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Determination of the Shear Failure Envelope by Adjusting with the Statistical Method of Error in Variables through the Relationship between the Principal Stresses

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Abstract

This work is based on developing the parametric representation of the failure envelope to Mohr's circles in intact rock as a function of the principal stresses. In the proposed method, the stresses are adjusted using the statistical method EIV (error-in-variables), which does not make artificial distinctions between the independent and dependent variables. To accomplish the transformation from the principal stress plane to the Mohr plane, Balmer's method was used by applying computational algebraic analysis. To illustrate and verify the application of this proposed methodology, the well-documented dataset collected from previous work by Pincus and Sheorey is used. To test the improvement provided by this method, the calculated objective function (likelihood of erroneous decision) have been compared with the parametric equation representation obtained using various least squares methods. It was found that our proposed methodology, and the transformation method of Balmer, has two advantages: i) It simplifies the process of creating a failure envelope for practical applications, and ii) It minimizes the likelihood of erroneous judgment during applications (*i.e.* indicating failure in a stable state or vice versa.

Keywords: EIV; failure envelope; objective function; principal stress plane; transformation.

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Determinación de la Envolvente de Falla por Corte mediante el Ajuste con el Método Estadístico de Error en Variables a través de la Relación entre las Tensiones Principales

Resumen

El presente trabajo se fundamentó en el desarrollo de la representación paramétrica de la envolvente de falla a los círculos de Mohr en roca intacta, en función de las tensiones principales. En el método propuesto, las tensiones se ajustan utilizando el método estadístico EIV (error en las variables), el cual no hace distinciones artificiales entre las variables independientes y dependientes. Para complementar la transformación desde el plano de esfuerzos principales al plano de Mohr, se utilizó el método de Balmer mediante la aplicación del análisis algebraico computacional. Para ilustrar y verificar la aplicación de la metodología propuesta, se usó el bien documentado conjunto de datos coleccionados de trabajos previos de Pincus y Sheorey. Para probar las mejoras provistas por este método, se comparó la función objetivo calculada (minimizar la probabilidad de una decisión errónea) con la representación de la ecuación paramétrica obtenida, usando varios métodos de mínimos cuadrados. Se encontró que la metodología propuesta y la transformación del método de Balmer, tienen dos ventajas: i) simplifica el proceso de crear una envolvente de falla para aplicaciones prácticas, y ii) minimiza la posibilidad de un juicio erróneo durante las aplicaciones (como es indicar falla en un estado estable o viceversa).

Palabras clave: EIV; envolvente de falla; función objetivo; plano de esfuerzos principales; transformación.

Determinação do Envelope de Falha por Cisalhamento através do Ajuste com o Método Estatístico de Erro nas Variáveis pela Relação entre Tensões Principais

Resumo

Este estudo concentrou-se no desenvolvimento da representação paramétrica do envelope de falha em círculos de Mohr em rocha íntegra com base nas tensões principais. O método proposto ajusta as tensões usando o método estatístico de Erro nas Variáveis (EIV), que não estabelece distinções artificiais entre variáveis independentes e dependentes. Para complementar a transformação do plano de tensões principais para o plano de Mohr, foi aplicado o método de Balmer por meio de análise algébrica computacional. Para ilustrar e verificar a aplicação da metodologia proposta, foi utilizado o conjunto de dados extensivamente documentado coletado de trabalhos anteriores de Pincus e Sheorey. Para testar as melhorias fornecidas por este método, a função objetivo calculada (minimizando a probabilidade de uma decisão incorreta) foi comparada com a representação da equação paramétrica obtida, utilizando vários métodos de mínimos quadrados. Verificou-se que a metodologia proposta e a transformação pelo método de Balmer oferecem duas vantagens: i) simplificam o processo de criação de um envelope de falha para aplicações práticas e ii) minimizam a possibilidade de julgamento incorreto durante as aplicações (como indicar falha em um estado estável ou vice-versa).

Palavras-chave: EIV; envelope de falha; função objetivo; plano de tensões principais; transformação.

Introduction

The parametric representation of the rock-strength failure envelope is used to characterize the mechanical behavior of a rock (Coulomb, 1776). When dealing with laboratory data, the physical limitation of the experimental setup mostly forces us to represent the failure criteria in the stress-state plane, neglecting the intermediate principal-stress influence (Hoek and Brown, 1980). Because a failure envelope represents the boundary between stable and unstable zones of the stress state, a closed-form representation of this envelope becomes of great importance in practical applications (Balmer, 1952). The delimitation of this boundary could be the key for modeling various near-wellbore region phenomena like subsidence, borehole stability and sanding propensity (Zambrano-Mendoza *et al.*, 2003). For instance, if the effect of the principal intermediate stress is considered having not to influence the rock strength (which is not always true) a failure criterion can be expressed in terms of the major (σ_1) and minor (σ_3) principal stresses, so that the criterion can be represented as (Coulomb, 1776; Balmer, 1952):

$$g(\sigma_1, \sigma_3) = 0 \tag{1}$$

Zambrano Mendoza *et al.* (2003), stated that because of the experimental errors and stochastic variations in the rock itself, there is no guarantee of creating a deterministic curve to represent the failure envelope. Rather, the problem consists of selecting a suitable algebraic form of Equation 1 and determining their unknown parameters from a suitable criterion related to minimizing possible false judgment of failure in future applications. The algebraic equations which have a relationship within the failure principal stress are Mohr-Coulomb for linear and parabolic form and Hoek & Brown to the quadratic case. In order to compare we also introduced the Ucar's method (Ucar, 2019).

Several empirical failure criteria have arisen over the past five decades in the attempt to simulate the triaxial behavior of in-situ (intact) rock specimens. The majority of those equations were proposed for a few particular rock types having in each case a limited number of data (Hobbs, 1964). Hoek and Brown (1980), developed a new failure equation and fitted it comprehensively to different rock types. Similarly to most empirical failure criteria, it was formulated in terms of major (σ_1), minor (σ_3) and independent of the intermediate principal stress (σ_2). Some of the empirical failure envelope equations were assumed only in the compression and did not necessarily exist in the tensile quadrant. This implies a limitation because a failure criterion should exist both in the tensile and compressive region to be comprehensive enough (Mostyn and Douglas, 2002). Sheorey (1997) provides a list of the most relevant empirical failure equations.

The least-squares method has been widely adopted to fit the failure envelope directly from the experimental data. Mostyn and Douglas (2002) presented different results fitting the different forms of the Hoek and Brown (1980) criterion. Recently, some variants of the least square method have been defined, such as the orthogonal regression method, and comparisons have been established with the traditional method to calculate the sum of the square of perpendicular distances (Keles and Altun, 2016; Recio-Lopez, 2021). This represents a significant improvement, but the concept of dependent and independent variables is still used. The non-linear character for the envelope in the Mohr-Coulomb failure criterion does not have a closed-form solution, which is why methods such as the Simplex Reflection Technique, which have also been used for curve fitting to evaluate the parameters of the envelope considering also the failure criterion for intact rocks of Hoek and Brown (Shah, 1992). The Simplex Regression Technique method consists of the minimization of a function of *n* parameters developed by Nelder and Mead (Nelder and Mead, 1965). A weakness of it is that it requires a large number of evaluation functions to locate a solution.

However, Zambrano Mendoza *et al.* (2003) emphasize that the statistical method of least-squares and some others derived from its, requires the distinction between independent and dependent variables, requiring that the former be known exactly. In that work, we proposed a new and efficient approach based on the statistical method (EIV) to fit the failure envelope in the Mohr plane. Also the EIV method to fit the failure envelope based on large experimental data have been further referred by some others researchers like (Jiefei and Puhui, 2018; Jiefei *et al.*, 2020). The quadratic form is frequently chosen as the non-linear failure function for isotropic materials due to its relatively good curve-fitting (Jiefei *et al.*, 2019).

Our earlier review of the use of the statistical method EIV found that Deming (1943) was the first to formulate the general EIV problem. See also (van Huffel and Lemmerling, 2013). His primary concern was how to obtain approximate solutions, appropriate for hand calculations. Other researchers (York, 1966; Willianson, 1968; O'Neil *et al*., 1969; Southwell, 1969), proposed exact solutions, extending the solutions to straight lines or higher-order polynomials. Britt and Luecke (1973), later suggest a general algorithm based on the concept of Lagrange multipliers. Whereas Peneloux *et al*. (1976), and Reilly and Patino-Leal (1981) provided computational improvements, Schwetlick and Tiller (1985), and Valkó and Vajda (1987) separated the parameter estimation and data reconciliation steps. Liebman and Edgar (1988) and Liebman *et al*. (1990), investigated the use of nonlinear parameter-estimation (NLP, not only in parameter estimation, but also in the data -reconciliation step. To avoid being trapped in a local minimum, Esposito and Floudas (1998) applied global optimization. Anand and Kumar (2015), present Multi-objective Optimization Techniques (MOT). Finally, Kumar and Kumar (2011), developed a parametric optimization of rock failure criterion using error-in-variables approach. We described the EIV method of curve fitting and developed a variant suitable for fitting failure envelopes in the Mohr plane according to Balmer (1952), and finally apply it to a well-documented set of data. In addition to obtaining the parametric representation of the failure envelope in the principal stress plane, a Mohr circle can be generated on the normal stress (σ), shear stress (τ) plane for every pair of stress state (σ_1, σ_3) producing a failure during the triaxial failure experiments according to (Coulomb, 1776):

$$\left(\sigma - \frac{\sigma_1 + \sigma_3}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 \tag{2}$$

Equation 2 represents family circles of variable radius through the minor principal stress. From this a series of failure circles (known as involutes). Sheorey (1997) stated that there are three way to obtain a Mohr envelope: i) analytical derivation using the original criterion the parametric function $\sigma_1 = f(\sigma_3)$, is not always possible, ii) curve fitting on the envelope resulting from the original equation $\sigma_1 = f(\sigma_3)$, or iii) direct regression of the σ, τ values obtained from $\sigma_1 = f(\sigma_3)$ using Balmer's equations, which are given by (Balmer, 1952):

$$\sigma = \sigma_3 + \frac{\sigma_1 - \sigma_3}{\frac{d\sigma_1}{d\sigma_3} + 1} \tag{3}$$

And

$$\tau = \frac{\sigma_1 - \sigma_3}{\frac{d\sigma_1}{d\sigma_3} + 1}\sqrt{\frac{d\sigma_1}{d\sigma_3}} \tag{4}$$

Where:

$$\frac{d\sigma_1}{d\sigma_3} = \frac{dg(\sigma_3)}{d\sigma_3} \qquad (5)$$

Also defining the major principal stress derivative $(d\sigma_1)$, minor principal stress derivative $(d\sigma_3)$, the derivative of parametric function of g $(dg(\sigma_3))$ and the slope ($^{d\sigma_1}\!/_{d\sigma_3}$). Using Balmer's solution of the Mohr's envelope, the failure envelope can be obtained in the Mohr plane via transformation from the principal-stress plane. Then the equation of the failure envelope in the Mohr plane can be represented by the parametric function (Coulomb, 1776):

$$f(\sigma,\tau) = 0 \qquad (6)$$

In this work, we propose to use the statistical method (EIV) to obtain the parametric representation of the failure envelope in the principal stress plane and to map out the resulting parametric representation into the Mohr plane. The earlier statistical method of error-in-variables has been previously defined elsewhere from York (1966) to Esposito and Floudas (1998) we presented its application in fitting the failure envelope in the Mohr plane (Zambrano Mendoza *et al.*, 2003). In the following; we describe first the application of the EIV method of curve-fitting in the principal stress plane to a well-documented set of data, then the transformation of the resulting envelope into the Mohr plane.

Materials and Methods

Application of EIV method used to fit the failure envelope in the principal stress plane

In the principal stress plane, the model can be written in its implicit form as:

$$g(\hat{\sigma}_3, \hat{\sigma}_1, \underline{\theta'}) = 0 \qquad (7)$$

Where:

$\hat{\sigma}_3$ = reconciled minor principal stress.

$\hat{\sigma}_1$ = reconciled major principal stress.

$\underline{\theta'}$ = vector of unknown parameters in the principal-stress plane.

Our goal is to find the optimum parameters at which the sum of necessary corrections squared, is minimum.

$$J = \sum_{i=1}^{n} d_{p_i}^2 \qquad (8)$$

Where:

J = sum of square distance.

d_{pi} = geometric distance in EIV method.

Where for each pair of measurements, the squared distance is given by:

$$d_{p_i}^2 = \left(\hat{\sigma}_{3i} - \sigma_{3i}\right)^2 + \left(\hat{\sigma}_{1i} - \sigma_{1i}\right)^2 \tag{9}$$

Where:

σ_{1i} = *i-th* measured major principal stress.

σ_{3i} = *i-th* measured minor principal stress.

$\hat{\sigma}_{1i}$ = *i-th* reconciled major principal stress.

$\hat{\sigma}_{3i}$ = *i-th* reconciled minor principal stress.

Thus minimizing Equation 8 subject to constraint Equation 7 constitutes the EIV formulation. The simple form of the objective function Equation 8 allows us to use the following algorithm.

EIV algorithm

In Figure 1 the measured points have been reconciled by using the EIV approach. From its, the tangent vector of the envelope at $(\hat{\sigma}_3, \hat{\sigma}_1)$ **a** is denoted by:

$$\mathbf{a} = -\frac{\partial g\left(\hat{\sigma}_{3i}\hat{\sigma}_{1i}, \underline{\theta'}\right)}{\partial \hat{\sigma}_3}\mathbf{i} + \frac{\partial g\left(\hat{\sigma}_{3i}\hat{\sigma}_{1i}, \underline{\theta'}\right)}{\partial \hat{\sigma}_1}\mathbf{j} \tag{10}$$

While **b** is the vector from the measured *i*-th point to the envelope:

$$\mathbf{b} = \left(\hat{\sigma}_{3i} - \sigma_{3i}\right)\mathbf{i} + \left(\hat{\sigma}_{1i} - \sigma_{1i}\right)\mathbf{j} \tag{11}$$

In the EIV method vectors **a** and **b** must be orthogonal each other, so that **a•b** = 0:

$$-\left(\hat{\sigma}_{3i} - \sigma_{3i}\right)\frac{\partial g\left(\hat{\sigma}_{3i}\hat{\sigma}_{1i}, \underline{\theta'}\right)}{\partial \hat{\sigma}_3} + \left(\hat{\sigma}_{1i} - \sigma_{1i}\right)_i \frac{\partial g\left(\hat{\sigma}_{3i}\hat{\sigma}_{1i}, \underline{\theta'}\right)}{\partial \hat{\sigma}_1} = 0 \tag{12}$$

Solving simultaneously the system of Equations 7 and 12 we obtain the reconciled pair of minor and major principal stresses $\left(\hat{\sigma}_3, \hat{\sigma}_1\right)$ at any parameter vector $\underline{\theta'}$, evaluating the objective function. Linear and parabolic functions can be considered to represent the failure envelope at the principal stress plane. Other algebraic forms may not be needed to fit the failure envelope in the principal stress plane. In this work only parabolic model is show.

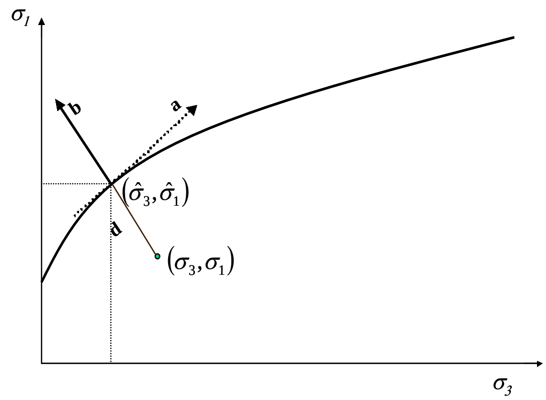

Figure 1. Graphic representation of the error-in-variables (EIV) algorithm in the principal stress plane.

Application to selected model

Parabolic envelope equation

For the parabolic approximation of the envelope in terms of the reconciled minor and major principal stresses $(\hat{\sigma}_3, \hat{\sigma}_1)$ at any parameter vector $\underline{\theta}'$, Equation 7 can be written as:

$$g(\hat{\sigma}_{3i}, \hat{\sigma}_{1i}, \theta') = \theta'_0 + \theta'_1 \hat{\sigma}_{3i} - \hat{\sigma}_{1i}^2 = 0 \tag{13}$$

Where, θ'_0 and θ'_1 are the unknowns parameters. The appropriate form of Equation 12 is:

$$\theta'_1(\hat{\sigma}_{1i} - \sigma_{1i}) - 2\hat{\sigma}_{1i}(\hat{\sigma}_{3i} - \sigma_{3i}) = 0 \tag{14}$$

Solving the system of Equations 13 and 14, to obtain the squared distance by Equation 9 to arrive at an unconstrained solution involving two unknown parameters, θ'_0 and θ'_1, when substituting Equation 9 into the objective function (Equation 7) It is important to emphasis that those parameters further up are related to the uniaxial compressive strength (σ_c) and tensile strength (σ_t). Due to the cumbersome nature of *i-th* reconciled minor principal stress ($\hat{\sigma}_{3i}$) roots, the squared distance (d_{pi}^2) is not shown for the parabolic envelope. Once we derived the parametric representation of the failure envelope using the EIV method through one of the algebraic forms already described above, we can generalize Balmer's solution of the Mohr' envelope to map out the resulting envelope into the Mohr plane.

Determination of the envelope of a family of plane curves applying the Balmer´s method and considering that the failure criterion is represented by a parabolic function

Using the resulting EIV parametric representation of the failure envelope in the principal stress plane (Equation 7) we can transform the algebraic solution to a parametric representation in the Mohr plane. To accomplish this task, we use Balmer's method, based on the principal stress components. Introducing the reconciled value of $\left(\hat{\sigma}_3, \hat{\sigma}_1\right)$ obtained from the EIV method in the principal stress plane, Equations 3, 4 and 5 can be expressed as:

$$\hat{\sigma} = \hat{\sigma}_3 + \frac{\hat{\sigma}_1 - \hat{\sigma}_3}{\dfrac{d\hat{\sigma}_1}{d\hat{\sigma}_3} + 1} \tag{15}$$

$$\hat{\tau} = \frac{\hat{\sigma}_1 - \hat{\sigma}_3}{\dfrac{d\hat{\sigma}_1}{d\hat{\sigma}_3} + 1}\sqrt{\frac{d\hat{\sigma}_1}{d\hat{\sigma}_3}} \tag{16}$$

$$\frac{d\hat{\sigma}_1}{d\hat{\sigma}_3} = \frac{dg\left(\hat{\sigma}_3, \hat{\sigma}_1, \underline{\theta'}\right)}{d\hat{\sigma}_3} \tag{17}$$

Also defining reconciled normal stress ($\hat{\sigma}$) and reconciled shear stress ($\hat{\tau}$). Solving simultaneously Equations 7, 15 and 16, considering (Equation 17) a parametric solution in $\left(\sigma, \tau\right)$ stress plane is obtained for the failure envelope in terms of reconciled normal stress $\left(\underline{\underline{\hat{\sigma}}}\right)$reconciled shear stress $\left(\underline{\underline{\hat{\tau}}}\right)$ vector of unknown parameters in the principal-stress plane $\left(\underline{\theta'}\right)$:

:

$$f\left(\underline{\underline{\hat{\sigma}}}, \underline{\underline{\hat{\tau}}}, \underline{\theta'}\right) = 0 \tag{18}$$

Which, represent the failure envelope for the Mohr circles, such as envelope is tangent to all the involutes. It is important to notice that a closed-form solution may or may not be achieved analytically. In this work we consider cases, when a closed form solution can be found using Computer Algebra software. To illustrate the transformation of the failure envelope from the principal stress plane to the Mohr plane, we present how the parametric representation of the parabolic model obtained in the principal stress plane is mapped out in the Mohr plane.

Transformation of selected model

Let us assume that we have obtained the EIV parametric representation of the failure envelope in the principal stress plane. That is, we have determined the optimum parameters of the parabolic model, Equation 13. To transform into the Mohr plane, we derive Equations 13 to 17 in terms of the reconciled minor $\left(\hat{\sigma}_3\right)$ and major $\left(\hat{\sigma}_1\right)$ principal stresses at any parameter vector $\left(\theta'\right)$ as:

$$\frac{d\hat{\sigma}_1}{d\hat{\sigma}_3} = \frac{dg\left(\hat{\sigma}_{3i}, \hat{\sigma}_{1i}, \theta'\right)}{d\hat{\sigma}_3} = \frac{\theta_1'}{2\sqrt{\theta_0' + \theta_1'\hat{\sigma}_{3i}}} \tag{19}$$

In this case, Equations 19 and 20 can be expressed as:

$$\hat{\sigma}_i = \hat{\sigma}_{3i} + \frac{\hat{\sigma}_{1i} - \hat{\sigma}_{3i}}{\dfrac{\theta_1'}{2\sqrt{\theta_0' + \theta_1'\hat{\sigma}_{3i}}} + 1} \tag{20}$$

And

$$\hat{\tau}_i = \frac{\hat{\sigma}_{1i} - \hat{\sigma}_{3i}}{\dfrac{\theta_1'}{2\sqrt{\theta_0' + \theta_1'\hat{\sigma}_{3i}}} + 1}\sqrt{\frac{\theta_1'}{2\sqrt{\theta_0' + \theta_1'\hat{\sigma}_{3i}}}} \tag{21}$$

Obtaining the reconciled normal stress $(\hat{\sigma}_i)$ and reconciled shear stress $(\hat{\tau}_i)$ of *i-th* Mohr circle. Solving Equations 13, 20 and 21 simultaneously, the resulting parametric solution in the (σ, τ) stress plane is expressed of reconciled normal stress $(\underline{\underline{\hat{\sigma}}})$ reconciled shear stress $(\underline{\underline{\hat{\tau}}})$ vector of unknown parameters in the principal-stress plane $(\underline{\theta'})$ as:

$$f\left(\underline{\underline{\hat{\sigma}_i}}, \underline{\underline{\hat{\tau}_i}}, \underline{\theta'}\right) = -\frac{2\hat{\sigma}_i^{\,2}}{3} + \frac{2\hat{\sigma}_i}{27\theta_1'} - \frac{2\hat{\sigma}_i\theta_0'}{3\theta_1'} + \frac{2\left(\hat{\sigma}_i^{\,2} + 3\theta_0' + 3\hat{\sigma}_i\theta_1'\right)^{3/2}}{27\theta_1'} - \tau_i^2 \tag{22}$$

Since the slope $\frac{d\hat{\sigma}_1}{d\hat{\sigma}_3}$ given by Equation 19 must exist, the square root term must be greater than or equal to zero. In addition, because the slope depends on the minor principal stress (σ_3) it should increase when the values of such variable increase; thus, the slope must be positive. This implies that such generate envelope can describe the failure criteria for the brittle region of the tested rock. Where we assumed, that the quantity under the square root is always positive, that is automatically satisfied, for example, in the brittle region.

Results and Discussion

To illustrate and verify the applicability of the EIV method to fit the failure envelope in the principal stress plane, the results of a previous ASTM (American Society for Testing and Materials) interlaboratory study (Pincus, 1993; 1994; 1996), were processed assuming linear and parabolic envelope models. In this ASTM interlaboratory testing study, the triaxial compressive strength of intact, uniformly oriented cylindrical specimens of Berea sandstone, were obtained. The goal was to obtain the failure envelopes for this rock in the principal stress plane, using all the available information provided by the various laboratories and then compare the failure envelopes to the ones obtained using least squares. For Berea sandstone, 107 and 147 (including tensile strength) pairs of axial and lateral stresses measurement (Pincus, 1993; 1994; 1996), were available. The optimal parameters for the parabolic model are θ_0'= 3291.76 MPa^2 and θ_1'= 1133.43 MPa, θ_0'= 3751.69 MPa^2 and θ_1'=1116.84 MPa for 107 and 147 pairs of Berea sandstone respectively.

The parameters describe the failure envelope located nearest to the 107 and 147 pairs of axial and lateral stress measurements obtained from data measured in various laboratories under various confining pressures. The standard deviation is the sum of the squared distances between the failure envelope and each pair of measurements (objective function values); the standard deviations for the parabolic model are 3.76 and 3.23 % using EIV while we get 12.93 and 12.68 % from LS (Least Squared) for 107 and 147 pairs of Berea sandstone respectively. For comparison, we calculated the standard deviation using least squares. The EIV model provides smaller "sum of squared distances" for the rock and all models. In addition, we compared our proposed method with the well-establishes method of Hoek and Brown (1980a; 1980b), and Balmer (1952), using a set of sandstone data present by Sheorey (1997) for comparison. Balmer (1952), original equation has been expressed in dimensionless form by Sheorey (1997) as follow:

$$\hat{\sigma}_1 = \hat{\sigma}_c \left(1 + \frac{\hat{\sigma}_3}{|\hat{\sigma}_t|} \right)^{b'} \tag{23}$$

And Hoek and Brown equation also could be rewrite as:

$$\hat{\sigma}_1 = \hat{\sigma}_3 + \hat{\sigma}_c \left(m \frac{\hat{\sigma}_3}{\hat{\sigma}_c} + s \right)^{1/2} \tag{24}$$

Also defining reconciled uniaxial compressive strength $(\hat{\sigma}_c)$, reconciled uniaxial tensile strength $(\hat{\sigma}_t)$, Balmers's method constant (b'), Hoek & Brown's method constant (*m*), Hoek & Brown's method constant (*s*= 1 intact rock) (*s*). The selected data contained at least 5 pairs of measurements and reported at least a tensile strength measurement. Table 1 shows the results. We note that even if the number of parameter (2) for the parabolic model using the EIV method is lower than the ones used by the Hoek & Brown and Balmer methods, and that the Balmer method look to have the best fitting, the resulting standard deviation when using EIV is too close enough than the one obtained by either one of those methods. Even better in most cases than Hoek and Brown equation. Highlighting in bolds letter the best standard deviation for each sample set in Table 1. The standard deviation, *S.Dev*, is defined by:

$$S.Dev = \sqrt{\frac{J}{n-2}} \tag{25}$$

For a failure criterion $\sigma_1 = g(\sigma_3)$, $(\sigma_{3i}, \sigma_{1i})$ being the *i*-th data pair and *n* the number of data pairs (Sheorey, 1997), and J the sum of square distance Equation 8. As an example for the parabolic model the uniaxial compressive strength (σ_c) and tensile strength (σ_t) can be obtain by the relation:

$$\hat{\sigma}_3 = 0 \Rightarrow \hat{\sigma}_1 = \hat{\sigma}_c = \sqrt{\theta_0'} \tag{26}$$

And

$$\hat{\sigma}_1 = 0 \Rightarrow \hat{\sigma}_3 = \hat{\sigma}_t = -\frac{\theta_0'}{\theta_1'} \tag{27}$$

Using sample number 32 from Sheorey (1997) and considering that the determined parameters for the parabolic model are θ_0'= 15285.8 MPa^2 and θ_1'= 1494.91 MPa then $\hat{\sigma}_c = 123.6358$ MPa and $\hat{\sigma}_t = -10.2252$ MPa, which are closer to the measured data $\hat{\sigma}_c = 125.4$ MPa and $\hat{\sigma}_t = -10.9$ MPa. Table 1 shown those value for each one of the samples sets considered. Also using the above mentioned methods the reconciled major principal stress $(\hat{\sigma}_1)$ have been calculated for comparison and show in Table 2. Values calculated from the three methods look to be close to the measured data. However, if we calculated for comparison the Hoek & Brown's parameter (*m*), by the given relationship *m*= |*σc/σt*| we obtained for the sample set No 32 the value of *m*= |123.6358/(-10.2252)|=12.0912.

If this value is compare with the Hoek & Brown value that is *m*= 7.017, we can say that our results is more realistic for sandstone rocks (*m* is approximately equal to 15) than the Hoek & Brown one. Further, we considered Ucar's (Ucar, 2019) equation for comparison. In that work the following equation was derived. Rewritten that equation in reconciled variables we get:

$$\left(\frac{\hat{\sigma}_1}{\hat{\sigma}_c}\right) = k_1\left(\frac{\hat{\sigma}_3}{\hat{\sigma}_c} - \xi\right) + k_2\left(\frac{\hat{\sigma}_3}{\hat{\sigma}_c} - \xi\right)^{1/2} \tag{28}$$

In this equation $\xi = \left(\frac{\hat{\sigma}_t}{\hat{\sigma}_c}\right)$ and k_1 and k_2 are:

$$k_1 = \frac{-(1+|\xi|) + \sqrt{(1+7|\xi|)(1-|\xi|)}}{2|\xi|} \tag{29}$$

$$k_2 = \frac{[1 - k_1(-\xi)]}{\sqrt{-\xi}} \tag{30}$$

Where: ξ , k_1 and k_2 are defining as Ucar' method constants. As an example we substituted values of sample No 32 (Sheorey, 1997), k_1= 0.719, k_2= 3.180 and $\xi =$ -0.087, while using our values we get k_1= 0.730, k_2= 3.27 and $\xi =$ -0.083, which are close enough. More recently, Ucar developed a new failure criterion for rock mass and concrete where consider: the relationship σ_t/σ_c (Ucar, 2021).

On the other hand, to verify the accuracy of the transform of the resulting parametric representation of the failure envelope in the principal stress plane to the Mohr plane, we used the obtained values of θ_0' and θ'_1, which are the parameters of the parabolic envelope given for Berea sandstone (Pincus, 1993; 1994; 1996), in the compressive region. For this case, if we use the parabolic model to represent the failure envelope in the Mohr plane (Zambrano Mendoza *et al.*, 2003), obtained a lower standard deviation, but the resulting envelope could cross out the plane close to the origin of coordinates, distorting lightly the estimation of the reconciled uniaxial compressive strength $\left(\hat{\sigma}_c\right)$ and tensile strength $\left(\hat{\sigma}_t\right)$. From the transformation, the resulting equation that represents the failure envelope in the Mohr plane can further minimize the sum of squared distance obtained when fitting the envelope directly in the Mohr plane using the EIV method with the parabolic model. Comparing both cases, we notice that the standard deviation given by equation 25, reduced from 3.05 % when fitting the failure envelope directly in the Mohr plane using the parabolic model to 2.91 % when using the transformation procedure. Figure 2 shows the failure envelope fit obtained from transformation of the parabolic model in the principal stress plane to a derived equation in the Mohr plane.

Table 1. Comparative analysis between measured data with error-in-variables (EIV), Balmer and Hoek & Brown methods (Sheorey, 1997).

Set	**P***	**Measured**		**EIV**			**Balmer**				**Hoek & Brown**			
		σ_c	**σ_t**	**σ_c**	**σ_t**	**S. Dev.**	**σ_c**	**σ_t**	**b**	**S. Dev.**	**σ_c**	**σ_t**	**m**	**S. Dev.**
32	12	125.40	-10.90	123.60	-10.22	14.18	125.90	-8.85	0.452	**13.90**	129.90	-18.15	7.017	14.85
33	9	115.40	-11.66	113.40	-9.59	7.64	113.30	-10.53	0.523	6.95	112.90	-14.68	7.561	**4.94**
49	5	104.00	-6.01	108.50	-5.44	5.16	104.30	-4.85	0.494	4.61	109.00	-8.11	13.37	9.44
57	5	28.10	-0.72	12.90	-0.21	9.80	21.70	-0.71	0.521	31.03	21.70	-0.88	24.537	**3.89**
58	6	127.50	-8.04	138.70	-8.25	20.34	127.10	-4.83	0.444	**18.44**	152.40	-16.54	9.110	29.01
68	5	42.00	-3.00	42.40	-2.56	4.30	41.20	-2.96	0.577	**4.10**	41.50	-2.92	14.123	4.11
134	7	115.20	-9.52	93.50	-3.40	18.24	106.00	-7.19	0.591	9.98	110.70	-7.26	15.174	**9.20**
135	9	80.80	-7.00	90.70	-4.56	12.66	85.00	-4.07	0.504	**12.21**	98.80	-8.09	12.125	14.62
136	9	83.90	-7.60	84.50	-4.34	23.99	91.60	-4.39	0.462	**23.45**	103.60	-10.63	9.643	24.81
137	11	91.08	-10.50	101.20	-7.92	9.97	91.10	-5.71	0.483	**9.04**	104.20	-13.39	7.652	11.49
162	12	41.40	-2.64	27.40	-0.78	4.07	37.80	-2.09	0.550	2.29	44.20	-3.39	12.972	**1.65**
163	8	57.20	-2.67	54.20	-2.84	**4.61**	54.20	-2.62	0.486	4.62	61.00	-5.75	10.51	6.62
164	6	44.30	-2.89	44.50	-1.66	3.31	44.30	-2.28	0.556	**1.02**	48.20	-2.87	16.759	2.25
165	7	95.50	-6.31	98.50	-5.16	4.97	94.00	-4.74	0.505	**4.09**	99.50	-7.39	13.379	6.29
166	7	179.10	-11.06	165.80	-13.20	7.46	174.90	-10.00	0.377	**5.04**	162.10	-16.47	9.741	10.53
167	6	97.00	-5.39	100.30	-4.15	5.40	96.30	-4.02	0.510	**4.59**	102.10	-5.82	17.498	5.60
168	6	92.00	-6.67	108.10	-4.42	11.05	91.50	-2.99	0.496	**4.25**	110.30	-6.33	17.382	12.68
177	7	24.30	-3.29	30.10	-2.87	2.85	27.10	-1.80	0.458	**1.86**	28.90	-3.93	7.261	3.16
183	12	63.20	-3.03	47.50	-0.91	12.39	64.80	-3.01	0.603	**3.03**	54.80	-1.56	35.107	4.39
191	6	67.70	-4.79	67.10	-4.52	2.03	66.20	-4.72	0.527	**1.82**	64.80	-5.02	12.824	1.96

Set: set number, P*: number of points, σ_c (MPa): compressive strength, σ_t (MPa): tensile strength, S. Dev.: standard deviation, b': Balmers's method constant, m: Hoek & Brown's method constant.

Table 2. Comparative values of the principal stresses of sample set No 32 using error-in-variables (EIV), Balmer and Hoek & Brown methods (Sheorey, 1997).

Measured		**EIV**	**Balmer**	**Hoek & Brown**
$\hat{\sigma}_3$ **MPa**	$\hat{\sigma}_1$ **MPa**	$\hat{\sigma}_1$ **MPa**	$\hat{\sigma}_1$ **MPa**	$\hat{\sigma}_1$ **MPa**
-10.900	0.000	0.000	0.000	0.000
0.000	125.400	123.636	125.900	129.000
6.300	164.400	157.174	160.529	156.688
12.100	193.000	182.686	185.859	179.143
18.100	204.500	205.776	208.267	200.781
24.200	229.100	226.854	228.388	221.513
30.200	214.200	245.829	246.275	240.917
37.200	283.000	266.264	265.330	262.549
41.400	292.900	277.804	276.007	275.089
47.900	282.800	294.774	291.608	293.939
53.900	315.300	309.615	305.161	310.813
60.800	313.300	325.847	319.895	329.675

$\hat{\sigma}_3$ (MPa): reconciled minor principal stress, and $\hat{\sigma}_1$ (MPa): reconciled major principal stress. Also $\hat{\sigma}_t$: reconciled uniaxial tensil strength are -10.22, -8.85 and -18.15 MPa for EIV, Balmer and Hoek & Brown method respectively.

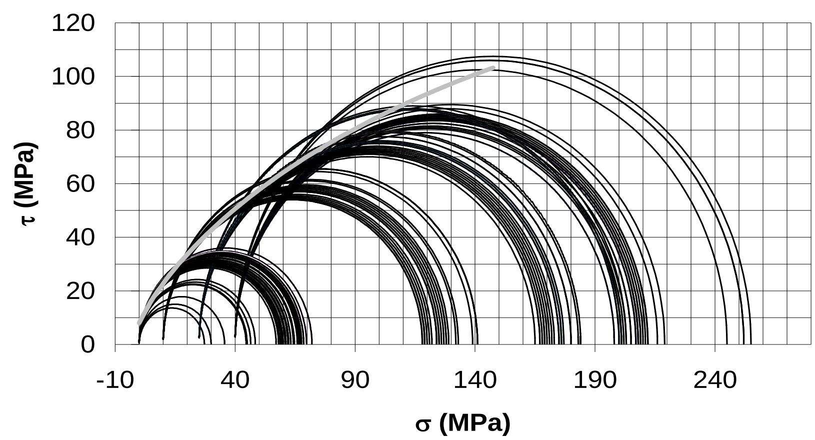

Figure 2. Failure envelope fitting through tansformation from the principal stress plane (parabolic model) to the Mohr plane (σ, τ) using error-in-variables (EIV) method.

Conclusions

The EIV method can provide a sound improvement for the parametric representation of the failure envelope in the principal stress plane. Compared with the methods of Balmer and Hoek and Brown, we obtained a very good curves fitting when considered the parametric equations of the parabolic type with the standard error values when comparing the methods. The resulting EIV nonlinear envelope equation can be easily transformed into the Mohr plane, providing a more appropriate approximation than those obtained when fitting the failure envelope directly in the Mohr plane using the parabolic model.

References

Anand, A., Kumar, S. (2015). *Application of multi-objective optimization techniques to geotechnical engineering problems*. M. Tech. Thesis. Rourkela: National Institute of Technology.

Balmer, G. (1952). A general analytical solution for Mohr's envelope. *Proceedings of American Society of Test Materials*, 52, 1260-1271.

Britt, H., Luecke, R. (1973). The estimation of parameters in nonlinear, implicit models. *Technometrics*, 15, 233.

Coulomb, C. (1776). Essai sur une application des regles des maximis et minimis a quelquels problemesde statique relatifs, a la architecture. *Memoires de L´Academie Royale des Sciences - Par Divers Savans*, 7, 343-387.

Deming, W. (1943). *Statistical adjustment of data*. New York: Willey.

Edgar, T., Liebman, M., Kim, I. (1990). Robust error-in-variables estimation using nonlinear programming techniques. *AIChE Journal*, 36(7), 985-993.

Esposito, W., Floudas, C. (1998). Parameter estimation in nonlinear algebraic models via global optimization. *Computers and Chemical Engineering*, 22, 213-220.

Hobbs, D. (1964). The strength and the stress-strain characteristics of coal in triaxial compression. *Journal of Geology*, 72, 214.

Hoek, E., Brown, T. (1980a). Empirical strength criterion for rock masses. *Journal of the Geotechnical Engineering Division - ASCE*, 106(GT9), 1013-1035.

Hoek, E., Brown, E. (1980b). *Underground excavation in rock*. London: CRC Press.

Jiefei, G., Puhui, C. (2018). A failure criterion for isotropic materials based on Mohr's failure plane theory. *Mechanics Research Communications*, 87, 1-6.

Jiefei, G., Ke, L., Lei, S. (2020). Modified nonlinear Mohr–Coulomb fracture criteria for isotropic materials and transversely isotropic UD composites. *Mechanics of Materials*, 151,103649.

Jiefei, G., Puhui, C., Ke, L., Lei. S. (2019). A macroscopic strength criterion for isotropic metals based on the concept of fracture plane. *Metal*, 9, 634-647.

Kumar, S., Kumar, P. (2011). Parameter optimization of rock failure criterion using error-in-variables approach. *International Journal of Geomechanics*, 11(1), 36-43.

Liebman, M., Edgar, T. (1988). *Data reconciliation for nonlinear processes*. Proceedings of the AIChE Annual Meeting. Washington, DC: American Institute of Chemical Engineers (AIChE), 137.

Mostyn, G., Douglas, K. (2002). *Strength of intact rock and rock masses* [on line] available from: http://geotle.t.u-tokyo.ac.jp/towhata/lecture/rock/mostyn.pdf [accessed: 1 August 2002].

Nelder, J., Mead, R. (1965). A simple method for function minimization. *Computer Journal*, 7, 308-313.

O'Neil, M., Sinclair, I., Smith, F. (1969). Polynomial curve fitting when abscissas and ordinates are both subject to error. *Computer Journal*, 12, 52-56.

Peneloux, A., Deyrieux, E., Neau, E. (1976). The maximum likelihood test and the estimation of experimental inaccuracies: Application to data reduction for vapor-liquid equilibrium. *Journal of Computers*, 73, 706-716.

Pincus, H. (1993). Interlaboratory testing program for rock properties, round one-longitudinal and transverse pulse velocities, unconfined compressive strength, uniaxial elastic modulus, and splitting tensile strength. *Geotechnical Testing Journal*, 16(1), 138-163.

Pincus, H. (1994). Addendum to interlaboratory testing program for rock properties, round one. *Geotechnical Testing Journal*, 17(2), 256-258.

Pincus, H. (1996). Interlaboratory testing program for rock properties, round two-confined compression: Young's modulus, Poisson's ratio, and ultimate strength. *Geotechnical Testing Journal*, 19(3), 321-336.

Reilly, P., Patino-Leal, H. (1981). Bayesian study of the error-in-variables model. *Technometrics*, 23(3), 221.

Schwetlick, H., Tiller, V. (1985). Numerical methods for estimating parameters in nonlinear models with error in the variables. *Technometrics*, 27(1), 17-24.

Sheorey, P. (1997). *Empirical rock failure criteria*. Rotterdam: A. A. Balkema.

Southwell, W. (1969). Fitting experimental data. *Journal of Computational Physics*, 4, 465-474.

Ucar, R. (2019). *La resistencia al corte en macizos rocosos y en el hormigón. Una metodología reciente de cálculo*. Madrid: Bellisco Ediciones.

Ucar, R. (2021). Determination of a new failure criterion for rock mass and concrete. *Geotechnical and Geological Engineering*, 39, 3795-3813.

Valkó, P., Vajda, S. (1987). An extended Marquardt-type procedure for fitting error-in-variables models. *Computational Chemical Engineering*, 11(1), 37-43.

van Huffel. S., Lemmerling. P. (2013). *Total least squares and errors-in-variables modeling: analysis, algorithms and applications*. Berlin: Springer Science & Business Media.

Willianson, J. H. (1968). Least squares fitting of a straight line. *Canadian Journal of Physics*, 46, 1845-1847.

York, P. (1966). Least squares fitting of a straight line. *Canadian Journal of Physics*, 44, 1079.

Zambrano Mendoza, O., Valkó, P., Russell, J. (2003). Error-in-variables for rock failure envelope. *International Journal of Rock Mechanics and Mining Sciences*, 40(1), 137-143.

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