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Sobolev Embedding Theorem for the Sobolev-Morrey spaces

In this paper we prove a Sobolev Embedding Theorem for Sobolev-Morrey spaces. The proof is based on the Sobolev Integral Representation Theorem and on a recent results on Riesz potentials in generalized Morrey spaces of Burenkov, Gogatishvili, Guliyev, Mustafaev and on estimates on the Riesz potentials. We mention that a Sobolev Embedding Theorem for Sobolev morrey spaces had been proved by Campanato, for a subspace of our Sobolev-Morrey space which corresponds to the closure of the set of smooth functions in our Sobolev-Morrey space. The methods of the present paper are considerably different from those of Campanato.

Key words: Morrey space, Sobolev-Morrey space, Sobolev Embedding Theorem .

Introduction

$\mathbb{N}$ denotes the set of all natural numbers including 0. Throughout the paper, n is a nonzero natural number. Let $B(x,r)$ be an open ball in $\mathbb{R}^n$ of radius $r>0$ centered at the point $x\in\mathbb{R}^n$.

Definition 1. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$. Let $0<p\leq+\infty$ and let w be a measurable function from $]0,+\infty[$ to $]0,+\infty[$. Denote by $\mathcal{M}_p^{w(\cdot)}(\Omega)$ the space of all real-valued measurable functions on Ω for which

$$\|f\|_{\mathcal{M}_p^{w(\cdot)}(\Omega)}=\sup_{x\in\Omega}\|w(\rho)\|f\|_{L_p(B(x,\rho)\cap\Omega)}\|_{L_\infty(0,\infty)}<\infty.$$

Definition 2. Let $0<p\leq+\infty$. Denote by $\Lambda_{p,\infty}$ the set of all measurable functions w from $]0,+\infty[$ to $]0,+\infty[$ which are not equivalent to 0 such that

$$\|w(\rho)\|_{L_\infty(1,\infty)}<\infty,\quad \|w(\rho)\rho^{\frac{n}{p}}\|_{L_\infty(0,1)}<\infty.$$

In [1, 2] it is proved that, if w is a non-negative measurable function from $]0,+\infty[$ to $]0,+\infty[$ which are not equivalent to 0, then the space $\mathcal{M}_p^{w(\cdot)}(\Omega)$ is non-trivial, *i.e.* consists not only of functions f equivalent to 0 on Ω if, and only if, $w\in\Lambda_{p,\infty}$.

Definition 3. If $w_\lambda(\rho)=\begin{cases}\rho^{-\lambda}, & \rho\in]0,1],\\ 1, & \rho\geq 1,\end{cases}$, then we set

$$M_p^\lambda(\Omega)\equiv\mathcal{M}_p^{w_\lambda}(\Omega)$$

and the condition $w_\lambda\in\Lambda_{p,\infty}$ means that $0\leq\lambda\leq\frac{n}{p}$.

Note that:

(i) $M_p^\lambda(\mathbb{R}^n)$ is continuously embedded into $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$;

(ii) If Ω is a bounded domain, then we have $\mathcal{M}_p^{r^{-\lambda}}(\Omega)=M_p^\lambda(\Omega)$ with equivalent norms.

Definition 4. Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $l \in \mathbb{N}$, $p \in [1, +\infty]$ and $\lambda \in \left[0, \frac{n}{p}\right]$. Then we define the Sobolev space of order l built on the Morrey space $M_p^\lambda(\Omega)$, as the set

$$W_p^{l,\lambda}(\Omega) \equiv \left\{ f \in M_p^\lambda(\Omega) : D_w^\alpha f \in M_p^\lambda(\Omega) \; \forall \, \alpha \in \mathbb{N}^n, |\alpha| \leq l \right\},$$

where $D_w^\alpha f$ is the weak derivative of f.

Then we set

$$\|f\|_{W_p^{l,\lambda}(\Omega)} = \sum_{|\alpha| \leq l} \|D_w^\alpha f\|_{M_p^\lambda(\Omega)} \quad \forall \, f \in W_p^{l,\lambda}(\Omega).$$

In particular, $W_p^{0,\lambda}(\Omega) = M_p^\lambda(\Omega)$ and $W_p^{l,0}(\Omega) = W_p^l(\Omega)$, where $W_p^l(\Omega)$ denotes the classical Sobolev space of exponents l, p in Ω. It is obvious that $W_p^{l,\lambda}(\Omega) \subset W_p^l(\Omega)$.

Preliminaries

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. Consider the Riesz potential

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

Theorem 1. Let $n \in \mathbb{N} \setminus \{0\}$. Let $1 \leq p \leq q < +\infty$. Let $0 \leq \lambda \leq \nu < \frac{n}{q}$. Let

$$\alpha \equiv \left(\nu - \frac{n}{q}\right) - \left(\lambda - \frac{n}{p}\right). \tag{1}$$

Then the following statements hold:

(i) If $\lambda < \nu$, then the operator I_α is bounded from $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ to $\mathcal{M}_q^{r^{-\nu}}(\mathbb{R}^n)$;

(ii) If $\lambda = \nu$ and if $1 < p < q$, then I_α is bounded from $\mathcal{M}_p^{r^{-\lambda}}(\mathbb{R}^n)$ to $\mathcal{M}_q^{r^{-\nu}}(\mathbb{R}^n)$;

(iii) If $\lambda = \nu$ and if $1 < p < q$, then I_α is bounded from $M_p^\lambda(\mathbb{R}^n)$ to $M_q^\nu(\mathbb{R}^n)$.

The proof of this theorem is based on [3, Theorem 1.3].

Remark 1. If $\nu = \lambda = 0$, then $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$, and this is the classical Hardy-Littlewood-Sobolev theorem.

Lemma 1. Let $p \in [1, +\infty[$, $\alpha \in]0, n[$, $\lambda \in [0, n/p]$. Let $q \in [1, p]$ be such that $(\alpha + \lambda) > \frac{n}{q}$. Let

$$\mu_{w_\lambda, q} \equiv \max\left\{1, \frac{1}{(\alpha+\lambda) - \frac{n}{q}}\right\}. \tag{2}$$

Then we have

$$\int_{E \cap \mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq m_n(E)^{\frac{1}{q}-\frac{1}{p}} \mu_{w_\lambda,q}(n+2-\alpha) v_n^{1-\frac{1}{q}} \|f\|_{M_p^\lambda(\mathbb{R}^n)} \quad \forall \, f \in M_p^\lambda(\mathbb{R}^n), \tag{3}$$

for all measurable subsets E of $\mathbb{R}^n$ of finite measure, and for all $x \in \mathbb{R}^n$.

Proof. The arguments of this proof are in part based on a development of the ideas of Campanato[4].

If $f \in M_p^\lambda(\mathbb{R}^n)$, then we know that $f\big|_{\mathbb{B}_n(x,r)} \in L_p(\mathbb{B}_n(x,r)) \subseteq L_1(\mathbb{B}_n(x,r))$ for all $x \in \mathbb{R}^n$ and $r \in]0, +\infty[$. In particular, $(\chi_E f)\big|_{\mathbb{B}_n(x,r)} \in L_1(\mathbb{B}_n(x,r))$ for all $x \in \mathbb{R}^n$ and $r \in]0, +\infty[$ and for all measurable subsets E of $\mathbb{R}^n$.

Now we fix $x \in \mathbb{R}^n$ and a measurable subset E of $\mathbb{R}^n$ of finite measure. The almost everywhere defined function from $]0,+\infty[$ to $[0,+\infty[$ which takes $s \in]0,+\infty[$ to $\int\limits_{\partial\mathbb{B}_n(x,s)} \chi_E|f|d\sigma$ is integrable in $]0,r[$ for all $r \in]0,+\infty[$. Then by the Fundamental Theorem of Calculus, the function $A_{E,x}$ from $[0,+\infty[$ to $[0,+\infty[$ defined by

$$A_{E,x}(\rho) \equiv \int\limits_0^\rho \int\limits_{\partial\mathbb{B}_n(x,s)} \chi_E|f|d\sigma\, ds, \quad \forall\, \rho \in [0,+\infty[,$$

is locally absolutely continuous and

$$A'_{E,x}(\rho) \equiv \int\limits_{\partial\mathbb{B}_n(x,\rho)} \chi_E|f|d\sigma, \quad x \in \mathbb{R}^n,$$

for almost all $\rho \in [0,+\infty[$ (cf. e.g., Folland [5, 3.35]). By the Monotone Convergence Theorem, we have

$$\int\limits_{E\cap\mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} = \int\limits_{\mathbb{B}_n(x,1)} \frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}} = \lim_{\varepsilon\to 0} \int\limits_{\mathbb{B}_n(x,1)\setminus\mathbb{B}_n(x,\varepsilon)} \frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}}. \tag{4}$$

Now let $\varepsilon \in]0,1[$. Then we have

$$\int\limits_{\mathbb{B}_n(x,1)\setminus\mathbb{B}_n(x,\varepsilon)} \frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}} = \int\limits_\varepsilon^1 s^{-n+\alpha} \int\limits_{\partial\mathbb{B}_n(x,s)} \chi_E|f|d\sigma\, ds = \int\limits_\varepsilon^1 s^{-n+\alpha} A'_{E,x}(s)ds. \tag{5}$$

Then by integrating by parts, we obtain

$$\int\limits_\varepsilon^1 s^{-n+\alpha} A'_{E,x}(s)ds = \left[s^{-n+\alpha}A_{E,x}(s)\right]_\varepsilon^1 - \int\limits_\varepsilon^1 (-n+\alpha)s^{-n+\alpha-1}A_{E,x}(s)ds, \tag{6}$$

(cf. e.g., Folland [5, ex.35, p.108]). Then the Hölder inequality and inequality (2) imply that

$$|A_{E,x}(\rho)| \le m_n(E\cap\mathbb{B}_n(x,\rho))^{1-\frac{1}{p}}\|f\|_{L_p(E\cap\mathbb{B}_n(x,\rho))} =$$

$$= m_n(E\cap\mathbb{B}_n(x,\rho))^{\frac{1}{q}-\frac{1}{p}}\, m_n(E\cap\mathbb{B}_n(x,\rho))^{1-\frac{1}{q}}\|f\|_{L_p(E\cap\mathbb{B}_n(x,\rho))} \le$$

$$\le m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{n-\frac{n}{q}}\|f\|_{L_p(\mathbb{B}_n(x,\rho))} \le m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{n-\alpha}\rho^{-n+\alpha}\rho^{n-\frac{n}{q}}w_\lambda^{-1}(\rho)w_\lambda(\rho)\|f\|_{L_p(\mathbb{B}_n(x,\rho))} \le$$

$$\le \rho^{n-\alpha}m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{\alpha-\frac{n}{q}}w_\lambda^{-1}(\rho)\|f\|_{M_p^\lambda(\mathbb{R}^n)} \le \rho^{n-\alpha}m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\rho^{(\alpha+\lambda)-n/q}\|f\|_{M_p^\lambda(\mathbb{R}^n)} \tag{7}$$

for all $\rho \in]0,1[$. Then by the second last line of inequality (7), we have

$$\left|\int\limits_\varepsilon^1 (-n+\alpha)s^{-n+\alpha-1}A_{E,x}(s)ds\right| \le$$

$$\le m_n(E)^{\frac{1}{q}-\frac{1}{p}}\, v_n^{1-\frac{1}{q}}\|f\|_{M_p^\lambda(\mathbb{R}^n)}\left|\int\limits_\varepsilon^1 (-n+\alpha)s^{-n+\alpha-1}s^{n-\alpha}s^{(\alpha+\lambda)-\frac{n}{q}}ds\right| \le$$

$$\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}} v_n^{1-\frac{1}{q}} \|f\|_{M_p^\lambda(\mathbb{R}^n)}(n-\alpha) \left| \int_\varepsilon^1 s^{(\alpha+\lambda)-\frac{n}{q}-1} ds \right| =$$

$$= m_n(E)^{\frac{1}{q}-\frac{1}{p}} v_n^{1-\frac{1}{q}} \|f\|_{M_p^\lambda(\mathbb{R}^n)}(n-\alpha) \frac{1}{(\alpha+\lambda)-n/q}(1-\varepsilon) =$$

$$= m_n(E)^{\frac{1}{q}-\frac{1}{p}} (n-\alpha) v_n^{1-\frac{1}{q}} \mu_{w_\lambda,q} \|f\|_{M_p^\lambda(\mathbb{R}^n)}(1-\varepsilon). \tag{8}$$

Then by combining (5)–(8), we deduce that

$$\int\limits_{\mathbb{B}_n(x,1)\setminus\mathbb{B}_n(x,\varepsilon)} \frac{\chi_E(y)|f(y)|dy}{|x-y|^{n-\alpha}} \leq \left| \int_\varepsilon^1 s^{-n+\alpha} A'_{E,x}(s) ds \right| \leq$$

$$\leq |A_{E,x}(1)| + |\varepsilon^{-n+\alpha} A_{E,x}(\varepsilon)| + \left| \int_\varepsilon^1 (-n+\alpha) s^{-n+\alpha-1} A_{E,x}(s) ds \right| \leq$$

$$\leq 1^{n-\alpha} m_n(E)^{\frac{1}{q}-\frac{1}{p}} v_n^{1-\frac{1}{q}} 1^{(\alpha+\lambda)-n/q} \|f\|_{M_p^\lambda(\mathbb{R}^n)} +$$

$$+ \varepsilon^{-n+\alpha} \varepsilon^{n-\alpha} m_n(E)^{\frac{1}{q}-\frac{1}{p}} v_n^{1-\frac{1}{q}} \varepsilon^{(\alpha+\lambda)-n/q} \|f\|_{M_p^\lambda(\mathbb{R}^n)} +$$

$$+ m_n(E)^{\frac{1}{q}-\frac{1}{p}} (n-\alpha) v_n^{1-\frac{1}{q}} \mu_{w_\lambda,q} \|f\|_{M_p^\lambda(\mathbb{R}^n)}(1-\varepsilon) \leq$$

$$\leq m_n(E)^{\frac{1}{q}-\frac{1}{p}} \mu_{w_\lambda,q} v_n^{1-\frac{1}{q}} \|f\|_{M_p^\lambda(\mathbb{R}^n)}[1+1+(n-\alpha)(1-\varepsilon)].$$

Then the limiting relation (4) immediately implies the validity of inequality (3).

Corollary 1. Let $p \in [1, +\infty[$, $\alpha \in]0, n[$. Let $\lambda \in [0, n/p]$. Let $(\alpha+\lambda) > \frac{n}{p}$. Let Ω be an open subset of $\mathbb{R}^n$. Then the following statements hold.

If $f \in M_p^\lambda(\mathbb{R}^n)$ and if $\int_\Omega |f| dx < \infty$, then the function from $\mathbb{R}^n$ to $\mathbb{R}$ which takes $x \in \mathbb{R}^n$ to

$$\int\limits_\Omega \frac{f(y)dy}{|x-y|^{n-\alpha}}$$

is bounded, and satisfies the following inequality

$$\sup_{x \in \mathbb{R}^n} \int\limits_\Omega \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq \max\{1, ((\lambda+\alpha)-(n/p))^{-1}\}(n+2-\alpha) v_n^{1-\frac{1}{p}} \|f\|_{M_p^\lambda(\mathbb{R}^n)} + \int\limits_\Omega |f| dx. \tag{9}$$

If Ω has finite measure, then the map $I_{\alpha,\Omega}$ from $M_p^\lambda(\Omega)$ to $B(\mathbb{R}^n)$ defined by

$$I_{\alpha,\Omega} f(x) \equiv \int\limits_\Omega \frac{f(y)dy}{|x-y|^{n-\alpha}} \quad \forall x \in \mathbb{R}^n,$$

for all $f \in M_p^\lambda(\Omega)$ is linear and continuous.

Proof. By applying Lemma 1 with $E = \Omega$, we deduce that

$$\int\limits_\Omega \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq \int\limits_{\Omega \cap \mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} + \int\limits_{\Omega \setminus \mathbb{B}_n(x,1)} \frac{|f(y)|dy}{|x-y|^{n-\alpha}} \leq$$

$$\leq m_n\left(\mathbb{B}_n(x,1)\right)^{\frac{1}{q}-\frac{1}{p}}\mu_{w_\lambda,q}(n+2-\alpha)v_n^{1-\frac{1}{q}}\|f\|_{M_p^\lambda(\mathbb{R}^n)}+\int\limits_{\Omega\setminus\mathbb{B}_n(x,1)}\frac{|f|}{1^{n-\alpha}}dx\leq$$

$$\leq \mu_{w_\lambda,q}(n+2-\alpha)v_n^{1-\frac{1}{q}}\|f\|_{M_p^\lambda(\mathbb{R}^n)}+\int\limits_{\Omega}|f|dx,$$

for all $x\in\mathbb{R}^n$. Hence, inequality (9) follows.

Definition 5. A domain $\Omega\subset\mathbb{R}^n$ is called star-shaped with respect to the ball $B\subset\Omega$ if for all $y\in B$ and for all $x\in\Omega$ we have $[x,y]\subset\Omega$. A domain $\Omega\subset\mathbb{R}^n$ is called star-shaped with respect to a ball if for some ball $B\subset\Omega$ it is star-shaped respect to the ball B.

Lemma 2. Let $l\in\mathbb{N}\setminus\{0\}$. Let $m\in\mathbb{N}$, $m<l$. Let $1\leq p,q\leq+\infty$, $0\leq\lambda\leq\frac{n}{p}$, $0\leq\nu\leq\frac{n}{q}$. Suppose that for each bounded domain $G\subset\mathbb{R}^n$ star-shaped with respect to a ball there exists $c_1>0$ such that for each $\beta\in\mathbb{N}^n$ satisfying $|\beta|\leq m$ and for all $f\in W_p^{l,\lambda}(G)$

$$\|D_w^\beta f\|_{M_q^\nu(G)}\leq c_1\|f\|_{W_p^{l,\lambda}(G)}.$$

Then for each open bounded set $\Omega\subset\mathbb{R}^n$ satisfying the cone condition there exists $c_2>0$ such that

$$\|D_w^\beta f\|_{M_q^\nu(\Omega)}\leq c_2\|f\|_{W_p^{l,\lambda}(\Omega)}$$

for each $\beta\in\mathbb{N}^n$ satisfying $|\beta|\leq m$ and for all $f\in W_p^{l,\lambda}(\Omega)$.

The proof of this Lemma is based on [6, lemma 4, Ch. 3.2] and Minkowski inequality for Morrey spaces.

Main result

First we introduce the following notation.

Definition 6. Let $p\in[1,+\infty]$, $l,n\in\mathbb{N}\setminus\{0\}$, $m\in\mathbb{N}$, $m\leq l$, $\lambda,\ \nu\ \in\ [0,+\infty[$. Let $l+\lambda-m-\nu\neq\frac{n}{p}$. Then we set

$$q^*(l,m,n,p,\lambda,\nu)\equiv\frac{n}{(n/p)-(l+\lambda-m-\nu)}.$$

If $\lambda=\nu=0$, then $q^*(l,m,n,p,\lambda,\nu)$ equals the classical Sobolev limiting exponent. If $\lambda,\nu\in[0,+\infty[$, then the exponent $q^*(l,m,n,p,\lambda,\nu)$ can be obtained from the classical one by replacing l by $l+\lambda$ and m by $m+\nu$.

We note that if $l+\lambda-\nu\neq\frac{n}{p}$, then the equality which defines $q^*(l,0,n,p,\lambda,\nu)$ is equivalent to the equality

$$l=\left(\nu-\frac{n}{q^*(l,0,n,p,\lambda,\nu)}\right)-\left(\lambda-\frac{n}{p}\right).$$

We also note that

$$\frac{q^*(l,0,n,p,\lambda,\nu)}{p}>1\quad\text{whenever}\quad\begin{cases} l+\lambda>\nu,\\ l+\lambda-\nu<\frac{n}{p}\end{cases}$$

and

$$q^*(l-m,0,n,p,\lambda,\nu)=q^*(l,m,n,p,\lambda,\nu).$$

We are now ready to prove the following Sobolev Embedding Theorem.

Theorem 2. Let $p\in[1,+\infty[$, $l,n\in\mathbb{N}\setminus\{0\}$, $m\in\mathbb{N}$, $m\leq l$, $\lambda\in\left[0,\frac{n}{p}\right]$. Let Ω be a bounded open subset of $\mathbb{R}^n$ which satisfies the cone property. Then the following statements hold.

(i) Let $l-m+\lambda<\frac{n}{p}$. Let $\nu\in]\lambda,(l-m)+\lambda]$. Then $W_p^{l,\lambda}(\Omega)$ is continuously embedded into $W^{m,\nu}_{q^*(l,m,n,p,\lambda,\nu)}(\Omega)$.

(ii) Let $l-m+\lambda<\frac{n}{p}$. If $p>1$, then $W_p^{l,\lambda}(\Omega)$ is continuously embedded into $W^{m,\lambda}_{q^*(l,m,n,p,\lambda,\lambda)}(\Omega)$.

(iii) Let $l-m+\lambda>\frac{n}{p}$. Then $W_p^{l,\lambda}(\Omega)$ is continuously embedded into $W^m_\infty(\Omega)$.

Proof. (i) First let $m=0$.

Let Ω be a bounded domain star-shaped with respect to the ball $B=B(x_0,r)$, $\bar{B}\subset\Omega$. Then by Sobolev's integral representation there exists $M_1>0$ such that

$$|f(x)|\le M_1\left(\int\limits_B |f|dy+\sum_{|\alpha|=l}\int\limits_{V_x}\frac{(D^\alpha_w f)(y)}{|x-y|^{n-l}}dy\right)$$

for almost all $x\in\Omega$ for each (cf. e.g., Burenkov [6, Ch.3, p.112]).

Hence,

$$\|f\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}\le M_1(\int\limits_B |f|dy\cdot\|1\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}+$$
$$+\sum_{|\alpha|=l}\left\|\int\limits_{\mathbb{R}^n}\frac{\Phi_\alpha(y)}{|x-y|^{n-l}}dy\right\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}),$$

where $\Phi_\alpha(y)=\begin{cases} D^\alpha_w f(y), & \text{if } \ y\in\Omega;\\ 0, & \text{if } \ y\notin\Omega.\end{cases}$

Note that

$$\|1\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}=\sup_{x\in\Omega}\sup_{\rho>0}\rho^{-\nu}\|1\|_{L_{q^*(l,0,n,p,\lambda,\nu)}(B(x,\rho)\cap\Omega)}\le$$
$$\le\sup_{x\in\Omega}\max\left\{\sup_{0<\rho\le(\operatorname{diam}\Omega)}\upsilon_n^{\frac{1}{q^*(l,0,n,p,\lambda,\nu)}}\rho^{-\nu+\frac{n}{q^*(l,0,n,p,\lambda,\nu)}},\sup_{\rho\ge(\operatorname{diam}\Omega)}\rho^{-\nu}m_n(\Omega)^{\frac{1}{q^*(l,0,n,p,\lambda,\nu)}}\right\}=$$
$$=\max\left\{\upsilon_n^{\frac{1}{q^*(l,0,n,p,\lambda,\nu)}}m_n(\Omega)^{-\nu+\frac{n}{q^*(l,0,n,p,\lambda,\nu)}},m_n(\Omega)^{-\nu+\frac{1}{q^*(l,0,n,p,\lambda,\nu)}}\right\}<\infty.$$

By Theorem 2 there exists $c>0$ depending only on n, l, p, $q^*(l,0,n,p,\lambda,\nu)$ such that

$$\left\|\int\limits_{\mathbb{R}^n}\frac{\Phi_\alpha(y)}{|x-y|^{n-l}}dy\right\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\mathbb{R}^n)}\le c\|\Phi_\alpha\|_{\mathcal{M}^{r-\lambda}_p(\mathbb{R}^n)}\le$$
$$\le 2^\lambda c\|D^\alpha_w f\|_{\mathcal{M}^{r-\lambda}_p(\Omega)}\le 2^\lambda c\|D^\alpha_w f\|_{M^\lambda_p(\Omega)}.$$

By Hölder inequality $\int\limits_B|f|dy\le m_n(B)^{\frac{1}{p'}}\|f\|_{L_p(\Omega)}$.

There fore, there exist $M_2>0$ and $M_3>0$ such that

$$\|f\|_{M^\nu_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}\le\max\{1,(\operatorname{diam}\Omega)^\nu\}\|f\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\Omega)}\le$$
$$\le\max\{1,(\operatorname{diam}\Omega)^\nu\}\|f\|_{\mathcal{M}^{r-\nu}_{q^*(l,0,n,p,\lambda,\nu)}(\mathbb{R}^n)}\le$$

$$\leq M_2\left(\|f\|_{L_p(\Omega)}+\sum_{|\alpha|=l}\left\|\int\limits_{\mathbb{R}^n}\frac{\Phi_\alpha(y)}{|x-y|^{n-l}}dy\right\|_{\mathcal{M}^{r-\nu}_{q^*}(\mathbb{R}^n)}\right)\leq$$

$$\leq M_3\left(\|f\|_{M^\lambda_p(\Omega)}+\sum_{|\alpha|=l}\|D^\alpha_w f\|_{M^\lambda_p(\Omega)}\right)=M_3\|f\|_{W^{l,\lambda}_p(\Omega)},\quad \forall\, f\in W^{l,\lambda}_p(\Omega).$$

Hence, by Lemma 1, the statement of Theorem 1 follows.

Now let $\alpha : |\alpha| = m$. Then $D^\alpha_w f \in W^{l-|\alpha|,\lambda}_p(\Omega) = W^{l-m,\lambda}_p(\Omega)$. Hence, there exists a constant $c_1 > 0$ such that

$$\|f\|_{W^{m,\nu}_{q^*(l,m,n,p,\lambda,\nu)}(\Omega)}=\sum_{|\alpha|\leq m}\|D^\alpha_w f\|_{M^\nu_{q^*(l,m,n,p,\lambda,\nu)}(\Omega)}=$$

$$=\sum_{|\alpha|\leq m}\|D^\alpha_w f\|_{M^\nu_{q^*(l-m,0,n,p,\lambda,\nu)}(\Omega)}\leq c_1\sum_{|\alpha|\leq m}\|D^\alpha_w f\|_{W^{l-m,\lambda}_p(\Omega)}\leq$$

$$\leq c_1\sum_{|\alpha|\leq m}\sum_{|\gamma|\leq l-m}\|D^{\gamma+\alpha}_w f\|_{M^\lambda_p(\Omega)}\leq$$

$$\leq c_1\sum_{|\alpha|\leq l}\|D^\alpha_w f\|_{M^\lambda_p(\Omega)}=c_1\|f\|_{W^{l,\lambda}_p(\Omega)},\quad \forall\, f\in W^{l,\lambda}_p(\Omega).$$

(ii) This case can be analized as case (i) by replacing $q^*(l,m,n,p,\lambda,\nu)$ by $q^*(l,m,n,p,\lambda,\lambda)$.

(iii) Now $(l-m+\lambda) > \frac{n}{p}$. Let Ω be a bounded domain star-shaped with respect to the ball $B = \mathbb{B}_n(\xi, r_0)$, $\bar{B} \subset \Omega$. Then by Sobolev's integral representation there exists $c > 0$ such that

$$|f(x)|\leq c\left(\int\limits_{\mathbb{B}_n(\xi,r_0)}|f|dx+\sum_{|\gamma|=l}\int\limits_{V_x}\frac{|(D^\gamma_w f)(y)|}{|x-y|^{n-l}}dy\right),\tag{10}$$

for almost all $x \in \Omega$ and for all $f \in W^{l,\lambda}_p(\Omega)$, and where V_x denotes the conical body based on $\mathbb{B}_n(\xi, r_0)$ and with vertex $x \in \Omega$ (cf. e.g., Burenkov [6, Ch.3 p.112]).

We first consider case $m = 0$. So we now assume that $(l + \lambda) > \frac{n}{p}$. We plan to estimate the supremum of $|f|$ by exploiting inequality (10). Since $\int\limits_{\mathbb{B}_n(\xi,r_0)}|f|dx$ is a constant, it defines an element of $C^0_b(\Omega) \subseteq L_\infty(\Omega)$. Next we prove that the sum in the right hand side of (10) is bounded if $f \in W^{l,\lambda}_p(\Omega)$.

We plan to treat separately case $l < n$ and case $l \geq n$.

Let $l < n$. Since $(l + \lambda) > \frac{n}{p}$, we can invoke Corollary 1 and conclude that $I_{l,\Omega}$ is linear and continuous from $M^\lambda_p(\Omega)$ to $B(\mathbb{R}^n)$.

Since $V_x \subseteq \Omega$ for all $x \in \Omega$, we deduce that

$$\left|\int\limits_{V_x}\frac{|h(y)|}{|x-y|^{n-l}}\,dy\right|\leq I_{l,\Omega}(|h|)\quad \forall\, x\in\Omega,$$

for all $h \in M^\lambda_p(\Omega)$. By the continuity of the restriction operator in Morrey spaces and by the above mentioned continuity of $I_{l,\Omega}$, we deduce that the map $J_{l,\Omega}$ from $M^\lambda_p(\Omega)$ to $B(\Omega)$ defined by

$$J_{l,\Omega}h(x)\equiv\int\limits_{V_x}\frac{h(y)}{|x-y|^{n-l}}dy\quad \forall\, x\in\Omega,$$

for all $h \in M_p^\lambda(\mathbb{R}^n)$ satisfies the inequality

$$|J_{l,\Omega}h(x)| \leq |I_{l,\Omega}(|h|)(x)| \leq \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))}\| \, |h| \, \|_{M_p^\lambda(\Omega)} \leq$$
$$\leq \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))}\|h\|_{M_p^\lambda(\Omega)} \quad \forall\, x \in \Omega, \tag{11}$$

for all $h \in M_p^\lambda(\Omega)$. Then we deduce that

$$|f(x)| \leq c\left(\int\limits_{\mathbb{B}_n(\xi,r_0)} |f|dx + \sum_{|\gamma|=l} |J_{l,\Omega}[D_w^\gamma f](x)|\right) \leq$$
$$\leq c\left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}}\|f\|_{L_p(\Omega)} + \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))}\sum_{|\gamma|=l}\|D_w^\gamma f\|_{M_p^\lambda(\Omega)}\right) \leq \tag{12}$$
$$\leq c\left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}} + \|I_{l,\Omega}\|_{\mathcal{L}(M_p^\lambda(\Omega),B(\mathbb{R}^n))}\right)\|f\|_{W_p^{l,\lambda}(\Omega)},$$

for almost all $x \in \Omega$ and for all $f \in W_p^{l,\lambda}(\Omega)$.

We now consider case $l \geq n$. The embedding of $M_p^\lambda(\Omega)$ into $L_p(\Omega)$ and inequality (10) and the Hölder inequality imply that

$$|f(x)| \leq c\left(\int\limits_{\mathbb{B}_n(\xi,r_0)} |f|dx + \sum_{|\gamma|=l}\int\limits_\Omega \frac{|(D_w^\gamma f)(y)|}{|x-y|^{n-l}}dy\right) \leq$$
$$\leq c\left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}}\|f\|_{L_p(\Omega)} + \sum_{|\gamma|=l}\|D_w^\gamma f\|_{L_1(\Omega)}(\operatorname{diam}\Omega)^{l-n}\right) \leq$$
$$\leq c\left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}}\|f\|_{L_p(\Omega)} + (\operatorname{diam}\Omega)^{l-n}[m_n(\Omega)]^{1-\frac{1}{p}}\sum_{|\gamma|=l}\|D_w^\gamma f\|_{L_p(\Omega)}\right) \leq \tag{13}$$
$$\leq c\left([m_n(\mathbb{B}_n(\xi,r_0))]^{1-\frac{1}{p}} + (\operatorname{diam}\Omega)^{l-n}[m_n(\Omega)]^{1-\frac{1}{p}}\right)\|f\|_{W_p^{l,\lambda}(\Omega)},$$

for almost all $x \in \Omega$ and for all $f \in W_p^{l,\lambda}(\Omega)$. By Lemma 2, by the inequality (12) for case $l < n$ and by the inequality (13) for case $l \geq n$, we deduce the validity of statement (iii) in case $m = 0$.

Next we prove the statement (iii) in case $m > 0$. If $f \in W_p^{l,\lambda}(\Omega)$, then $D_w^\beta f \in W_p^{l-m,\lambda}(\Omega)$ for all $|\beta| \leq m$. Now by assumption, we have $(l-m)+\lambda > \frac{n}{p}$. Hence, case $m = 0$ with l replaced by $l-m$ implies that $W_p^{l-m,\lambda}(\Omega) \subseteq L_\infty(\Omega)$ and that there exists $c_1 > 0$ such that

$$\|g\|_{L_\infty(\Omega)} \leq c_1\|g\|_{W_p^{l-m,\lambda}(\Omega)} \quad \forall\, g \in W_p^{l-m,\lambda}(\Omega).$$

Hence, $D_w^\beta f \in L_\infty(\Omega)$ for all $\beta \in \mathbb{N}^n$ such that $|\beta| \leq m$ and

$$\|f\|_{W_\infty^m(\Omega)} \leq \sum_{|\beta|\leq m}\|D_w^\beta f\|_{L_\infty(\Omega)} \leq c_1\sum_{|\beta| m}\|D_w^\beta f\|_{W_p^{l-m,\lambda}(\Omega)} \leq$$
$$\leq c_1\sum_{|\beta|\leq m}\sum_{|\gamma|\leq l-m}\|D_w^{\gamma+\beta}f\|_{M_p^\lambda(\Omega)} \leq c_1\left(\sum_{|\beta|\leq m}\sum_{|\gamma|\leq l-m}1^{|\gamma+\beta|}\right)\|f\|_{W_p^{l,\lambda}(\Omega)}$$

for all $f \in W_p^{l,\lambda}(\Omega)$. Hence, the proof of statement (iii) is complete.

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Соболев-Морри кеңістігі үшін Соболевтің ішіне салу теоремасы

Мақалада Соболев-Морри кеңістіктері үшін Соболевтің ішіне салу теоремасы дәлелденді. Дәлелдеу Соболевтің интегралдық бейнелеуі мен В.И.Буренковтің, А.Гогатишвилидің, В.С.Гулиевтің, Р.Мустафаевтың жалпыланған Морри кеңістігіндегі Рисс потенциалдарына арналған соңғы нәтижелеріне негізделген. Жұмыста қарастырылып отырған жалпыланған Соболев-Морри кеңістігінің ішкі кеңістігі үшін ішіне салу теоремасын С.Капанато дәлелдеген болатын. Біздің ішіне салу теоремасын дәлелдеу әдісіміз С.Кампанатоның дәлелдеу жолынан мүлдем басқа.

В.И.Буренков, Н.А.Кыдырмина

Теорема вложения Соболева для пространств Соболева-Морри

В статье доказана теорема вложения Соболева для пространств Соболева-Морри. Доказательство основано на интегральном представлении Соболева и последних результатах В.И.Буренкова, А.Гогатишвили, В.С.Гулиева, Р.Ч. Мустафаева касательно потенциала Рисса в обобщенных пространствах Морри. Теорема вложения Соболева для пространств Соболева Морри была доказана впервые С.Кампанато для подпространства нашего пространства Соболева-Морри, которое представляет собой замыкание множества гладких функций в нашем пространстве Соболева-Морри. Методы доказательства, представленные в нашей работе, существенно отличаются от методов С.Кампанато.