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Steklov problem for a linear ordinary fractional delay differential equation with the Riemann-Liouville derivative

This paper studies a nonlocal boundary value problem with Steklov's conditions of the first type for a linear ordinary delay differential equation of a fractional order with constant coefficients. The Green's function of the problem with its properties is found. The solution to the problem is obtained explicitly in terms of the Green's function. A condition for the unique solvability of the problem is found, as well as the conditions under which the solvability condition is satisfied. The existence and uniqueness theorem is proved using the representation of the Green's function and its properties, as well as the representation of the fundamental solution to the equation and its properties. The question of eigenvalues is investigated. The theorem on the finiteness of the number of eigenvalues is proved using the notation of the solution in terms of the generalized Wright function, as well as the asymptotic properties of the generalized Wright function as $\lambda \to \infty$ and $\lambda \to -\infty$.

Keywords: fractional differential equation, delay differential equation, Steklov's boundary value problem, Green function, generalized Mittag-Leffler function, generalized Wright function.

Introduction

In this paper, we consider the equation

$$D_{0t}^{\alpha}u(t) - \lambda u(t) - \mu H(t-\tau)u(t-\tau) = f(t), \quad 0 < t < 1, \tag{1}$$

where D_{0t}^{α} is the Riemann–Liouville fractional derivative [1], $1 < \alpha \leq 2$, λ, μ are the arbitrary constants, τ is the fixed positive number, $H(t)$ denotes the Heaviside function.

In [1–6], the theory of fractional calculus is studied (see also the references in these works). Barrett [7] investigated a linear ordinary differential equation of fractional order. For a fractional order differential equation the existence and uniqueness theorem is proved in [8], and the boundary value problem with the Sturm-Liouville type conditions was considered in [9]. In paper [10], the initial value problem for a linear ordinary differential equation of fractional order was studied.

To the theory of delay differential equations were devoted the following works [11–15].

The Cauchy problem for Eq.(1) was solved in [16], and the solutions to the Dirichlet and the Neumann problems were obtained in [17]. The boundary value problem with Sturm-Liouville type conditions was founded in [18].

The papers [19], [20] are devoted to the study of the Steklov problem for a fractional order differential equation. In this paper, we construct the solution to the first-type Steklov boundary value for Eq.(1) and prove the existence and uniqueness theorem and the finiteness theorem for the number of real eigenvalues of the problem under study.

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Main results

A function $u(t)$ is called *a regular solution* of equation (1) if $D_{0t}^{\alpha-2}u(t) \in C^2(0,1)$, $u(t) \in L(0,1)$ and $u(t)$ satisfies Eq. (1) for all $0 < t < 1$.

The problem we solve here is to find the regular solution to equation (1) satisfying the conditions

$$\begin{aligned} a_1 \lim_{t\to 0} D_{0t}^{\alpha-2}u(t) + a_2 \lim_{t\to 0} D_{0t}^{\alpha-1}u(t) + a_3 \lim_{t\to 1} D_{0t}^{\alpha-2}u(t) + a_4 \lim_{t\to 1} D_{0t}^{\alpha-1}u(t) = 0, \\ b_1 \lim_{t\to 0} D_{0t}^{\alpha-2}u(t) + b_2 \lim_{t\to 0} D_{0t}^{\alpha-1}u(t) + b_3 \lim_{t\to 1} D_{0t}^{\alpha-2}u(t) + b_4 \lim_{t\to 1} D_{0t}^{\alpha-1}u(t) = 0. \end{aligned} \tag{2}$$

In the case $a_2b_4 - a_4b_2 \neq 0$ the conditions (2) can be write out in the form

$$\begin{aligned} D_{0t}^{\alpha-1}u(t)\big|_{t=1} = c_1 D_{0t}^{\alpha-2}u(t)\big|_{t=0} + c_2 D_{0t}^{\alpha-2}u(t)\big|_{t=1}, \\ D_{0t}^{\alpha-1}u(t)\big|_{t=0} = c_3 D_{0t}^{\alpha-2}u(t)\big|_{t=0} + c_4 D_{0t}^{\alpha-2}u(t)\big|_{t=1}, \end{aligned}$$

where

$$c_1 = \frac{a_1b_2 - a_2b_1}{a_2b_4 - a_4b_2}, \quad c_2 = \frac{-a_2b_3 + a_3b_2}{a_2b_4 - a_4b_2}, \quad c_3 = \frac{-a_1b_4 + a_4b_1}{a_2b_4 - a_4b_2}, \quad c_4 = \frac{-a_3b_4 + a_4b_3}{a_2b_4 - a_4b_2}. \tag{3}$$

Previously, in work [21], it was defined the function

$$W_\nu(t) = W_\nu(t,\tau;\lambda,\mu) = \sum_{m=0}^{\infty} \mu^m (t - m\tau)_+^{\alpha m+\nu-1} E_{\alpha,\alpha m+\nu}^{m+1}(\lambda(t - m\tau)_+^{\alpha}), \nu \in \mathbb{R}, \tag{4}$$

where

$$E_{\alpha,\beta}^{\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\Gamma(\alpha k + \beta)k!}$$

is the generalized Mittag-Leffler function [22], $\Gamma(z)$ is the Gamma function, $(\rho)_k = \frac{\Gamma(\rho+k)}{\Gamma(\rho)}$ is the Pochhammer symbol,

$$(t - m\tau)_+ = \begin{cases} t - m\tau, & t - m\tau > 0, \\ 0, & t - m\tau \leq 0. \end{cases}$$

Function (4) satisfies the following properties [21]:

1) *for some* m *the expression* $t - m\tau < 0$, *therefore the series in (4) contains a finite number of the terms* $N \leq [\frac{t}{\tau}] + 1$*;*

2) it follows from (4) that

$$W_k^{(i)}(0) = \begin{cases} 0, k \neq i + 1, \\ 1, k = i + 1; \end{cases}$$

3) it holds true the integrodifferentional formula

$$D_{0t}^{\alpha} W_\nu(t) = W_{\nu-\alpha}(t), \quad \alpha \in \mathbb{R}, \quad \nu > 0 \tag{5}$$

and the autotransformation formula

$$W_\nu(t) = \lambda W_{\nu+\alpha}(t) + \mu W_{\nu+\alpha}(t - \tau) + \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad \alpha > 0, \quad \nu \in \mathbb{R}. \tag{6}$$

The solution to the Cauchy problem to the equation (1) was found in the paper [16] and has the form

$$u(t) = \int_0^t f(\xi) W_\alpha(t - \xi) d\xi + D_{0t}^{\alpha-1}u(t)\big|_{t=0} W_\alpha(t) + D_{0t}^{\alpha-2}u(t)\big|_{t=0} W_{\alpha-1}(t). \tag{7}$$

Using formula (7) we can define $D_{0t}^{\alpha-1}u(t)\big|_{t=1}$ and $D_{0t}^{\alpha-2}u(t)\big|_{t=1}$:

$$D_{0t}^{\alpha-1}u(t)\big|_{t=1}=\int\limits_0^1 f(\xi)W_1(1-\xi)d\xi+D_{0t}^{\alpha-1}u(t)\big|_{t=0}W_1(1)+D_{0t}^{\alpha-2}u(t)\big|_{t=0}\left[\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\right], \quad (8)$$

$$D_{0t}^{\alpha-2}u(t)\big|_{t=1}=\int\limits_0^1 f(\xi)W_2(1-\xi)d\xi+D_{0t}^{\alpha-1}u(t)\big|_{t=0}W_2(1)+D_{0t}^{\alpha-2}u(t)\big|_{t=0}W_1(1). \quad (9)$$

Inserting (8) and (9) into the first formula of the system (2), we have

$$D_{0t}^{\alpha-2}u(t)\big|_{t=0}\Big[a_1+a_3W_1(1)+a_4\Big(\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\Big)\Big]+D_{0t}^{\alpha-1}u(t)\big|_{t=0}\Big[a_2+a_3W_2(1)+a_4W_1(1)\Big]+$$

$$+\int\limits_0^1 f(\xi)\Big[a_3W_2(1-\xi)+a_4W_1(1-\xi)\Big]d\xi=0,$$

or

$$A_1D_{0t}^{\alpha-2}u(t)\big|_{t\to 0}+A_2D_{0t}^{\alpha-1}u(t)\big|_{t\to 0}+F_1=0,$$

where

$$A_1=a_1+a_3W_1(1)+a_4\Big(\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\Big),\quad A_2=a_2+a_3W_2(1)+a_4W_1(1),$$

$$F_1=\int\limits_0^1 f(\xi)\Big[a_3W_2(1-\xi)+a_4W_1(1-\xi)\Big]d\xi.$$

In the same way, substituting (8) and (9) into the second formula of the system (2), we have

$$B_1D_{0t}^{\alpha-2}u(t)\big|_{t=0}+B_2D_{0t}^{\alpha-1}u(t)\big|_{t=0}+F_2=0,$$

where

$$B_1=b_1+b_3W_1(1)+b_4\Big(\lambda W_\alpha(1)+\mu W_\alpha(1-\tau)\Big),\quad B_2=b_2+b_3W_2(1)+b_4W_1(1),$$

$$F_2=\int\limits_0^1 f(\xi)\Big[b_3W_2(1-\xi)+b_4W_1(1-\xi)\Big]d\xi.$$

Thus, we get the system:

$$\begin{array}{l} A_1D_{0t}^{\alpha-2}u(t)\big|_{t=0}+A_2D_{0t}^{\alpha-1}u(t)\big|_{t=0}=-F_1,\\ B_1D_{0t}^{\alpha-2}u(t)\big|_{t=0}+B_2D_{0t}^{\alpha-1}u(t)\big|_{t=0}=-F_2,\end{array} \quad (10)$$

and the solution to that system (10) equals:

$$D_{0t}^{\alpha-2}u(t)\big|_{t\to 0}=\frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1},\quad D_{0t}^{\alpha-1}u(t)\big|_{t\to 0}=\frac{-A_1F_2+B_1F_1}{A_1B_2-A_2B_1}. \quad (11)$$

Using (11) and the Cauchy problem solution (7), we get the equality:

$$u(t)=\int\limits_0^t f(\xi)W_\alpha(t-\xi)d\xi+\frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}W_{\alpha-1}(t)+\frac{-A_1F_2+B_1F_1}{A_1B_2-A_2B_1}W_\alpha(t)=$$

$$=\int\limits_0^t f(\xi)W_\alpha(t-\xi)d\xi+\frac{B_1W_\alpha(t)-B_2W_{\alpha-1}(t)}{A_1B_2-A_2B_1}\int\limits_0^1 f(\xi)[a_3W_2(1-\xi)+a_4W_1(1-\xi)]d\xi-$$

$$-\frac{A_1W_\alpha(t)-A_2W_{\alpha-1}(t)}{A_1B_2-A_2B_1}\int\limits_0^1 f(\xi)[b_3W_2(1-\xi)+b_4W_1(1-\xi)]d\xi,$$

or

$$\int\limits_0^1 f(\xi)\left[H(t-\xi)W_\alpha(t-\xi)+W_\alpha(t)\left(\frac{a_4B_1-b_4A_1}{\triangle}W_1(1-\xi)+\frac{a_3B_1-b_3A_1}{\triangle}W_2(1-\xi)\right)-\right.$$

$$\left.-W_{\alpha-1}(t)\left(\frac{a_4B_2-b_4A_2}{\triangle}W_1(1-\xi)+\frac{a_3B_2-b_3A_2}{\triangle}W_2(1-\xi)\right)\right],$$

where

$$\triangle=A_1B_2-A_2B_1. \tag{12}$$

Green function

Assume $G(t,\xi)$ is given by

$$G(t,\xi)=H(t-\xi)W_\alpha(t-\xi)+W_\alpha(t)\left(\frac{a_4B_1-b_4A_1}{\triangle}W_1(1-\xi)+\frac{a_3B_1-b_3A_1}{\triangle}W_2(1-\xi)\right)-$$

$$-W_{\alpha-1}(t)\left(\frac{a_4B_2-b_4A_2}{\triangle}W_1(1-\xi)+\frac{a_3B_2-b_3A_2}{\triangle}W_2(1-\xi)\right) \tag{13}$$

with λ and μ satisfying the condition (12). Here the function $W_\nu(t)$ is defined via (4).

Function $G(t,\xi)$ (13) satisfies the following properties.

1. The function $G(t,\xi)$ is continuous for all values of t and ξ from the closed interval $[0,1]$.
2. The function $G(t,\xi)$ satisfies the conditions

$$\lim_{\varepsilon\to 0}[D_{0t}^{\alpha-2}G_\xi(t,\xi)|_{\xi=t+\varepsilon}-D_{0t}^{\alpha-2}G_\xi(t,\xi)|_{\xi=t-\varepsilon}]=1. \tag{14}$$

3. The function $G(t,\xi)$ is the solution to the equation

$$\partial_{1\xi}^\alpha G(t,\xi)-\lambda G(t,\xi)-\mu H(1-\tau-\xi)G(t,\xi+\tau)=0. \tag{15}$$

Here ∂_{0t}^α is the Caputo derivative [1; 11] defines as

$$\partial_{1t}^\alpha v(t)=D_{1t}^{\alpha-2}v''(t)=\frac{1}{\Gamma(2-\alpha)}\int\limits_1^t\frac{v''(\xi)d\xi}{(t-\xi)^{\alpha-1}}.$$

4. The function $G(t,\xi)$ satisfies the boundary conditions

$$\begin{cases} \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=0} = -c_1 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=1} + c_3 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=0}, \\ \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=1} = c_2 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=1} - c_4 D_{0t}^{\alpha-2} G(t,\xi)\big|_{\xi=0}. \end{cases} \tag{16}$$

Here the coefficients c_1, c_2, c_3, c_3 were defined via formula (3).

These properties obviously are implied from formula (13), condition (12) and the relations (5), (6).

The function $G(t,\xi)$ that possesses properties 1–4 is called Green function for problem (1), (2).

Existence and uniqueness theorem

Theorem 1. Assume the function $f(t) \in L(0,1) \cap C(0,1)$ and the condition (12) is satisfied. Then there exists a regular solution to problem (1), (2) in the form of

$$u(t) = \int\limits_0^1 f(\xi) G(t,\xi) d\xi \tag{17}$$

and the solution to problem (1), (2) is unique if and only if condition (12) is satisfied.

Proof. We show that it holds true the representation of the solution to problem (1), (2) in the form (17). For this, we multiply both sides of Eq. (1) (given in terms of variable ξ) by $D_{0t}^{\alpha-2} G(t,\xi)$ and integrate it with respect to variable ξ ranging from ε to $1-\varepsilon$ ($\varepsilon \to 0$):

$$\int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha} u(\xi) d\xi - \lambda \int\limits_{\varepsilon}^{1-\varepsilon} u(\xi) D_{0t}^{\alpha-2} G(t,\xi) d\xi -$$

$$-\mu \int\limits_{\varepsilon}^{1-\varepsilon} H(t-\tau) u(\xi-\tau) D_{0t}^{\alpha-2} G(t,\xi) d\xi = \int\limits_{\varepsilon}^{1-\varepsilon} f(\xi) D_{0t}^{\alpha-2} G(t,\xi) d\xi, \quad 0 < t < 1. \tag{18}$$

Integrate by parts the first term of equality (18):

$$\int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha} u(\xi) d\xi = D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-1} u(\xi)\Big|_{\varepsilon}^{1-\varepsilon} - \int\limits_{\varepsilon}^{t-\varepsilon} \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-1} u(\xi) d\xi -$$

$$-\int\limits_{t+\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2} G_{\xi}(t,\xi) D_{0\xi}^{\alpha-1} u(\xi) d\xi = D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-1} u(\xi)\Big|_{\varepsilon}^{1-\varepsilon} - \frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-2} u(\xi)\Big|_{\varepsilon}^{t-\varepsilon} -$$

$$-\frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi) D_{0\xi}^{\alpha-2} u(\xi)\Big|_{t+\varepsilon}^{1-\varepsilon} + \int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2} G_{\xi\xi}(t,\xi) D_{0\xi}^{\alpha-2} u(\xi) d\xi =$$

$$= D_{0t}^{\alpha-2} u(t) \left[\frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi)\Big|_{t+\varepsilon} - \frac{d}{dx} D_{0t}^{\alpha-2} G(t,\xi)\Big|_{t-\varepsilon} \right] + \int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2} G_{\xi\xi}(t,\xi) D_{0\xi}^{\alpha-2} u(\xi) d\xi +$$

$$+ D_{0\xi}^{\alpha-2} u(\xi)\Big|_{\xi=0} \left[c_1 D_{0t}^{\alpha-2} G(t,\xi)\Big|_{\xi=1-\varepsilon} - c_3 D_{0t}^{\alpha-2} G(t,\xi)\Big|_{\xi=\varepsilon} + \frac{d}{d\xi} D_{0t}^{\alpha-2} G(t,\xi)\Big|_{\xi=\varepsilon} \right] +$$

$$+D_{0\xi}^{\alpha-2}u(\xi)\Big|_{\xi=1}\left[c_2 D_{0t}^{\alpha-2}G(t,\xi)\Big|_{\xi=1-\varepsilon} - c_4 D_{0t}^{\alpha-2}G(t,\xi)\Big|_{\xi=\varepsilon} - \frac{d}{d\xi}D_{0t}^{\alpha-2}G(t,\xi)\Big|_{\xi=1-\varepsilon}\right].$$

Using properties of the Green function (14) and (16), we get the identity

$$\int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2}G(t,\xi)D_{0\xi}^{\alpha}u(\xi)d\xi = D_{0t}^{\alpha-2}u(t) + \int\limits_{\varepsilon}^{1-\varepsilon} D_{0t}^{\alpha-2}G_{\xi\xi}(t,\xi)D_{0\xi}^{\alpha-2}u(\xi)d\xi. \tag{19}$$

In the third integral of the equality (18) we replace ξ into $\xi-\tau$:

$$\int\limits_0^1 H(\xi-\tau)u(\xi-\tau)G(t,\xi)d\xi = \int\limits_0^1 H(1-\tau-\xi)u(\xi)G(t,\xi+\tau)d\xi. \tag{20}$$

Substituting (19) and (20) into Eq. (18) and using the formula for fractional integration by parts [20; 15]

$$\int\limits_a^b g(s)D_{as}^{\alpha}h(s)ds = \int\limits_a^b h(s)D_{bs}^{\alpha}g(s)ds,$$

we have the identity

$$D_{0t}^{\alpha-2}u(\xi) + D_{0t}^{\alpha-2}\int\limits_0^1 u(\xi)\left[D_{1\xi}^{\alpha-2}G_{\xi\xi}(t,\xi) - \lambda G(t,\xi) - \mu H(1-t-\xi)G(t,\xi+\tau)\right]d\xi =$$

$$= D_{0t}^{\alpha-2}\int\limits_0^1 f(\xi)G(t,\xi)d\xi,$$

and,using the Green function property (15) and finding the derivative of order $D_{0t}^{2-\alpha}$ we get solution (17).

Next, we show that the function (17) is the solution to equation (1). Formula (17) can be written out in the form of bellow:

$$u(t) = \nu_1 + \nu_2 + \nu_3,$$

where

$$\nu_1 = \int\limits_0^t f(\xi)W_\alpha(t-\xi)d\xi, \quad \nu_2 = \frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}W_{\alpha-1}(t), \quad \nu_3 = \frac{-A_1F_2+B_1F_1}{A_1B_2-A_2B_1}W_\alpha(t).$$

Denote $D_{0t}^{\alpha}\nu_1$, $D_{0t}^{\alpha}\nu_2$ and $D_{0t}^{\alpha}\nu_3$. We have

$$D_{0t}^{\alpha}\nu_1 = \frac{d}{dt}D_{0t}^{\alpha-1}\int\limits_0^t f(\xi)W_\alpha(t-\xi)d\xi = \frac{d}{dt}\int\limits_0^t f(\xi)W_1(t-\xi)d\xi =$$

$$= \int\limits_0^t f(\xi)\frac{d}{dt}(\lambda W_{\alpha+1}(t-\xi)+\mu W_{\alpha+1}(t-\xi-\tau))d\xi + f(t) = \int\limits_0^t f(\xi)(\lambda W_\alpha(t-\xi)+\mu W_\alpha(t-\xi-\tau))d\xi + f(t);$$

$$D_{0t}^{\alpha}\nu_2 = \frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}\frac{d^2}{dt^2}D_{0t}^{\alpha-2}W_{\alpha-1}(t) = \frac{-F_1B_2+F_2A_2}{A_1B_2-A_2B_1}\frac{d^2}{dt^2}W_1(t) =$$

$$= \frac{-F_1B_2 + F_2A_2}{A_1B_2 - A_2B_1} \frac{d^2}{dt^2}(\lambda W_{\alpha+1}(t) + \mu W_{\alpha+1}(t-\tau)) = \frac{-F_1B_2 + F_2A_2}{A_1B_2 - A_2B_1} (\lambda W_{\alpha-1}(t) + \mu W_{\alpha-1}(t-\tau))\,;$$

$$D^{\alpha}_{0t}\nu_3 = \frac{-A_1F_2 + B_1F_1}{A_1B_2 - A_2B_1} \frac{d}{dt} D^{\alpha-}_{0t} W_\alpha(t) = \frac{-A_1F_2 + B_1F_1}{A_1B_2 - A_2B_1} \frac{d}{dt}(\lambda W_{\alpha+1}(t) + \mu W_{\alpha+1}(t-\tau)) =$$

$$= \frac{-A_1F_2 + B_1F_1}{A_1B_2 - A_2B_1}(\lambda W_\alpha(t) + \mu W_\alpha(t-\tau)).$$

Next, using formulas (5), (6) we obtain by the previous relation that

$$D^{\alpha}_{0t}u(t) = f(t) + \lambda \int_0^1 f(\xi)G(t,\xi)d\xi + \mu \int_0^1 f(\xi)G(t,\xi-\tau)d\xi,$$

that is that (17) satisfies (1).

Remark. For $\lambda = 0, \mu > 0$ and

$$a_1 > b_1, \quad a_2 < b_2, \quad a_3 = b_3, \quad a_4 = b_4$$

condition (12) is always satisfied.

On the finiteness of the number of real eigenvalues

Definition. The eigenvalues of problem (1), (2) are the values λ, such that problem (1), (2) has a regular solution that is not the identically zero.

The set of real eigenvalues for problem (1), (2) coincides with the set of real zeros for the function

$$\Phi(\lambda) = \Big[a_1 + a_3W_1(1) + a_4\Big(\lambda W_\alpha(1) + \mu W_\alpha(1-\tau)\Big)\Big]\Big[B_2 = b_2 + b_3W_2(1) + b_4W_1(1)\Big] -$$

$$- \Big[a_2 + a_3W_2(1) + a_4W_1(1)\Big]\Big[b_1 + b_3W_1(1) + b_4\Big(\lambda W_\alpha(1) + \mu W_\alpha(1-\tau)\Big)\Big]. \qquad (21)$$

Theorem 2. Problem (1), (2) has only a finite number of real eigenvalues.

The function $W_\nu(\lambda)$ can be written out as [3; 45]

$$W_\nu(1,\tau;\lambda,\mu) = \sum_{m=0}^{\infty} \frac{\mu^m}{m!}(1-m\tau)_+^{\alpha m+\nu-1} {}_1\Psi_1 \left[\begin{matrix} (m+1,1) \\ (\alpha m+\nu,\alpha) \end{matrix} \middle| \lambda(1-m\tau)_+^{\alpha} \right], \qquad (22)$$

where

$${}_p\Psi_q \left[\begin{matrix} (a_l,\alpha_l)_{1,p} \\ (b_l,\beta_l)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p}\Gamma(a_l+\alpha_l k)}{\prod_{l=1}^{q}\Gamma(b_l+\beta_l k)} \frac{z^k}{k!}$$

is the generalized Wright function [23].

As $\lambda \to +\infty$ the following asymptotic formula holds true for the generalized Wright function [23], [24]:

$${}_1\Psi_1 \left[\begin{matrix} (m+1,1) \\ (\alpha m+\nu,\alpha) \end{matrix} \middle| \lambda(1-m\tau)_+^{\alpha} \right] = \alpha^{-m} \lambda^{\frac{m(1-\alpha)-\nu+1}{\alpha}} (1-m\tau)_+^{m(1-\alpha)-\nu+1} e^{\lambda^{1/\alpha}(1-m\tau)_+} \left[1 + O\left(\frac{1}{\lambda^{\frac{1}{\alpha}}}\right)\right],$$

and the asymptotic formula for the generalized Wright function as $\lambda \to -\infty$ has form [23], [24]

$${}_1\Psi_1 \left[\begin{matrix} (m+1,1) \\ (\alpha m+\nu,\alpha) \end{matrix} \middle| \lambda(1-m\tau)_+^{\alpha} \right] = \sum_{l=0}^{n} \frac{(-1)^{m+l+1}(l+m)!(1-m\tau)_+^{-\alpha(m+l+1)}}{|\lambda|^{m+l+1}\Gamma(\nu-\alpha-\alpha l)(m+l+1)!} + O\left(\frac{1}{|\lambda|^m}\right).$$

Let N be the maximum value of m that satisfies the inequality $(1 - m\tau) > 0$. From these formulas we get the asymptotic formulas for function (22) as $\lambda \to +\infty$

$$W_\nu(1, \tau; \lambda, \mu) = \sum_{m=0}^{N} \frac{\mu^m \alpha^{-m}}{m!} \lambda^{\frac{m(1-\alpha)-\nu+1}{\alpha}} (1 - m\tau)_+^m e^{\lambda^{\frac{1}{\alpha}}(1-m\tau)_+} \left[1 + O\left(\frac{1}{\lambda^{\frac{1}{\alpha}}}\right)\right], \quad (23)$$

and $\lambda \to -\infty$ in the form

$$W_\nu(1, \tau; \lambda, \mu) = \sum_{m=0}^{N} \mu^m \sum_{l=0}^{n} \frac{(-1)^{m+l+1}(m+1)_l (1 - m\tau)_+^{-\alpha(l+1)+\nu-1}}{|\lambda|^{m+l+1}\Gamma(\nu - \alpha - \alpha l)(m + l + 1)!} + O\left(\frac{1}{|\lambda|^m}\right). \quad (24)$$

From the representation (21) and asymptotic formula (23) we see that letting $\lambda \to \infty$ the function (21) increases without limit.

As $\lambda \to -\infty$, since $\Phi(\lambda)$ is an entire function of the variable λ, it follows from asymptotic formula (24) that the function (21) may have only a finite number of real zeros.

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Аргументі кешігетін Риман-Лиувилль бөлшек туындысы бар сызықтық қарапайым дифференциалдық теңдеу үшін Стеклов есебі

Мақалада тұрақты коэффициенттері бар аргументі кешігетін бөлшек ретті сызықты қарапайым дифференциалдық теңдеу үшін бірінші типті Стеклов шарттарымен жергілікті емес шеттік есептер зерттелген. Грин функциясы табылып, оның қасиеттері дәлелденді. Зерттелетін есептің шешімі Грин функциясы тұрғысынан айқын түрде алынды. Есептің бірегей шешілу шарты, сондай-ақ шешілу шарты сөзсіз орындалатын шарттар табылды. Бар болу және жалғыздық теоремасы дәлелденді. Теорема Грин функциясын және оның қасиеттерін, сондай-ақ теңдеудің іргелі шешімін және оның қасиеттерін көрсету арқылы дәлелденген. Меншікті мәндер сұрағы зерттелді. Зерттелетін есеп нақты меншікті мәндердің шектеулі санына ғана ие болуы мүмкін екендігі теоремамен дәлелденді. Теорема шешімнің жалпыланған Райт функциясы тұрғысынан белгіленуді қолданып, сондай-ақ $\lambda \to \infty$ және $\lambda \to -\infty$ үшін жалпыланған Райт функциясының асимптотикалық қасиеттері арқылы дәлелденді.

Кілт сөздер: бөлшек ретті сызықты дифференциалдық теңдеу, аргументі кешігетін дифференциалдық теңдеу, Стеклов шеттік есебі, Грин функциясы, Миттаг-Леффлераның жалпыланған функциясы, Райттың жалпыланған функциясы.

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Задача Стеклова для линейного обыкновенного дифференциального уравнения с дробной производной Римана–Лиувилля с запаздывающим аргументом

В статье исследована нелокальная краевая задача с условиями Стеклова первого типа для линейного обыкновенного дифференциального уравнения дробного порядка с запаздывающим аргументом с постоянными коэффициентами. Найдена функция Грина и доказаны ее свойства. Решение исследуемой задачи получено в явном виде в терминах функции Грина. Найдено условие однозначной разрешимости задачи, а также условия, при которых условие разрешимости заведомо выполняется. Доказана теорема существования и единственности, с использованием представления функции Грина, ее свойств, а также фундаментального решения уравнения и ее свойств. Исследован вопрос о собственных значениях. Доказана теорема о том, что исследуемая задача может иметь только конечное число действительных собственных значений. Теорема доказана с применением записи решения в терминах обобщенной функции Райта, а также асимптотических свойств обобщенной функции Райта при $\lambda \to \infty$ и $\lambda \to -\infty$.

Ключевые слова: дифференциальное уравнение дробного порядка, дифференциальное уравнение с запаздывающим аргументом, краевая задача Стеклова, функция Грина, обобщенная функция Миттаг-Леффлера, обобщенная функция Райта.

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