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On the solvability of the boundary value problems for the elliptic equation of high order on a plane

For the elliptic equation of $2l-$th order with of constant (and only) real coefficients we consider boundary value problem of the normal derivatives $(k_j - 1)$ order, $j = 1, \ldots, l$, where $1 \leq k_1 < \ldots < k_l \leq 2l - 1$. When $k_j = j$ it moves into the Dirichlet problem, and when $k_j = j + 1$ it moves into the Neumann problem. In this paper, the study is carried out in space $C^{2l,\mu}(\overline{D})$. We found the condition for Fredholm solvability of this problem and computed the index of this problem.

Keywords: elliptic equation, boundary value problem, Dirichlet problem, Neumann problem, solvability of BVP.

Introduction

From the viewpoint of an explicit description of the conditions of solvability of Fredholm and of index for this problem has been studied [1] in the class

$$u \in C^{2l}(D) \cap C^{2l-1,\mu}(\overline{D}), \qquad \sum_{0 \leq r \leq 2l} a_{r,2l} \frac{\partial^{2l} u}{\partial x^{2l-r} \partial y^r} \in C^{\mu}(\overline{D}).$$

In this paper, under the assumption that $\Gamma \in C^{2l,\mu}$ obtained in the paper [1] results extend to a standard class $C^{2l,\mu}(\overline{D})$, which no longer depends on the equation (1).

In [2–8], an explicit form of the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball is constructed. The paper [9, 10] is devoted to the investigation of the solvability of various boundary value problems for a polyharmonic equation in a multidimensional ball. In this paper we obtain a necessary and sufficient condition for the problem to be Fredholm in terms of the original data, that is, from the right-hand side of the inhomogeneous polyharmonic equation and from the right-hand sides of the inhomogeneous boundary conditions. The correct restrictions of the stationary Navier-Stokes equation in a three-dimensional cube are described in [11], and the correct boundary conditions for the pressure in the medium are determined. In [12], initial-boundary value problems for the equations of motion of a viscous heat-conducting gas are studied with allowance for a magnetic field with cylindrical and spherical symmetry. In this paper, we prove theorems on the existence and uniqueness of solutions as a whole with respect to the time of initial-boundary value problems. In [13], a brief summary of the theory of extensions and contractions of operators in Hilbert space is given, and certain classes of well-posed boundary value problems for the bi-Laplace operator are written out. The Green function of the Neumann problem for the Poisson equation in a multidimensional ball is constructed in [14].

Formulation of the problem

In simply connected region D in the plane bounded by a simple smooth contour Γ, we consider the elliptic equation

$$\sum_{0 \leq r \leq k \leq 2l} a_{rk}(z) \frac{\partial^k u}{\partial x^{k-r} \partial y^r} = g(z), \quad z = x + iy \in D, \tag{1}$$

with real coefficients $a_{rk} \in C^{\mu}(\overline{D}),\ 0 < \mu < 1$, constant at $k = 2l$. Without loss of generality we can assume that $a_{2l,2l} = 1$.

The Generalized Dirichlet - Neumann problem for this equation is determined by the boundary conditions

$$\frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}\bigg|_\Gamma = f_j, \quad j=1,\dots,l, \tag{2}$$

where $1\le k_1<k_2<\ldots<k_l\le 2l$, $n=n_1+in_2$ means the unit external normal and under normal derivative $k-$th order we mean the expression

$$\left(n_1\frac{\partial}{\partial x}+n_2\frac{\partial}{\partial y}\right)^k u=\sum_{r=0}^{k}\binom{k}{r}n_1^r n_2^{k-r}\frac{\partial^k u}{\partial x^r\partial y^{k-r}}.$$

Fredholm solvability of the problem

As usual Fredholm property and the index of the problem are understood in relation toward its restricted operator

$$C^{2l,\mu}(\overline{D})\to C^{\mu}(\overline{D})\times\prod_{j=1}^{l}C^{2l-k_j+1,\mu}(\Gamma). \tag{3}$$

For derivatives of $v\in C^{r,\mu}(\Gamma),\ 1\le r\le 2l-1$, with respect to the parameter arc length we have the expression

$$\left(\frac{d}{ds}\right)^r v=\frac{\partial^r v}{\partial e^r}+\ldots$$

where $e=e_1+ie_2=-in$ is the unit tangent vector to the contour Γ, tangential derivative of $r-$ order $\partial^r v/\partial e^r$ is understood as analogous (2) and the dots denote a linear differential operator of order $r-1$, whose coefficients are expressed through the function e_1,e_2 and derivatives of order $r-1$ inclusive. In virtue of the assumptions about the smoothness of the contour Γ coefficients of the operator belong to the class $C^{2l-r,\mu}(\Gamma)$. Therefore, similar to [1] boundary conditions (2) can be rewritten in the equivalent form

$$\left(e_1\frac{\partial}{\partial x}+e_2\frac{\partial}{\partial y}\right)^{2l-k_j}\left(n_1\frac{\partial}{\partial x}+n_2\frac{\partial}{\partial y}\right)^{k_j-1}u+L_j^0u=f_j^0,\quad 1\le j\le l, \tag{4}$$

with the right-hard side

$$f_j^0=f_j^{(2l-k_j)}+\int_\Gamma f_j(t)d_1t,$$

where the symbol d_1t is an element of arc length, and operators

$$L_j^0u=\sum_{0\le r\le s\le 2l-2}a_{j,rs}\frac{\partial^s u}{\partial x^{s-r}\partial y^r}+\int_\Gamma\frac{\partial^{k_j-1}u}{\partial n^{k_j-1}}d_1t,$$

with some coefficients $a_{j,rs}(z)\in C^{1,\mu}(\Gamma)$. It is clear that the operator $L^0=(L_1^0,\dots,L_l^0)$ is compact $C^{2l,\mu}(\overline{D})\to C^{1,\mu}(\Gamma)$.

Consider the map

$$\mathcal{D}u=(U_1,\dots,U_{2l}),\quad U_j=\frac{\partial^{2l-1}u}{\partial x^{2l-j}\partial y^{j-1}},$$

that acts from $C^{2l,\mu}(\overline{D})$ in the space $C^{1,\mu}(\overline{D})$ of vector-functions satisfying the relations

$$\frac{\partial U_j}{\partial y}=\frac{\partial U_{j+1}}{\partial x},\quad 1\le j\le 2l-1. \tag{5}$$

The core of this operators $\ker\mathcal{D}$ is the class P_{2l-2} of all polynomials of degree at most $2l-2$, which is equal to the dimension of $l(2l-1)$.

As in [1] introduce the right-hand operator $\mathcal{D}^{(-1)}$, so that any function $u\in C^{2l,\mu}(\overline{D})$ uniquely represented in the form

$$u=\mathcal{D}^{(-1)}U+p,\quad p\in P_{2l-2}, \tag{6}$$

where the vector-function $U\in C^{1,\mu}(\overline{D})$ satisfying the relations (5).

Substituting this representation in (1) and using (4), from the elliptic equation can come to the equivalent first order system

$$\frac{\partial U}{\partial y} - A\frac{\partial U}{\partial x} + L^1(\mathcal{D}^{(-1)}U + p) = g^1 \tag{7}$$

with $2l \times 2l-$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{2l-1} \end{pmatrix}, \quad a_r = a_{r,2l},$$

with the right-hand side $g^1 = (0, \dots, 0, g)$ and the operator

$$L^1 v = (0, \dots, 0, L^1_{2l} v), \quad L^1_{2l} v = \sum_{0 \le r \le k \le 2l-1} a_{rk} \frac{\partial^k v}{\partial x^{k-r} \partial y^r}.$$

Note that the operator L^1 is compact $C^{2l,\mu}(\overline{D}) \to C^{\mu}(\overline{D})$.

With respect to the matrix $C = (C_{jk}) \in C^{2l-1,\mu}(\Gamma)$, the elements of which are defined by the relations

$$\sum\nolimits_{k=1}^{2l} C_{jk}(t) z^{k-1} = [e_1(t) + e_2(t)z]^{2l-k_j}[-e_2(t) + e_1(t)z]^{k_j-1}, \ 1 \le j \le l, \tag{8}$$

the boundary conditions (4) can be written in the form

$$CU^+ + L^0(\mathcal{D}^{(-1)}U + p) = f^0, \tag{9}$$

where the symbol + indicates the limit value functions. Recall that appearing here the operator $L\mathcal{D}^{(-1)}$ is compact $C^{1,\mu}(\overline{D}) \to C^{1,\mu}(\Gamma)$.

We write the characteristic polynomial equation (1) in the form

$$\sum\nolimits_{r=0}^{2l} a_{r,2l} z^r = \prod\nolimits_{k=1}^{m} [(z - \nu_k)(z - \overline{\nu_k})]^{l_k}, \quad \operatorname{Im} \nu_k > 0, \tag{10}$$

and with each vector-function $g(z) = (g_1(z), \dots, g_n(z))$, analytic in the neighborhood of the point $\nu_1, \dots, \nu_m$. We introduce block $n \times l-$ matrix

$$W_g(\nu_1, \dots, \nu_m) = (W_g(\nu_1), \dots, W_g(\nu_m)),$$

where the matrix $W_g(\nu_k) \in \mathbb{C}^{n \times l_k}$ is composed of column - vectors

$$g(\nu_k), g'(\nu_k), \dots, \frac{1}{(l_k - 1)!} g^{(l_k-1)}(\nu_k).$$

We introduce block $2l \times 2l-$ matrix

$$\widetilde{B} = (B, \overline{B}), \quad B = W_h(\nu_1, \dots, \nu_m) \in \mathbb{C}^{2l \times l},$$
$$\widetilde{J} = \operatorname{diag}(J, \overline{J}) \quad J = \operatorname{diag}(J_1, \dots, J_m), \tag{11}$$

where $h(z) = (1, z, \dots, z^{2l-1})$ and

$$J_k = \begin{pmatrix} \nu_k & 1 & 0 & \dots & 0 \\ 0 & \nu_k & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \nu_k \end{pmatrix} \in \mathbb{C}^{l_k \times l_k}$$

is a Jordan cell, corresponding to the eigenvalue ν_k.

As shown in [1], the matrix $\widetilde{B}$ is reversible and transfers in A to Jordan form $\widetilde{J}$, i.e. we have the equality

$$\widetilde{B}^{-1} A \widetilde{B} = \widetilde{J}.$$

Obviously, the operation of multiplication by a matrix $\widetilde{B}^{-1}$ transforms real $2l-$ vector-functions U in the complex vector-function $\widetilde{\phi}$ block form $(\phi,\overline{\phi})$. Wherein

$$(\widetilde{B}^{-1}L_A\widetilde{B})\widetilde{\phi}=(L_J\phi,\overline{L_J\phi}), \tag{12}$$

where for brevity

$$L_A=\frac{\partial}{\partial y}-A\frac{\partial}{\partial x},\quad L_J=\frac{\partial}{\partial y}-J\frac{\partial}{\partial x}.$$

Recall that the operator $\mathcal{D}^{-1}$, appearing in (7), (8), is defined on $2l-$ the vector-functions $U\in C^{1,\mu}(\overline{D})$, satisfying the conditions (5). In terms of projector Q, acting according to the formula

$$(QU)_j=\begin{cases} U_j, & 1\le j\le 2l-1;\\ 0, & j=2l,\end{cases}$$

these conditions can be described in the form $QL_AU=0$. As shown in [1], there is limited to $C^{1,\mu}(\overline{D})$ projector P with the image $\operatorname{im}P=\{U\in C^{1,\mu}(\overline{D}),\ QL_AU=0\}$. This operator is constructed as follows [1].

We choose ρ so large that the closed region $\overline{D}$ is contained in the disc $D_0=\{|z|<\rho\}$. Then there is a bounded operator $C^{\mu}(\overline{D})\to C^{\mu}(\overline{D}_0)$ continuation, denoted by $\varphi\to\widehat{\varphi}$, with properties

$$\widehat{\varphi}\big|_D=\varphi,\quad \widehat{\varphi}\big|_{\partial D_0}=0.$$

To every non-zero complex number $z=x+iy$ we associate an invertible matrix $z_J=x1+yJ$, where 1 is a single $l\times l-$ matrix. We introduce the integral operator

$$(I^1\varphi)(z)=\frac{1}{\pi i}\int_D (t-z)_J^{-1}\widehat{\varphi}(t)d_2t,\quad z\in D,$$

where d_2t is the area element. This expression is the bounded mapping $C^{\mu}(\overline{D})\to C^{1,\mu}(\overline{D})$ and is a right-hard inverse of L_J, i.e.

$$L_JI^1\varphi=\varphi. \tag{13}$$

Taking into account

$$(\widetilde{B}^{-1}I\widetilde{B})\widetilde{\varphi}=(I^1\varphi,\overline{I^1\varphi}),\quad \widetilde{\varphi}=(\varphi,\overline{\varphi}),$$

obtain an operator I, acting in the space $C^{\mu}(\overline{D})$ of real $2l-$ vector-functions, which in view of (12) has a similar property in relation to L_A. In our notation the desired projector P is defined by $P=1-IQL_A$.

As in [1] via this projector from (7), (8) we can move on to the problem

$$L_AU+L^1(\mathcal{D}^{(-1)}PU+p)=g^1,\quad CU^++L^0(\mathcal{D}^{(-1)}PU+p)=f^0, \tag{14}$$

which is already considered in the whole space $C^{1,\mu}(\overline{D})$. Since $QL^0=0$, from the first equation of this problem it follows $QL_AU=Qf^1$. Therefore, if the right side f^1 has the property $Qf^1=0$, i.e. $f^1_j=0,\ 1\le j\le 2l-1$, then any solution U problems (14) satisfies the condition (5). In other words, for the given right-hand side f^1 problem (14) is equivalent to (7), (8).

We use further substitution

$$U=\widetilde{B}\widetilde{\phi},\quad \widetilde{\phi}=(\phi,\overline{\phi}), \tag{15}$$

according to which we introduce the operators $L^{(1)}:C^{1,\mu}(\overline{D})\times P_{2l-2}\to C^{\mu}(\overline{D})$ and $L^{(0)}:C^{1,\mu}(\overline{D})\times P_{2l-2}\to$ $\to C^{\mu}(\Gamma)$, acting according to the formulas

$$(L^{(1)}(\phi,p),\overline{L^{(1)}(\phi,p)})=\widetilde{B}^{-1}L^1(\mathcal{D}^{(-1)}P\widetilde{B}\widetilde{\phi}+p),\quad L^{(0)}(\phi,p)=L^0(\mathcal{D}^{(-1)}P\widetilde{B}\widetilde{\phi}+p).$$

Then, taking into account (11), (12) the substitution of (15) leads (14) to the following equivalent problem

$$L_J\phi+L^{(1)}(\phi,p)=f^1,\quad 2\mathrm{Re}\,(CB\phi)+L^{(0)}(\phi,p)=f^0, \tag{16}$$

where we put $\widetilde{B}^{-1}g^1=(f^1,\overline{f^1})$, which is considered in the class $C^{1,\mu}(\overline{D})$ $l-$ complex vector-functions ϕ.

So far all reviews have been carried out in the same way as [1] with the difference that in this work problem (16) is considered in the class of functions $\phi \in C^{\mu}(\overline{D}) \cap C^1(D)$, for which $L_J\phi \in C^{\mu}(\overline{D})$. Following [2], we introduce the generalized Cauchy type integrals

$$(I^0\psi)(z) = \frac{1}{2\pi i}\int_{\Gamma}(t-z)_J^{-1}dt_J\psi(t), \quad z \in D,$$

with a density $\varphi \in C^{1,\mu}(\Gamma)$, where witch respect to the point $t = t_1 + it_2$ on the curve dt_J is a complex matrix differential $dt_1 1 + dt_2 J$ and contour Γ positively oriented with respect to D. It is important to note that it has the property

$$L_J I^0\varphi^0 = 0. \tag{17}$$

The Cauchy type integrals answer corresponding singular integral

$$(S^0\psi)(t_0) = \frac{1}{\pi i}\int_{\Gamma}(t-t_0)_J^{-1}dt_J\psi(t), \quad t_0 \in \Gamma,$$

which is understood in the sense of the Cauchy principal value. Note that in the case of a scalar matrix $J = i$ the operator S^0 becomes classic singular Cauchy operator, denoted by S. As shown in [3], operators S and S^0 are bounded in the spaces $C^{\mu}(\Gamma)$, $C^{1,\mu}(\Gamma)$, and the difference $S^0 - S$ is a compact operator. In addition, by the differentiation formulas given in [3] the operator L^0 is bounded $C^{1,\mu}(\Gamma) \to C^{1,\mu}(\overline{D})$ and just corresponds to an analogue of Sokhotskii - Plemelj

$$(I^1\varphi)^+ = (\varphi + S^1\varphi)/2. \tag{18}$$

Based on these results, similarly to the classical theory of singular operators [4] we show that under the assumption of

$$\det[C(t)B] \neq 0, \quad t \in \Gamma, \tag{19}$$

the operator

$$N^0\psi = \mathrm{Re}\,[CB(\psi + S^0\psi)], \tag{20}$$

acting in the space of real $l-$ of vector-functions $\psi \in C^{1,\mu}(\Gamma)$, is Fredholm and its index is given by

$$\mathrm{ind}\, N^0 = -\frac{1}{\pi}[\arg\det(CB)]\big|_{\Gamma}. \tag{21}$$

Further arguments are similar to those given in [1]. As this paper shows any function $\phi \in C^{1,\mu}(\overline{D})$ can by uniquely represented in the form

$$\phi = I^1\varphi^1 + I^0\varphi^0 + i\xi, \quad \xi \in \mathbb{R}^l,$$

with some complex $l-$ vector-function $\varphi^1 \in C^{\mu}(\overline{D})$ and real $\varphi^0 \in C^{1,\mu}(\Gamma)$. The substitution of this representation in (16) given (13), (17), (18) reduces the problem to an equivalent system of integral equations

$$\varphi^1 + L_J(I^0\varphi^0 + i\xi) + L^{(1)}(I^1\varphi^1 + I^0\varphi^0 + i\xi, p) = f^1;$$

$$\mathrm{Re}\,[CB(\varphi^0 + S^0\varphi^0)] + 2\mathrm{Re}\,[CB(I^1\varphi^1 + i\xi)] + L^{(0)}(I^1\varphi^1 + i\xi, p) = f^0.$$

In the notation (20) we write it briefly in the operator form

$$N^0\varphi^0 + M^{00}\varphi^0 + M^{01}\varphi^1 + T^0(p,\xi) = f^0, \quad \varphi^1 + M^{10}\varphi^0 + M^{11}\varphi^1 + T^1(p,\xi) = f^1, \tag{22}$$

with the relevant operators T^i and

$$M^{00}\varphi^0 = L^{(0)}I^0\varphi^0, \quad M^{01}\varphi^1 = 2\mathrm{Re}\,(CBI^1\varphi^1) + L^{(0)}I^1\varphi^1,$$

$$M^{10}\varphi^0 = L^{(1)}I^0\varphi^0, \quad M^{11}\varphi^1 = L^{(1)}I^1\varphi^1.$$

Since the operators $L^{(0)}$ and $L^{(1)}$ are compact, in the operator matrix

$$M = \begin{pmatrix} M^{00} & M^{01} \\ M^{10} & M^{11} \end{pmatrix},$$

acting in the space $C^{1,\mu}(\Gamma) \times C^{\mu}(\overline{D})$, all elements except M^{01} are compact. Therefore, by the general theory of Fredholm operators [5] the operators $N = \mathrm{diag}\,(N^0, 1)$ and $N + M$ are Fredholm equivalent and their indices coincide. Recalling that $\dim P_{2l-2} = l(2l-1)$ and $\xi \in \mathbb{R}^l$, taking into account (20) and the corresponding properties of Fredholm operators we conclude that the next theorem is proved.

Theorem. Suppose that condition

$$\det[C(t)B] \neq 0, \quad t \in \Gamma$$

is satisfied. Then the problem (1), (2) is Fredholm in the class $C^{2l,\mu}(\overline{D})$, and its index æ is calculated by the formula

$$æ = -\frac{1}{\pi}[\arg\det(CB)]\big|_\Gamma + 2l^2,$$

where the increment of a continuous branch of the argument on the contour Γ is taken in the counterclockwise direction.

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Б.Д. Қошанов, А.П. Солдатов

Жазықтықта жоғары дәрежелі эллиптикалық теңдеулер үшін шеттік есептердің шешімділігі туралы

Мақалада тұрақты (тек жоғары дәрежелері) нақты коэффициентті $2l$-дәрежелі, шекарада $(k_j - 1)$-дәрежелі нормал туындылары берілген шеттік есептер қарастырылған, $j = 1, \dots, l$, $1 \leq k_1 < \dots < k_l \leq 2l-1$. Бұл есеп $k_j = j$ болған кезде — Дирихле есебі, ал $k_j = j+1$ кезде Нейман есебі болады. Авторлар осы есептің фредгольмді шешімділігінің шартын тауып, индексін есептеген.

Кілт сөздер: эллиптикалық теңдеулер, шеттік есептер, Дирихле есебі, Нейман есебі, шеттік есептердің шешімділігі.

Б.Д. Кошанов, А.П. Солдатов

О разрешимости краевых задач для эллиптического уравнения высокого порядка на плоскости

В статье для эллиптического уравнения 2 l-го порядка с постоянными (и только старшими) вещественными коэффициентами рассмотрена краевая задача, заключающаяся в задании нормальных производных $(k_j - 1)$−го порядка, $j = 1, \ldots, l$, где $1 \leq k_1 < \ldots < k_l \leq 2l - 1$. При $k_j = j$ она переходит в задачу Дирихле, а при $k_j = j + 1$ – в задачу Неймана. Авторами найдено условие фредгольмовой разрешимости этой задачи и вычислен индекс.

Ключевые слова: эллиптическое уравнение, краевые задачи, задача Дирихле, задача Неймана, разрешимость краевых задач.