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On unit P-Groups in Group Algebra

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Abstract: The aim of this paper we have define the group of units U(F(G)), where F(G) is the group algebra with G is finite group over a field F. Now if char F=0 and G nonabelian or F is a nonabsolute field of characterstic p > 0 and G/ Op (G) is nonabelian, then it is well known that the group of unit U(K[G]) contains a nonabelain P-group.There for we will prove that there are two cyclic subgroups X and Y of G of prime power order and units uX ∈ U(K[X]) and uY∈ U(K[X]) such that (uX,uY) contain nonabelian P-subgroups in linear group.

Keywords: units U(F(G)), P-Groups , Algebra

Introduction

Let K[G] denote the group algebra of a finite group G over a field K. In this paper we are concerned with the existence of nonabelian p-subgroup of the group of units U(K[G]). For convenience and following [4] we say that an arbitrary group J is 2-related if it contains no nonabelian p-subgroup. Thus J is 2-related if and only if every homomorphism from the 2-generator p-group V_2 into J has nontrivial kernel and hence if and only if every tow elements of J are related that is satisfy a nontrivial word in V_2. Obviously the property of being 2_related is closed under taking subgroups and homomorphic images.

If G is a belian then U(K[G]) is commutative and if G/ Op (G) is abelian where char K= p >0 and Op (G) is the largest normal p -subgroup of G then U(K[G]) is a solvable since the kernel of the natural homomorphism K[G]→ G/ Op (G) is a nilpotent ideal. Furthermore if K is an absolute field that is algebraic over a finite field then U(K[G]) is a periodic group certainly in all of these three situations U(K[G]) cannot contain a nonabelian p-group and consequently it is 2_related on the other hand if K[G] dose not satisfy the above then U(K[G]) dose contain a nonabelian p-group. For the most part this result of [2] follows from the fact that GL2(K) contains such a p-subgroup. See [5] for analogous problem in integral group rings.

If G has a nonnormal subgroup then specific generators for a nonabelian p-subgroup of the unit group of the integral group ring Z(G) were given in [9]. A similar result for group algebras in positive characteristic can be found in [3]. In this paper we consider units of a different nature namely.

Definition 1.1. Let K[G] be the group algebra of g over a nonabsolute field K, and let X=‹x› be cyclic subgroup of G of prime power order. Then we say that uX ∈ U(K[X]) is special unit depending upon the generator x if one of the following three conditions is satisfied

1. char K= 0 ,|X|p >0 and ux=(x-r)(x-s) for suitable integers r,s∈ Z ⊆K with r,s ≥ 2.
2. char K=p >0 |X|is prime to p , and ux=(x-r)(x-s) for suitable r,s∈ K that are positive powers of a fixed element t∈ K transcendental over the subfield K0=GF(p).
3. char K=p >0 X is a p -group and ux=1+t(1+x+…+x^{p-1}) where t∈ K is transcendental over K0.
4. In part (ii) and (iii) above we say more precisely that ux is special based on t. Using this notation our main result is.[4]

Theorem 1.2. Assume that we say char K=0 and G is nonabelian or that K is a nonabsolute field of characteristic p >0 and G/ Op (G) is nonabelian. Then there are two cyclic subgroups X and Y of G of prime power order and two special units uX ∈ U(K[X]) and

$uY \in U(K[X])$ (based on the same reselected transcendental element if char K>0), such that ‹ux,uy› is not 2_related. [8]

Corollary 1.3. Assume that either char K=0 and G is nonabelain or that K is a nonabsolute field of characteristic p >0 and G/ Op (G) is nonabelain. Then the subgroup of U(K[G]) generated by units of the form x-r with $x \in G$ and $r \in K$ has a nonabelian p-subgroup.[8]

Definition1.4. A group G is said to be a p-group if the order of each element of G is a power of a fixed prim p.[2]

Example1.5. Any group of order pn (p prime) is p-group since the order of each element must divide the order of the group. In particular the group of symmetries of square is p-group where p=2.[2]

Example1.6. Let G be a commutative group and the set H consist of those element whose order are powers of a fixed prime p (quite possible H {e}). Then H forms subgroup of G which by its definition a p-group.[2]

Lemma1.7. If G is finite commutative group whose order is divisible by a prime p then G contains an element of order p.[2]

Corollary1.8 Let G be a finite commutative group and p prime dividing o(G). Then G has subgroup of order p. [2]

Theorem1.9. Let G be a finite group and let p be a fixed prime. Suppose that G/ Op (G) is nonabelian but that H/ Op (H) is abelian for every proper subgroup and every proper homomorphic image H of G. Then we the following two possibilities.

5. (The p-group case) G is a p-group with p $\neq p$ its center Z(G) is cyclic of index p2and ׀ G'׀=p. Furthermore either ׀G׀=p3 or G=X ×Y where X is cyclic and ׀Y׀=p.
6. (The Frobenius case) G=A×X where A is on elementary abelian q-group with the prime q different from p , X is cyclic of prime order p ≠ q and X acts faithfully and irreducibly on A.

Proof: (i) It is clear that Op (G)=1. suppose first that Z(G) ≠ 1 and choose Z to be a central subgroup of prime order p. Since Op (G)=1 we have p $\neq p$ and it follows easily that Op (G/Z)=1. Hence G/Z is an abelian p '-group by hypothesis. Thus G is nilpotent of class 2 and Z=G' . In particular Z is unique so Z(G) must be a cyclic p-group, and since G nilpotent we see that G is a minimal nonabelian p-group and by [9] either׀G׀=p3 or G = X×Y with X and ׀Y׀=p.

(ii)We can now assume that Z(G) =1 and in particular that G is not nilpotent. Suppose next by way of contradiction that G is simple and let p $\neq p$ be prime divisor of ׀G׀. If P is any nonidentity p-subgroup of G then NG(P) is proper and therefore has a normal p-complement by hypothesis. Frobenius theorem (see [6]) now implies that G has a normal p-complement and this contradicts the assumption that G is simple group that not nilpotent. Consequently G is not simple and we conclude from the hypothesis that G is solvable.

Finally, let A be a minimal normal subgroup of G. Then A is a elementary abelian q-group for some prime p $\neq p$ and A is central. In particular we can choose $x \in G$ to be an element of minimal order not centralizing A. certainly x has prime power order say ׀x׀=pn. Note that the group‹ A,x› has nontrivial commutator subgroup contained in a so G =‹ A,x› by hypothesis. The minimal natural of ׀x׀ now implies that $xp \in Z(G)$ =1 and hence X =‹ x › is cyclic of prime order p. Clearly G= A×Y and since A is a minimal normal subgroup of G.

we conclude that X acts faithfully and irreducibly on A.

As we will see the p-groups above are fairly easy to handy but the Frobenius group work is much more difficult. The proof of our main theorem uses techniques from [4]. However the objective of that paper was somewhat different from the problem here. In particular since we were not concerned with a precise description of the unitary units in K[G] we were able to finesse a serious study of the Frobenius group G= A×Y in [4]. Here we have to come to grips with the representations theory of such groups. Surprisingly there are interesting open questions concerning these representations especially in positive characteristic. We start with a few simple properties see[7] for basic information on this subject. We do have to be a bit careful below to allow for possibility that p is the characteristic of K.

Lemma 1.10. Let G= A×X where A is an elementary abelian q-group, X is cyclic of prime order p and X acts faithfully and irreducibly on A. Let K be a field of characteristic ≠q and assume that K contains a primitive qth root of unity.

7. If $m : K[A] \to K$ is a nonprincipal linear character of A, that is a nontrivial one-dimensional character then the induced representation $q = m^G$ is an absolutely irreducible representation of K[G].

8. Conversely if q is a nonlinear irreducible representation of K[G] and if $m : K[A] \rightarrow K$ is constituent of the restriction q_A then μ≠1 and $q = m^G$.

In either situation, q is faithful on the group G and deg$q$ =p, Furthermore, q is injective on the group ring K[X] and by conjugating if necessary we can assume that q (α)=diag(μ(α), μ(αx),…, μ(αx^{p-1})) for all α∈ K[A].

Proof: Since G= A×Y is Frobenius group, X acts in a fixed-point-free manner on the dual group of A. Thus each nonprincipal character of K[A] has p=$|X|$ conjugates under the action of X.

9. Let $m : K[A] \rightarrow K$ be nonprincipal character and set $q = m^G$. Then deg$q$ =p and θA=μ1+μ2+…+μp is the sum of the p distinct conjugates of μ. If y is an irreducible sub representation of q then y_A must contain some μi and hence it contains the entire X-orbit of μ. In particular we have $p = \deg q \geq \deg y \geq p, \ so \ \ q = y \quad irreducible$.

10. Conversely, let q be a nonlinear irreducible representation of K[G] and let μ be irreducible constituent of q A. If μ=1 then G'=A⊆ ker$q$ and q is linear a contradiction. Thus μ≠1 and hence by (i) above μG is irreducible. In particular since q is quotient of (q A)G and since the latter is direct sum of copies of μG, we conclude that q = μG, as required.
The remaining observations follow from the definition of induced representation and the fact that A= G' is the unique nontrivial normal subgroup of G.

FROBENIUS GROUPS

As we indicated in the introduction our proof relies on certain special case considerations. Indeed the p-groups are easy handle while the Frobenius groups are much more of a challenge. The following result is well. Known. We include it here as motivation for later work.

Lemma 2.1. Let G be a nilpotent group of class ≤2 and let $q : K[G] \rightarrow M_n(K)$ a G-faithful absolutely irreducible representation. If T is trivial for Z(G) in G. then q (T) is a K-basis for Mn(K) and hence n2=$|T| = |G : Z(G)|$.

Proof : Since q absolutely irreducible q (K[G])=Mn(K). Now for each g∈ G, let c (g) ∈ K be the matrix trace of q (g). $c : G \rightarrow K$ is the character of G associated with q. If g∈ Z(G) then q (g)=λI is a scalar matrix and hence χ(g)= λn. If g∉ Z(G) then since G has class ≤2, there exists x∈ G with x-1gx=gz for some 1≠z∈ Z(G). Thus θ(x)-1θ(g)θ(x)=θ(g)θ(z)=μθ(g), where θ(z)=μI, and μ ≠1 since z≠1and θ is faithful. Taking matrix traces and using the fact that similar matrices have same trace we obtain χ(g)=0 In other words, χ vanishes off Z(G). Now all matrices in θ(Z(G)) are scalar so it follows that θ(T). spans Mn(K). Furthermore since there are matrices in Mn(K) with nonzero trace, we see that χ cannot vanish on G and in particular we have n≠0 in K. Finally suppose $\sum_{g \in T} k_g q(g)$ =0 is linear dependence relation for θ(T). If x∈ T then multiplying this equation by θ(x-1) and taking traces yields kxn=0, sine gx-1∈ Z(G) if and only if g=x. Thus kx=0 for all x∈ T and θ(T) is K-linearly independent as required.

Next we consider the necessary Frofebiuns groups. Specifically, let G=A×X, where A is an elementary abelian q-group X=‹x› is cyclic of prime order p, and X acts faithfully and irreducible on A. Assume that K is a field of characteristic different from p and q and that K contains a primitive (pq)th root of unity. We fix this notation throughout the remainder of section.

If θ is a nonlinear irreducible representation of K[G], then by lemma 1.10, θ is faithful on G and θ(K[G])=Mp(K) has dimension p2. In analogy with lemma 2.1 it is appropriate to ask whether there is a natural basis for this matrix built from certain group elements. For example if 1≠a∈ A then Y=aXa-1 is acyclic subgroup of G of order p disjoint from X. Thus XY is a basis for Mp(K). As it turns out this indeed the case if either char K=0 or char K is positive and sufficiently there exists an appropriate

K[G] such that for all θ and X,Y, the set θ(XY) is a basis for the matrix ring.
Returning to the general group G, we know that θ(K[G]) may be taken to be the set of diagonal matrices in Mp(K) and hence this image has dimension p. On the other hand each nonidentity G-conjugacy class contained in A has size p and we ask whether there exists such a class Aa with θ(Aa) a basis for the diagonal matrices. This question turns out to be precisely equivalent to the preceding one and hence has the sam positive and negative. Fortunately we are able to partially finesse the negative answers and prove a result just strong enough to enable us construct the unit we require.
We now start the formal considerations. Since X acts on A it also acts on K[G] and for each linear character $l : K[X] \to K$ we define the λ-trace $tr_l : K[A] \to K[A]$ to be the K-linear map given by

$$tr_l a = \sum_{i=0}^{p-1} l(x^{-i}) a^{x^i} = \sum_{i=0}^{p-1} l(x^i) a^{x^{-i}} \qquad for\ \ all\ a \in K[A].$$

Basic properties are as follows.
Lemma 2.2 With the above notation we have (trλα)x=λ(x)trλα and

$$tr_l a)(tr_m b) = \sum_{k=0}^{p-1} l(x^{-k}) tr_{lm}(a^{x^k} b) = \sum_{k=0}^{p-1} m(x^{-k}) tr_{lm}(a b^{x^k}).$$

Proof: For the first fact note that

$$(tr_l a)^x = \sum l(x^{-i}) a^{i+1} = l(x) \sum l(x^{-(i+1)}) a^{i+1} = l(x) tr_l a.$$

Foe the second write i=j+k and observe that

$$(tr_l a)(tr_m b) = \sum_{i,j} l(x^{-i}) a^{x^i} m(x^{-j}) b^{x^j} = \sum_{j,k} l(x^{-(j+k)}) m(x^{-j}) a^{x^{j+1}} b^{x^j}$$
$$= \sum_k l(x^{-k}) \sum lm(x^{-j})(a^{x^k} b)^{x^j} = \sum_k l(x^{-k}) tr_{lm}(a^{x^k} b)$$

The third formula follows from the above by interchanging the factors.
Now suppose µ:K[X]→K is a linear character. Then the idempotent eµ∈ K[X] associated with µ is given by

$$e_m = \frac{1}{p}\sum_{i=0}^{p-1} m(x^{-i}) x^i = \frac{1}{p}\sum_{i=0}^{p-1} m(x^i) x^{-i}.$$

Indeed we have

$$x e_m = \frac{1}{p}\sum_{i=0}^{p-1} m(x^{-i}) x^{i+1} = \frac{m(x)}{p}\sum_{i=0}^{p-1} m(x^{-i-1}) x^{i+1} = m(x) e_m.$$

The basic relation between these idempotents and λ-traces is as follows.

Lemma 2.3 Let µ,η:K[X]→K be linear characters and let α∈ K[A]. Then

$$e_m a\, e_h = \frac{1}{p}(tr_l\ a) e_h = \frac{1}{p} e_m (tr_l\ a) \qquad where\ \ l = m^{-1} h.$$

Proof: To start with we have

$$e_m\, a\, e_h = \frac{1}{p}\sum_{i=0}^{p-1} m(x^i) x^{-i} a\, e_h = \frac{1}{p}\sum_{i=0}^{p-1} m(x^i) x^{-i} a\, x^i x^{-i} e_h$$
$$= \frac{1}{p}\sum_{i=0}^{p-1} m(x^i)\, h(x^{-i}) x^{-i} a\, x^i e_h$$

Since $x^{-i} e_h = h(x^{-i}) e_h.$ Thus setting λ = ηµ-1 we obtain

$$e_m\, a\, e_h = \frac{1}{p}\sum_{i=0}^{p-1} l(x^{-i}) a^{x^i} e_h = \frac{1}{p}(tr_l\ a) e_h.$$

The second formula follows in a similar fashion.
Recall from lemma 1.10 that every nonlinear irreducible representation θ of K[G] has degree p. Furthermore according to that lemma we can always assume that θ(A) consists of diagonal matrices.
Lemma 2.4 Let θ be a nonlinear irreducible representation of K[G] and let µ:K[A]→K be a constituent of the restriction θA. If α∈ K[A] then θ(trλa) is either zero or an invertible element in Mp(K) =θ(K[G]). It is invertible if and only if $\sum_{i=0}^{p-1} l(x^{-i}) m(a^{x^i}) \neq 0.$
Proof: Since trλ α commutes with A and since(trλ α)x = λ(x) trλ α we see that θ(trλa) Mp(K) is two-sided ideal of the matrix ring Mp(K) = θ(K[G]. With this it is clear that θ(trλa) is either zero or invertible. Furthermore since θ(trλa) is a diagonal matrix it is invertible if and only if its (1,1)-entry is not zero and according to lemma 1.10 this entry is equal to $\sum_{i=0}^{p-1} l(x^{-i}) m(a^{x^i})$.
We can prove the equivalence of the various problem.
Lemma 2.5 Let θ be a nonlinear irreducible representation of K[G] and let µ be an irreducible constituent of θA. Fix 1≠a∈ A and set Y=aXa-1. The following are equivalent:

1. θ(XY)= θ(X) θ(Y) is a basis for Mp(K)= θ(K[G]).
2. θ(Aa) is a basis for the diagonal matrices in Mp(K).
3. θ(trλa) ≠0 for each λ:K[X]→K.

4. $\sum_{i=0}^{p-1} l\,(x^{-i})\,m\,(a^{x^i}) \neq 0$ for each λ:K[X]→K.

Proof: We show that each of these condition is equivalent to (iii) and note that (iv)↔(iii) from the previous lemma.

(ii) ↔(iii). If θ(Aa)={θ(a),θ(ax),…, $q\,(a^{x^{p-1}})$} is K-linearly independent then certainly θ(trλa)≠0 for each λ. Conversely suppose that each θ(trλa)≠0 and note that by lemma 2.2 each of these is an eigenvector for the conjugation action of θ(x) with distinct eigenvalue λ(x). Thus the various θ(trλa) are linearly independent and span a K-vector space of dimension p. Since this space is contained in the span of θ(Aa), we conclude that the latter span has dimension p and is equal to the set of diagonal matrices in Mp(K).

(i) ↔(iii). Let μ,η:K[X]→K , let eμ be the idempotent of K[X] associated with μ, and let fμ=aeμa-1 be the idempotent of K[Y] associated with η. Then, by lemma 2.3,

eμfη=(eμaeη)a-1 =1/p (trλa)eηa-1,

where λ =μ-1η. θ is faithfule on K[X] and K[Y], we know that θ(eμ) and θ(fη) are not zero. If θ(X) θ(Y) is linearly independent, then it follows immediately that θ(eμ) θ(fη) ≠ 0 for all λ, then since θ(trλa) and θ(a-1) are invertible, we see that θ(eμ) θ(fη) ≠ 0 for all μ,η. The orthogonality of the sets { eμ| all μ} and { fη| all η} now clearly implies that the set {θ(eμ) θ(fη)| all μ,η} of size p2 is linearly independent and hence spans Mp(K). Therefore θ(X) θ(Y) also spans Mp(K).

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دراسة نظائر الزمرة الأولية في الزمرة الجبرية

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الخلاصة

الهدف من هذا البحث هو تعريف زمرة النظائر حيث F(G) هو جبر الزمرة المنتهية G المعرفة على الحقل F . اذا كان المميز هو صفر والزمرة G غير ابدالية و F حقل غير مطلق ومميز p اكبر من الصفر (G) G/Op غير ابدالية فأنه معرفة تعريفا حسنا لزمرة النظائر المحتوية من الزمرة الاولية غير ابدالية من G ذي الرتب الاولية والنظائر (u_x, u_y) بحيث ان $u_x \in U(K[X])$ and $u_y \in U(K[Y])$ تحتوي الزمرة الجزئيه الاولية غير ابدالية من الزمر الخطية.