods for function minimization, Math. Comp., 24, 1970, pp. 647-656
12. **Dennis J.E., Jr. and Jorge J. More**. Quasi-Newton Methods, Motivation and Theory. SIAM Review, Vol. 19, No. 1, January 1977. pp. 46-89
13. **H. Matthies and G. Strang.** The Solution of Nonlinear Finite Element Equations. International Journal for Numerical Methods in Engineering, Vol. 14, 1979, p. 1613-1626
14. **Willam K.J., Warnke E.P.** Constitutive Model for the Triaxial Behavior of Concrete. Proceedings of IABSE. Structural Engineering Report 19, Section III, 1975, pp. 1-30
15. СП 63.13330.2018 Бетонные и железобетонные конструкции. Основные положения. Актуализированная редакция СНиП 52-01-2003. М.: Минрегион России, 2018.148 с.
16. CEB, CEB-FIP Model Code 1990 / CEB Bulletin d'Information № 213/214, Comite Euro-International du Beton, Lausanne, Switzerland, 1993. 437 p.
17. Рекомендации по определению прочностных и деформационных характеристик бетона при неодноосных напряженных состояниях. М.: НИИЖБ Госстроя СССР, 1985, 72 с.
18. **Kupfer, H., Hilsdorf, H., Rusch, H.** Behavior of Concrete under Biaxial Stresses, ACI Journal, Proceedings Vol. 66, No. 8, August, 1969, pp. 656-666
19. **Launay P., Gachon H.** Strain and Ultimate Strength of Concrete under Triaxial Stress, Am. Concrete Inst. Spec. Publ. 34, pap. 13, 1972
20. **Mills L.L., Zimmerman R.M.** Compressive Strength of Plain Concrete under Multiaxial Loading Conditions, ACI Journal, Vol. 67, No. 10, 1970, October, pp. 802-807
21. **Корсун В.И.** Сопоставительный анализ критериев прочности для бетонов / В.И. Корсун, А.В. Недорезов, С.Ю. Макаренко // Современное промышленное и гражданское строительство. 2014. Т. 10. № 1. С. 65-78. EDN THXXCZ.
22. **Hansen T.C.** Triaxial test with concrete and cement paste: Report № 319 / T. C. Hansen. – Lyngby : Technical University of Denmark, 1995. – 54 p
23. **Agapov V.P. and Markovich A.S.** The family of multilayered finite elements for the analysis of plates and shells of variable thickness: La familia de elementos finitos multicapa para el análisis de placas y cascos de espesor variable. South Florida Journal of Development. 2021;2(4):034-5048. https://doi.org/10.46932/sfjdv2n4-007
24. **Agapov V.P., Markovich A.S.** Dynamic method for determining critical loads in the PRINS computer program. Structural

Mechanics of Engineering Constructions and Buildings. 2020;16(5):380-389. (In Russ.) DOI: 10.22363/1815-5235-2020-16-5-380-389

25. **Agapov V.P., Markovich A.S.** Investigation of the accuracy and convergence of the results of thin shells analysis using the PRINS program. Structural Mechanics of Engineering Constructions and Buildings. 2021; 17(6):617-627. (In Russ.) DOI: 10.22363/1815-5235-2021-17-6-617-627
26. **Obernikhin D.V., Nikulin A.I.** Experimental studies of strength, crack resistance and deformability of reinforced concrete beams of trapezoidal and rectangular cross sections. Innovative science. 2016; 8(2):73-77.
27. **Rimshin V.I., Amelin P.A.** Numerical calculation of bent reinforced concrete elements of rectangular section in the Abaqus software. Structural Mechanics of Engineering Constructions and Buildings. 2022;18(6):552-563. (In Russ.) http://doi.org/10.22363/1815-5235-2022-18-6-552-563
28. **Cedolin L., S. Deipoli.** Finite element studies of shear-critical R/C beams. ASCE Journal of the Engineering Mechanics Division. June 1977. Vol. 103. No. EM3, Pp. 395-410.
29. **von Mises R.** Mechanik der festen Körper im plastisch-deformablen Zustand. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-Physikalische Klasse. 1913; 1:582-592.

Vladimir P. Agapov, doctor of technical sciences, professor of the Department of Civil Engineering, Engineering Academy; RUDN University, 117198, Russia, Moscow, Miklukho-Maklaya, 6, tel. +7(499)9368787, e-mail: agapovpb@mail.ru.

Alexey S. Markovich, Ph. D., Associate Professor of the Department of Civil Engineering, Engineering Academy; RUDN University, 117198, Russia, Moscow, Miklukho-Maklaya, 6, tel. +7(499)9368787, e-mail: markovich-as@rudn.ru, Associate Professor of the Department of Fundamental Education, Moscow State University of Civil Engineering; 129337, Russia, Moscow, Yaroslavskoe Shosse, 26, tel. +7 (499) 936-87-87, e-mail: markovich-as@rudn.ru

Агапов Владимир Павлович, доктор технических наук, профессор департамента строительства инженерной академии, Российский университет дружбы народов им. Патриса Лумумбы, 117198, Россия, г. Москва, ул. Миклухо-Маклая, д.6, тел. +7(499)9368787, e-mail: agapovpb@mail.ru.

Маркович Алексей Семенович, кандидат технических наук, доцент департамента строительства инженерной академии, Российский университет дружбы народов им. Патриса Лумумбы, 117198, Россия, г. Москва, ул. Миклухо-Маклая, д.6; доцент кафедры фундаментального образования, Национальный исследовательский Московский государственный строительный университет; 129337, Россия, г. Москва, Ярославское шоссе, д. 26, тел. +7 (499) 936-87-87, e-mail: markovich-as@rudn.ru