DOI 10.31489/2020M1/104-110

UDC 510.67

A.R. Yeshkeyev, A.K. Issayeva, N.V. Popova

Ye.A. Buketov Karaganda State University, Kazakhstan
(E-mail: dandn@mail.ru)

Core Jonsson theories

The article concerns the description the new concept as core of Jonsson theories, also their combinations, which admit a core model in the class of existentially closed models of this theory. Along with core the property of an existentially algebraically prime theory is considered as an additional property to core Jonsson theory. This article also discusses some combinations of Johnson's theories, where the authors tried to transfer some results from [1] to Johnson's theories that satisfy the definition of core or EAP, or their combinations. From the definition of the core and the existentially algebraic primeness of Johnson theory, it can be noted that the core model from [1] in the framework of the study of any Johnson theory will be a unique and rigidly embedded model of the theory were considered. And thus, such a solution to the problem with respect to core models is considered for the first time.

Keywords: convex theory, strongly convex theory, center of Jonsson theory, semantic model, algebraically prime model, core model, core theory.

This article in its content refers to those issues of model theory that are related to the themes defined by A. Robinson and are related to the study of the theory's convexity. On the one hand, a full description of a concept of convexity is given in [1] . In work [1], a close relationship between the concepts of the theory's convexity and the concept of core model of this theory is studied. The concepts were considered in [1] are defined for arbitrary theories that, generally speaking, are not complete. Also, we note that, in the proofs of the main results of this article, the considered formulas in their complexity have a prenix length of no more than two.

In the present article, we will restrict the studied klass, generally speaking, of incomplete theories to the class of Jonsson theories. On the other hand, we will consider the concept of core models in a more general context, namely in the class of existentially closed models of the considered Jonsson theory. Due to the inductance of the Jonsson theory, its class of existentially closed models of theory is always not empty. In [1], the core structures (a model of the signature of this theory) are actually considered , not the models of theory, and as its special case, the concept of a rigidly embedded model was considered. In our case, we will consider the core model and, by definition, this concept will coincide with the concept of a rigidly embedded model, as in [1]. Accordingly, all properties of the above models of theory will be translated into the concept of the core model in our sense. The purpose of this work is to relate the results of the study of Jonsson theories [3, 4] with the study of model-theoretical properties of core. Jonsson theories that admit a core model in the class of existentially closed models of this theory will be called the core Jonsson theories. It is clear that any core model is an algebraically prime model [2]. In [3, 4], new types of atomic and prime countable models of the corresponding types of Jonsson theories were considered.

To obtain the main results of this article, we give the necessary definitions of concepts and their model-theoretical properties. For more in-depth information on Jonsson theories, please refer to the following sources [3, 4]. Nevertheless, we give some basic definitions and related results.

Consider the following definitions:

Definition 1. A theory T is called a Jonsson theory if:

1) the theory T has infinite models;
2) the theory T is inductive;
3) the theory T has the joint embedding property (JEP);
4) the theory T has the property of amalgam (AP).

Examples of Jonsson theories are:

1) the group theory,
2) the theory of Abelian groups;
3) the theory of fields of fixed characteristics;
4) the theory of Boolean algebras;

5) the theory of polygons over a fixed monoid;
6) the theory of modules over a fixed ring;
7) the theory of linear order.

When studying the model-theoretic properties of Jonsson theory, the semantic method plays an important role, i.e. the elementary properties of the center of Jonsson theory are in a certain sense associated with the corresponding first-order properties of Jonsson theory itself. The center of Jonsson theory is a syntactic invariant and its properties are well defined in the case when Jonsson theory is perfect. The following concepts define the essence of the semantic model and the center of Jonsson theory [6].

Definition 2. Let $\kappa \geq \omega$. Model M of theory T is called κ-universal for T, if each model T with the power strictly less κ isomorphically imbedded in M; κ- homogeneous for T, if for any two models A and A_1 of theory T, which are submodels of M with the power strictly less then κ and for isomorphism $f : A \to A_1$ for each extension B of model A, wich is a submodel of M and is model of T with the power strictly less then κ there is exist the extension B_1 of model A_1, which is a submodel of M and an isomorphism $g : B \to B_1$ which extends f.

Definition 3. A model C of a Jonsson theory T is called semantic model, if it is ω^+-homogeneous-universal.

Definition 4. The center of a Jonsson theory T is an elementary theory $T*$ of the semantic model C of T, i.e. $T^* = Th(C)$ [8].

Fact 1 [6]. Each Jonsson theory T has k^+-homogeneous-universal model of power 2^k. Conversely, if a theory T is inductive and has infinite model and ω^+-homogeneous-universal model then the theory T is a Jonsson theory.

Fact 2 [6]. Let T is a Jonsson theory. Two k-homogeneous-universal models M and M_1 of T are elementary equivalent.

Definition 5. A model A of theory T is called existentially closed if for any model B and any existential formula $\varphi(\overline{x})$ with constants of A we have $A \models \exists\overline{x}\varphi(\overline{x})$ provided that A is a submodel of B and $B \models \exists\overline{x}\varphi(\overline{x})$.

We denote by E_T the class of all existentially closed models of the theory T.

In connection with this definition in the frame of the study of inductive theories, the following two remarks are true:

Remark 1: For any inductive theory E_T is not empty.

Remark 2: Any countable model of the inductive theory is isomorphically embedded in some countable existentially closed model of this theory.

An analogue of a prime model (in the sense of a complete theory) for an inductive model, generally speaking, incomplete theory, is the concept of an algebraically prime model, which introduced A. Robinson [5].

Theorem 1 [9]. Let T be a Jonsson theory. Then the following conditions are equivalent:
1) the theory T is perfect;
2) the theory T^* is a model companion of T.

Theorem 2 [10]. If T is a perfect Jonsson theory then $E_T = ModT^*$.

When studying the model-theoretic properties of an inductive theory, so called existentially closed models play an important role. Recall their definitions.

Definition 6. A model of theory is called an algebraically prime, if it is isomorphically embedded in each model of the considered theory.

Note that since the class of Jonsson theories of a fixed signature is a subclass of inductive theories of this signature, then the above remarks 1,2 are true for Jonssons theories and, by criterion of Jonsson theory's perfectness, class of existentially closed models of considered Jonsson theory coincides with the class of center's model of this theory.

In connection with the interest to the AAP problem in the frame of the study of Jonsson theory in [1] a new class of theories was defined, in which there is an algebraically prime model which is existentially closed.

Recall the definition of this class.

Definition 7. A theory is called **convex** if for any its model A and for any family $\{B_i \mid i \in I\}$ of substructures of A, which are models of the theory T, the intersection $\bigcap_{i\in I} B_i$ is a model of T, provided it is non-empty. If in addition such an itnersection is never empty, then T is called **strongly convex**.

Definition 8. The model A of theory T is called core if it is isomorphically embedded in any model of a given theory and this isomorphism exactly one.

Recall the definition of a rigidly embedded model from [1].

Definition 9. The model A of theory T is rigidly embeddable in model B of theory T, if there is exactly one isomorphism of A into B. It is clear that A is rigidly embeddable in every model of T if and only if A is a core model for T and has no proper automorphisms except identical. Thus, any core model of the core Jonsson theory is rigidly embeddable in any existentially closed model of this theory.

All the new definitions of theories given below distinguish a fairly wide natural subclass of theories among the class of inductive theories. The relevance of studying this class of theories is expressed by the fact that each of the above classes of theories is determined by a natural concept that generalizes the well-known concepts of core, algebraic simplicity, and their combinations. At the same time, new classes of theories become interesting for studying their model-theoretical properties of incomplete theories as part of the study of Johnson theories that are associated with the above concepts.

Since, we will deal with, generally speaking, incomplete Jonsson theories and their classes of existentially closed models in connection with the study of core models, we distinguish a natural subclass of the class of all Jonsson theories, which is naturally connected with the concept of a core model. We give the following definition.

Definition 10. An inductive theory T is called a core theory if there exists a model $A \in E_T$ such that for any model $B \in E_T$ there exists a unique isomorphism from A to B.

Since, by definition, any core model is an algebraically prime model, we distinguish a natural subclass of all Jonsson theories' class, namely, the class of such Jonsson theories that necessarily have an algebraically prime model. We give the following definition.

Definition 11. Theory T is called existentially algebraically prime (EAP) if it has a model $A \in E_T$ such that for any $B \in E_T$, A is isomorphically embedded in B.

Definition 12. The inductive theory T is called the existentially prime if:

1) it has a algebraically prime model, the class of its AP (algebraically prime models) denote by AP_T;

2) class E_T non trivial intersects with class AP_T, i.e. $AP_T \bigcap E_T \neq 0$.

On the other hand, in [1] the notion of core model was studied in the framework of the study of convex theory or strongly convex theory. Therefore, later in this article we will consider the property of convex and strongly convex of the considering theory, as an additional concept to the core Jonsson theory.

The following fact about the realization of existential formulas with respect to extensions is well known.

Lemma 1. Assume that $A \subseteq B$ are models of $\exists^{=n} x\phi$, where $\phi(x)$ is existential formula. Then

$$\{a \in A : A \models \phi[a]\} = \{b \in B : B \models \phi[b]\}.$$

Proof follows from the fact that existential formulas are closed with respect to extensions.

Sometimes, we will need structures with special properties, and we will deal with theories that satisfy certain model-theoretical conditions. In the remainder of this section, we determine the properties that we will use and state some elementary facts concerning them. For more information, see [7] and [5].

To denote that B satisfies every true sentence of an existential sentence on A we write $A \exists B$. $Th(C)$, the complete theory of C, is the set of all sentences true on C.

The convex theories have an important algebraic property: let T be a convex theory, then for any model A of T, any nonempty subset $B \subseteq A$ generates a single substructure, which is a model of the T. In particular, the intersection of all models of T contained in this model and which contain this set B. If the theory of T is strongly convex, then the intersection of all models of T contained in this model of T is also a model of T. This intersection is called the core model of T. In [1] noted that if T satisfies a joint embedding property and it is strongly convex, then the core model of this theory is unique up to isomorphism.

In the remainder of this article, we will deal with the above mentioned combinations of Jonsson theories. Hence, we will try to transfer some results from [1] to Jonsson theories that satisfy Definition 10 or Definition 11, or their combinations. What is the meaning here? The fact is that from the definition of core and the existentially algebraically primeness of Jonsson theory, it can be noted that a core model from [1] in the framework of the study of any Jonsson theory will be unique and rigidly embeddable model of the considering theory. And thus, such statement of the problem regarding of the core models is considered for the first time.

Theorem 3. For any core perfect Jonsson theory T, the following conditions are equivalent:

1. A is core model of T

2. A is rigidly embeddable in any existentially closed of model T

3. A is a model of center of T^* and exist an existential formulas $\phi_i(x)$ and $k_i \in \omega$ for $i \in I$ exist , such that

$$A, T^* \models \exists^{=k_i} x\phi_i \ \forall i \in I,$$

and

$$A \models \forall x \bigvee_{i\in I} \phi_i$$

Proof.

The equivalence of items (1) and (2) follows from the fact that the theory is core and perfect. Let us prove from (3) to (2) Let B be some model of the theory T^*, then there exists B' such that B' is an existentially closed model of T^*, where B is elementary embedded in B' relative to existential formulas. Such B' exists due to the inductance of T^*. Moreover, the power of B' can be any power that less or equal to the power of the semantic model of this theory. It is clear that A is elementary embedded in B' with respect to existential formulas, then in B' there is an existentially closed submodel A' such that A is isomorphic to A'.

$$A' = \{a \in A' : A' \models \bigvee_{i \in I} \phi_i[a]\}$$

where ϕ_i are existential formulas such that A', B' and B are models of $\exists^{=k_i} x\phi_i$. By Lemma 1, therefore

$$A' = \{b \in B : B' \models \bigvee_{i \in I} \phi_i[b]\} = \{b \in B : B \models \bigvee_{i \in I} \phi_i[b]\}$$

Hence $A' \subseteq B$ and therefore A is the core model of T.

(2) $\to$ (3)

By virtue of the perfectness of theory follows that A is a model of the center and, due to of its coreness, it is embedded exactly once in any model of this center. Further, due to the fact that the center is a model companion of T (since the theory T is perfect), and the model companion is a model-complete theory. Accordingly, in a model-complete theory any formula is equivalent to some existential formula. This implies condition (3)

Corollary 1. C is the core model of some perfect core Jonsson theory T if and only if C is the core model of the Kaiser Hull T^0 of T.

Proof: Let us prove the necessity of the statement.

Let C be the core model of the above perfect core Jonsson theory T.

Let M be semantic model of T. Let T^0 be the Kaiser hull of T, i.e.

$$T^0 = \{\varphi \in L_0 \ : \varphi \in \forall\exists \, sentences \, and \, M \models \varphi\}$$

where φ are the set of all sentences of the signature language of the theory T.

Such that T is perfect, then T^* is a model companion of T and hence, a model complete theory. As a consequence of this, any formula in T^* is equivalent to some existential formula. Since M is a semantic model of T and a model of T^*, then the Kaiser Hull T^0 will be equal to the center of the theory T, i.e. T^*, where

$$T^* = \{\varphi \in L_0 \text{ (the set of all sentences of the signature's language of T)} : \, in \, M \models \varphi\}.$$

The condition (3) of Theorem 3 holds for the model C and T^*, and it follows that C is a core model of the theory T^*. Let A be an arbitrary model of T^*, then C is isomorphically embedded in A in a unique way, that is, there is C' such that $C' \subseteq A$, where C' is isomorphic to C. Moreover, C' is isomorphically embedded in every model of T^*.

$$C' = \cap\{B : B \subseteq A \, and \, B \models T^*\}$$

Thus, it turns out that T^* is strongly convex and that C is the core model of theory T^*. $\sharp$.

Let us prove the sufficiency of the condition. Suppose that T^* is a strongly convex theory and that C is the core model of theory T^*. Let A be a model of T and let M be a semantic model of theory T. Then $C \in M$ and $C \cong C_1$ for some $C_1 \subseteq M$. Since C, which is a model of theory T^*, has no proper submodel we can say that there is a model.

$$C_1 = \cap\{B : B \subseteq M \, and \, B \models T\}.$$

In particulary, $C_1 \subseteq A$. If C is isomorphic to some other $C_2 \subseteq A$, then it is easy to show in a similar way, that $C_1 = C_2$, and therefore C is a core model of the theory T.

Corollary 2. Let C be the core model of the strongly convex perfect core Jonsson theory T. Then there exist existential formulas $\phi_i(x)$ for $i \in I$, such that

$$T^* \models \exists^{<\omega} x\phi_i \, for \, all \, i \in I,$$

and

$$C \models \forall x \bigvee_{i \in I} \phi_i$$

Corollary 3. The core model C is rigidly embeddable in each model of T if and only if condition (3) of Theorem 3 is satisfied with $k_i = 1$ for all $i \in I$.

Theorem 4. Let C be the core model for some existentially algebraically simple theory T. Then the following conditions are equivalent.

(1) C is embedded in every existentialy closed model of the center of this theory.

(2) C is an algebraically prime model of the theory T.

Proof: We prove from (1) to (2). Let model C be embedded in each existential closed model of the center of this theory. Suppose that model A does not belong to E_T and suppose that model C is not embedded in model A. Since T is a Jonsson theory, then by the inductance of this theory there exists a model $B \in E_T$ such that A is isomorphically embedded in B, but model B is isomorphically embedded in the semantic model M of T. The model B also belongs to set of existentially closed models of the theory T^*.

Let C' be an isomorphic image of the model C in the model B. The model A' is an isomorphic image of the model A in the model B, if C' is embedded in A', then we get a contradiction with our assumption that C' is not embedded in A. Therefore, suppose that C' is not embedded in A' and they are not isomorphic. From this it follows that there is a formula that distinguishes them. Let this formula be $\varphi(x)$. Without loss of generality, suppose that in $C' \models \varphi[c]$ where $c \in C'$, that is, in $C' \models \exists x\varphi(x)$ but by assumption in the model A', which is isomorphic to the model A, it will be true that $A' \models \neg\varphi[a]$ where $a \in A'$. But both A' and C' belong to the model B, which is existentially closed in accordance with the above. Then $B \models \exists x(\varphi(x)\&\neg\varphi(x))$ we get a contradiction. So the model C is an algebraically prime model of the theory T

From 2 we prove 1. Let C be an algebraically prime model of the theory T. Thus C is isomorphically embedded into any model $B \in ModT$, but since $T \subseteq T^*$ we have $ModT^* \subseteq ModT$. It follows that the model C is algebraically prime for the theory T^*, and from this we conclude that C is isomorphically embedded in any model from E_T^*, because $E_T \subseteq ModT^*$

References

1 Kueker D.W. Core structures for theories / D.W. Kueker // Fundamenta Mathematicae LXXXIX. — 1973. — P. 154–171.

2 Baldwin J.T. Algebraically prime models / J.T. Baldwin, D.W. Kueker // Ann. Math. Logic. — 1981. — 20. — P. 289–330.

3 Yeshkeyev A.R. ∇-cl — atomic and prime sets / A.R. Yeshkeyev, A.K. Issaeva // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 93, No. 1. — P. 88–94. DOI 10.31489/2019M1/88-94

4 Yeshkeyev A.R. The atomic definable subsets of semantic model / A.R. Yeshkeyev, A.K. Issaeva, N.M. Mussina // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 94, No. 2. — C. 84–91. DOI 10.31489/2019M2/84-91

5 Robinson A. Introduction to Model Theory and to the Mathematics of Algebra / A. Robinson. — Amsterdam. North-Holland, 1963.

6 Ешкеев А.Р. Йонсоновские теории и их классы моделей: моногр. / А.Р. Ешкеев, М.Т. Касыметова. — Караганда: Изд-во КарГУ, 2016. — 370 с.

7 Кейслер Х.Дж. Теория моделей / Х.Дж. Кейслер, Ч.Ч. Чэн. — М.: Мир, 1977. — С. 667.

8 Yeshkeyev A.R. Central types of convex fragments of the perfect Jonsson theory / A.R. Yeshkeyev, M.T. Omarova // Bulletin of the Karaganda University. Mathematics series. — 2019. — Vol. 93, No. 1. — P. 95–101. DOI 10.31489/2019M1/95-101

9 Yeshkeyev A.R. Properties of lattices of the existential formulas of Jonsson fragments / A.R. Yeshkeyev, M.T. Kasymetova // Bulletin of the Karaganda University.Mathematics series. — 2015. — Vol. 79, No. 3. — P. 25–32.

10 Yeshkeyev A.R. On lattice of existential formulas for fragment of Jonsson set / A.R. Yeshkeyev, O.I. Ulbrikht // Bulletin of the Karaganda University. Mathematics series. — 2015. — Vol. 79, No. 3. — P. 33–39.

А.Р. Ешкеев, А.К. Исаева, Н.В. Попова

Ядролық йонсондық теориялар

Мақаладағы негізгі нәтижелер ол йонсондық теориялардың ядролылығы сияқты жаңа ұғымды және де осы теориядағы экзистенциалды тұйық модельдер класындағы ядролы модельді рұқсат ететін комбинацияларды сипаттау болып табылады. Ядролықпен қатар осы теориядағы ядролы йонсондық теориялардың экзистенциалды алгебралық жай теориялардың қасиеттері қарастырылған. Сонымен қатар, авторлар [1] кейбір нәтижелерді ядролық немесе EAP анықтамаларын немесе олардың комбинацияларын қанағаттандыратын Йонсон теорияларына көшіріп көрді, яғни Йонсон теорияларының кейбір комбинацияларын қарастырды. Йонсон теорияларының ядролылық және экзистенциалды жай анықтамаларынан [1] ядролық модель кез келген Йонсондық теорияның зерттеу аясы тұрғысынан алсақ та, қарастырылып отырған теорияның ерекше және қатаң енгізілімді моделі болатынын атап кетуге болады. Сондықтан, ядролық модельдерге қатысты қойылған мәселе алғашқылардың бірі болып табылады.

Кілт сөздер: дөңес теория, қатты дөңес теория, йонсон теориясының центрі, семантикалық модель, алгебралық жай модель, ядролы модель, ядролы теория.

А.Р. Ешкеев, А.К. Исаева, Н.В. Попова

Ядерные йонсоновские теории

Основным результатом статьи является описание нового понятия как ядерность йонсоновских теорий, а также их комбинации, которые допускают ядерную модель в классе экзистенциально замкнутых моделей этой теории. Наряду с ядерностью рассмотрено свойство экзистенциально алгебраически простой теории, как дополнительное свойство ядерной йонсоновской теории. Также авторами изучены некоторые комбинации теорий Йонсона, где они попытались перенести некоторые результаты из [1] в теории Йонсона, которые удовлетворяют определению ядерности или EAP, или их комбинациям. Из определения ядерности и экзистенциально алгебраической простоты теории Йонсона можно отметить, что ядерная модель из [1] в рамках изучения любой теории Йонсона будет уникальной и жестко вложимой моделью рассматриваемой теории. И в заключении отметим, что такая постановка проблемы относительно ядерных моделей изучена впервые.

Ключевые слова: выпуклая теория, сильно выпуклая теория, центр йонсоновской теории, семантическая модель, алгебраически простая модель, ядерная модель, ядерная теория.

References

1 Kueker, D.W. (1973). Core structures for theories. *Fundamenta Mathematicae LXXXIX,* 154–171.

2 Baldwin, J.T., & Kueker, D.W. (1981). Algebraically prime models. *Ann. Math. Logic., 20,* 289–330.

3 Yeshkeyev, A.R., & Issaeva, A.K. (2019). ∇-cl — atomic and prime sets. *Bulletin of the Karaganda University. Mathematics series, Vol. 93, 1,* 88–94. DOI 10.31489/2019M1/88-94

4 Yeshkeyev, A.R., & Issaeva, A.K. & Mussina, N.M. (2019). The atomic definable subsets of semantic model. *Bulletin of the Karaganda University. Mathematics series, Vol. 94, 2,* 84–91. DOI 10.31489/2019M2 /84-91

5 Robinson, A. (1963). *Introduction to Model Theory and to the Mathematics of Algebra.* Amsterdam: North-Holland.

6 Yeshkeyev, A.R., & Kassymetova, M.T. (2016). *Yonsonovskie teorii i ikh klassy modelei [Jonsson theories and their classes of models].* Karaganda: Izdatelstvo KarHU [in Russian].

7 Keisler, H.J., & Chang, C.C. (1977). *Teoriia modelei [Model theory].* Moscow: Mir [in Russia].

8 Yeshkeyev, A.R., & Omarova, M.T. (2019). Central types of convex fragments of the perfect Jonsson theory. *Bulletin of the Karaganda University. Mathematics series, Vol. 93, 1,* 95–101. DOI 10.31489 /2019M1/95-101

9 Yeshkeyev, A.R., & Kasymetova, M.T. (2015). Properties of lattices of the existential formulas of Jonsson fragments. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 25–32.

10 Yeshkeyev, A.R., & Ulbrikht, O.I. (2015). On lattice of existential formulas for fragment of Jonsson set. *Bulletin of the Karaganda University. Mathematics series, Vol. 79, 3,* 33–39.