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On the solution of boundary value problems by the spectral element method

In spectral element method approximate solution of the original differential operator is found in the form of a combination of the linearly independent system of orthogonal functions on the unit interval. Using the spectral decomposition for sufficiently smooth functions, one can obtain an exponential rate of convergence of the approximate solution to the exact solution and the approximation error will decrease exponentially as n grows. In the article the application of spectral element method to the solution of the boundary value problem for the Poisson equation is presented.

Key words: spectral element method, the Poisson equation, Dirichlet boundary conditions, Galerkin formulation, orthogonal functions, orthogonal polynomials, Legendre polynomials, Bubble functions, Lagrange polynomials.

For solving boundary value problems a spectral element method is used and we will demonstrate this numerical method on the Poisson equation with one variable. Let's consider the Poisson equation

$$\frac{d^2\tilde{u}}{dx^2} = -f(x);$$
$$\tilde{u}(0) = \tilde{u}(1) = 0.$$

We seek an approximate solution u from a finite-dimensional trial space X_0^n,

$$u \in X_0^n = span\{\varphi_1(x),...,\varphi_n(x)\};$$
$$\varphi_j(0) = \varphi_j(1) = 0.$$

We use the subscript on X_0^n to indicate that functions in this space satisfy the homogeneous Dirichlet boundary conditions. The trial solution has the form [1]

$$u(x) = \sum_{j=1}^{n} \varphi_j(x)\hat{u}_j,$$

where the $\varphi_j(x)$ are the basis functions and the $\hat{u}_j$ are the basis coefficients. We define the residual,

$$r(x,u) = r(x),$$

as

$$r(x,u) = f(x) + \frac{d^2\tilde{u}}{dx^2}.$$

In the weighted residual technics, we do not require $r = 0$. Rather, we insist that r be L^2-orthogonal to a set of functions v belonging to the test space X_0^n

$$\int_0^1 vrdx = 0;\ \forall v \in X_0^n.$$

Convergence is attained as we complete the approximation space, that is, as we let $n \to \infty$ for a reasonable set of $\varphi_j(x)$. Consider the Galerkin formulation, find $u \in X_0^n$ such that

$$-\int_0^1 v\frac{d^2u}{dx^2}dx = \int_0^1 vfdx;\ \forall v \in X_0^n.$$

It appears that u must be twice differentiable. However, if we integrate by parts, we can reduce the continuity requirements on u.

Let I denote the left-hand side of the preceding equation

$$I = -\int_0^1 v\frac{d^2u}{dx^2}dx = \int_0^1 \frac{dv}{dx}\cdot\frac{du}{dx}dx.$$

For a variety of technical reasons, it is generally a good idea to balance the continuity requirements of v and u, to the extent possible.

Using the integration-by-parts trick, we arrive at the weighted residual statement for u.

Find $u \in X_0^n$ such that

$$\int_0^1 \frac{dv}{dx} \cdot \frac{du}{dx} dx = \int_0^1 vf dx, \ \forall v \in X_0^n.$$

Convergence is attained by taking the limit $n \to \infty$ for an appropriate set of functions in X_0^n.

An essential property of the Galerkin formulation for the Poisson equation is that the solution is the best fit in the approximation space, with respect to the energy norm.

Specifically, we consider the bilinear form

$$a(v,u) = \int_0^1 \frac{dv}{dx} \cdot \frac{du}{dx} dx$$

and associated semi-norm

$$\|u\|_a^2 = a(u,u),$$

which is in fact a norm for all u satisfying the boundary conditions.

We define the sets by following formulas [2]

$$L_\Omega^2 = \left\{ v : \int_\Omega v^2 dx < \infty \right\};$$

$$H^1 = \left\{ v : v \in L_\Omega^2, \int_\Omega (v')^2 dx < \infty \right\};$$

$$H_0^1 = \left\{ v : v \in H^1, v|_{\partial\Omega} = 0 \right\}.$$

We can now easily generate our discrete system that allows us to compute the set of basis coefficients. Let

$$\underline{u} = (u_1, ..., u_n)^T;$$
$$\underline{v} = (v_1, ..., v_n)^T,$$

then

$$I = \int_\Omega v'u' dx = \int_\Omega \left(\sum_{i=1}^n \varphi_i'(x) v_i \right) \left(\sum_{j=1}^n \varphi_j'(x) u_j \right) dx =$$

$$= \sum_{i=1}^n \sum_{j=1}^n v_i \left(\int_\Omega \varphi_j'(x) \varphi_i'(x) dx \right) u_j = \sum_{i=1}^n \sum_{j=1}^n v_i A_{ij} u_j = \underline{v}^T A \underline{u};$$

with the (global) stiffness matrix A, given by

$$A_{ij} = \int_\Omega \varphi_j'(x) \varphi_i'(x) dx.$$

We proceed in similar way with the right-hand side. Assuming

$$f(x) = \sum_{j=1}^n \varphi_j(x) f_j$$

then

$$I = \int_\Omega vf dx = \int_\Omega \left(\sum_{i=1}^n \varphi_i(x) v_i \right) \left(\sum_{j=1}^n \varphi_j(x) f_j \right) dx =$$

$$= \sum_{i=1}^n \sum_{j=1}^n v_i \left(\int_\Omega \varphi_j(x) \varphi_i(x) dx \right) f_j = \sum_{i=1}^n \sum_{j=1}^n v_i B_{ij} f_j = \underline{v}^T B \underline{f}$$

with the (global) mass matrix B, given by

$$B_{ij} = \int_\Omega \varphi_i(x) \varphi_j(x) dx.$$

Combining the result, we have

$$I = \underline{v}^T A\underline{u} = \underline{v}^T B\underline{f},$$

which implies

$$A\underline{u} = B\underline{f}.$$

Since A is symmetric positive definite, this system is solvable.

Now we get specific and choose the space X_0^n and associated basis $\{\varphi_i\}$. For the spectral element method in R^1, we choose X^n to be the space of piecewise polynomials of degree N on each element Ω^e, $e = 1, 2, ..., E$. Then we typically use nodal bases on the Gauss-Lobatto-Legendre (GLL) quadrature points. However, we often map back and forth between GLL-based nodal values and Legendre or bubble function modal bases with minimal information loss. Examples of stable bases are

1) Orthogonal polynomials;

2) Legendre polynomials: $L_k(x)$;

3) Bubble functions: $\varphi_k(x) = L_{k+1}(x) - L_{k-1}(x)$;

4) Lagrange (nodal) polynomials based on Gauss quadrature points (e.g. Gauss-Legendre, Gauss-Chebyshev, Gauss-Lobato-Legendre) [3].

The GLL points are the zeros of $(1-x^2)L_N'(x)$. The Legendre polynomials are orthogonal with respect to the L^2 inner product

$$\int_{-1}^{1} L_i(x)L_j(x)dx = \delta_{ij};\ L_i(x) \in P_i.$$

They can be efficiently and stably computed using the 3-term recurrence

$$L_0(x) = 1;\ L_1(x) = x;$$

$$L_k(x) = \frac{1}{k}\left[(2k-1)xL_{k-1}(x) - (k-1)L_{k-2}(x)\right].$$

On Ω^e we have

$$u(x)\big|_{\Omega^e} = \sum_{j=0}^{N} u_j^e h_j(r),\ r \in \hat{\Omega} = [-1;1];$$

$$x^e(r) = x_{\Omega^e} = \tilde{x}^{e-1} + \frac{\tilde{x}^e - \tilde{x}^{e-1}}{2}(r+1);$$

$$h_j(r) \in P_N(r);$$

$$h_j(\xi_i) = \delta_{ij},\ i, j \in [0, ..., N]^2;$$

$$\xi_{ij} \in [-1;1],$$

where ξ_{ij} — GLL quadrature points.

Return to the weighted residual techniques and consider $v, u \in X^n$. Let

$$I = \int_{\Omega} \frac{dv}{dx} \cdot \frac{du}{dx} dx = \sum_{e=1}^{E} \int_{\Omega^e} \frac{dv}{dx}\frac{du}{dx} dx = \sum_{e=1}^{E} I^e,$$

where E — number of spectral elements. With

$$L^e = \tilde{x}^e - \tilde{x}^{e-1}$$

we have

$$I^e = \int_{\Omega^e} \frac{dv}{dx}\frac{du}{dx} dx = \frac{2}{L^e}\int_{-1}^{1} \frac{dv}{dr}\frac{du}{dr} dr.$$

Using

$$u\big|_{\Omega^e} = u^e(r) = \sum_{j=0}^{N} u_j^e h_j(r),\ v\big|_{\Omega^e} = v^e(r) = \sum_{i=0}^{N} v_i^e h_i(r),$$

we can readily compute the derivatives

$$\frac{du^e}{dr} = \sum_{j=0}^{N} u_j^e \frac{dh_j}{dr};$$

$$\frac{dv^e}{dr} = \sum_{i=0}^{N} v_i^e \frac{dh_i}{dr}.$$

Inserting the local basis into the local integral yields

$$I^e = \int_{\Omega^e} \frac{dv}{dx}\frac{du}{dx}dx = \frac{2}{L^e}\int_{-1}^{1} \left(\sum_{i=0}^{N} v_i^e \frac{dh_i}{dr}\right)\left(\sum_{j=0}^{N} u_j^e \frac{dh_j}{dr}\right)dr =$$

$$= \sum_{i=0}^{N}\sum_{j=0}^{N} v_i^e \left(\frac{2}{L^e}\int_{-1}^{1} \frac{dh_i}{dr}\frac{dh_j}{dr}dr\right)u_j^e = \sum_{i=0}^{N}\sum_{j=0}^{N} v_i^e A_{ij}^e u_j^e,$$

where

$$A_{ij}^e = \frac{2}{L^e}\hat{A}_{ij}$$

and

$$\hat{A}_{ij} = \int_{-1}^{1} \frac{dh_i}{dr}\frac{dh_j}{dr}dr.$$

If we define

$$\underline{u}^e = \left(u_0^e, u_1^e, ..., u_N^e\right)^T$$

and similarly for $\underline{v}^e$, we have

$$I^e = \sum_{i=0}^{N}\sum_{j=0}^{N} v_i^e A_{ij}^e u_j^e = \left(\underline{v}^e\right)^T A^e \underline{u}^e.$$

Let

$$u_L = \begin{pmatrix} \underline{u}^1 \\ \underline{u}^2 \\ . \\ \underline{u}^e \\ . \\ \underline{u}^E \end{pmatrix}, \; A_L \begin{pmatrix} A^1 & & & & & \\ & A^2 & & & & \\ & & . & & & \\ & & & A^e & & \\ & & & & . & \\ & & & & & A^E \end{pmatrix}$$

Define $\underline{v}$ similarly using $\underline{v}^e$.

The left-hand side of our weighted residual statement reads

$$I = \sum_{e=1}^{E} I^e = \sum_{e=1}^{E} (\underline{v}^e)^T A^e \underline{u}^e = \begin{pmatrix} \underline{v}^1 \\ \underline{v}^2 \\ . \\ \underline{v}^e \\ . \\ \underline{v}^E \end{pmatrix}^T \begin{pmatrix} A^1 & & & & & \\ & A^2 & & & & \\ & & . & & & \\ & & & A^e & & \\ & & & & . & \\ & & & & & A^E \end{pmatrix} \begin{pmatrix} \underline{u}^1 \\ \underline{u}^2 \\ . \\ \underline{u}^e \\ . \\ \underline{u}^E \end{pmatrix} = \underline{v}_L^T A_L \underline{u}_L^T.$$

A_L has precisely the same structure in higher space dimensions. In 2D and 3D problems, we work exclusively with A^e and $\underline{u}^e$, $e = 1, 2, ..., E$. In fact, we never even form A^e, but just compute the action of A^e on u^e. It is clear that $A^e \underline{u}^e$ $e = 1, 2, ..., E$ can be computed independently, in parallel. $\underline{u}^e$ is simply the set of local basis coefficients on Ω^e.

This is the approach used in the method of spectral elements. Therefore, the spectral element method is based on the weighted residual technique, which is essentially a method of undetermined coefficients.

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Спектралды элементтер әдісімен шеттік есептерді шешу туралы

Спектралды элементтер әдісінде берілген дифференциалды оператордың жуықталған шешімі ортогоналды функциялардың бірлік кесіндідегі комбинациясында болады. Жеткілікті тегіс функциялар үшін спектралды жүктелуін қолданып, жуықталған шешімнің экспоненциалды жинақталу жылдамдығын нақты алуға болады. Аппроксимация қателігі экспоненциалды *n* кеміп отырады. Мақалада Пуассон теңдеуінің шеттік шешімдеріне спектралды элементтер әдісіне қосымша келтірілген.

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О решении краевых задач методом спектральных элементов

В методе спектральных элементов приближенное решение исходного дифференциального оператора находится в виде комбинации по линейно независимой системе ортогональных функций на единичном отрезке. Используя спектральное разложение для достаточно гладких функций, можно получить экспоненциальную скорость сходимости приближенного решения к точному, причём ошибка аппроксимации будет уменьшаться экспоненциально с ростом *n*. В статье представлено приложение метода спектральных элементов к решению краевой задачи для уравнения Пуассона.

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