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Statistical convergence in vector lattices

The statistical convergence is defined for sequences with the asymptotic density on the natural numbers, in general. In this paper, we introduce the statistical convergence in vector lattices by using the finite additive measures on directed sets. Moreover, we give some relations between the statistical convergence and the lattice properties such as the order convergence and lattice operators.

Keywords: statistical convergence of nets, order convergence, vector lattice, directed set measure.

Introduction

The statistical convergence of sequences is handled together with the asymptotic (or, natural) density of subsets on the natural numbers $\mathbb{N}$. On the other hand, Connor introduced the notion of statistical convergence of sequences with finitely additive set functions [1, 2]. After then, some similar works have been done [3–5]. Also, several applications and generalizations of the statistical convergence of sequences have been investigated by several authors [6–13]. However, as far as we know, the concept of statistical convergence related to nets has not been done except for the paper [14], in which the asymptotic density of a directed set $(D, \leq)$ was introduced by putting a special and strong rule on the directed sets such that the set $\{\alpha \in D : \alpha \leq \beta\}$ is finite and the set $\{\alpha \in D : \alpha \geq \beta\}$ is infinite for each element β in $(D, \leq)$. We aim to introduce a general concept of statistical convergence for nets with a new notion called a directed set measure.

Recall that a binary relation "$\leq$" on a set A is called a *preorder* if it is reflexive and transitive. A non-empty set A with a preorder binary relation "$\leq$" is said to be a *directed upwards* (or, for short, *directed set*) if for each pair $x, y \in A$ there exists $z \in A$ such that $x \leq z$ and $y \leq z$. Unless otherwise stated, we consider all directed sets as infinite. For given elements a and b in a preorder set A such that $a \leq b$, the set $\{x \in A : a \leq x \leq b\}$ is called an *order interval* in A. A subset I of A is called an *order bounded set* whenever I is contained in an order interval.

A function domain of which is a directed set is said to be a *net*. A net is briefly abbreviated as $(x_\alpha)_{\alpha \in A}$ with its directed domain set A. Let $(A, \leq_A)$ and $(B, \leq_B)$ be directed sets. Then a net $(y_\beta)_{\beta \in B}$ is said to be a *subnet* of a net $(x_\alpha)_{\alpha \in A}$ in a non empty set X if there exists a function $\phi : B \to A$ such that $y_\beta = x_{\phi(\beta)}$ for all $\beta \in B$, and also, for each $\alpha \in A$ there exists $\beta_\alpha \in B$ such that $\alpha \leq \phi(\beta)$ for all $\beta \geq \beta_\alpha$ (Definition 3.3.14 [15]). It can be seen that $\{\phi(\beta) \in A : \beta_\alpha \leq \beta\} \subseteq \{\alpha' \in A : \alpha \leq \alpha'\}$ holds for subnets.

A real vector space E with an order relation "$\leq$" is called an *ordered vector space* if, for each $x, y \in E$ with $x \leq y$, $x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold for all $z \in E$ and $\alpha \in \mathbb{R}_+$. An ordered vector space E is called a *Riesz space* or *vector lattice* if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \quad \text{and} \quad x \vee y = \sup\{x, y\}$$

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exist in E, respectively. A vector lattice is called *Dedekind complete* if every nonempty bounded from the above set has a supremum (or, equivalently, whenever every nonempty bounded below subset has an infimum). A subset I of a vector lattice E is said to be a *solid* if, for each $x \in E$ and $y \in I$ with $|x| \leq |y|$, it follows that $x \in I$. A solid vector subspace is called an *order ideal*. A vector lattice E has the *Archimedean* property provided that $\frac{1}{n}x \downarrow 0$ holds in E for each $x \in E_+$. In this paper, unless otherwise stated, all vector lattices are assumed to be real and Archimedean. We remind the following crucial notion of vector lattices [16–20].

Definition 1. A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E is called *order convergent* to $x \in E$ if there exists another net $(y_\alpha)_{\alpha\in A} \downarrow 0$ (i.e., $\inf y_\alpha = 0$ and $y_\alpha \downarrow$) such that $|x_\alpha - x| \leq y_\alpha$ holds for all $\alpha \in A$.

We refer the reader to some different types of order convergence and some relations among them [21]. Throughout this paper, the vertical bar of a set will stand for the cardinality of the given set and $\mathcal{P}(A)$ is the power set of A.

1 *The μ-statistical convergence*

We remind that a map from a field $\mathcal{M}$ (i.e., $M_1, M_2, \dots \in \mathcal{M}$ implies $\cup_{i=1} M_n \in \mathcal{M}$ and $A^c \in \mathcal{M}$ for all $A \in \mathcal{M}$) to $[0,\infty]$ is called *finitely additive measure* whenever $\mu(\emptyset) = 0$ and $\mu(\cup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$ for all finite disjoint sets $\{E_i\}_{i=1}^n$ in $\mathcal{M}$ [22; 25]. Now, we introduce the notion of measuring on directed sets.

Definition 2. Let A be a directed set and $\mathcal{M}$ be a subfield of $\mathcal{P}(A)$ (i.e., it satisfies the properties of field). Then

(1) an order interval $[a, b]$ of A is said to be a *finite order interval* if it is a finite subset of A;

(2) $\mathcal{M}$ is called an *interval field* on A whenever it includes all finite order intervals of A;

(3) a finitely additive measure $\mu : \mathcal{M} \to [0, 1]$ is said to be a *directed set measure* if $\mathcal{M}$ is an interval field and μ satisfies the following facts: $\mu(I) = 0$ for each finite order interval $I \in \mathcal{M}$ and $\mu(A) = 1$.

It is clear that $\mu(C) = 0$ whenever $C \subseteq B$ and $\mu(B) = 0$ holds for $B, C \in \mathcal{M}$ because μ is finitely additive.

Example 1. Consider the directed set $\mathbb{N}$ and define a measure μ from $2^{\mathbb{N}}$ to $[0, 1]$ denoted by $\mu(A)$ as the Banach limit of $\frac{1}{k}|A \cap \{1, 2, \dots, k\}|$ for all $A \in 2^{\mathbb{N}}$. Then one can see that $\mu(I) = 0$ for all finite order interval sets because of $\frac{1}{k}|I \cap \{1, 2, \dots, k\}| \to 0$. Also, it follows from the properties of the Banach limit that $\mu(\mathbb{N}) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for disjoint sets A and B. Thus, μ is finitely additive, and so, it is a directed set measure.

Let's give an example of a directed set measure for an arbitrary uncountable set.

Example 2. Let A be an uncountable directed set. Consider a field $\mathcal{M}$ consisting of countable or co-countable (i.e., the complement of set is countable) subsets of A. Then $\mathcal{M}$ is an interval field. Thus, a map μ from $\mathcal{M}$ to $[0, 1]$ defined by $\mu(C) := 0$ if C is a countable set, otherwise $\mu(C) = 1$. Hence, μ is a directed set measure.

In this paper, unless otherwise stated, we consider all nets with a directed set measure on interval fields of the power set of the index sets. Moreover, in order to simplify the presentation, a directed set measure on an interval field $\mathcal{M}$ of directed set A will be expressed briefly as a measure on the directed set A. Motivated from [23; 302], we give the following notion.

Recall that the asymptotic density of a subset K of natural numbers $\mathbb{N}$ is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}| .$$

We refer the reader for an exposition on the asymptotic density of sets in $\mathbb{N}$ to [24, 25]. We give the following observation.

Remark 1. It is clear that the asymptotic density of subsets on $\mathbb{N}$ satisfies the conditions of a directed set measure when $\mathcal{P}(\mathbb{N})$ is considered as an interval field on the directed set $\mathbb{N}$. Thus, it can be seen that the directed set measure is an extension of the asymptotic density.

Remind that a sequence (x_n) in a vector lattice E is called *statistically monotone decreasing* to $x \in E$ if there exists a subset K of $\mathbb{N}$ such that $\delta(K) = 1$ and the subsequence $(x_k)_{k\in K}$ is decreasing to x, i.e., $x_k \downarrow$ and $\inf\limits_{k\in K} x_k = x$ (see for example [8]). Now, by using the notions of measure on directed sets and the statistical monotone decreasing which was introduced in [25] for real sequences, we introduce the concept of statistical convergence of nets on vector lattices.

Definition 3. Let E be a vector lattice and $(p_\alpha)_{\alpha\in A}$ be a net in E with a measure μ on the index set A. Then $(p_\alpha)_{\alpha\in A}$ is said to be *μ-statistical decreasing* to $x \in E$ whenever there exists a subnet $q_\delta = (p_{\phi(\delta)})_{\beta\in\Delta}$ such that $\mu(\Delta) = 1$ and $(q_\delta)_{\delta\in\Delta} \downarrow x$. Then it is abbreviated as $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} x$.

We denote the class of all μ-statistical decreasing nets on a vector lattice E by $E_{st_\mu\downarrow}$, and also, the set $E_{st_\mu\downarrow}\{0\}$ denotes the class of all μ-statistical decreasing null nets on E. It is clear that $\mu(\Delta^c) = \mu(A-\Delta) = 0$ whenever $\mu(\Delta) = 1$ because of $\mu(A) = \mu(\Delta\cup\Delta^c) = \mu(\Delta)+\mu(\Delta^c)$. We consider Example 2 for the following example.

Example 3. Let E be a vector lattice and $(p_\alpha)_{\alpha\in A}$ be a net in E. Take $\mathcal{M}$ and μ from Example 2. Thus, if $(p_\alpha)_{\alpha\in A} \downarrow x$ then $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} x$ for some $x \in E$.

For the general case of Example 3, we give the following work proof of which follows directly from the basic definitions and results.

Proposition 1. If $(p_\alpha)_{\alpha\in A}$ is an order decreasing null net in a vector lattice then $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$.

Now, we introduce the crucial notion of this paper.

Definition 4. A net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E is said to be *μ-statistical convergent* to $x \in E$ if there exists a net $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$ with a subnet $q_\delta = (p_{\phi(\delta)})_{\beta\in\Delta}$ such that $\mu(\Delta) = 1$ and $(q_\delta)_{\delta\in\Delta} \downarrow 0$ and $|x_{\phi(\delta)} - x| \leq q_\delta$ for every $\delta \in \Delta$. Then it is abbreviated as $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$.

It can be seen that $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ in a vector lattice means that there exists another sequence $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$ such that $\mu\big(\{\alpha \in A : |x_\alpha - x| \nleq p_\alpha\}\big) = 0$. It follows from Remark 1 that the notion of statistical convergence of sequence coincides with the notion of μ-statistical convergence in the reel line. We denote the set E_{st_μ} as the family of all st_μ-convergent nets in E, and $E_{st_\mu}\{0\}$ is the family of all μ-statistical null nets in E.

Lemma 1. Every μ-statistical decreasing net is μ-statistical convergent.

Remark 2. Recall that a net $(x_\alpha)_{\alpha\in A}$ in a vector lattice E *relatively uniform converges* to $x \in E$ if there exists $u \in E_+$ such that, for any $n \in \mathbb{N}$, there is an index $\alpha_n \in A$ so that $|x_\alpha - x| \leq \frac{1}{n}u$ for all $\alpha \geq \alpha_n$ (Lemma 16.2 [18]). It is well known that the relatively uniform convergence implies the order convergence on Archimedean vector lattices (Lemma 2.2 [20]). Hence, it follows from Proposition 1 and Lemma 1 that every decreasing relatively uniform null net is μ-statistical convergent in vector lattices.

2 *Main Results*

Let μ be a measure on a directed set A. Following from Exercise 9. in [22; 27], it is clear that $\mu(\Delta\cap\Sigma) = 1$ for any $\Delta, \Sigma \subseteq A$ whenever $\mu(\Delta) = \mu(\Sigma) = 1$. We begin the section with the following proposition and skip its simple proof.

Proposition 2. Assume $x_\alpha \leq y_\alpha \leq z_\alpha$ satisfies in a vector lattices for each index α. Then $y_\alpha \xrightarrow{\mathrm{st}_\mu} x$ whenever $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $z_\alpha \xrightarrow{\mathrm{st}_\mu} x$.

It can be seen from Proposition 2 that if $0 \leq x_\alpha \leq z_\alpha$ satisfies for each index α and $(z_\alpha)_{\alpha\in A} \in E_{st_\mu}\{0\}$ then $(x_\alpha)_{\alpha\in A} \in E_{st_\mu}\{0\}$. We give a relation between the order and the μ-statistical convergences in the next result.

Theorem 1. Every order convergent net is μ-statistical convergent to its order limit.

Proof. Suppose that a net $(x_\alpha)_{\alpha\in A}$ is order convergent to x in a vector lattice E. Then there exists another net $(y_\alpha)_{\alpha\in A} \downarrow 0$ such that $|x_\alpha - x| \leq y_\alpha$ holds for all $\alpha \in A$. It follows from Proposition 1 that $(y_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$. So, we obtain the desired result, $(x_\alpha)_{\alpha\in A} \xrightarrow{\mathrm{st}_\mu} x$.

The converse of Theorem 1 need not to be true. To see this, we consider Example 3. [26].

Example 4. Let us consider the set of all real numbers $\mathbb{R}$ with the usual order. Define a sequence (x_n) in $\mathbb{R}$ as n^2 whenever $n = k^2$ for some $k \in \mathbb{N}$ and $\frac{1}{n+1}$ otherwise. It is clear that (x_n) is not an order convergent sequence. However, if we choose another sequence (p_n) as n whenever $n = k^2$ for some $k \in \mathbb{N}$ and $\frac{1}{1}$ otherwise. Then we have $p_n \downarrow^{st_\mu} 0$. Setting $K = \{n \in \mathbb{N} : \text{n is not a square}\} \cup \{1\}$. Then we get $\mu(K) = 1$ and $|x_k| \leq p_k$ for each $k \in K$. Thus we have $x_n \xrightarrow{\mathrm{st}_\mu} 0$.

Moreover, following from Theorem 23.2 [18], we observe the following result.

Corollary 1. Every order bounded monotone net in a Dedekind complete vector lattice is μ-statistical convergent.

By the definition of subnet given at the beginning of the paper, a subnet is based on some other set B, where the measure μ is not defined. However, for a subnet $y_\beta = x_{\phi(\beta)}$ of a net $(x_\alpha)_{\alpha\in A}$ with a measure μ on the index set A, we can consider the measure of a subset Δ of B as the measure of $\mu(\phi(\Delta))$ in A.

Proposition 3. The st_μ-convergence of subnets implies the st_μ-convergence of nets.

Proof. Let $(x_\alpha)_{\alpha\in A}$ be a net in a vector lattice E. Assume that a subnet $(x_{\phi(\delta)})_{\delta\in\Delta}$ of $(x_\alpha)_{\alpha\in A}$ μ-statistical converges to $x \in E$. Then there exists a net $(p_\alpha)_{\alpha\in A} \in E_{st_\mu\downarrow}\{0\}$ such that $|x_{\phi(\sigma)} - x| \leq p_{\phi(\sigma)}$ for all some $\sigma \in \Sigma \subseteq \Delta$, $(p_{\phi(\sigma)})_{\sigma\in\Sigma} \downarrow 0$ and $\mu(\Sigma) = 1$. Since $\Sigma \subseteq A$ and $(x_{\phi(\sigma)})_{\sigma\in\Sigma}$ is also a subnet of $(x_\alpha)_{\alpha\in A}$, we can obtain the desired result.

Since every order bounded net has an order convergent subnet in atomic KB-spaces (Remark 6. [27]), we give the following result by considering Theorem 1 and Proposition 3.

Corollary 2. If E is an atomic KB-space then every order bounded net is μ-statistical convergent in E.

The lattice operations are μ-statistical continuous in the following sense.

Theorem 2. If $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $w_\alpha \xrightarrow{\mathrm{st}_\mu} w$ then $x_\alpha \vee w_\alpha \xrightarrow{\mathrm{st}_\mu} x \vee w$.

Proof. Assume that $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $w_\alpha \xrightarrow{\mathrm{st}_\mu} w$ hold in a vector lattice E. So, there are nets $(p_\alpha)_{\alpha\in A}, (q_\alpha)_{\alpha\in A} \in E_{st_\mu\downarrow}\{0\}$ with $\Delta, \Sigma \in \mathcal{M}$ and $\mu(\Delta) = \mu(\Sigma) = 1$ such that

$$|x_{\phi(\delta)} - x| \leq p_{\phi(\delta)} \text{ and } |w_{\rho(\sigma)} - w| \leq q_{\rho(\sigma)}$$

satisfy for all $\delta \in \Delta$ and $\sigma \in \Sigma$. On the other hand, it follows from Theorem 1.9(2) [17] that the inequality $|x_\alpha \vee w_\alpha - x \vee w| \leq |x_\alpha - x| + |w_\alpha - w|$ holds for all $\alpha \in A$. Therefore, we have

$$|x_{\phi(\delta)} \vee w_{\phi(\sigma)} - x \vee w| \leq p_{\phi(\delta)} + q_{\phi(\sigma)}$$

for each $\delta \in \Delta$ and $\sigma \in \Sigma$. Take $\Gamma := \Delta\cap\Sigma \in \mathcal{M}$. So, we have $\mu(\Gamma) = 1$, and also, $|x_{\phi(\gamma)} \vee w_{\phi(\gamma)} - x \vee w| \leq p_{\phi(\gamma)} + q_{\phi(\gamma)}$ holds for all $\gamma \in \Gamma$. It follows from $(p_{\phi(\gamma)} + q_{\phi(\gamma)})_{\gamma\in\Gamma} \downarrow 0$ that $x_\alpha \vee w_\alpha \xrightarrow{\mathrm{st}_\mu} x \vee w$.

Corollary 3. If $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ and $w_\alpha \xrightarrow{\mathrm{st}_\mu} w$ in a vector lattice then
(i) $x_\alpha \wedge w_\alpha \xrightarrow{\mathrm{st}_\mu} x \wedge w$;
(ii) $|x_\alpha| \xrightarrow{\mathrm{st}_\mu} |x|$;
(iii) $x_\alpha^+ \xrightarrow{\mathrm{st}_\mu} x^+$;
(iv) $x_\alpha^- \xrightarrow{\mathrm{st}_\mu} x^-$.

We continue with several basic results that are motivated by their analogies from vector lattice theory.

Theorem 3. Let $(x_\alpha)_{\alpha\in A}$ be a net in a vector lattice E. Then the following results hold:
(i) $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ iff $(x_\alpha - x) \xrightarrow{\mathrm{st}_\mu} 0$ iff $|x_\alpha - x| \xrightarrow{\mathrm{st}_\mu} 0$;
(ii) the μ-statistical limit is linear;
(iii) the μ-statistical limit is uniquely determined;
(iv) the positive cone E_+ is closed under the statistical μ-convergence;
(v) $x_{\phi(\delta)} \xrightarrow{\mathrm{st}_\mu} x$ for any subnet $(x_{\phi(\delta)})_{\delta\in\Delta}$ of $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ with $\mu(\Delta) = 1$.

Proof. The properties (i), (ii) and (iii) are straightforward.

For (iv), take a non-negative μ-statistical convergent net $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ in E. Then it follows from Corollary 3 that $x_n = x_n^+ \xrightarrow{\mathrm{st}_\mu} x^+$. Moreover, by applying (ii), we have $x = x^+$. So, we obtain the desired result $x \in E_+$.

For (v), suppose that $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$. Then there is a net $(p_\alpha)_{\alpha\in A} \in E_{st_\mu\downarrow}\{0\}$ with $\Delta \in \mathcal{M}$ and $\mu(\Delta) = 1$ such that $|x_{\phi(\delta)} - x| \le p_{\phi(\delta)}$ for each $\delta \in \Delta$. Thus, it is clear that $x_{\phi(\delta)} \xrightarrow{\mathrm{st}_\mu} x$. However, it should be shown that it is provided for all arbitrary elements in field under the assumption. Thus, take an arbitrary element $\Sigma \in \mathcal{M}$ with $\Sigma \neq \Delta$ and $\mu(\Sigma) = 1$. We show $(x_{\phi(\sigma)})_{\sigma\in\Sigma} \xrightarrow{\mathrm{st}_\mu} x$. Consider $\Gamma := \Delta \cap \Sigma \in \mathcal{M}$. So, we have $\mu(\Gamma) = 1$. Therefore, following from $|x_{\phi(\gamma)} - x| \le p_{\phi(\gamma)}$ for each $\gamma \in \Gamma$, we get the desired result.

Proposition 1 shows that a decreasing order convergent net is μ-statistical convergent. For the converse of this fact, we give the following result.

Proposition 4. Every monotone μ-statistical convergent net is order convergent.

Proof. We show that $x_\alpha \downarrow$ and $x_\alpha \xrightarrow{\mathrm{st}_\mu} x$ implies $x_\alpha \downarrow x$ in any vector lattice E. To see this, choose an arbitrary index α_0. Then $x_{\alpha_0} - x_\alpha \in E_+$ for all $\alpha \ge \alpha_0$. It follows from Theorem 3 that $x_{\alpha_0} - x_\alpha \xrightarrow{\mathrm{st}_\mu} x_{\alpha_0} - x$, and also, $x_{\alpha_0} - x \in E_+$. Hence, we have $x_{\alpha_0} \ge x$. Then x is a lower bound of $(x_\alpha)_{(\alpha\in A)}$ because α_0 is arbitrary. Suppose that z is another lower bound of $(x_\alpha)_{\alpha\in A}$. So, we obtain $x_\alpha - y \xrightarrow{\mathrm{st}_\mu} x - y$. It means that $x - y \in E_+$, or equivalent to saying that $x \ge y$. Therefore, we get $x_\alpha \downarrow x$.

Remark 3. Let $x := (x_\alpha)_{\alpha\in A}$ be a net in a vector lattice. If $x\mathcal{X}_\Delta \xrightarrow{o} 0$ holds for some $\Delta \in \mathcal{M}$ with $\mu(\Delta) = 1$ and characteristic function $\mathcal{X}_\Delta$ on Δ then $x \xrightarrow{\mathrm{st}_\mu} 0$. Indeed, suppose that there exists $\Delta \in \mathcal{M}$ with $\mu(\Delta) = 1$ and $x\mathcal{X}_\Delta \xrightarrow{o} 0$ satisfies in a vector lattice E for the characteristic function $\mathcal{X}_\Delta$ of Δ. Thus, there is another net $(p_\alpha)_{\alpha\in A} \downarrow 0$ such that $|x\mathcal{X}_\Delta| \le p_\alpha$ for all $\alpha \in A$. So, it follows from Proposition 1 that $(p_\alpha)_{\alpha\in A} \downarrow^{\mathrm{st}_\mu} 0$. Then there exists a subset $\Sigma \in \mathcal{M}$ such that $\mu(\Sigma) = 1$ and $(p_{\phi(\sigma)})_{\sigma\in\Sigma} \downarrow 0$. Take $\Gamma := \Delta \cap \Sigma$. Hence, we have $\mu(\Gamma) = 1$. Following from $|x\mathcal{X}_\Gamma| \le p_{\phi(\gamma)}$ for each $\gamma \in \Gamma$, we obtain $x\mathcal{X}_\Delta \xrightarrow{\mathrm{st}_\mu} 0$. Therefore, by applying Theorem 3 (v) and Remark 3, we obtain $(x_\alpha)_{\alpha\in A} \xrightarrow{\mathrm{st}_\mu} 0$.

Proposition 5. The family of all st_μ-convergent nets E_{st_μ} is a vector lattice.

Proof. Let $(x_\alpha)_{\alpha\in A} \xrightarrow{\mathrm{st}_\mu} x$ and $(y_\beta)_{\beta\in B} \xrightarrow{\mathrm{st}_\mu} y$ in E. Then it follows from Theorem 3(ii) that $(x_\alpha + y_\beta)_{(\alpha,\beta)\in A\times B} \xrightarrow{\mathrm{st}_\mu} x+y$. So E_{st_μ} is a vector space. Take an element $x := (x_\alpha)_{\alpha\in A}$ in E_{st_μ}. Then we have $x \xrightarrow{\mathrm{st}_\mu} z$ for some $z \in E$. Thus, it follows from Corollary 3 that $|x| \xrightarrow{\mathrm{st}_\mu} |z|$. It means that $|x| \in E_{st_\mu}$, i.e., E_{st_μ} is a vector lattice subspace Theorem 1.3 and Theorem 1.7 [16].

Proposition 6. The set of all order bounded nets in a vector lattice E is an order ideal in $E_{st_\mu}\{0\}$.

Proof. By the linearity of μ-statistical convergence, $E_{st_\mu}\{0\}$ is a vector space. Now, assume that $|y| \leq |x|$ hold for arbitrary $x := (x_\alpha)_{\alpha\in A} \in E_{st_\mu}\{0\}$ and for an order bounded net $y := (y_\alpha)_{\alpha\in A}$. Since $x \xrightarrow{\mathrm{st}_\mu} 0$, we have $|x| \xrightarrow{\mathrm{st}_\mu} 0$. Then it follows from Proposition 2 that $|y| \xrightarrow{\mathrm{st}_\mu} 0$, and so, it follows from Theorem 3(i) that $y \xrightarrow{\mathrm{st}_\mu} 0$. (Therefore, we get the desired result, $y \in E_{st_\mu}\{0\}$).

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Векторлық торлардағы статистикалық жинақталу

Статистикалық жинақталу, жалпы жағдайда, натурал сандардағы асимптотикалық тығыздығы бар тізбектер үшін анықталады. Мақалада бағытталған жиындардағы ақырлы аддитивті өлшемдерді қолдана отырып, векторлық торларға статистикалық жинақталу енгізілген. Сонымен қатар, статистикалық жинақталу мен тордың қасиеттері арасындағы кейбір қатынастар келтірілген, мысалы, реттік жинақталу және тор операторлары.

Кілт сөздер: желілердің статистикалық жинақталуы, реттік жинақталу, векторлық тор, бағытталған жиынның өлшемі.

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Статистическая сходимость в векторных решетках

Статистическая сходимость, в общем случае, определена для последовательностей с асимптотической плотностью на натуральных числах. В статье мы вводим статистическую сходимость в векторных решетках, используя конечные аддитивные меры на направленных множествах. Кроме того, приводим некоторые соотношения между статистической сходимостью и свойствами решетки, такими как сходимость порядка и операторы решетки.

Ключевые слова: статистическая сходимость сетей, порядковая сходимость, векторная решетка, мера направленного множества.